Nucleon-Nucleon Scattering Near 50 MeV. I. Phase-Shift Analysis of the Data*

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We carry out a phase-shift analysis of p-p and n-p elastic scattering data in the laboratory kinetic energy range of 47.5 to 60.9 MeV. Despite the inclusion of new n-p total and differential cross-section data since the phase-shift analyses of MacGregor, Arndt, and Wright (papers VI and X in particular), the χ^2 vs ϵ_1 curve retains its double minimum, and the anomalous value for the phase parameter $\delta(^1P_1)$ persists. The data yield a range of solutions rather than a unique $I=0$ phase-shift solution, and, in fact, the χ^2 vs ϵ_1 curve is even flatter than it was in paper VI. The allowed range of ϵ_1 is found to be from $-10°$ to $+3°$, approximately. To find a unique solution, we constrain ϵ_1 to have a reasonable theoretical value of $+2.78°$, and present the corresponding constrained phase-shift solution. This yields $\delta(^1P_1)=-3.52°\pm1.04°$, which is 4.5 standard deviations or more above predictions by meson-theoretical models. In fact, throughout the allowed range of ϵ_1, the searched value of $\delta(^1P_1)$ remains at least this far above theoretical predictions. We determine that the Harwell n-p $d\sigma/d\Omega$ data at extreme forward and extreme backward angles are responsible for the high value of $\delta(^1P_1)$, and recommend that these measurements be retaken. We emphasize that contrary to popular belief, it is not sufficient just to fix the relative backward n-p $d\sigma/d\Omega$, because incorrect forward values will result in a wrong value for $\delta(^1P_1)$. With regard to forward data, good extreme forward absolute $d\sigma/d\Omega$ data will be more effective than relative forward data spanning the 0°–90° range. Finally, with respect to ϵ_1, we emphasize that the existing types of n-p data ($d\sigma/d\Omega$, σ_{tot}, and P) will not remove the ambiguity in this phase parameter. Some other type of experiment must be done, as we intend to discuss in a succeeding paper.

I. INTRODUCTION

In this paper we present the results of a phase-shift analysis of all currently available nucleon-nucleon elastic scattering data falling in the laboratory kinetic-energy range of 47.5 to 60.9 MeV. This analysis was motivated in part by a desire to help the experimental nucleon-nucleon group at the Cyclotron Institute at Texas A & M University to select the most useful n-p experiments to carry out. The analysis was further motivated by the well-known observation[1,2] that the $\delta(^1P_1)$ phase parameter[3] determined by the Livermore group in a previous analysis at 50 MeV appears to be grossly out of line both with experiment at neighboring energies and with theory; in particular, it is several standard deviations too positive.

What we discover in analyzing the old data and also new data which have become available since the Livermore analysis is that there actually exists a continuum of solutions corresponding to a range in ϵ_1 of $-10°$ to $+3°$. Furthermore, throughout this range, $\delta(^1P_1)$ remains at least 4.5 standard deviations above theoretical predictions (see Sec. II). Thus we are left with two problems: the ϵ_1 problem and the $\delta(^1P_1)$ problem.

We have found that the types of data which are currently available (σ_{tot}, $d\sigma/d\Omega$, and P) are not sufficient to pin down ϵ_1. Some other kind of experiment must be performed. In a succeeding paper, we intend to investigate in greater detail which other experiments are most effective in determining ϵ_1.

In this paper, we discuss the $\delta(^1P_1)$ problem, pinpoint the data responsible, and determine what experiments should be performed in order to best fix this parameter. Furthermore, lacking a determination of ϵ_1 which is necessary to narrow the range of phase-shift solutions to a single solution, we pick what we believe is a reasonable theoretical value for ϵ_1 and perform a phase-shift analysis on the remaining parameters. The phase shifts obtained in this constrained solution are plotted as open circles in Fig. 1(a) and Fig. 1(b).

II. DISCUSSION OF THE $\delta(^1P_1)$ PROBLEM

In Fig. 1 we plot the N-N phase parameters that have been found by the Livermore group (MacGregor, Arndt, and Wright[4]) in several energy bands spanning the 20–450-MeV range. The bands are centered at 25, 50, 95, 142, 210, 330, and 425 MeV. The Livermore points appear as plain error bars in the figure, and are designated MAW-X in the legend. For comparison we also plot the energy-dependent phase-shift solution that has been found by the Yale group (Seamon, Friedman, Breit, Haracz, Holt, and Prakash[5]). This phase-shift solution is plotted in dashed curves, and is referred to in the legend as Y-IV.

One may observe that at 50 MeV, the MacGregor-Arndt-Wright (MAW-X) value for $\delta(^1P_1)$ seems out of line with their values for $\delta(^1P_1)$ at 25 and 95 MeV. (Ignore the open-circle error bar at 55 MeV for the moment.) This might be written off as due to expected scatter in the phase-shift values, except that all the other MAW-X phase parameters (with the exception of ϵ_1 at 330 MeV) interpolate very smoothly from one energy to the next, particularly for the $I=1$ states, but to some

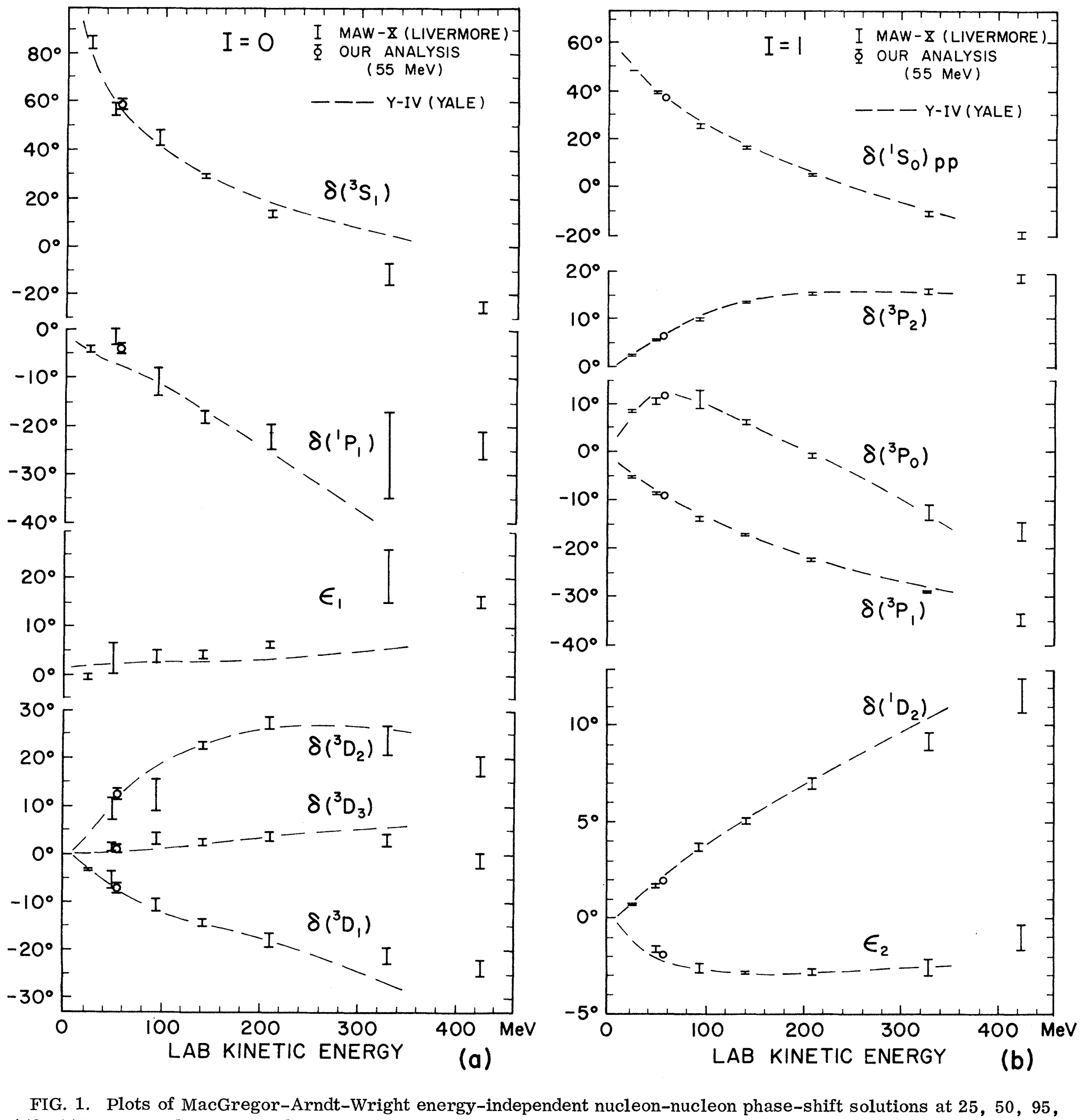

FIG. 1. Plots of MacGregor-Arndt-Wright energy-independent nucleon-nucleon phase-shift solutions at 25, 50, 95, 142, 210, 330, and 425 MeV, designated MAW-X and indicated by error bars (see Ref. 4). Also plotted are the phase parameters at 55 MeV for the new solution reported in this article, indicated by error bar plus open circle, and the Yale group's solution $(Y-IV)_{pp+np}$, derived for the 0–350 MeV range, indicated by dashed lines (see Ref. 5).

degree also in the $I=0$ states, suggesting that both the data and the analyses are very good at these other energies. Thus the aberrant $\delta(^1P_1)$ of MacGregor *et al.* at 50 MeV suggests that something is wrong, either in the data or in the analysis.

One might respond that the Yale $\delta(^1P_1)$ phase-shift curve interpolates smoothly enough between 25 and 95 MeV. However, we believe that this is because the Yale group imposed a low-energy threshold condition on $\delta(^1P_1)$, constraining it to resemble the one-pion-exchange contribution (OPEC) at 50 MeV, while the Livermore group left $\delta(^1P_1)$ free to follow the data (in their energy-independent analyses only). The Livermore group also presents two energy-dependent solutions to the 0–450-MeV N-N data in Ref. 4. We concern ourselves here only with their single-energy solutions as we believe these more accurately reflect the data. Of course all groups set the higher-partial-wave phase shifts to the OPEC prediction, so there is some constraint on $\delta(^1P_1)$ and other searched lower-partial-wave phase parameters because of this requirement. However, we believe the effect of this on P waves is small.

As stated above, the MAW-X 50-MeV $\delta(^1P_1)$ seems out of line with theory, as well as out of line with the single-energy analyses at adjacent energies. In Fig. 2 we plot the predictions of several different theoretical N-N models for $\delta(^1P_1)$. As one can see from the graph, the MAW-X $\delta(^1P_1)$ at 50 MeV, indicated by the plain error bar, disagrees by several standard deviations with each of the models. (Ignore the open-circle error bar for the time being.) The models are chosen to be representative of different theoretical approaches to the N-N problem, but all have in common the one-pion-exchange contribution for the long-range part of the N-N force. They also have in common the fact that they achieve a reasonably good fit to the N-N data over the 0–350-MeV range. The models are as follows: Model LF is a boundary-condition model put forth by Lomon and Feshbach.[6] It incorporates one-pion-exchange and two-pion-exchange contributions, plus heavy-boson exchanges, treated as a potential which is inserted in the Schrödinger equation. The potential is supplemented by boundary conditions on the wave functions at an inner distance. Model BG(D) is a one-boson-exchange potential model developed by Bryan and Gersten.[7] It is typical of the whole class of one-boson-exchange potentials.[8] Like the Lomon-Feshbach model, it treats the one-meson-exchange contribution as a potential which is inserted in the Schrödinger equation. In this case, the mesons are the π, η, ρ, ω, δ, and ϵ. However, it does not include two-

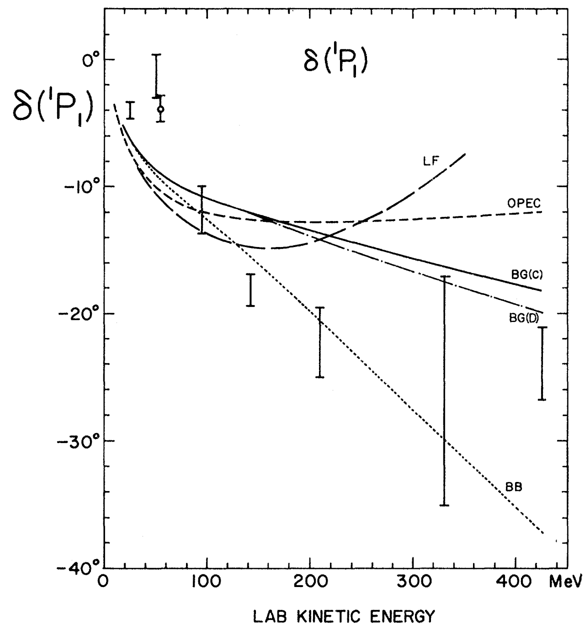

FIG. 2. Predictions for $\delta(^1P_1)$ by the nucleon-nucleon models due to Lomon and Feshbach (LF) (Ref. 6), Binstock and Bryan (BB) (Ref. 9), and Bryan and Gersten [BG(C) and BD(D)] (Ref. 7). Also plotted is the relativistic one-pion-exchange contribution projection in the $^1P_1$ state, set equal to $\delta(^1P_1)$, with $g_\pi^2=14.4$ and $m_\pi=135.04$ MeV/c^2. Also plotted are the MAW-X values for $\delta(^1P_1)$ at 25, 50, 95, 142, 210, 330, and 425 MeV, shown as error bars, and our own phase-shift analysis value for $\delta(^1P_1)$, computed and plotted at 55 MeV as an open circle plus error bar.

pion exchange contributions, apart from the two-pion-exchange ladder graph generated from the one-pion-exchange contribution by the Schrödinger equation. Also unlike the LF model, it does not incorporate boundary conditions (or even a hard core). Model BG(C) is also due to Bryan and Gersten.[7] It is similar to model BG(D) in all respects except that it utilizes the Blankenbecler-Sugar approximation to the Bethe-Salpeter equation in place of the Schrödinger equation to unitarize the one-meson-exchange amplitudes. Model BB is due to Binstock and Bryan.[9] Like the LF model, it incorporates one-pion-exchange, two-pion-exchange, and heavy-meson-exchange contributions. The two-pion exchange term involves both $N(938)$ and $\Delta(1236)$ baryons in the intermediate states. Unlike any of the previous models, it does not employ a Schrödinger-type equation to unitarize the amplitudes, but rather employs simple geometric unitarization, that is, each partial-wave projection of the sum of contributing amplitudes is equated with the phase shift δ_{JLS}.

In Fig. 2 we also graph for completeness the $^1P_1$ partial-wave projection of the one-pion-exchange contribution, set equal to the phase shift and labeled OPEC. As it happens, all the theoretical models' predictions for $\delta(^1P_1)$ are close to OPEC at 50 MeV, but this is partly coincidental. [The theoretical models' predictions for another P wave, $\delta(^3P_0)$, are far from OPEC.]

As stated earlier, the value for $\delta(^1P_1)$ that MacGregor, Arndt, and Wright find at 50 MeV agrees with none of the theoretical models. The models all predict a smooth downward curve for $\delta(^1P_1)$ in the region at 50 MeV, not an upward bump.

The theoretical evidence for $\delta(^1P_1)$ at 50 MeV, plus the mismatch of the MAW-X 50-MeV $\delta(^1P_1)$ point with $\delta(^1P_1)$ at neighboring energies, suggested that the data used in the MAW-X analysis might be in error. Meanwhile, new n-p total and differential cross-section data have become available in the 50-MeV energy region. Therefore, we decided to redo the phase-shift analysis of the 50-MeV p-p and n-p data. We used the Livermore analysis code, adapted to the IBM 7094 at the TAMU Cyclotron Institute. In Sec. III we present the results of our analysis. In Sec. IV we scrutinize the data more carefully, and in Sec. V we give our interpretation of the physical consequences of the current data, should they prove to be correct. Section VI contains some recommendations for future experiments.

III. PHASE-SHIFT ANALYSIS OF CURRENT DATA IN THE 50-MeV REGION

The data we took into account in the phase-shift analysis were those measurements falling within the energy range 47.5–60.9 MeV. The p-p data considered comprised 18 experiments, all listed in Table V of MAW-X. For the n-p data we considered 17 experiments, all either n-p total or differential cross-section data, or else polarization data. These experiments are listed in Table I of this paper. Five of these n-p experiments became available after publication of the MAW-X analysis. These new data consist of three σ_{tot} data points,[10] twelve unpublished Davis $d\sigma/d\Omega$ data points,[11] and nine unpublished Oak Ridge $d\sigma/d\Omega$ data points.[12] The n-p total cross section data used in the analysis (including the older Harwell[13] data) are plotted in Fig. 3(a). The n-p differential cross-section data are plotted in Fig. 3(b) and Fig. 3(c), where one may see the new preliminary Oak Ridge and Davis data, as well as the older Harwell[14] data used in the MAW-X analysis. The Davis and Oak Ridge backward (proton detection) $d\sigma/d\Omega$ data were analyzed with the normalizations floated[15] separately. These data are tabulated in Table II and Table III.

The polarization data used in this analysis are from the Harwell[16] group, and are plotted in Fig. 3(d). The only major difference between the way we handled these data and the way MacGregor, Arndt, and Wright handled the data in MAW-X is that we treated the forward and backward n-p polarization as one experiment with one over-all normalization. (Actually these are asymmetry measurements, with an initially polarized beam.) We made this change because the detector efficiencies cancel out in measuring an asymmetry so that the forward (neutron detection) and backward (proton detection) polarization should have the same over-all normalization.

We fitted the data at the several energies where it is to be found, by assuming that the energy derivatives of the searched (non-OPEC) phase parameters are constant, and taking these derivatives to be those of the nucleon-nucleon potential model of Bryan and Gersten, Model C referred to in Sec. II. Previous experience indicates that the phase-shift solutions are not sensitive to the energy derivatives of the phase parameters, so we adopted the BG(C) derivatives. Another possible choice might have been the derivatives from one of the energy-dependent solutions of the Livermore group[4] or the Yale group.[5]

We searched on all the phase parameters appearing in Table IV except for $\delta(^1S_0)_{np}$ and ϵ_1. The value of $\delta(^1S_0)_{np}$ was fixed at 38.98° since this parameter is not well determined by the data. (See the end of this section for an explanation of how this number was arrived at.) The result of the χ^2 search as a function of ϵ_1 is shown in Fig. 4. Note the very flat χ^2 vs ϵ_1 curve with the two very shallow minima indicated by arrows. The allowed values of ϵ_1 appear to range from −10° to +3°, approximately. The corresponding continuous phase-parameter solutions are indicated in Fig. 4 along with their error corridors. Note that $\delta(^1P_1)$ remains far from the expected value (at least 4.5 standard deviations above the predictions shown in Fig. 2) everywhere within the allowed range for ϵ_1.

To reduce this continuous range of phase-shift solutions to a single phase-shift set, additional data are necessary in order to pin down ϵ_1. Lacking these data at present, we defined a constrained solution by setting ϵ_1 equal to 2.78°, the value predicted at 50 MeV by Model C of Ref. 7. We believe this to be a theoretically reasonable value. Other values that might alternatively be chosen are 2.62°, predicted by the Yale potential,[17] 2.36°, predicted by the Reid potential (soft core),[18] and 2.37°, predicted by the Hamada-Johnston potential.[19]

The phase parameters of this constrained search on the data are recorded in Table IV and are

TABLE I. The n-p data used in our $pp+np$ analysis, together with the M values (χ^2 per data point) and the predicted normalizations.

Energy [a,b] (MeV)	Number, type data	M value	Predicted [c] normalization	Source [d]
47.5	11 $d\sigma/d\Omega$, forward	0.65	0.987	Harwell (1963)
47.5	11 $d\sigma/d\Omega$, backward	1.29	0.973	Harwell (1963)
48.8	1 σ_{tot}	1.73	...	Harwell (1961)
50.0	15 P	0.60	1.035	Harwell (1965)
52.5	12 $d\sigma/d\Omega$, forward	0.44	1.009	Harwell (1963)
52.5	11 $d\sigma/d\Omega$, backward	0.86	1.014	Harwell (1963)
52.5	1 σ_{tot}	1.61	...	Harwell (1961)
56.6	1 σ_{tot}	0.01	...	Harwell (1961)
57.5	12 $d\sigma/d\Omega$, forward	0.44	1.007	Harwell (1963)
57.5	11 $d\sigma/d\Omega$, backward	1.66	0.990	Harwell (1963)
58.8	1 σ_{tot}	0.16	...	Harwell (1961)
60.0	16 P	1.72	0.911	Harwell (1965)
49.06	1 σ_{tot}	4.69	...	Davis (1970)
54.37	1 σ_{tot}	2.55	...	Davis (1970)
59.35	1 σ_{tot}	0.04	...	Davis (1970)
50.0	12 $d\sigma/d\Omega$, backward	0.62	1.016	Davis (1971)
60.9	9 $d\sigma/d\Omega$, backward	1.09	0.927	Oak Ridge (unpublished)

[a] All data listed were used in MAW-X (Ref. 4), except for new data which appear in the last five lines of this table.

[b] Because of memory-space limitations, our code treated some of the data as if they occurred at slightly different energies from those reported by the experimentalists, although no energy shift was greater than 0.8 MeV. To decrease the effect of this energy discrepancy, we added a 1% error to the normalization of the total cross sections reported in Ref. 10. This seemed desirable because the very small errors reported for these measurements would otherwise require very precise energies (considering the energy slope of the n-p total cross section). A later computation by one of the authors (R. A. A.), where the energies were represented exactly, reproduced the results of the phase-shift analysis in this paper to within $\frac{1}{2}$ standard deviation for all phase parameters searched.

[c] The data are renormalized by dividing by this quantity.

[d] Harwell (1961), Ref. 13; (1963), Ref. 14; and (1965), Ref. 16. Davis (1970), Ref. 10; (1971), Ref. 11. Oak Ridge (unpublished), Ref. 12.

graphed as open-circle error bars in Fig. 1 [$\delta(^1P_1)$ is also graphed in Figs. 2 and 6]. The corresponding goodness of fit for each experiment is listed in Table I. The standard of comparison is taken to be the M value, which is the χ^2 per data point. Also given in Table I is the theoretical normalization for each experiment. This is the quantity by which the experimental observables are to be divided in order to yield a minimum-χ^2 contribution, determined both by the errors quoted for the over-all normalization and by the errors for the individual data points. We have plotted in Fig. 3 curves of the predictions of the Table IV phase shifts for the n-p observables employed in the analysis; these curves are labeled E, for experiment. The agreement with the data is good, as is to be expected since the χ^2 per data point is less than one (see Table IV).

Returning to Fig. 1, we observe that the phase parameters of our constrained search (computed and plotted at 55 MeV to avoid overlap with the 50-MeV points) agree rather well with the 50-MeV MAW-X parameters. This is to be expected in the case of the $T=1$ phase shifts, as the same p-p data were included in both searches and these data dominate the $T=1$ solutions. However, this agreement in the case of the $T=0$ phase parameters is mostly fortuitous, for the MAW-X solution is seen to merely correspond to a local χ^2 minimum in hyper-phase-parameter space; had the minimum occurred for ϵ_1 equal to, say $-10°$, a considerably different picture would have emerged (see Fig. 4).

Note that our errors are somewhat smaller than the MAW-X errors; this is mostly because we did not search ϵ_1, $\delta(^1S_0)_{np}$, $\delta(^3F_2)$, $\delta(^3F_3)$, or $\delta(^3F_4)$, while MacGregor *et al.* did. We did not search the triplet-F phase shifts because they are not well determined by the data, yet are quite consistent with OPEC. This can be seen in Fig.

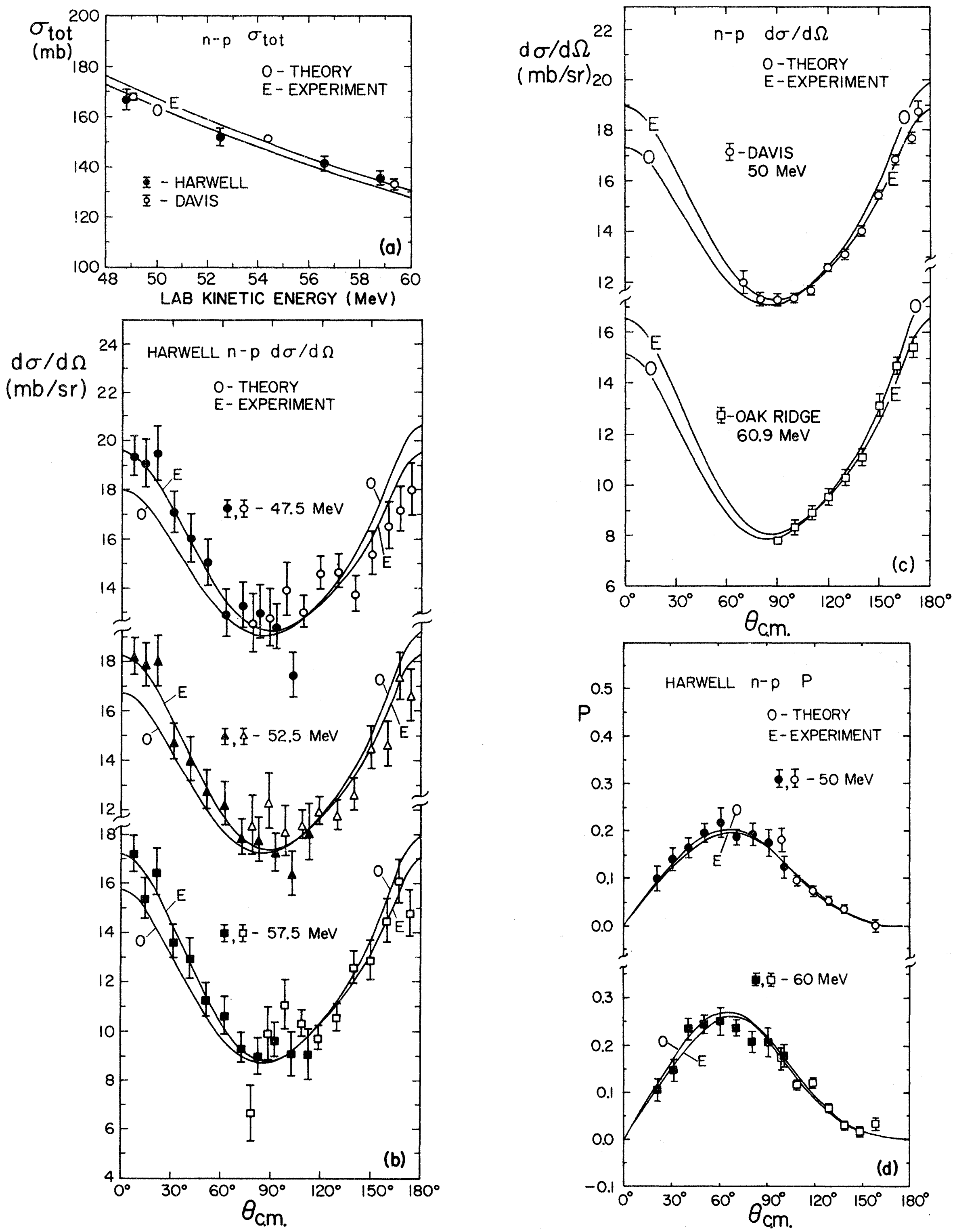

FIG. 3. Existing n-p data in the energy range 47.5–60.9 MeV. These are the data used in our phase-shift analysis. The curves labeled E are the predictions for σ_{tot}, $d\sigma/d\Omega$, and P by our constrained phase-shift analysis (Table IV). The curves labeled O are the predictions for σ_{tot}, $d\sigma/d\Omega$, and P by the phase parameters of a potential model due to Bryan and Gersten (Table V). The experimental differential cross-section and polarization data shown have been renormalized for minimum-χ^2 contribution in our phase-shift analysis, and thus go with curve E.

1(d) and Fig. 1(e) of Ref. 9, where the MAW-X error bars are shown, along with the curves for OPEC, OPEC$+2\pi+\rho+\omega$-exchange, and OPEC $+2\pi+\rho+\omega+\epsilon$-exchange. These curves, as can be seen in that reference, are so close together at 50 MeV that OPEC appears to be a reasonable approximation for the triplet-F phase shifts. With regard to $\delta(^1S_0)_{np}$, since this parameter is also not well determined by the data, we used the results of a potential model calculation of the Yale group reported in Table II of Ref. 17 to fix the $\delta(^1S_0)_{np} - \delta(^1S_0)_{pp}$ difference at 1.4°.

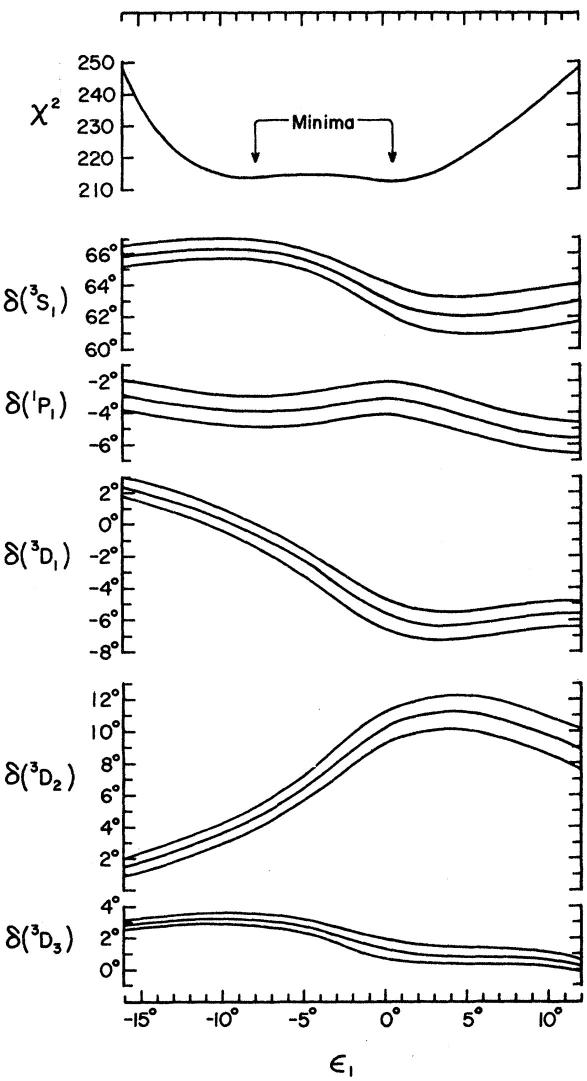

FIG. 4. Range of nuclear-bar phase shift solutions predicted by the 47.5- to 60.9-MeV world nucleon-nucleon data, plotted as a function of the unsearched coupling parameter ϵ_1. Plotted vs ϵ_1 are χ^2, and also the searched $T=0$ phase parameters with their error corridors.

TABLE II. Preliminary Davis neutron-proton differential cross-section measurements at 50 MeV.[a] (Presented by permission of T. C. Montgomery, F. P. Brady, and B. E. Bonner.)

Proton detection
Searched normalization[b] =1.016

c.m. angle (deg)	$d\sigma/d\Omega$ (unnormalized) (mb/sr)
173.34	19.07 ± 0.44
169.18	18.01 ± 0.23
159.42	17.15 ± 0.19
149.42	15.71 ± 0.18
139.36	14.27 ± 0.20
129.30	13.34 ± 0.22
119.24	12.80 ± 0.16
109.21	12.33 ± 0.19
99.20	11.89 ± 0.20
89.16	11.51 ± 0.26
79.20	11.10 ± 0.29
69.24	12.22 ± 0.45

[a] Reference 11. We have recently received from the same authors a report which quotes the above data plus new forward (neutron detection) data. We are informed that these forward data are probably not yet in final form, due to uncertainty in the calibration of the neutron detector. These new data points have 5% relative errors, and when included in our search, in a test run, have little effect on the phase shifts listed in Table IV, for reasons mentioned in Sec. VI of our paper.

[b] Data are divided by this factor.

TABLE III. Preliminary Oak Ridge neutron-proton differential cross-section measurements at 60.9 MeV.[a] (Presented by permission of M. J. Saltmarsh, C. R. Bingham, M. L. Halbert, C. A. Ludemann, and A. van der Woude.)

Proton detection
Searched normalization[b] =0.927

c.m. angle (deg)	$d\sigma/d\Omega$ (unnormalized) (mb/sr)
170[c]	14.31 ± 0.38
160	13.64 ± 0.32
150	12.19 ± 0.41
140	10.31 ± 0.31
130	9.56 ± 0.30
120	8.85 ± 0.31
110	8.26 ± 0.27
100	7.72 ± 0.28
90	7.23 ± 0.15

[a] Reference 12.

[b] Data are divided by this factor.

[c] The 170° point is still being analyzed by the above-mentioned investigators.

IV. A CLOSER LOOK AT THE *n-p* DATA

Since $\delta(^1P_1)$ in our search came out in disagreement with theory, we suspected that the data might be in error, and furthermore that the erroneous data might be brought to light if we would recompute the n-p σ_{tot}, $d\sigma/d\Omega$, and P curves with $\delta(^1P_1)$ set to the theoretically expected value of approximately $-9°$. We did this recomputation, but set all the phase parameters of Table IV to a theoretical prediction and not just $\delta(^1P_1)$. The theoretical model we elected for the phase parameters is the BG(C) model,[7] whose parameters were fitted to nucleon-nucleon data spanning the 0-450-MeV energy range. These "theoretical" phase parameters are listed in Table V.

The results of this computation of σ_{tot}, $d\sigma/d\Omega$, and P are interesting. The numbers are plotted in Fig. 3 as curves labeled O, and are referred to in the legend as Theory. These may conveniently be compared with the curves labeled E which represent the predictions of the 47.5-60.9-MeV experimental points via the phase-shift analysis as given in Table IV.[20] Note that in the case of σ_{tot}, while O disagrees with E, it still overlaps most of the data points. In the case of the polarization, O and E agree even more closely. However, in the case of $d\sigma/d\Omega$, curves O and E disagree qualitatively: The experimental values are, on the average, too high in the forward direction and too low in the backward direction of scattering. Thus it appears to be the differential cross-section data which are responsible for the aberrant value of $\delta(^1P_1)$.

Now, on closer inspection, where we separately analyze the data with only one laboratory's measurement of n-p $d\sigma/d\Omega$ present at a time, we find that the discrepancy between the theoretical and experimental $d\sigma/d\Omega$ curves is due to the Harwell $d\sigma/d\Omega$ data alone. The Harwell data also happen to be the largest and most overwhelming n-p $d\sigma/d\Omega$ data set. Note in Fig. 3(c) that the Davis and Oak Ridge $d\sigma/d\Omega$ data have no extreme forward points at all to establish the distinction between curves O and E in the forward direction. As for the backward direction, the Davis and Oak Ridge $d\sigma/d\Omega$ data are not in conflict with theoretical predictions, but are overwhelmed by the Harwell data. Analysis with only Davis data for n-p $d\sigma/d\Omega$ yields $\delta(^1P_1) = -11.75 \pm 3.58°$, which agrees with the theoretical predictions in Fig. 2; similarly an analysis with only Oak Ridge data for n-p $d\sigma/d\Omega$ yields $\delta(^1P_1) = -15.69 \pm 3.60°$. The errors are so large here because of the absence of extreme forward n-p $d\sigma/d\Omega$ data.

It would seem therefore that the measurements to make would be a careful redetermination of n-p $d\sigma/d\Omega$ at far forward and far backward angles.

TABLE IV. Searched nuclear-bar phase parameters [a] at 50 MeV.

No. of p-p data	98	
No. of n-p data	127	
p-p χ^2	97.8	
n-p χ^2	116.5	
Energy band	47.5–60.9 MeV	
	Phase parameters [b,c] (deg)	Energy slopes [d] $d\delta/dE$ (deg/MeV)
$I=1$		
$\delta(^1S_0)_{pp}$	$+38.98 \pm 0.28$	-0.390
$\delta(^3P_0)$	$+11.71 \pm 0.35$	$+0.000$
$\delta(^3P_1)$	-8.26 ± 0.15	-0.117
$\delta(^3P_2)$	$+5.90 \pm 0.09$	$+0.125$
$\delta(^1D_2)$	$+1.71 \pm 0.06$	$+0.041$
ϵ_2	-1.73 ± 0.07	-0.032
$\delta(^1S_0)_{np}$	$+40.38$	-0.390
$I=0$		
$\delta(^3S_1)$	$+62.22 \pm 1.12$	-0.648
$\delta(^1P_1)$	-3.52 ± 1.04	-0.081
ϵ_1	$+2.78$	$+0.025$
$\delta(^3D_1)$	-6.37 ± 0.90	-0.141
$\delta(^3D_2)$	$+11.24 \pm 1.06$	$+0.240$
$\delta(^3D_3)$	$+0.95 \pm 0.54$	$+0.019$

[a] Values used in the analysis are $g_\pi^2 = 14.43$, $m_\pi = 135.04$ MeV/c^2, and $M_N = 938.211$ MeV/c^2. The errors correspond to an increase in χ^2 of one when the other searched phase parameters have been readjusted for minimum χ^2. The phase parameters not appearing in this table were set to the OPEC values.

[b] The $\delta(^1S_0)_{np}$ phase shift was fixed at the value of $\delta(^1S_0)_{pp}$ plus 1.4°. This difference of 1.4° was taken from a potential-model calculation by the Yale group reported in Ref. 17.

[c] The coupling parameter ϵ_1 was fixed at the Bryan-Gersten (Ref. 7, model C) value of +2.78°.

[d] The lab kinetic-energy slopes of the phase parameters were not searched on, but were taken from a potential model of Bryan and Gersten (Ref. 7, model C).

V. THEORETICAL CONSEQUENCES IF THE 50-MeV DATA ARE CORRECT

Up until now we have tended to assume that the 50-MeV $\delta(^1P_1)$ resulting from our search, as well as from that of MacGregor *et al.*, is incorrect. This is because to distort one of the meson-theoretical curves for $\delta(^1P_1)$ in Fig. 2 to fit the 25- and 50-MeV points, one would have to add in a very attractive long-range potential to produce the upward bump in $\delta(^1P_1)$ at 50 MeV, and then probably increase the potential's repulsive strength at an even longer range to retain the negative value of $\delta(^1P_1)$ at 25 MeV. But even if a mechanism could be found for the long-range attraction, a mechanism which increases the even-longer-range repulsion will be in conflict with pion theory. Pion theory predicts that the longest-range part of the two-nucleon interaction is given by the one-pion-exchange potential, and this happens to be only mildly repulsive in the $^1P_1$ state at large distances. We show this in Fig. 5, where the one-pion-exchange potential is plotted in dashes and labeled OPEP. There it can be compared with the centrifugal barrier for P waves.

However, after we completed the phase-shift analysis of the data, we happened to come across a fit to $\delta(^1P_1)$ by a phenomenological potential due to Reid (soft-core version).[18] To our surprise, we found that the Reid potential predicts a $\delta(^1P_1)$, which is more or less in accord with the MAW-X analysis and our own analysis. Reid's $\delta(^1P_1)$ skirts above the MAW-X 25-MeV point and falls a little below the MAW-X and our own 50-MeV points, but over all fits the $\delta(^1P_1)$ phase-shift-analysis points over the 25- to 350-MeV range reasonably well. This one may see in Fig. 6, where we have plotted the predictions for $\delta(^1P_1)$ of the Reid potential vs the 25-425-MeV MAW-X values and our own value at 55 MeV. We were surprised that the Reid potential more or less fits the data because this potential reduces to the one-pion-exchange potential at large distances. It is not a meson-theoretical potential in the sense of other models described thus far in this paper, in that the more short-range parts of the potential are not directly related to specific mesonic

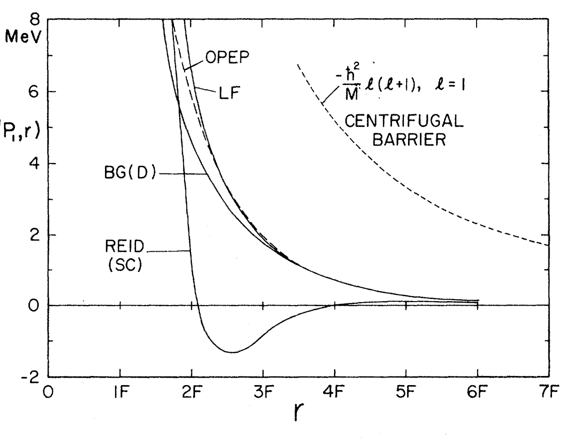

FIG. 5. Comparison of Lomon-Feshbach (LF), Reid soft-core [REID(SC)], Bryan-Gersten [BG(D)], and nonrelativistic one-pion-exchange potentials (OPEP) acting in the $^1P_1$ state. All potentials are to be used in conjunction with the Schrödinger equation. The Bryan-Gersten potential shown is the effective potential acting at 50 MeV. Parameters for the nonrelativistic one-pion-exchange potential are $g_\pi{}^2=15$ and $m_\pi = 138.1$ MeV/c^2. The centrifugal barrier for P waves is also shown.

TABLE V. Nuclear-bar phase parameters and energy slopes at 50 MeV predicted by a potential model due to Bryan and Gersten (Ref. 7, model C).

	Phase parameters [a] (deg)	Energy slopes $d\delta/dE$ (deg/MeV)
$I=1$		
$\delta(^1S_0)_{pp}$	+38.78	−0.390
$\delta(^3P_0)$	+11.97	+0.000
$\delta(^3P_1)$	−8.51	−0.117
$\delta(^3P_2)$	+6.00	+0.125
$\delta(^1D_2)$	+1.73	+0.041
ϵ_2	−1.81	−0.032
$\delta(^1S_0)_{np}$	+39.61	−0.390
$I=0$		
$\delta(^3S_1)$	+60.08	−0.648
$\delta(^1P_1)$	−8.76	−0.081
ϵ_1	+2.78	+0.025
$\delta(^3D_1)$	−6.83	−0.141
$\delta(^3D_2)$	+10.37	+0.240
$\delta(^3D_3)$	0.41	+0.019

[a] This model is an energy-dependent fit to 0–450-MeV data. Bryan and Gersten fix the $\delta(^1S_0)_{np} - \delta(^1S_0)_{pp}$ difference to be 0.83°, taken from the Yale potential calculations reported in Ref. 5.

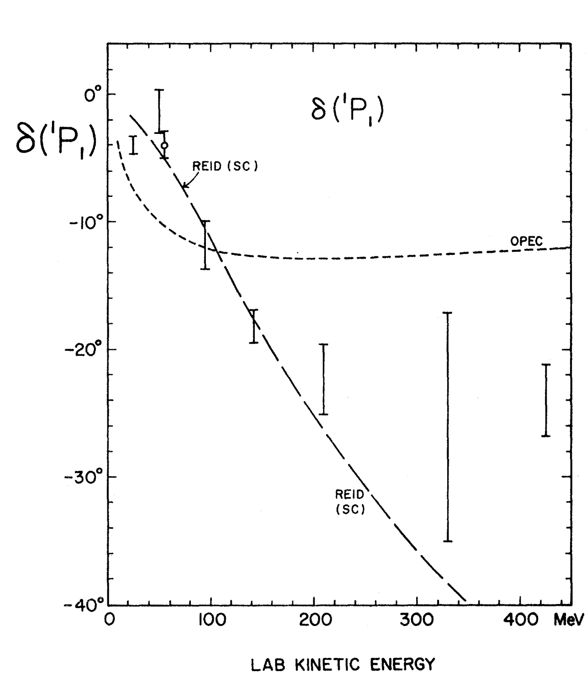

FIG. 6. Prediction for $\delta(^1P_1)$ by the phenomenological–plus–one-pion-exchange potential model of Reid [Ref. 18, soft-core (SC) model]. Also plotted is the OPEC prediction for $\delta(^1P_1)$, identical with that explained in the caption of Fig. 2. Also plotted are the MAW-X predictions and our own 55-MeV prediction for $\delta(^1P_1)$, likewise referred to in the caption of Fig. 2.

processes, but it does properly reduce to OPEP asymptotically.

To see how the Reid potential succeeds in fitting the $\delta(^1P_1)$ data, we made a plot of his potential and compared it with other potentials. We have reproduced these plots in Fig. 5. It becomes apparent that the Reid potential fits the data because it has a strong intermediate-range attraction, far stronger than that encountered in, e.g., the Lomon-Feshbach or the Bryan-Gersten (D) potentials, also plotted in Fig. 5.

This intermediate-range attraction of the Reid $^1P_1$ potential comes from a term which goes as $(e^{-2m_\pi r})/r$, so as far as the exponent is concerned, it can be identified with two-pion-exchange processes. However, in real two-pion-exchange processes, the effective mass of the exchanged two-pion system ranges from two-pion masses on up, so that the effective long-range part of any two-pion exchange potential is likely to go more like $(e^{-3m_\pi r})/r$, or to have even shorter range. What the Reid intermediate-range attraction really resembles is not two-pion exchange but exchange of a scalar meson ($J^P=0^+$) or perhaps a tensor meson ($J^P=2^+$), with mass approximately equal to two pion masses. One is reminded of the old ABC (Abashian, Booth, and Crowe) meson,[21] which, if it exists, has a mass $\approx$300 MeV and $J^P=0^+$. The exchange of this meson will give rise to an attractive potential of range $(2m_\pi)^{-1}$, exactly like the $(2m_\pi)^{-1}$ range attractive potential term which appears in the Reid $^1P_1$ potential. If such a meson exists, and if it couples strongly enough to the nucleon to produce the intermediate-to-long-range attraction in the $^1P_1$ state depicted by Reid, one can say that a very interesting new feature in nucleon forces has been uncovered by the n-p differential cross-section data near 50 MeV. However, one would expect that if a scalar meson produced a strong attraction in one P state, it would produce an attraction in all P states, whereas a glance at Fig. 1 reveals that there is no evidence near 50 MeV of an attractive force in either the $^3P_0$, $^3P_1$, or $^3P_2$ states. Furthermore, the existence of the ABC meson has never been confirmed, nor has the existence of any other meson near 300 MeV ever been confirmed, if indeed reported.

Another possible explanation of the $\delta(^1P_1)$ anomaly that has occurred to us is that it might be a mistake to assume that charge independence is exact in the P states. In carrying out the analysis of the n-p data to determine the $I=0$ phase parameters, we simultaneously search on the p-p data and assume that the $I=1$ phase parameters are identical in the p-p and n-p system, apart from a mild $\delta(^1S_0)_{pp}-\delta(^1S_0)_{np}$ splitting reported in Sec. II. Indeed, if we searched on the n-p data alone to get the $I=0$ parameters, it would not be possible to determine them because too few n-p experiments have been carried out. However, assuming that charge independence is exactly satisfied in the p-p and n-p systems (except for the $^1S_0$ state) may be too stringent a condition. We have performed a rough calculation which shows that if the n-p triplet-P phase shifts were uniformly 2° more positive than the p-p triplet-P phase shifts, then the $^1P_1$ phase shift would search several degrees more negative and agree with the predictions of the theoretical models, such as are graphed in Fig. 2. Still another way the data could be correct while $\delta(^1P_1)$ agreed with theoretical predictions would be if charge independence were violated to a significant extent due to effects like the splitting of the charged and uncharged pion masses in the OPE contribution to the long-range part of the nucleon-nucleon force. This would affect P waves and higher partial waves as well.

On the other hand, at energies other than 50 MeV, where phase-shift analyses have been performed assuming charge independence, $\delta(^1P_1)$ does not appear unreasonable. [Perhaps the MAW-X $\delta(^1P_1)$ is a little unreasonable at 25 MeV.] It would seem to be a strange mechanism which would produce a breaking of charge independence in the $l\geq 1$ waves at 50 MeV and not at other energies. Still, we intend to explore charge-independence breaking as a possible explanation of the $\delta(^1P_1)$ anomaly.

VI. EXPERIMENTAL RECOMMENDATIONS

We have studied the sensitivity of various n-p observables ($d\sigma/d\Omega$, P, triple scattering, and correlations) to small variations in $\delta(^1P_1)$ and find that the differential cross section is indeed the best type of measurement to pin down $\delta(^1P_1)$. Therefore, we recommend that the n-p differential cross section near 50 MeV be remeasured to determine if the $\delta(^1P_1)$ anomaly really exists. We want to emphasize that, contrary to popular belief, it is not sufficient merely to measure the relative differential cross section at 90° and at far backward angles, because incorrect $d\sigma/d\Omega$ measurements at far *forward* angles will throw off the $^3D_1$, $^3D_2$, and $^3D_3$ phase shifts, even if the backward-angle data are correct, and this in turn will throw off the value of $\delta(^1P_1)$.

If the Harwell $d\sigma/d\Omega$ data are dispensed with, we find that a single n-p $d\sigma/d\Omega$ experiment with *absolute* measurement of $d\sigma/d\Omega$ at, say, 10° and 20° c.m. to ±1%, will, in conjunction with the remaining σ_{tot}, non-Harwell $d\sigma/d\Omega$, and P data, be sufficient to fix the value of $\delta(^1P_1)$ quite well.[22] In this case new backward $d\sigma/d\Omega$ data can be dispensed with. We note, incidentally, that absolute measurements of $d\sigma/d\Omega$ at 10° and 20° will be far more effective in pinning down $\delta(^1P_1)$ than relative measurements of $d\sigma/d\Omega$ to ±3% accuracy at 10° intervals from 10° to 90° c.m. with unknown normalization.

If the Harwell $d\sigma/d\Omega$ measurements are retained in a phase-shift analysis, then in order to override the Harwell data, a good relative measurement of $[d\sigma/d\Omega(180°)]/[d\sigma/d\Omega(90°)]$ should be added to the absolute forward measurement of $d\sigma/d\Omega$ previously suggested.

Finally, we emphasize that more n-p data of the kinds now in existence (σ_{tot}, $d\sigma/d\Omega$, and P) will be insufficient to pin down ϵ_1. Some other type of experiment must be carried out. We intend to investigate this in a succeeding paper.

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[22]To test this statement, we carried out a trial phase-shift analysis wherein the Harwell $d\sigma/d\Omega$ data were deleted, and two hypothetical $d\sigma/d\Omega$ points were inserted at 10° and 20° c.m. falling on curve O [see Fig. 3(b)] with absolute accuracy ±1%. This resulted in a continuum of solutions as before, with χ^2 vs ϵ_1 nearly flat in the range $-10° \leq \epsilon_1 \leq +3°$, also as before. Thus the hypothetical data left ϵ_1 as poorly determined as ever. However, they pinned down $\delta(^1P_1)$ to a value which was nearly constant throughout the allowed range of ϵ_1. This value was $\delta(^1P_1) = -8° \pm 1°$. Thus it is possible to experimentally determine $\delta(^1P_1)$, and compare it with theory, even though ϵ_1 remains undetermined.

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A Statistical Measure of Clustering in Multiparticle Final States*

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A quantitative measure of clustering effects in many-particle final states is defined and its significance discussed. The results consist of a single curve and a number, $\langle \omega_n^2 \rangle$, which may be extracted from inclusive or exclusive data where the longitudinal momenta (or some other variable) have been measured for n particles in each event. The curve is a measure of the average fluctuation of each event away from the over-all distribution, and is defined strictly in terms of experimental quantities. The results appear to provide a sensitive test for models of hadron production. Comparison with Monte Carlo calculations, or with a statistical reference model which is described, allow one to interpret the results in a fairly model-independent manner. The analysis is then applied to some 13-GeV/c K^-p data.

I. INTRODUCTION

One of the most important issues concerning multiparticle production in high-energy hadron collisions is the character of the clustering of final-state particles within the allowed phase space. The depopulation of phase space at large transverse momenta is a well-known and apparently universal signature of high-energy collision processes. However, the identification and detailed study of clustering effects in the longitudinal variables has been carried out only for specific low-multiplicity final states where exclusive analyses are feasible.[1,2] Attempts to gain more global information regarding the importance and character of clustering in the longitudinal variables have proved to be inconclusive for two reasons:

(i) The averaging inherent in measurements of inclusive cross sections appears to obscure the longitudinal clustering behavior to such a degree that models based on very different pictures of particle production are able to account equally well for much of the observed behavior of the data.

(ii) For events of high final-state multiplicity, it is difficult to make a precise operational definition of clustering. The interpretation of the longitudinal behavior is strongly colored by assumptions about the clustering effects in the transverse-momentum variables, and by the constraints imposed by energy and momentum conservation.

In this paper we recast the problem into a form which suggests a method for analyzing clustering effects in a general and model-independent manner. We analyze here some particular low-multiplicity ($n \leq 8$) data at low energy ($E \leq 30$ GeV), and point out the ease with which this analysis can be extended to the highest available energies and multiplicities.

Presently, experimental evidence for longitudinal clustering in n-body hadronic final states is obtained from studies of correlations among two or more of the $3n-4$ independent kinematic variables. Specifically, one examines a Dalitz plot, or a longitudinal phase-space plot (or prism plot),[1] or employs some other device for determining whether or not the final-state particles tend to bunch in isolated regions of the allowed phase space for some subset of the available kinematic variables.[2] An example from the class of 4-body final states for which such analyses have been carried out is $K^-p \rightarrow K^-p\pi^+\pi^-$. (See Ref. 3.) Two nearly incoherent mechanisms contribute to the final state, with each resulting in a clustering of events in separated regions of phase space. This clustering may be examined by projecting the data onto a two-dimensional plot, provided the right variables are chosen (Fig. 1). Dissociation