Self-Consistent Linearly Rising ρ Regge Trajectory

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A bootstrap calculation of a linearly rising ρ Regge trajectory is performed using the Balázs method of decoupling the N/D equations, with the left-hand cut approximated by 7 poles. The input ρ mass and width are taken as the experimental values of 750 MeV and 100 MeV, respectively. The value of the bootstrapped slope is 0.701 GeV^{-2} corresponding to an input intercept of $[\alpha_\rho(0)]_{\mathrm{in}} = 0.605$, an output intercept of $[\alpha_\rho(0)]_{\mathrm{out}} = 0.608$, an output ρ mass of $(m_\rho)_{\mathrm{out}} = 747$ MeV, and an output ρ width of $(\Delta m_\rho)_{\mathrm{out}} = 71$ MeV. The sensitivity of the results to the details of the calculation is studied extensively. The details of the Balázs method are given as well as some techniques of handling the method in an approximation with a large number of poles.

I. INTRODUCTION

Previous work[1-3] on the bootstrap of the ρ and f^0 mesons by the Balázs method[4] has shown that the N/D method, in addition to bootstrapping masses, can also produce narrow widths in reasonable agreement with experiment. The above results depended on the use of the Williamson-Everett (WE) criterion[5] for choosing the optimum matching point, and the approximation of the left-hand cut by a large number of poles. The above-mentioned results seemed to indicate that in a 7-pole approximation, the Balázs method, in conjunction with the WE criterion, might be sophisticated enough to handle not only bootstraps of resonances but also of whole Regge trajectories. In this paper we report on one such successful bootstrap calculation of a linearly rising ρ Regge trajectory for small values of angular momentum. Several other bootstrap calculations of the ρ trajectory have been performed by various methods.[6] The present calculation seems to cure some of the difficulties encountered by these calculations, such as a large or indeterminate ρ width, a nonlinear trajectory, and an uncertain trajectory intercept.

The general calculational methods are similar to those of Refs. 1 and 2. In the principal portion of this work, the part of the input force due to the low-energy absorptive part of the crossed-channel amplitude is taken as being due entirely to the contribution of the ρ meson; i.e., we take the long-range interaction between two pions as being dominated by ρ exchange. The effect of including f^0 exchange is partially investigated. The crossed-channel absorptive part at high energy, i.e., the short-range force, is taken as being given by the contribution of the direct-channel ρ trajectory, which we take to be real and linear. The chief difference here from our previous calculations is that we are interested in determining the output ρ Regge trajectory, rather than just the output ρ mass and width, so that we can require that the input and output trajectories be self-consistent. A point on the output trajectory $\alpha_\rho(s)$ is given by the pair of values (l, s_l), where $\mathrm{Re}\, D_l^1(s_l) = 0$ when the optimum value of the matching point is chosen; $D_l^1(s)$ is the denominator function for the $I=1$, $l=1$ partial wave. Although one could, in principle, carry out the calculation for arbitrary l, we confine ourselves to integral or half-integral values, for which we can evaluate the integrals for the denominator function, given a suitable approximation for the inelasticity parameter $R_l^1(s)$, in closed form without numerical integrations.

Since the approximations involved in a one-channel calculation get worse with increasing energy, the reliability of the present calculation is best at low energies and consequently low angular momentum. As such, only a bootstrap of the ρ trajectory in the range $0 \leq l \leq 2.5$ is attempted.

The Balázs method and the procedure followed in bootstrapping the ρ trajectory are discussed in Secs. II and III, respectively, and the numerical results of the bootstrap are given in Sec. IV.

II. THE BALÁZS METHOD

This section is based on Refs. 1–4, and combined with Appendixes A–C gives a description of the Balázs method as applied to the π-π scattering problem, with a discussion of the approximations employed. We will start by defining the problem and the different kinematical variables.

Four pions with masses m_π converge to the same space-time point and annihilate.[7] p_i is the four-momentum of the ith particle and is timelike, and

I_z^i is the third component of the isotopic spin of the ith particle. The c.m. energy squared in the s, t, and u channels is given, respectively, by

$$\begin{aligned} s &= (p_1+p_2)^2 , \\ t &= (p_1+p_3)^3 , \\ u &= (p_1+p_4)^2 . \end{aligned} \tag{1}$$

We have

$$\begin{aligned} |E_i| &= \tfrac{1}{2}\sqrt{s} = (k^2+m_\pi^2)^{1/2} \\ &= (\nu+1)^{1/2} , \end{aligned} \tag{2}$$

where $k=|p_i|$ is the magnitude of the c.m. momentum in the s channel, $\vec{p}_i$ and E_i are, respectively, the c.m. momentum and energy of the ith particle in the s channel, and $\nu=k^2$. The s-channel scattering angle is defined by

$$\vec{p}_1\cdot\vec{p}_3 = -\nu\cos\theta . \tag{3}$$

The partial-wave amplitude for s-channel scattering with total angular momentum l and total isospin I is denoted by $A_l^I(\nu)$ and is normalized to

$$\mathrm{Im}[A_l^I(\nu)]^{-1} = -\left(\frac{\nu}{\nu+1}\right)^{1/2} R_l^I(\nu) , \tag{4}$$

where $R_l^I(\nu)$ is the ratio of total to elastic partial-wave cross section.

A. The N/D Equations

The threshold and asymptotic behavior of the partial-wave amplitude can be factored out of the N/D equations according to

$$A_l^I(\nu) = \nu^l(\nu-\nu_K)^{1-l}H_l^I(\nu) . \tag{5}$$

As usual, one writes

$$H_l^I(\nu) = N_l^I(\nu)/D_l^I(\nu) , \tag{6}$$

with N_l^I and D_l^I having, respectively, only left- and right-hand cuts. $|\nu_K|$ must be large enough not to destroy the threshold behavior $\sim\nu^l$ of the scattering amplitude, and small enough to allow the asymptotic behavior $\sim\nu$ to set in at a point which is consistent with the assumed starting point of asymptotic behavior in the rest of the equations. We will consistently assume throughout this paper that asymptotic behavior sets in at a fixed value of ν which we designate by ν_{DR}. Thus it seems reasonable to set $\nu_K = -\nu_{DR}$.

The starting point of Balázs is to separate the dispersion relation for $N_l^I(\nu)$ into two parts:

$$\begin{aligned} N_l^I(\nu) = \frac{1}{\pi}\int_{\nu_L}^{-1} d\nu' \, \frac{D_l^I(\nu')\,\mathrm{Im}H_l^I(\nu')}{\nu'-\nu} \\ -\frac{1}{\pi}\int_0^{x_L}\frac{dx}{x}\,\frac{D_l^I(1/x)\,\mathrm{Im}H_l^I(-1/x)}{1+x\nu} , \end{aligned} \tag{7}$$

where $x=-1/\nu'$ and $x_L=-1/\nu_L$. If we now assume that the first resonance in the t channel is at $\nu_t=\nu_\rho$, where the mass of the ρ meson is $m_\rho=2(\nu_\rho+1)^{1/2}$, then in the t channel the right-hand cut runs from ν_ρ to ∞ if nonresonance contributions to the partial-wave amplitude are neglected. If we now choose $\nu_L=-\nu_\rho-1$, the first integral in Eq. (7) vanishes, and $N_l^I(\nu)$ is given by

$$N_l^I(\nu) = -\frac{1}{\pi}\int_0^{(\nu_\rho+1)^{-1}}\frac{dx}{x}\,\frac{D_l^I(-1/x)\,\mathrm{Im}H_l^I(-1/x)}{1+x\nu} . \tag{8}$$

The effect of neglecting the first integral in Eq. (7) has been discussed by Dilley and Gibbons.[8,9]

A central step in the Balázs method is approximating the kernel in Eq. (8) by an interpolation formula:

$$\frac{1}{1+x\nu} = \sum_{i=1}^{n}\frac{G_i^n(x)}{1+x_i\nu} , \tag{9a}$$

where

$$G_i^n(x) = \prod_{\substack{j=1\\ j\neq i}}^{n}\frac{(x-x_j)}{(x_i-x_j)} . \tag{9b}$$

The values of the x_i are chosen so as to make the approximation of Eq. (9a) as good as possible in the range of integration of Eq. (8). A rather simple method of evaluating the x_i's is given in Appendix A. *A priori* the larger the value of n, the better is the kernel approximation.

Substituting Eq. (9a) into Eq. (8), we find

$$N_l^I(\nu) = \sum_{i=1}^{n}\frac{f_{lI}^i}{(\nu+w_i)} , \tag{10}$$

where $w_i=1/x_i$. Thus the numerator function is approximated by n poles whose positions are determined by Eq. (9). The residues are determined by requiring that the partial-wave amplitude and $n-1$ of its derivatives, as given by the N/D equations, be in agreement at some matching point ν_F chosen in between the right- and left-hand cuts, with the expression for the partial wave amplitude $A_l^I(\nu)$, and $n-1$ of its derivatives, as obtained by projection from a fixed-energy dispersion relation for the scattering amplitude.

Using Eq. (10), one finds the denominator function is given by

$$D_l^I(\nu) = 1 - \frac{\nu-\nu_0}{\pi}\sum_{i=1}^{n} f_{lI}^i \int_0^\infty d\nu'\left(\frac{\nu'}{\nu'+1}\right)^{1/2}\frac{\nu'^l R_l^I(\nu')}{(\nu'-\nu_0)(\nu'-\nu)(\nu'+w_i)(\nu'-\nu_K)^{l-1}} . \tag{11}$$

The function $R_l^I(\nu)$ is defined by Eq. (4) and is explicitly evaluated in Appendix A of Ref. 2, for values of $\nu > \nu_{DR}$. To simplify Eq. (11) we introduce a parameter ν_D defined as the point at which inelastic effects become appreciable.[10] We take ν_D in the range $20 \leq \nu_D \leq 30$. We then approximate $R_l^I(\nu)$ by

$$R_l^I(\nu) = \begin{cases} 1 & \text{for } \nu < \nu_D \\ R_l^I & \text{for } \nu \geq \nu_D \end{cases}, \tag{12}$$

where R_l^I is a constant estimated as a weighted average over ν from a graph of $R_l^I(\nu)$ such as Fig. 1 of Ref. 2.

Making use of Eq. (12) the expression for $D_l^I(\nu)$ becomes

$$D_l^I(\nu) = 1 + \sum_{i=1}^{n} C_{lI}^i(\nu) f_{lI}^i \ , \tag{13}$$

where

$$C_{lI}^i(\nu) = -\frac{\nu - \nu_0}{\pi} \left[I_l^i(\nu, 0) + (R_l^I - 1) I_l^i(\nu, \nu_D) \right] \ , \tag{14}$$

with

$$I_l^i(\nu, y) = \int_y^\infty d\nu' \left(\frac{\nu'}{\nu' + 1} \right)^{1/2} \frac{\nu'^l}{(\nu' - \nu_0)(\nu' - \nu)(\nu' + w_i)(\nu' - \nu_K)^{l-1}} \ . \tag{15}$$

The integral of Eq. (15) is evaluated in Appendix B. A closed-form solution is obtainable for non-negative integer and half-integer values of l, while for other positive values of l a numerical integration is necessary.

B. The Input Amplitude $A_l^I(\nu)$

The Froissart-Gribov[11] interpolation formula for the partial-wave amplitude $A_l^I(\nu)$ for physical and non-physical values of l is given by

$$A_l^I(\nu) = \frac{1}{\pi\nu} \int_4^\infty dt A_t^I(s,t) Q_l\left(1 + \frac{t}{2\nu}\right) , \tag{16}$$

where $A_t^I(s,t)$ is the s-channel absorptive part of the definite isospin scattering amplitude. Through the interval $-9 < \nu < 0$, the absorptive and imaginary parts of the amplitude are equal, and hence $A_t^I(s,t)$ can be replaced by $\mathrm{Im}A^I(s,t)$ in Eq. (16). The range of integration in Eq. (16) is split into two parts at t_{DR}. For $t \geq t_{DR}$ we assume that the scattering amplitude can be represented by the top-lying Regge poles. t_{DR} is related to ν_{DR} by

$$t_{DR} = 4(\nu_{DR} + 1) \approx 4\nu_{DR} \approx 4\nu_D \ . \tag{17}$$

The last equality in Eq. (17) means that we approximate the amplitude by its Regge form above approximately the energy of the f meson. We can now write Eq. (16) as

$$A_l^I(\nu) = A_l^{I(L)}(\nu) + A_l^{I(H)}(\nu) \ , \tag{18a}$$

where

$$A_l^{I(L)}(\nu) = \frac{1}{\pi\nu} \int_4^{t_{DR}} dt \, \mathrm{Im}A^I(s,t) Q_l\left(1 + \frac{t}{2\nu}\right) \tag{18b}$$

and

$$A_l^{I(H)}(\nu) = \frac{1}{\pi\nu} \int_{t_{DR}}^\infty dt \, \mathrm{Im}A^I(s,t) Q_l\left(1 + \frac{t}{2\nu}\right) \ . \tag{18c}$$

To evaluate $A_l^{I(L)}(\nu)$ we expand $\mathrm{Im}A^I(t,s)$ in t-channel partial waves. We obtain

$$A^{I(L)}(\nu) = \frac{1}{\pi\nu} \sum_{I'=0}^{2} \beta_{II'} \sum_{l'=0}^{\infty} \tfrac{1}{2}\left[1 + (-1)^{I'+l'}\right](2l'+1) \int_4^{t_{DR}} dt \, \mathrm{Im}A_{l'}^{I'}(\tfrac{1}{4}t - 1) P_{l'}\left(1 + \frac{2s}{t-4}\right) Q_l\left(1 + \frac{2t}{s-4}\right) \ , \tag{19}$$

where $\beta_{II'}$ is the $\pi\pi$ crossing matrix. Equation (19) involves no approximations and is true for $-9 < \nu < 0$. To evaluate it we need an explicit expression for $\mathrm{Im}A_{l'}^{I'}(\frac{1}{4}t - 1)$. To obtain this we assume that for $t < t_D$, only partial waves corresponding to physical resonances give appreciable contributions to the amplitude;

these we treat in the zero-width approximation.

Let the set of values $(\nu_i, \Gamma_i, l_i, I_i)$ describe, respectively, c.m. momentum squared, reduced half-width, angular momentum, and isospin of a low-energy t-channel resonance whose contribution to the amplitude is to be included. Making the change of variable $t \to \nu' = (\frac{1}{4}t - 1)$ in Eq. (19) we obtain, summing over resonances,

$$A_l^{I(L)}(\nu) = \frac{4}{\pi\nu} \sum_i \beta_{II'}(2l_i+1) \int_0^{\nu_{DR}} d\nu' \, \mathrm{Im}A_i(\nu') P_{l_i}\left(1 + \frac{2(\nu+1)}{\nu'}\right) Q_l\left(1 + \frac{2(\nu'+1)}{\nu}\right) . \tag{20}$$

To evaluate the integral in Eq. (20) we use the narrow-width approximation for $\mathrm{Im}A_i(\nu)$. Thus $\mathrm{Im}A_i(\nu)$ is zero except for ν near ν_i. Remembering that

$$\mathrm{Re}D_i(\nu_i) = 0 , \tag{21}$$

one has for $\nu > 0$ and near ν_i

$$A_i(\nu) = \nu^{l_i}(\nu - \nu_K)^{1-l_i} \left\{ \frac{-\Gamma_i}{(\nu-\nu_i) + i[\nu/(\nu+1)]^{1/2} \nu^{l_i}(\nu-\nu_K)^{1-l_i} R_i(\nu)\Gamma_i} \right\} , \tag{22}$$

where the reduced half-width,

$$\Gamma_i = -N_i(\nu_i) \Big/ \frac{\partial \mathrm{Re}D_i(\nu)}{\partial \nu}\bigg|_{\nu=\nu_i} , \tag{23}$$

is the residue of $H_l^I(\nu)$ at $\nu = \nu_i$. Making the zero-width approximation and performing the resulting integration over δ functions, we find

$$A_l^{I(L)}(\nu) = \frac{4}{\nu} \sum_i \beta_{II_i}(2l_i+1) \Gamma_i \nu_i^{l_i} (\nu_i - \nu_K)^{1-l_i} P_{l_i}\left(1 + \frac{2(\nu+1)}{\nu_i}\right) Q_l\left(1 + \frac{2(\nu_i+1)}{\nu}\right) . \tag{24}$$

From Eq. (22) we find that the half-width $(\Delta\nu_i)$ in ν is related to the reduced half-width Γ_i by

$$\Delta\nu_i = [\nu_i/(\nu_i+1)]^{1/2} \nu_i^{l_i} (\nu_i - \nu_K)^{1-l_i} \Gamma_i , \tag{25}$$

where we have set $R_i(\nu_i) = 1$, and hence the full width in energy of the resonance is given by

$$\Delta m_i = 2\nu_i^{l_i+1/2} (\nu_i - \nu_K)^{1-l_i} \Gamma_i/(\nu_i+1) . \tag{26}$$

To evaluate $A_l^{I(H)}(\nu)$, $A^I(s,t)$ is approximated by the highest-lying s-channel trajectories with isospin I. We obtain

$$A^I(s,t) = -\tfrac{1}{2}\pi \sum_j \frac{[2\alpha_j(s)+1]\, \beta_j(s)\, C_1(\alpha_j(s))\, [e^{-i\pi\alpha_j(s)} + (-1)^I] e^{-\alpha_j(s)\ln 2\nu}}{\sin \pi\alpha_j(s)} \, t^{\alpha_j(s)} , \tag{27}$$

where $\alpha_j(s)$ and $\beta_j(s)$ are trajectory and corresponding residue functions, $C_1(\alpha) = 2^\alpha \Gamma(\alpha + \frac{1}{2})/\sqrt{\pi}\,\Gamma(\alpha+1)$, and we have set $1 + t/2\nu \sim t/2\nu$. Substituting Eq. (27) in Eq. (18c) and using the asymptotic form of Q_l we find

$$A_l^{I(H)}(\nu) = \sum_j \frac{[2\alpha_j(s)+1]\beta_j(s)}{(2\nu)^{\alpha_j(s)-l}} C_1(\alpha_j(s))\, C_2(l) \times \int_{t_{DR}}^{\infty} dt' (t')^{\alpha_j(s)-l-1} , \tag{28}$$

where $C_2(l) = \sqrt{\pi}\,\Gamma(l+1)/[2^{l+1}\Gamma(l+\frac{3}{2})]$. For $l > \alpha_j(s)$ the integral in Eq. (28) exists, and we find

$$A_l^{I(H)}(\nu) = -\nu^l \sum_j [2\alpha_j(s)+1] \frac{\beta_j(s)}{\nu^{\alpha_j(s)}} \times \frac{C_1(\alpha_j(s))\, C_2(l)}{\alpha_j(s) - l} (2\nu_{DR})^{\alpha_j(s)-l} . \tag{29}$$

Equation (29) can now be analytically continued to $l \leq \alpha_j(s)$.

We assume that all the trajectories are linear and real:

$$\alpha_j(s) = l_j + \epsilon_j(\nu - \nu_j) \tag{30a}$$

or

$$\alpha_j(s) = \alpha_j(0) + \alpha_j' s , \tag{30b}$$

where

$$\alpha_j' = \tfrac{1}{4}\epsilon_j . \tag{30c}$$

Furthermore we parametrize the residue function according to[12]

$$\beta_j(s) = \mathrm{const} \times \nu^{\alpha_j(s)} e^{c_j s} . \tag{31a}$$

This can be written as

$$\beta_j(s) = [\beta_j(s_j)/\nu_i^{\alpha_j(s_j)}]\, \nu^{\alpha_j(s)} e^{c_j(s-s_j)} , \tag{31b}$$

where c_j is a parameter to be determined experi-

mentally. The residue function is related to the trajectory slope and the reduced half-width $\Gamma_l^{I_j}$ of a resonance on the trajectory, having mass $\sqrt{s_j}$ and spin $l_j = \alpha_j(s_j)$, by

$$\beta_j(s_j) = \nu_j^{\alpha_j(s_j)} (\nu_j - \nu_K)^{1-\alpha_j(s_j)} \Gamma^{I_i}_{\alpha_j(s_j)} [d\alpha_j(s)/d\nu]_{s=s_j} . \tag{32}$$

Finally we make the approximation

$$[2\alpha_j(s)+1]\, C_1(\alpha_j(s)) \approx \text{const} , \tag{33a}$$

and evaluate the constant at $s = s_j$ with the help of Eq. (30) to obtain

$$[2\alpha_j(s)+1]\, C_1(\alpha_j(s)) \approx (2l_i+1)\, C_1(l_i) . \tag{33b}$$

Substituting these equations into Eq. (29), we obtain the expression

$$A_l^{I(H)}(\nu) = \nu^l \sum_j K_j(l) \frac{\beta_j(s_j)\, e^{4c_j(\nu-\nu_j)} (2\nu_{DR})^{l_j - l + \epsilon_j(\nu-\nu_j)}}{\nu_j^{l_j} [l_j - l + \epsilon_j(\nu-\nu_j)]} . \tag{33c}$$

C. Matching the Amplitudes

To complete the solution of the N/D equations, all that is left is to determine the n residues f^i_{lI}. This is done by matching the function $\nu^l(\nu-\nu_K)^{1-l} \times N_l^I(\nu)/D_l^I(\nu)$ and $n-1$ of its derivatives to $A_l^I(\nu)$ and $n-1$ of its derivatives at some matching point ν_F in the region $-9 < \nu < 0$.

As is clear from the above discussion, the input information is stored in the residues $\{f^i_{lI}, i = 1, \ldots, n\}$, and thus the larger n is, the greater is the part of the input information which is retained by the N/D amplitude. This is also clear from the fact that the larger is n, the more the number of derivatives matched and the closer is the functional form of the N/D equations to that of $A_l^I(\nu)$ in their common range of validity.

In between the cuts where the matching is performed, $N_l^I(\nu)$, $D_l^I(\nu)$, and $(\nu-\nu_K)^{1-l}$ are all real, but ν^l is complex for noninteger values of l. Hence it is convenient to factor out ν^l from the amplitude before matching. To this end we define the function $B_l^I(\nu)$ by

$$B_l^I(\nu) = \nu^{-l} A_l^I(\nu) . \tag{34}$$

Then the matching equation for the amplitude can be written as

$$B_l^I(\nu_F) D_l^I(\nu_F) = (\nu_F - \nu_K)^{1-l} N_l^I(\nu_F) , \tag{35}$$

and making use of Eqs. (10) and (13), we obtain

$$\sum_{i=1}^{n} S^i_{lI}(\nu_F) f^i_{lI} = B_l^I(\nu_F) , \tag{36}$$

where

$$S^i_{lI}(\nu) = \frac{(\nu-\nu_K)^{1-l}}{\nu+w_i} - B_l^I(\nu)\, C^i_{lI}(\nu) . \tag{37}$$

III. PROCEDURE AND CHOICE OF PARAMETERS

In performing a bootstrap of the ρ trajectory we have considered the effect of retaining the contributions of the ρ, as well as the ρ and f^0, terms in $A_l^{1(L)}(\nu)$. These terms are given by

$$[A_l^{1(L)}(\nu)]_\rho = 12\beta_{11}\Gamma_\rho \left(\frac{\nu_\rho + 2(\nu+1)}{\nu}\right) Q_l\left(1 + \frac{2(\nu_\rho+1)}{\nu}\right) \tag{38a}$$

and

$$[A_l^{1(L)}(\nu)]_f = \frac{20\beta_{10}\Gamma_f \nu_f^2}{(\nu_f - \nu_K)} \frac{1}{\nu} P_2\left(1 + \frac{2(\nu+1)}{\nu_f}\right) \times Q_l\left(1 + \frac{2(\nu_f+1)}{\nu}\right) . \tag{38b}$$

In $A_l^{1(H)}(\nu)$ we only retain the contribution of the highest-lying trajectory, which for isospin $I=1$ is the ρ trajectory.

$$A_l^{1(H)}(\nu) = K_\rho(l)\, \beta_\rho(s_\rho) \frac{e^{4c_\rho(\nu-\nu_\rho)} (2\nu_{DR})^{1-l+\epsilon_\rho(\nu-\nu_\rho)}}{\nu_\rho[1-l+\epsilon_\rho(\nu-\nu_\rho)]} \nu^l , \tag{39}$$

where

$$K_\rho(l) = -\frac{3\sqrt{\pi}}{2^{l+2}} \frac{\Gamma(l+1)}{\Gamma(l+\frac{3}{2})} , \tag{40}$$

and from Eqs. (30) and (32),

$$\beta_\rho(s_\rho) = \nu_\rho \Gamma_\rho \epsilon_\rho . \tag{41}$$

Furthermore we use the intercept $\alpha_\rho(0)$ of the ρ trajectory as input rather than the slope. Using Eq. (30b), and remembering that $\alpha_\rho(s_\rho) = 1$, we have

$$\epsilon_\rho = [1 - \alpha_\rho(0)]/(\nu_\rho + 1) . \tag{42}$$

We further set

$$c_\rho = 0 , \tag{43}$$

which, according to Eqs. (31b) and (32), is equivalent to the following parametrization of the ρ residue function:

$$\beta_\rho(s) = \Gamma_\rho \epsilon_\rho \nu^{\alpha_\rho(s)} . \tag{44}$$

Substituting the above results in Eq. (39) we obtain

$$A_l^{1(H)}(\nu) = -\frac{3\sqrt{\pi}\,\Gamma(l+1)}{2^{l+1}\Gamma(l+\frac{3}{2})} \frac{\Gamma_\rho(1-\alpha_\rho(0))\,(2\nu_D)^{1-l+[1-\alpha_\rho(0)](\nu-\nu_\rho)/(1+\nu_\rho)}}{(\nu_\rho+1)(1-l)+[1-\alpha_\rho(0)](\nu-\nu_\rho)} \nu^l . \tag{45}$$

The pole positions $\{w_i; i=1, \ldots, n\}$ as a function of n are given by Table I. The method by which this table was obtained is discussed in Appendix A. The central set of parameters about which variations are considered is $n=7$, $R_l^I=3$, $\nu_0=-2$, $\nu_K=-25$, $\nu_D=\nu_{DR}=25$, $\alpha_\rho(0)=0.58$ and the experimental values of the mass and width of the ρ and f^0 mesons as given by Table II. Bond[13] has shown that the positions and widths of the output resonances are independent of the subtraction point ν_0. Consequently we will not vary the value of ν_0.

Due to the form of the integral in Eq. (15), an analytic expression for the denominator function can only be obtained for non-negative integer and half-integer values of l. To avoid numerical integration we will only calculate the points on the trajectory corresponding to $l=0$, 0.5, 1, 1.5, 2.0, and 2.5.

The roots of the real part of the denominator function $\mathrm{Re}D_l^I(\nu)$ are denoted by ν_R. For a given value of l, the value of ν_R corresponding to the optimum value $(\nu_F)_{\mathrm{opt}}$ of the matching point ν_F will be denoted ν_l. The pair of points $(l,\nu_l)\equiv(\alpha_\rho(s_l), \nu_l)$ determine a point on the Regge trajectory.

The WE criterion essentially requires choosing for the matching point ν_F the value that gives best agreement between the input and output forms of the partial-wave amplitude in their common range of validity. Since the first $n-1$ derivatives of the two forms of the partial wave amplitude are already made equal at ν_F by the Balázs method, the WE criterion is equivalent to requiring that the nth derivatives be also equal at the optimum value of ν_F, or, in case this is not possible in the allowed range of ν_F, that at least their difference be minimized. We can thus express the WE criterion as the requirement of minimizing the percentage difference $P_l^I(n,\nu_F)$ between the nth derivatives, evaluated at ν_F, of the two forms of the partial-wave amplitude, where[2]

$$P_l^I(n,\nu_F)=\left|\frac{[A_l^I(\nu)]_{\nu=\nu_F}^{(n)}-\left[\frac{\nu^l}{(\nu-\nu_K)^{l-1}}\frac{N_l^I(\nu)}{D_l^I(\nu)}\right]_{\nu=\nu_F}^{(n)}}{[A_l^I(\nu)]_{\nu=\nu_F}^{(n)}}\right|\times 100\,. \tag{46}$$

We have discovered the following alternate procedure for choosing the optimum matching point, which in practice turns out always to be equivalent to the WE criterion, and is much more convenient to use. As the matching point varies, of course the roots ν_R at which $\mathrm{Re}D_l^I(\nu)=0$ vary also, as well as the width of the produced resonance. It turns out that the optimum value of ν_F, as given by the WE criterion, is that which minimizes $d\nu_R/d\nu_F$, i.e., minimizes the sensitivity of the position of the output resonance to ν_F. (Minimizing the sensitivity of the width to ν_F yields, in practice, essentially equivalent results.) We refer to this procedure for choosing ν_F as the minimum slope criterion. We have not succeeded in giving a convincing proof of the equivalence of the two procedures for choosing the optimum ν_F, although we illustrate this equivalence in Figs. 2 and 3, discussed below. The minimum slope criterion is considerably simpler to use since the optimum matching point is determined simultaneously with ν_R without the need to evaluate the nth derivatives which appear in $P_l^I(n,\nu_F)$. Furthermore, the above form of the WE criterion shows that, in general, when there is a need to make a choice among the different values of ν_F a choice can be made. The minimum slope criterion fails in two special cases: when $d\nu_R/d\nu_F$ is a constant different from zero and also when the curve $\nu_R(\nu_F)$ has more than one stationary point. In both of the above cases recourse needs to be had to the normal form of the WE criterion as expressed by Eq. (46).

TABLE I. Variation of the pole positions $w_i(n, \nu, \nu_0)$ $(i=1, \ldots, n)$ with n for $\nu_\rho=5.5$ and $\nu=-2$. w_i is in units of $(m_\pi)^2$, with $\hbar=c=m_\pi=1$.

n	w_1	w_2	w_3	w_4	w_5	w_6	w_7
2	8	35	...	...	...	...	...
3	7.3	13	60	...	...	...	...
4	6.8	9.5	19	105	...	...	...
5	6.6	8.4	13	28	170	...	...
6	6.6	7.7	10	17	39	260	...
7	6.6	7.3	9.1	13	22	52	370

IV. RESULTS

A reasonably self-consistent ρ Regge trajectory, shown in Fig. 1, is obtained by using the central set of parameters given in Sec. III, with only the ρ term as given by Eq. (38a) retained in $A_l^{I(L)}(\nu)$. For the ρ mass and width we use the experimental values as given in Table II. These correspond to $(\nu_\rho)_{\mathrm{in}}=6.18$ and $(\Gamma_\rho)_{\mathrm{in}}=0.167$, and lead, in the 7-pole approximation, to output values of $(\nu_\rho)_{\mathrm{out}}=6.02$, and $(\Gamma_\rho)_{\mathrm{out}}=0.111$, where $(\Gamma_\rho)_{\mathrm{out}}$ is given by Eq. (36). The input intercept is $[\alpha_\rho(0)]_{\mathrm{in}}=0.58$; the input slope $(\epsilon_\rho)_{\mathrm{in}}=0.058$.[12] The output values are $[\alpha_\rho(0)]_{\mathrm{out}}=0.63$, and $(\epsilon_\rho)_{\mathrm{out}}=0.061$. The difference between input and output is about 5% for the slopes and about 8% for the intercepts. In

TABLE II. Values used for the input, or experimental ρ and f^0 masses and widths. ν_ρ and ν_f are in units of $\hbar = m_{\pi^\pm} = c = 1$, Γ_ρ and Γ_f are dimensionless, m_ρ and m_f are the masses in MeV, and Δm_ρ and Δm_f are the full widths in MeV. Γ_f is computed with $\nu_K = -25$ in Eq. (25).

ν_ρ $[(m_\pi)^2]$	m_ρ (MeV)	Γ_ρ (dimensionless)	Δm_ρ (MeV)	ν_f $[(m_\pi)^2]$	m_f (MeV)	Γ_f (dimensionless)	Δm_f (MeV)
618	750	0.167	100	10.02	1253	0.279	140

evaluating $(\epsilon_\rho)_{\text{out}}$, heaviest weight is given for the section of the trajectory in between $l = 0.5$ and $l = 2.0$. Although for all practical purposes this constitutes a bootstrap of the ρ trajectory, we will still attempt an exact bootstrap of the slope. The results of this bootstrap are shown in Fig. 8 and will be discussed later on.

In doing a bootstrap of the ρ meson, closer agreement than that obtained above would normally be required between the input and output values of the mass and width before the ρ is considered as bootstrapped.[1-3] On the other hand, the above discrepancy between input and output values of the ρ mass and width is negligible when a trajectory, rather than a single resonance, is to be bootstrapped, as is clear from Fig. 1. Furthermore, as will be shown later on, small variations in the input ρ mass and width have a modest effect on the parameters of the output trajectory. Hence we can safely use the experimental values of ν_ρ and Γ_ρ throughout and consider them as bootstrapped values.[14]

The zeros ν_R of $D_l^1(\nu)$ as a function of l and ν_F are given in Fig. 2. The zeros $\nu_l = (\nu_R)_{\text{opt}}$ correspond to the optimum value of ν_F. The values of ν_l of Fig. 1 as well as the corresponding values of $(\nu_F)_{\text{opt}}$ are given in columns 8 and 9 of Table III. As seen from Fig. 3, the WE criterion leads to the same choice of ν_l as does the minimum slope criterion.

A. The Slope and Intercept

The variation of the ρ trajectory with the value of the intercept $\alpha_\rho(0)$ is shown in Fig. 4. Experimentally the intercept of the ρ trajectory lies in the range[15] $0.46 \leq \alpha_\rho(0) \leq 0.58$, which corresponds to $0.0752 \geq \epsilon_\rho \geq 0.0585$. For this range of values

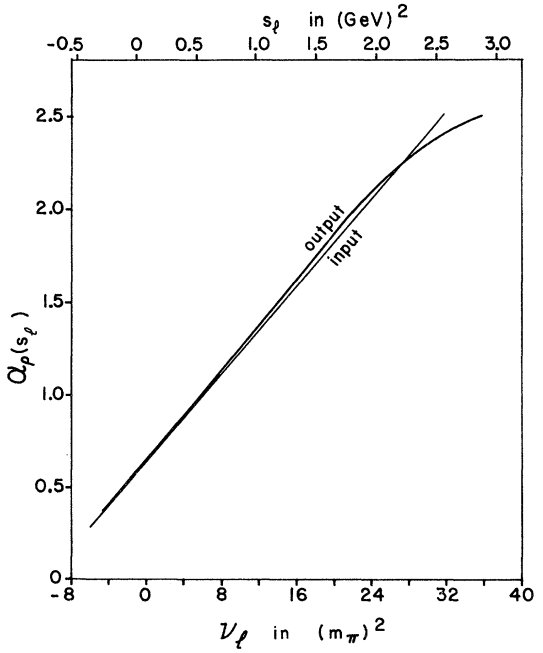

FIG. 1. The input and output ρ Regge trajectories corresponding to the central set of parameters $n = 7$, $R_l^1 = 3$, $\nu_K = -25$, $\nu_D = \nu_{DR} = 25$, $\nu_\rho = 6.18$, $\Gamma_\rho = 0.167$, $\alpha_\rho(0) = 0.58$ and only a ρ term retained in $A_l^{1(L)}(\nu)$. ν_l is the optimum root of $\text{Re}D_l^1(\nu)$; that is, for $\nu_F = (\nu_F)_{\text{opt}}$, $\text{Re}D_l^1(\nu_l) = 0$; $s_l = 4(\nu_l + 1)$.

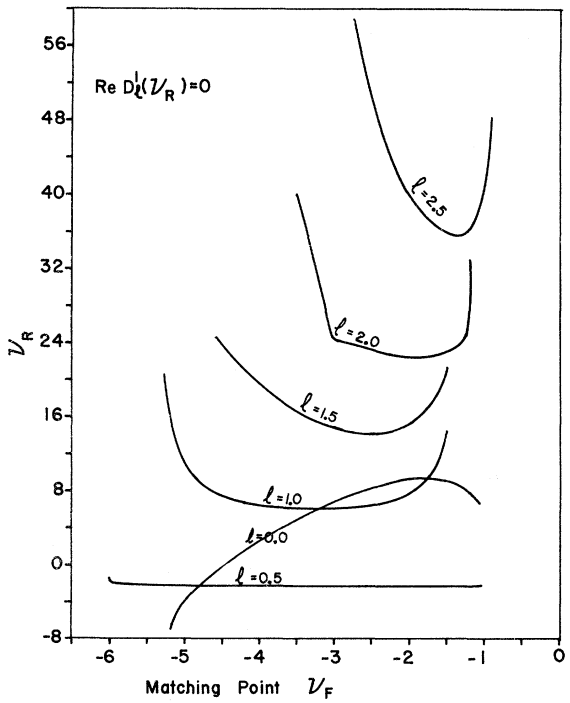

FIG. 2. The roots ν_R of $\text{Re}D_l^1(\nu)$ as a function of l and ν_F. The parametrization is the same as in Fig. 1.

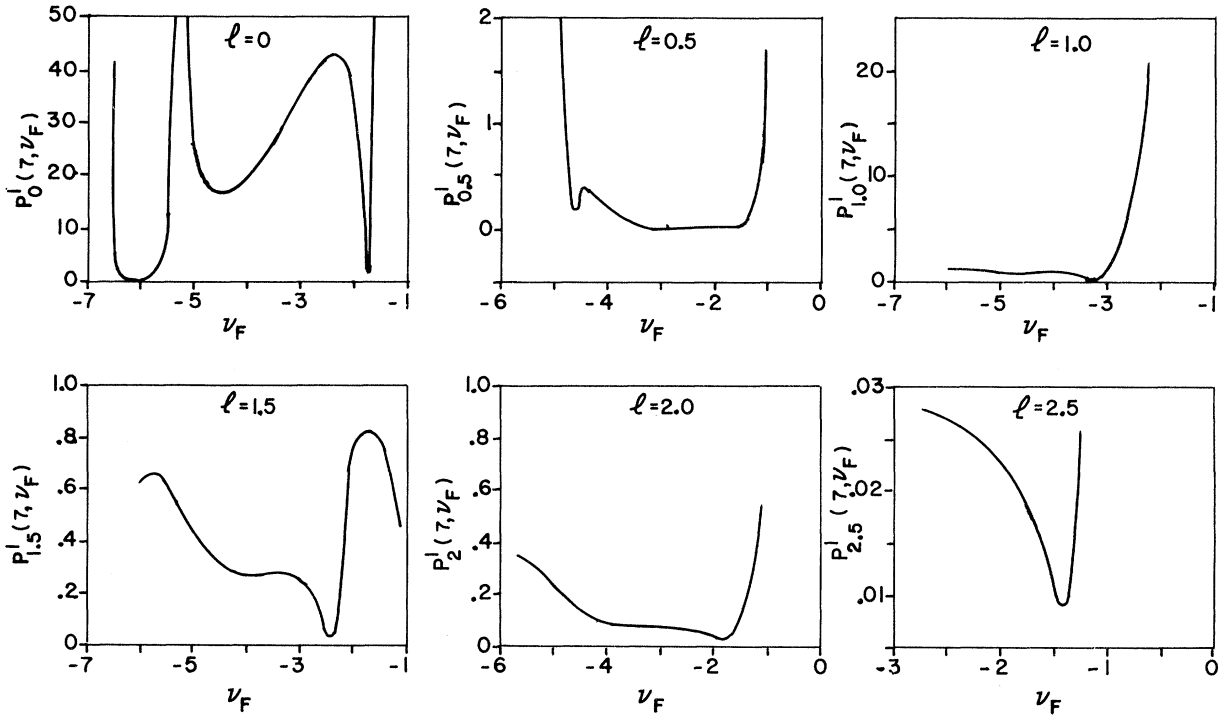

FIG. 3. $P_l^1(7,\nu_F)$ as given by Eq. (46) is plotted as a function of l and ν_F. The parametrization is the same as in Fig. 1. $P_l^I(n,\nu_F)$ is the percentage difference between the values of the input and output partial-wave scattering amplitudes at ν_F.

of $[\alpha_\rho(0)]_{\text{in}}$, the input and output values of the slope and intercept of the trajectory are in good agreement as can be seen from Figs. 5 and 6, respectively. We see however from Fig. 7 that the output values of the ρ mass and width are most nearly self-consistent for values of $[\alpha_\rho(0)]_{\text{in}}$ lying in the high end of this range. The slope seems to be bootstrapped most accurately for $(\epsilon_\rho)_{\text{in}}=(\epsilon_\rho)_{\text{out}}=0.055$, which corresponds to $[\alpha_\rho(0)]_{\text{in}}=0.605$. From Figs. 6 and 7 we then find that $[\alpha_\rho(0)]_{\text{out}}=0.608$, $(\nu_\rho)_{\text{out}}=6.12$, and $(\Gamma_\rho)_{\text{in}}=0.167$. The results are summarized in Table IV, and Fig. 8. Self-consistency is obviously excellent except for the ρ width. This can be improved somewhat by taking $[\alpha_\rho(0)]_{\text{in}}\approx 0.68$, although there are then

TABLE III. The optimum roots of $\text{Re}D_l^1(\nu)$, $\nu_l=(\nu_R)_{\text{opt}}$, corresponding to the optimum values of the matching point, $(\nu_F)_{\text{opt}}$, as a function of the intercept $\alpha_\rho(0)$ and the angular momentum l. The other parameters are the same as in Fig. 1 and only a ρ term is retained in $A_l^{1(L)}(\nu)$. For $\alpha_\rho(0)=0.58$ these values can be obtained from Fig. 2 by the minimum slope criterion, or equivalently, from Figs. 2 and 3 by the WE criterion. All quantities are in units with $\hbar=c=m_\pi=1$.

	$\alpha_\rho(0)=0.48$		$\alpha_\rho(0)=0.58$		$\alpha_\rho(0)=0.68$	
l	$(\nu_F)_{\text{opt}}$	ν_l	$(\nu_F)_{\text{opt}}$	ν_l	$(\nu_F)_{\text{opt}}$	ν_l
0.0	−6.00	<−7	−6.00	<−7	−6.25	<−7
0.5	−1.75	−0.724	−2.75	−2.37	−4.50	−5.04
1.0	−3.00	5.68	−3.25	6.02	−3.50	6.51
1.5	−2.25	11.73	−2.50	14.37	−2.75	18.94
2.0	−1.75	18.19	−1.75	22.41	−2.25	37.10
2.5	−1.25	24.45	−1.35	35.64	−1.70	69.35

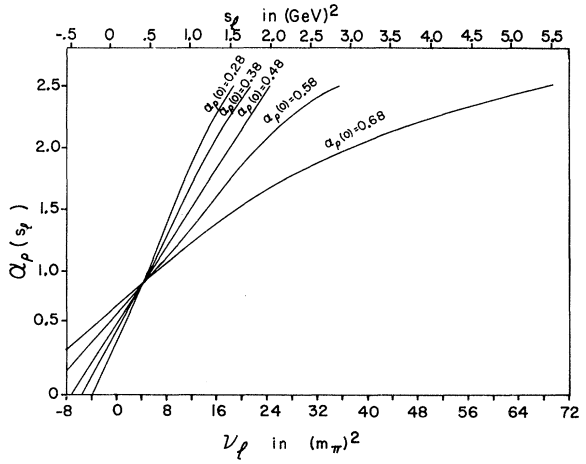

FIG. 4. The ρ Regge trajectory as a function of the input value of the intercept $\alpha_\rho(0)$. The other parameters are the same as in Fig. 1, $\text{Re}D_l^1(\nu_l)=0$ for $\nu_F=(\nu_F)_{\text{opt}}$, and $s_l=4(\nu_l+1)$.

TABLE IV. Input and output values of the parameters that bootstrap the ρ Regge trajectory. $(\epsilon_\rho)_{in} = (\epsilon_\rho)_{out}$ is obtained from Fig. 5. $[\alpha_\rho(0)]_{in}$ is then calculated through Eq. (42). Subsequently $[\alpha_\rho(0)]_{out}$, $(\nu_\rho)_{out}$, and $(\Gamma_\rho)_{out}$ are read off from Figs. 6 and 7. The other input parameters are $n=7$, $R_I^I=3$, $\nu_K=-25$, $\nu_D=\nu_{DR}=25$, $\nu_\rho=6.18$, $\Gamma_\rho=0.167$. The pole positions w_i are given in Table I and only a ρ term is retained in $A_I^{I(L)}(\nu)$. m_ρ and Δm_ρ are the ρ mass and full width in MeV, and $\alpha_\rho' = d\alpha_\rho(s)/ds$ is given in $(\text{GeV})^{-2}$.

ϵ_ρ $[(m_\pi)^{-2}]$		$\alpha_\rho(0)$ $(\hbar)$		ν_ρ $[(m_\pi)^2]$		Γ_ρ (dimensionless)		m_ρ (MeV)		Δm_ρ (MeV)		α_ρ' $[(\text{GeV})^{-2}]$	
in	out	in	out	in	out	in	out	in	out	in	out	in	out
0.055	0.055	0.605	0.608	6.18	6.12	0.167	0.119	750	747	100	71	0.701	0.701

slightly larger discrepancies in the input and output values of the other quantities, as well as poorer agreement with the experimental value. The bootstrapped value of the slope is somewhat small compared to the value obtained from fitting scattering data.[12]

B. The Left-Hand Poles

As was pointed out previously, the Balázs method should be used with a large number n of poles approximating the left-hand cut. In bootstrapping a resonance this is advisable, but in bootstrapping a trajectory it is absolutely necessary. From Fig. 9 it can be seen that the effect of increasing n is to "straighten out" the trajectory. As n was varied, the input mass and width of the ρ meson were kept fixed at their experimental values. Although this is justified for $n=7$, as discussed above, it is not justified for low n values where the bootstrapped ρ width is several times larger than the experimental value.[1,2] On the other hand, using the bootstrapped values of the mass and width of the ρ meson as input improves the low-n trajectories only in the vicinity of $l=1$ but further distorts these trajectories as a whole. Although, *a priori*, there are no objections to nonlinear trajectories, nevertheless, since the input trajectory is a linear one, we cannot even start to hope for a bootstrap before the output trajectory is also linear.

It is interesting to note that the variation of ν_ρ with n appreciably levels off to the experimental

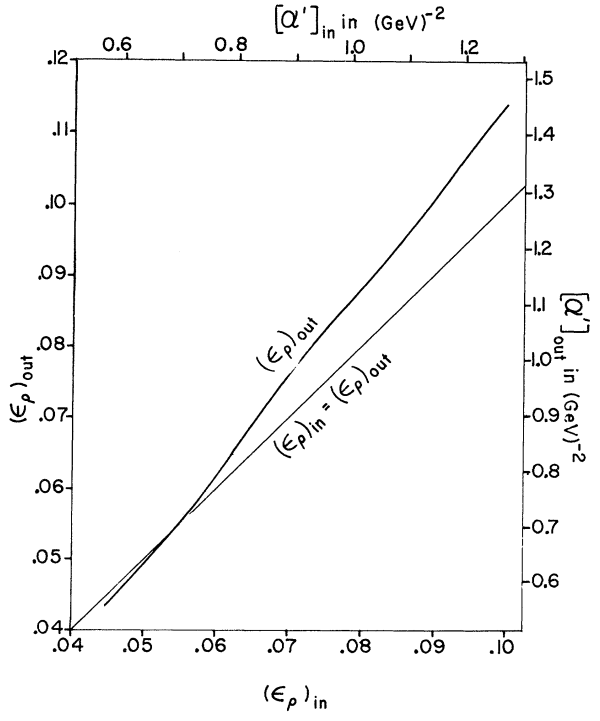

FIG. 5. The input vs output values of the slope, ϵ_ρ, of the ρ Regge trajectory. $(\epsilon_\rho)_{in}$ is obtained through Eq. (42) and $(\epsilon_\rho)_{out}$ is obtained from Fig. 4. Other than $[\alpha_\rho(0)]_{in}$, the input parameters are the same as in Fig. 1.

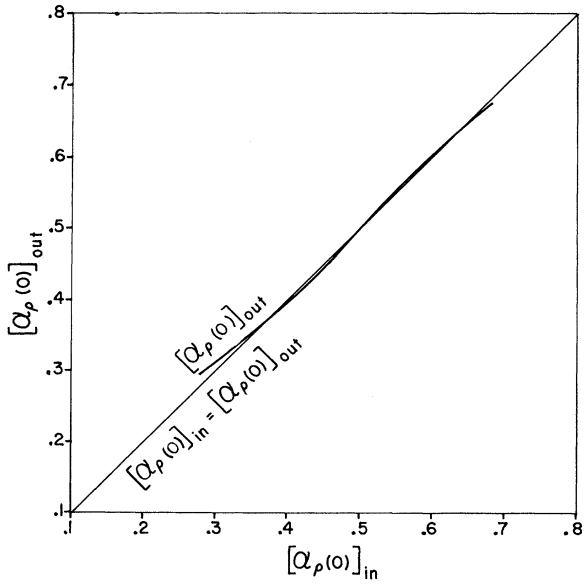

FIG. 6. The input vs output values of the intercept $\alpha_\rho(0)$ of the ρ Regge trajectory. $[\alpha_\rho(0)]_{out} = [\alpha_\rho(\nu=-1)]_{out}$ can be obtained from Fig. 4. Other than $[\alpha_\rho(0)]_{in}$. Other than $[\alpha_\rho(0)]_{in}$, the input parameters are the same as in Fig. 1.

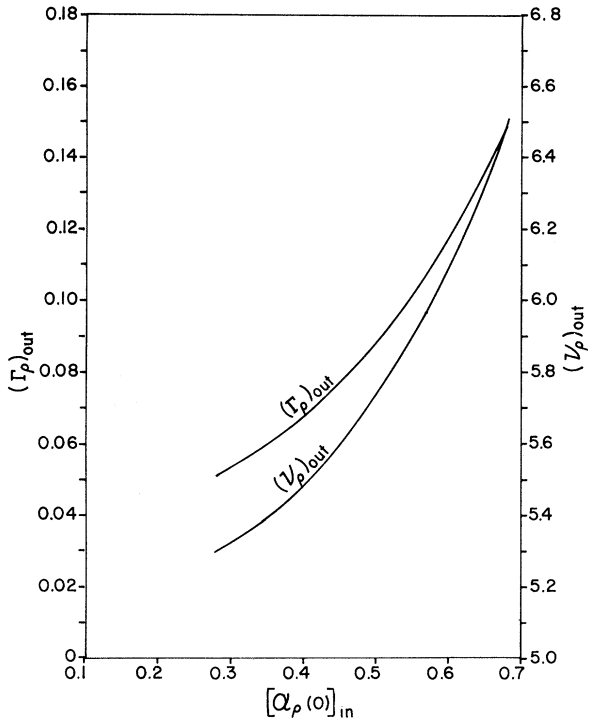

FIG. 7. The output values $(\nu_\rho)_{out}$ and $(\Gamma_\rho)_{out}$ as a function of the value of the input intercept $[\alpha_\rho(0)]_{in}$. Other than $[\alpha_\rho(0)]_{in}$ the input parameters are the same as in Fig. 1.

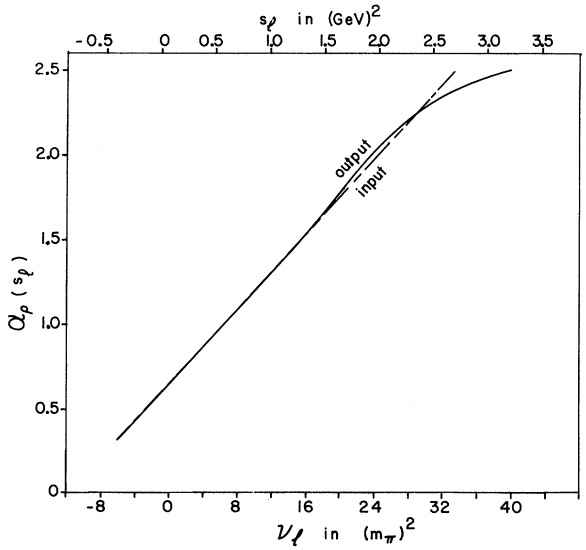

FIG. 8. The input and output ρ Regge trajectories which give an optimum bootstrap in the angular momentum range $0 \le \alpha_\rho(s) \le 2.5$. The input parameters are $n=7$, $R_l^I=3$, $\nu_K=-25$, $\nu_D=\nu_{DR}=25$, $(\nu_\rho)_{in}=6.18$, $(\Gamma_\rho)_{in}=0.167$ and $[\alpha_\rho(0)]_{in}=0.605$. Only a ρ term is retained in $A_l^{I(L)}(\nu)$ The output values are $(\nu_\rho)_{out}=6.12$, $(\Gamma_\rho)_{out}=0.119$, and $[\alpha_\rho(0)]_{opt}=0.608$. The slope is exactly bootstrapped with the value $(\epsilon_\rho)_{in}=(\epsilon_\rho)_{out}=0.055$. $\mathrm{Re}D_l^I(\nu_l)=0$ for $\nu_F=(\nu_F)_{opt}$.

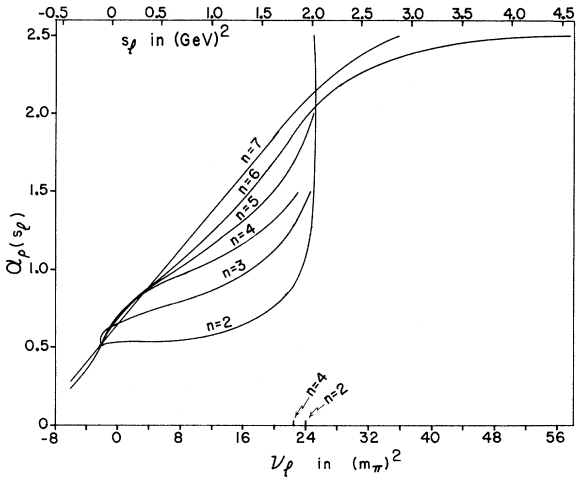

FIG. 9. The ρ Regge trajectory as a function of n, the number of poles approximating the left-hand cut. The pole positions corresponding to the different values of n are those given in Table I. The other input parameters are the same as in Fig. 1, $\mathrm{Re}D_l^1(\nu_l)=0$ for $\nu_F=(\nu_F)_{opt}$.

value at $n=7$. For $l=0$, the form of the function $D_0^1(\nu)$ suggests that $\nu_\rho < -7$ for $n=7$ and $n=6$. This would indicate a possible linear decrease of the $n=6$ and $n=7$ trajectories below $l=0.5$. Unfortunately the range of validity of the N/D equations in the present calculation is $\nu > -(\nu_\rho+1) = -7.18$, and thus $\nu_{l=0}$ cannot be determined in this case. For $n=2$ and $n=4$, $\nu_{l=0}$ is large and positive, and hence we did not venture to extrapolate these trajectories in between $l=0.5$ and $l=0$. For $n=5$ and $l=0$, the optimum matching point is $\nu_F=-5.10$, and there is no corresponding zero of the denominator function; instead $D_0^1(\nu)$ has a minimum at about $\nu=25$. For $n=3$ and $l=0$ the situation is even worse, and it is not at all clear what is the optimum value of ν_F or the corresponding value of $\nu_{l=0}$.

Figure 9 leads to the conclusion that the effect of the various approximations in the Balázs method can be smoothed out by the use of a large number of poles to approximate the left-hand cut. Furthermore it is seen that $n=7$ is a large enough value to produce appreciable leveling off of the variation of the results with n.

C. Effect of Other Parameters

We have studied the effect on the output trajectories of varying the parameters ν_K, ν_D, and ν_R. These chiefly affect the degree to which the trajectories tend to depart from linearity and turn over for $l>1.5$. In particular, there is almost no sensitivity of the $l=0.5$ intercept to any of the variations. This is, in fact, easily understandable. From Eq. (29), we note that the input form of the

amplitude analytically continued in l has a pole at $\alpha(s)=l$. For $\alpha=\frac{1}{2}$, the form of our input trajectory assures us that this occurs for $\nu\approx-2$, in the region in which the matching points are chosen. Thus the matching procedure rather directly forces the denominator function for $\alpha=\frac{1}{2}$ to have a zero for $\nu\approx-2$, i.e., the output trajectory satisfies $\alpha(\nu\approx-2)=\frac{1}{2}$, regardless of the values of the various parameters. For larger values of l, the point at which $\alpha(s)=l$ lies relatively far from the matching point, so that consistency between the input and output trajectory for larger l is not directly enforced by the matching process, and may depend on the values of the various parameters. Since, as is well known and as we have seen in our earlier papers, the effect of inelasticity tends to be attractive, it is not surprising that one finds that reducing the effect of inelasticity by either raising ν_D or lowering R_l^I causes the output trajectory to turn over more quickly.

D. Variation with the ρ Mass and Width

As can be seen from Figs. 10 and 11, respectively, the variation of the trajectory with the ρ mass and width is rather small except near $l=0$. This justifies using the experimental values of the ρ mass and width as input rather than the bootstrapped values which are slightly different. It should nonetheless be noted that with the bootstrapped value of the ρ mass corresponding to $\nu_\rho=5.49$ as input, the trajectory is no longer straight for $l<0.5$.

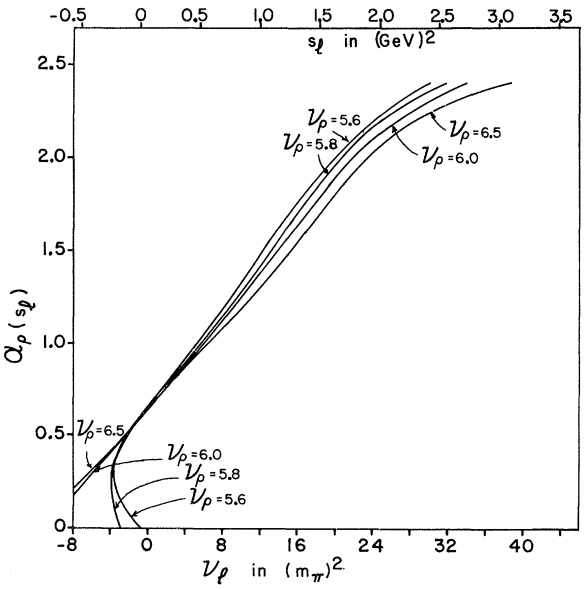

FIG. 10. The ρ Regge trajectory as a function of ν_ρ. The other input parameters are the same as in Fig. 1, $\mathrm{Re}D_l^1(\nu_l)=0$ for $\nu_F=(\nu_F)_{\mathrm{opt}}$.

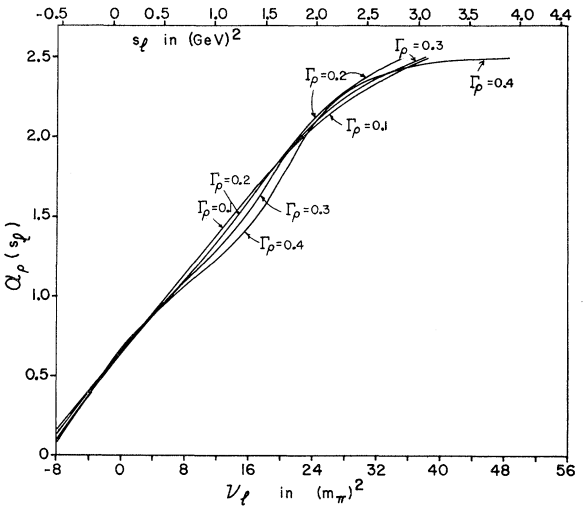

FIG. 11. The ρ Regge trajectory as a function of Γ_ρ. The other input parameters are the same as in Fig. 1. $\mathrm{Re}D_l^1(\nu_l)=0$ for $\nu_F=(\nu_F)_{\mathrm{opt}}$.

In Fig. 10 the ρ mass is varied while the reduced half-width is fixed at the central value of $\Gamma_\rho=0.167$. Thus, as can be seen from Eq. (26), the input ρ width does not remain constant as the input ρ mass is varied. If Γ_ρ is constrained to vary with ν_ρ so that the input ρ width is held constant, the resulting trajectories are hardly any different from those of Fig. 10.

The strange and unphysical behavior near $\nu=0$ of the trajectories corresponding to $\nu_\rho=5.6$ and 5.8 is presumably to be attributed to the fact, discussed above, that the trajectories are essentially constrained to pass through $l=\frac{1}{2}$ near $\nu=-2$. If it were not for this constraint, these trajectories would presumably pass through $l=\frac{1}{2}$ farther to the right, so that one would have the normal situation of a trajectory with positive slope and negative second derivative throughout the region of interest. In any event, this difficulty is not present when ν_ρ has its physical (and bootstrapped) value of 6.18.

E. The f_0 Term

Adding the f_0 term, as given by Eq. (38b), to $A_l^{I(L)}(\nu)$ produces a more complicated input amplitude. That is, the f_0 term increases the information content of the input. To store this extra input information, more residues f_{lI}^i are needed. Thus the left-hand cut will have to be approximated by a larger number of poles. As a working hypothesis we assume that when the input information is properly transmitted to the N/D equations the resulting output trajectory will be linear, at least in the

range of ν under consideration. If we now consider Fig. 9, we find that according to the above assumptions, seven residues f_{lI}^{i} are needed to properly store the information contained in the input amplitude with only a ρ term included. Hence more than seven residues are needed to handle the more complicated $\rho - f^0$ input.

We do not attempt such a calculation here, but only note that if the f_0 term is added to the input, then the central set of values combined with an input intercept of 0.605, which with only a ρ term produced a bootstrapped trajectory, now leads to $(\nu_\rho)_{\text{out}} = 7.90$, and $(\Gamma_\rho)_{\text{out}} = 0.229$. These values correspond to a ρ mass and full width of 835 MeV and 160 MeV, respectively. Both of these values are expected to diminish with increasing n.

V. CONCLUSION

The present calculation gives considerable support to both the bootstrap approach to elementary particles in general, and the N/D method in particular. In addition it is one more example[1-3] of the utility of the Balázs method as a practical calculational procedure. On the other hand, it is clear that the more sophisticated the calculation, that is, the more input information there is and the more the output information required, the larger will be the number of poles needed to approximate the left-hand cut. As has been discussed, the residues of these poles store the input information and transmit it to the output amplitude. With only the contribution of the ρ term included in the low energy part of the input amplitude, it is seen that the results level off as a function of n around $n = 7$. Thus a 7-pole approximation to the left-hand cut is sufficient. On the other hand, if an f^0 term is added to the input then larger values of n are needed. A Balázs-type calculation with such a large number of poles may at first sight look prohibitive. But with the calculational techniques developed in the Appendixes and particularly the method of differentiation of Appendix C, the computer program can be written in a general form as a function of n, the number of poles approximating the left-hand cut. Thus in principle a calculation for a large value of n is as easy to handle as one for a small value of n. What needs to be supplied for a given calculation are the pole positions corresponding to the value of n desired.

Another conclusion of the present calculation is that when the approximations are improved, the method tends to underestimate particle widths rather than overestimate them as in the case with many bootstrap calculations. It is not clear whether this remains true if the f^0 is included.

Finally it is found that a linearly rising ρ Regge trajectory in the angular momentum range $0 \leq l \leq 2.5$ can be bootstrapped, leading to a self-consistent slope of 0.701 $(\text{GeV})^{-2}$ and a corresponding input and output intercept of $[\alpha_\rho(0)]_{\text{in}} = 0.605$ and $[\alpha_\rho(0)]_{\text{out}} = 0.608$, respectively. This is reasonably consistent with the experimental evidence on the intercept of the ρ trajectory, which allows a value of $\alpha_\rho(0)$ as large as 0.58.[12,15] The output values of the ρ mass and full width are 747 MeV and 71 MeV, respectively. These are to be compared with the input experimental values of $m_\rho = 750$ MeV and $\Delta m_\rho = 100$ MeV.

It should nevertheless be admitted that although the variation of the results with n levels off appreciably at $n = 7$, still the values of m_ρ and Δm_ρ continue to decrease, though slightly, with n beyond $n = 7$. On the other hand, and by the same mechanism, the changes in m_ρ and Δm_ρ due to the contribution of the f^0 term are expected to decrease with increasing n and level off for large enough n to a value that hopefully compensates for the major part of the discrepancy between the input experimental values and the output values due to an input ρ term.

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APPENDIX A: THE KERNEL APPROXIMATION

According to the Balázs method, part of the process of decoupling the N/D equations is accomplished by approximating the kernel $(1 + x\nu)^{-1}$ of the numerator function integral, by an interpolation formula such as Eq. (9a). For a given value of n, the interpolation points $\{x_i,\ i = 1, \ldots n\}$ are evaluated by the requirement that approximation (9a) be as good as possible in the range of integration $0 \leq x \leq (1 + \nu_\rho)^{-1}$.

Clearly the right side of Eq. (9a) is a polynomial of degree $n - 1$ in x, and we designate it by $T_{n-1}(x)$. Thus

$$\sum_{i=1}^{n} \frac{G_i^n(x)}{1 + x_i \nu} \equiv T_{n-1}(x) = \sum_{k=0}^{n-1} a_k x^k . \tag{A1}$$

Furthermore, since $G_i(x_j) = \delta_{ij}$, we have

$$T_{n-1}(x_j) = \frac{1}{1 + x_j \nu} . \tag{A2}$$

It is then clear that the interpolating points $\{x_i\}$ are the roots of the polynomial $S_n(x)$, of degree n in x, which is defined by

$$S_n(x) = T_{n-1}(x)[1+x\nu] - 1 = \sum_{k=0}^{n} b_k x^k . \tag{A3}$$

The above considerations lead to the following simple procedure for determining the set of values $\{x_i\}$ and the corresponding pole positions $\{w_i\}$:

(i) The coefficients $\{a_k;\ k=1,\ldots,n-1\}$ of the polynomial $T_{n-1}(x)$ are determined by a least-squares fit of $T_{n-1}(x)$ to $(1+x\nu)^{-1}$ in the range $0 \leq x \leq (1+\nu_\rho)^{-1}$.

(ii) The coefficients $\{b_k;\ k=1,\ldots,n\}$ of $S_n(x)$ can then be determined using the relation

$$b_k = \begin{cases} a_0 - 1, & k=0 \\ a_k + \nu a_{k-1}, & 1 \leq k \leq n-1 \\ \nu a_{n-1}, & k=n . \end{cases} \tag{A4}$$

(iii) The n parameters $\{x_i\}$ are then found as the n roots of the equation $S_n(x)=0$.

The pole positions $w_i = 1/x_i$ are then seen to depend on three parameters: n, ν, and the range of matching $[0,\ (1+\nu_\rho)^{-1}]$. *A priori*, n should be chosen as large as is practical, and the range of matching is rather well defined by the mass of the ρ meson. The variation of the pole positions with the remaining parameter ν turns out to be very slight.

APPENDIX B: THE INTEGRAL OF THE DENOMINATOR FUNCTION

To evaluate the denominator function $D_l^I(\nu)$ we need to evaluate the integral

$$I_l^i(\nu, t) = \int_t^\infty d\nu' \left(\frac{\nu'}{\nu'+1}\right)^{1/2} \frac{\nu'^l}{(\nu'-\nu_0)(\nu'-\nu)(\nu'+w_i)(\nu'-\nu_K)^{l-1}} \tag{B1}$$

For non-negative integer or half-integer values of l we show how this integral can be evaluated in closed form; for other positive values of l a numerical integration is necessary.

By performing a partial-fraction expansion, this integral can be transformed to

$$I_l^i(\nu, t) = p_1^i(\nu, N) G_1(\nu_0, a_l, b_l, c_l, t) + p_2^i(\nu, N) G_1(\nu, a_l, b_l, c_l, t) + p_3^i(\nu, N) G_1(-w_i, a_l, b_l, c_l, t) + \sum_{m=1}^{N-1} q_{N-m}^i(\nu, N) G_m(\nu_K, a_l, b_l, c_l, t), \tag{B2}$$

where

$$N = \text{integer part of } l, \tag{B3}$$

and

$$\begin{aligned} p_1^i(\nu, N) &= F_N(\nu_0, \nu, -w_i, \nu_K), \\ p_2^i(\nu, N) &= F_N(\nu, -w_i, \nu_0, \nu_K), \\ p_3^i(\nu, N) &= F_N(-w_i, \nu, \nu_0, \nu_K), \end{aligned} \tag{B4}$$

with

$$F_N(x, y, z, u) = \frac{x^{N+1}}{(x-y)(x-z)(x-u)^{N-1}} . \tag{B5}$$

Also

$$G_m(\alpha, a_l, b_l, c_l, t) = \int_t^\infty \frac{d\nu'}{(\nu'-\alpha)^m [Z_l(\nu')]^{1/2}}, \tag{B6}$$

where

$$Z_l(\nu) = a_l \nu^2 + b_l \nu + c_l , \tag{B7}$$

with

	a_l	b_l	c_l
zero or integer	1	1	0
half-integer	1	$(1-\nu_K)$	$-\nu_K$

(B8)

Furthermore,

$$q_1^i(\nu, N) = \frac{\nu_K^{N+1}}{(\nu_K-\nu_0)(\nu_K-\nu)(\nu_K+w_i)}, \tag{B9}$$

and

$$q_{m+2}^i(\nu, N) = \frac{1}{m+1} \sum_{k=0}^{m} q_{k+1}^i(\nu, N) \xi_{m-k}(\nu, N), \quad m = 0, 1, 2, \ldots , \tag{B10}$$

where

$$\xi_{m-k}(\nu, N) = (-1)^{m-k} \left[\frac{N+1}{(\nu_K)^{m-k+1}} - \frac{1}{(\nu_K-\nu_0)^{m-k+1}} - \frac{1}{(\nu_K-\nu)^{m-k+1}} - \frac{1}{(\nu_K+w_i)^{m-k+1}} \right] . \tag{B11}$$

Finally $G_m(\alpha, a, b, c, t)$ is given recursively by

$$\frac{1}{(m-1)k} \left[\frac{[Z(t)]^{1/2}}{(t-\alpha)^{m-1}} - \sqrt{a}\,\delta_{m2} - \frac{(2m-3)}{2} \beta G_{m-1}(\alpha, a, b, c, t) - (m-2) a G_{m-2}(\alpha, a, b, c, t) \right] \text{ for } k \neq 0 ,$$

and

$$\frac{2}{(2m-1)\beta} \left[\frac{[Z(t)]^{1/2}}{(t-\alpha)^m} - (m-1) a G_{m-1}(\alpha, a, b, c, t) \right] \text{ for } k=0 \text{ and } \beta \neq 0 ; \tag{B12}$$

and $G_1(\alpha, a, b, c, t)$ is given by

$$\begin{aligned}
&\frac{1}{\sqrt{k}}\ln\left[\frac{(\gamma-\sqrt{a})|t-\alpha|}{\gamma t+\delta-[Z(t)]^{1/2}}\right]+\frac{i\pi}{\sqrt{k}}\theta(\alpha-t) \text{ for } k>0,\ \Delta\neq 0,\\
&\frac{2}{\sqrt{-k}}\left[\tan^{-1}\left(\frac{\sqrt{\Delta}-(\Delta-\beta^2)^{1/2}}{\beta}\right)-\tan^{-1}\left(\frac{(t-\alpha)\sqrt{\Delta}-\{(t-\alpha)^2\Delta-[\beta(t-\alpha)+2k]^2\}^{1/2}}{\beta(t-\alpha)+2k}\right)\right] \text{ for } k<0,\ \Delta>0,\\
&-\frac{2}{\beta}\left(\sqrt{a}-\frac{[Z(t)]^{1/2}}{(t-\alpha)}\right) \text{ for } k=0,\ \beta\neq 0,\ \Delta\neq 0\\
&\frac{2\sqrt{a}}{\beta}\ln\left[\frac{2at+b}{2a(t-\alpha)}\right] \text{ for } \Delta=0,
\end{aligned} \tag{B13}$$

where

$$\begin{aligned}
\theta(\alpha-t)&=\begin{cases}0 \text{ for } \alpha<t\\ 1 \text{ for } \alpha>t\end{cases},\\
k&=Z(\alpha)\\
&=a\alpha^2+b\alpha+c,\\
\beta&=\frac{\partial k}{\partial\alpha}\\
&=2a\alpha+b,\\
\gamma&=\beta/(2\sqrt{k}),\\
\delta&=(b\alpha+2c)/(2\sqrt{k}),\\
\Delta&=b^2-4ac.
\end{aligned} \tag{B14}$$

$G_m(\alpha, a, b, c, t)$ can be calculated recursively in m.

APPENDIX C: RECURSIVE ANALYTIC DIFFERENTIATION

When the Balázs method is used in an n-pole approximation, the first $n-1$ derivatives (n derivatives if the WE criterion is used) of essentially all functions entering into the calculation are needed. For large values of n this can become a prohibitive endeavor. To bypass this difficulty, attempts have been made to modify the Balázs procedure of determining the residues of the left-hand poles. Instead of matching the two forms of the amplitude and $n-1$ of their derivatives at a matching point ν_F in the gap between the right- and left-hand cuts, Bond[13] matched the two forms of the amplitude at n points in the gap, while Gibbons and Dilley[9] made a least-squares fit of the two forms of the amplitude in the gap. The matching procedure of Bond simplifies the Balázs method considerably, and like the matching procedure of Balázs leads to n simultaneous linear equations in the n residues, but unlike the Balázs method it makes the choice of an optimum set of matching points a prohibitive job, especially for large values of n. On the other hand, the least-squares fit of Gibbons and Dilley is an accurate procedure for determining the residues, and implicitly incorporates the WE criterion.

Although the matching procedure of Gibbons and Dilley is a viable one, we would like to point out that the original procedure of Balázs can be rather easily handled by the following method of recursive analytic differentiation. The utility of the method is due to the fact that for a class of functions consisting of products and quotients of polynomials and exponentials, the general form of the nth derivative of the logarithm of the function is very easy to write down explicitly, while that of the function itself may be hopelessly complicated.

The mathematics involved consists essentially of using the Leibnitz rule for the differentiation of a product,

$$[u(x)v(x)]^{(n)}=\sum_{k=0}^{n}\binom{n}{k}u^{(n-k)}(x)v^{(k)}(x), \tag{C1}$$

and the identity

$$F^{(1)}(x)=F(x)G(x), \tag{C2}$$

where the logarithmic derivative $G(x)$ is defined by

$$G(x)=\frac{d\ln F(x)}{dx}. \tag{C3}$$

We have used the following notation for differentiation:

$$f^{(n)}(x)=\frac{d^n f(x)}{dx^n}\ ,\quad n=0, 1, 2, \ldots . \tag{C4}$$

From Eqs. (C1) and (C2) we obtain a recursive analytic formula for the $(n+1)$th derivative of $F(x)$,

$$F^{(n+1)}(x)=\sum_{k=0}^{n}\binom{n}{k}F^{(n-k)}(x)G^{(k)}(x)\quad n=0, 1, 2. \tag{C5}$$

As an example, if $F(x)$ is given by

$$F(x)=e^{bx}\prod_i (x-a_i)^{q_i}, \tag{C6}$$

then

$$G^{(k)}(x) = (-1)^k k! \left\{ b\delta_{k0} + \sum_i \frac{q_i}{(x-a_i)^{k+1}} \right\}, \qquad \text{(C7)}$$

and the derivatives of $F(x)$ can be recursively obtained through Eq. (C5) by using Eqs. (C6) and (C7). Hence the problem of differentiation has been reduced to an algebraic one of summation and multiplication.

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Some Consequences of an SU(2)⊗U(1) Gauge Model*

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The experimental consequences of an SU(2) ⊗ U(1) model in which the intermediate-boson mass m_W is bounded below by 13 GeV are discussed. For the special choice of $m_W = 18$ GeV neutral-current effects essentially disappear. The model includes a heavy neutral lepton of the muon type whose mass is bounded above and below by ~1.2 GeV and 390 MeV, respectively. The model is thus relatively more accessible to experimental tests than other gauge models. Other aspects of the model are discussed.

I. INTRODUCTION

The construction of gauge models of weak and electromagnetic interactions, initiated by Weinberg[1] and Salam,[2] has now mushroomed into a booming industry.[3] The available models more often than not contain intermediate bosons too massive to be produced easily with present-day machines and/or leptons with mass in the range of several GeV. We have considered an SU(2)⊗U(1) model[4] which contains bosons and leptons of relatively low mass and which thus may have the (dubious) distinction of being an early casualty in a confrontation of gauge theories with experiment. (Our previous paper will be referred to as I.)

We list those features of this model that are relevant for experimental investigations.

(a) The lower bound for m_W, the mass of the charged intermediate bosons, is 13 GeV. This may be contrasted with the value of 39 GeV for Weinberg's model[1] and a number of other models. While 13 GeV is the smallest mass allowed in this