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Semimonotonic Bounds for K_{l3} Scalar Form Factor

Susumu Okubo*† and Yoshiaki Ueda
Institute of Theoretical Physics, Fack, S-402 20 Göteborg 5, Sweden
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On the basis of analyticity and the soft-pion theorem, various exact bounds for the scalar K_{l3} form factor $D(t)$ have been obtained, assuming that $D(t)$ satisfies at most a once-subtracted dispersion relation, and that $\operatorname{Im} D(t+i0)$ can change its sign at most only once on the cut. These bounds do not contain any arbitrary free parameters. Neither do we use any explicit Hamiltonian nor any approximation such as the pole-dominance model for its derivation. The results obtained disagree with present experiment. A connection of this method with the phase representation of $D(t)$ is also discussed, and we find that under some simple conditions, the result is at variance with present experiment.

I. INTRODUCTION AND SUMMARY OF PRINCIPAL RESULTS

The K_{l3} decay problem is interesting theoretically as well as experimentally. In this paper we will specifically consider the scalar form factor $D(t)$ defined by

$$D(t) = (m_K^2 - m_\pi^2) f_+(t) + t f_-(t) , \tag{1.1}$$

where $f_\pm(t)$ is the standard form factor.[1,2] It is convenient to set

$$d(t) = \frac{D(t)}{D(0)} . \tag{1.2}$$

Experimentally, we can measure values of $D(t)$ in the physical range:

$$m_l^2 \leq t \leq (m_K - m_\pi)^2 , \tag{1.3}$$

where m_l is the lepton mass. So far, the majority of experimental data suggests[2,3] that we have

$$d(t) < 1, \quad d'(0) < 0 , \tag{1.4}$$

although a recent preliminary datum[4] appears to give a result contradictory to this. Note that the derivative $d'(0)$ is related to the conventional K_{l3} parameters λ_+ and ξ by

$$m_\pi^2 d'(0) = \lambda_+ + \xi (m_K^2 - m_\pi^2)^{-1} m_\pi^2 . \tag{1.5}$$

The result (1.4) is very difficult for us to explain theoretically[2] because of the following facts: First, the soft-pion theorem[5] demands

$$d(\Delta) \approx \frac{f_K}{f_\pi f_+(0)} \approx 1.28 , \tag{1.6}$$

where the soft-pion point Δ is taken[6] to be

$$\Delta = m_K^2 - m_\pi^2 . \tag{1.7}$$

Second, $D(t)$ is a real analytic function of t with a cut on the real axis at

$$t_0 = (m_K + m_\pi)^2 \leq t < \infty . \tag{1.8}$$

Here by reality, we imply validity of

$$D^*(t^*) = D(t) , \tag{1.9}$$

so that $D(t)$ is real below threshold t_0. We note that the soft-pion point Δ lies nearly in the middle between the threshold t_0 of the cut and the extreme end of the physical region specified by (1.3), i.e., we have

$$(m_K - m_\pi)^2 < \Delta < t_0 . \tag{1.10}$$

Now, if we assume temporarily that $d(t)$ can be approximated by a linear function of t in the range $0 \leq t < \Delta$, then we must have

$$d(t) \geq 1 , \quad d'(0) > 0 \tag{1.11}$$

in that range because of the soft-pion theorem (1.6). This contradicts the experimental result (1.4). Although the linear interpolating procedure may look very suspicious at this point, the analyticity property of $D(t)$ is so stringent that we can actually justify the final answer (1.11) on the basis of exact inequalities,[7,8] if the chiral SW(3) theory[9] of Gell-Mann *et al.* is assumed in addition to some unspecified technical assumptions. However, the validity of the chiral SW(3) theory is very far from being established and it is desirable to consider the problem from other angles.

In a previous paper,[10] a new simpler approach, which is based upon a purely dispersion-theoretical technique, has been suggested to obtain various exact inequalities for $D(t)$, which do not contain any arbitrary free parameters. An additional advantage of the method is the fact that we need *not* assume any explicit form for the strong-interaction Hamiltonian. In this paper, we will investigate the same method in greater detail and show that the same difficulty with the experimental result (1.4) persists.

Anticipating a possible error of the soft-pion theorem up to 20%, we shall relax it by a weaker one,

$$d(\Delta) \geq 1 \ , \tag{1.12}$$

unless it is otherwise stated. Also, we note that we must have the normalization condition

$$d(0) = 1 \tag{1.13}$$

from the definition (1.2).

Now, suppose first that $d(t)$ satisfies an unsubtracted dispersion relation (hereafter referred to as USDR)

$$d(t) = \frac{1}{\pi}\int_{t_0}^{\infty} dt' \frac{1}{t'-t} \operatorname{Im} d(t'+i0) \ . \tag{1.14}$$

Moreover, if $\operatorname{Im} d(t'+i0)$ does not change its sign on the entire cut $t' > t_0$, then we can derive an inequality

$$0 \leq A_1(t) \leq \frac{1}{t}[d(t)-1] \leq \operatorname{Min}\{A_2(t), A_3(t)\} \tag{1.15}$$

for the values of t satisfying

$$0 \leq t \leq \Delta \ , \tag{1.16}$$

which includes especially the physical range (1.3). In the above, $A_j(t)$ $(j=1,2,3)$ are given by

$$A_1(t) = \frac{t_0-\Delta}{t_0-t}\ \frac{1}{\Delta}[d(\Delta)-1] \ , \tag{1.17a}$$

$$A_2(t) = \frac{d(\Delta)-1}{(\Delta-t)\,d(\Delta)+t} \ , \tag{1.17b}$$

$$A_3(t) = \frac{1}{t_0-t} \ . \tag{1.17c}$$

Hereafter, the range of t inside any inequality for $d(t)$ is automatically understood to satisfy the condition (1.16).

Note that (1.15) gives a result contradicting (1.4). Even if $\operatorname{Im} d(t'+i0)$ does change its sign only once on the cut, we can still derive a weaker inequality whose explicit form slightly varies according to numerical values assumed for $d(\Delta)$ [see Eqs. (2.14) and (2.15)]. If we accept the exact soft-pion value of $d(\Delta) = 1.28$, then it is given by

$$0 \leq A_1(t) \leq \frac{1}{t}[d(t)-1] \leq B_1(t) \ , \tag{1.18}$$

$$B_1(t) = \frac{(t_0-\Delta)^2}{(t_0-t)^2}\ \frac{1}{\Delta}[d(\Delta)-1] + \frac{\Delta-t}{(t_0-t)^2} \ . \tag{1.19}$$

Again, this leads to a result contradicting (1.4). Moreover, if $d(t)$ is super-convergent, i.e., if it satisfies a stronger condition

$$\lim_{t\to\infty} t\,d(t) = 0 \tag{1.20}$$

as the electromagnetic form factors of the nucleon, then we can derive stronger results. Now allowing $\operatorname{Im} d(t'+i0)$ to change its sign up to twice on the cut, we find

$$0 \leq A_1(t) \leq \frac{1}{t}[d(t)-1] \leq \operatorname{Max}\{B_2(t), B_3(t)\} \ , \tag{1.21}$$

where $B_j(t)$ $(j=2,3)$ are defined by

$$B_2(t) = \frac{1}{\Delta}[d(\Delta)-1] \ , \tag{1.22a}$$

$$B_3(t) = \frac{1}{\Delta}\left(\frac{t_0-\Delta}{t_0-t}\right)^3 d(\Delta) + \frac{1}{t}\left[\frac{\Delta-t}{\Delta}\left(\frac{t_0}{t_0-t}\right)^3 - 1\right]. \tag{1.22b}$$

In this derivation we have assumed

$$\frac{t_0}{t_0-\Delta} \geq d(\Delta) \geq 1 \ ,$$

which is numerically satisfied by (1.6).

Next, consider the case that $d(t)$ satisfies a once-subtracted dispersion relation (hereafter referred to as OSDR):

$$d(t) = 1 + \frac{t}{\pi}\int_{t_0}^{\infty} dt' \frac{1}{t'(t'-t)} \operatorname{Im} d(t'+i0) \ . \tag{1.23}$$

First, if $\operatorname{Im} d(t'+i0)$ does not change its sign on the cut, then the following inequality can be obtained:

$$0 \leq A_1(t) \leq \frac{1}{t}[d(t)-1] \leq \frac{1}{\Delta}(d(\Delta)-1) = B_2 \ . \tag{1.24}$$

It is interesting to note that all inequalities derived so far have a common lower bound. Explicitly, inequalities (1.15), (1.18), (1.21), and (1.24) give

$$0 \leq A_1(t) \leq \frac{1}{t}[d(t)-1] \ . \tag{1.25}$$

Setting $t=0$ in the above, we find also a common lower bound for $d'(0)$,

$$0 \leqslant A_1(0) \leqslant d'(0) \ .$$

Using the soft-pion value $d(\Delta)=1.28$, bounds for $d'(0)$ are estimated to be

$$0.01 \leqslant m_\pi^{\,2} d'(0) \leqslant \{0.018,\ 0.03,\ 0.066,\ 0.023\} \ , \tag{1.26}$$

corresponding to (1.15), (1.18), (1.21), or (1.24), respectively.

These values are comparable to the bounds found in Ref. 7. These bounds are in conflict with (1.4). Hence if the experimental result (1.4) is correct, then $\mathrm{Im}\,d(t'+i0)$ must change its sign either three times, or twice, or once, according to whether $d(t)$ satisfies a superconvergent, unsubtracted, or once-subtracted dispersion relation, respectively. In terms of pole models, this implies that either four, three, or two κ poles must exist to account for the behavior of $d(t)$, respectively.

The case when $D(t)$ has a zero point on the cut itself is interesting. Assuming that $\mathrm{Im}\,D(t'+i0)$ changes its sign only at that zero point and that $D(t)$ satisfies a once-subtracted or twice-subtracted dispersion relation, we find

$$A_4(t) \leqslant \frac{1}{t}\,[d(t)-1] \leqslant B_4(t) \ , \tag{1.27}$$

$$B_4(t) = \frac{t_0-t}{t_0-\Delta}\ \frac{1}{\Delta}\ [d(\Delta)-1] + \frac{1}{t_0}\,\frac{\Delta-t}{t_0-\Delta} \ , \tag{1.28a}$$

$$A_4(t) = A_1(t) - t_0\frac{\Delta-t}{t_0-t}\left(\frac{1-x}{\Delta}\right)^2, \tag{1.28b}$$

$$x = \left[\frac{(t_0-\Delta)}{t_0}\ d(\Delta)\right]^{1/2}, \tag{1.28c}$$

where we have assumed $x<1$, which is numerically satisfied by (1.6).

When $D(t)$ has no zero point, we can also make the following predictions: Assuming that the inverse function $1/D(t)$ satisfies USDR, then we can show first that $\mathrm{Im}\,d(t'+i0)$ must change its sign at least once, since otherwise it will lead to a result incompatible with (1.12). Hence, assuming now that $\mathrm{Im}\,d(t'+i0)$ changes its sign only once on the cut, we find

$$A_5(t) \leqslant \frac{1}{t}\left[1-\frac{1}{d(t)}\right] \leqslant B_5(t) \ , \tag{1.29}$$

$$A_5(t) = \left(\frac{t_0-\Delta}{t_0-t}\right)^2 \frac{1}{\Delta}\left[1-\frac{1}{d(\Delta)}\right] - \frac{(\Delta-t)}{(t_0-t)^2} \ , \tag{1.30a}$$

$$B_5(t) = \frac{1}{\Delta}\left[1-\frac{1}{d(\Delta)}\right] \tag{1.30b}$$

under the same conditions for $1/D(t)$.

Note that the inequality (1.29) can be compatible with (1.4). However, as we see from Fig. 2, the deviation of $d(t)$ from the unity is rather small to be compatible with the experimental data of Ref. 2. The inequality (1.29) is not compatible with the preliminary data of Ref. 4 either.

Finally, we have investigated the connections of our method with the phase representation:

$$d(t) = P(t)\exp\left[\frac{1}{\pi}\,t\int_{t_0}^{\infty} dt'\,\frac{\delta(t')}{t'(t'-t)}\right] , \tag{1.31}$$

where $P(t)$ is a real polynomial of t, and $\delta(t)$ is the $I=\frac{1}{2}$, S-wave pion-kaon scattering phase shift. We have also studied Eq. (1.31) for cases where $P(t)$ is at most a linear function of t. Again, we find that it leads to a result incompatible with (1.4), suggesting $P(t)$ to be at least quadratic in t.

Summarizing, we conclude that it is very difficult to explain the experimental result (1.4), unless either we give up the soft-pion theorem (1.6) or $d(t)$ requires at least two subtractions for its dispersion relation. However, it must be kept in mind that the present experimental situation is perhaps by no means final. Hence, we have plotted our inequalities in Figs. 1 and 2 (see Refs. 2 and 4) and compared them with experimental data available, assuming the exact soft-pion value of $d(\Delta)=1.28$.

II. DERIVATION OF INEQUALITIES

Let $f(t)$ always represent a real holomorphic function of t in a cut t plane with a cut on the real positive axis at $t_0 \leqslant t < \infty$. The reality property of $f(t)$ means

$$f^*(t^*) = f(t) \ . \tag{2.1}$$

Moreover, we assume that its imaginary part never changes its sign on the entire cut, i.e., we have

$$\epsilon\,\mathrm{Im}f(t'+i0) \geqslant 0 \ , \quad \epsilon = \pm 1 \tag{2.2}$$

for all $t' \geqslant t_0$, where ϵ is a constant sign function independent of t' with values $\epsilon=\pm 1$. Then, in accordance with the terminology of Ref. 10, we shall call such a function $f(t)$ semimonotonic. We remark that its inverse $[f(t)]^{-1}$ is also semimonotonic, if $f(t)$ has no zero point in the cut plane.

First suppose that $f(t)$ satisfies OSDR (once-subtracted dispersion relation):

$$f(t) = f(0) + \frac{t}{\pi}\int_{t_0}^{\infty} dt'\,\frac{1}{t'(t'-t)}\,\mathrm{Im}f(t'+i0). \tag{2.3}$$

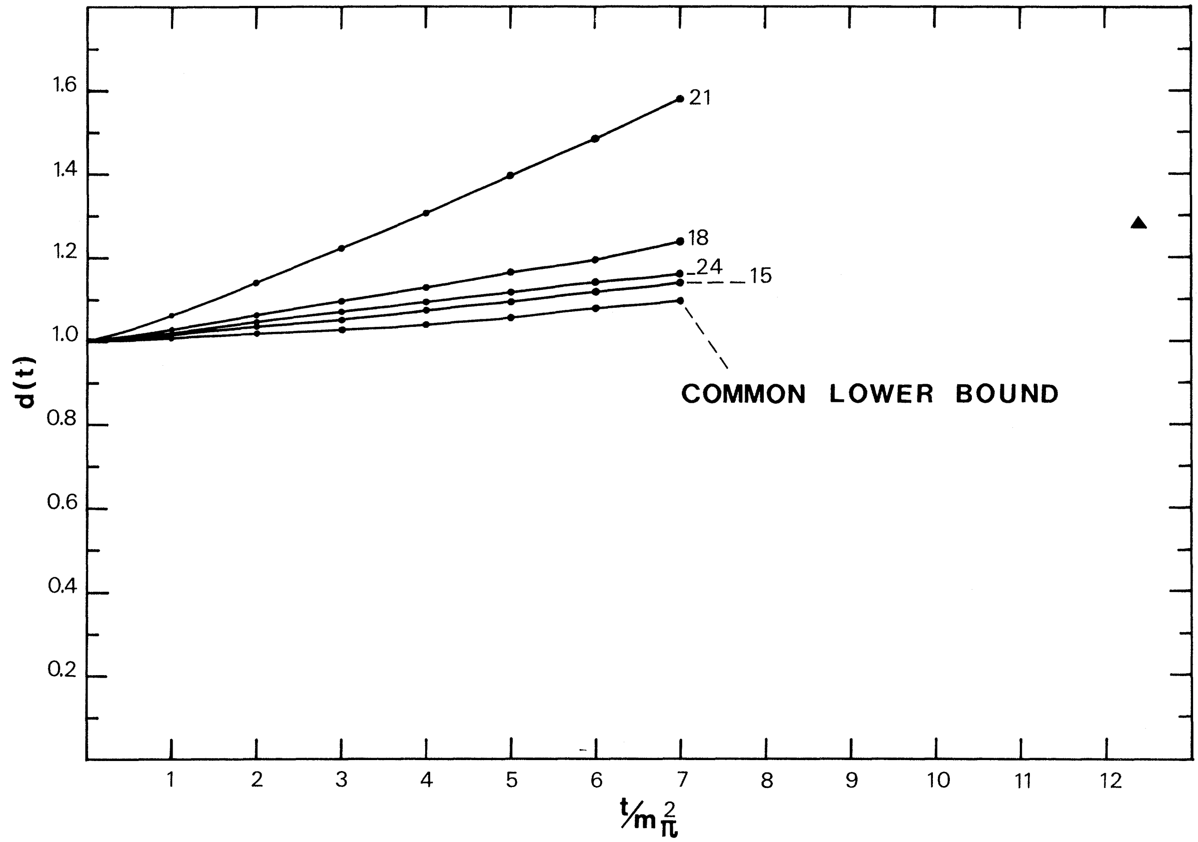

FIG. 1. Various theoretical bounds for the normalized scalar form factor $d(t)$, corresponding to inequalities (1.15), (1.18), (1.21), and (1.24). The number 15, 18, 21, or 24 represents an upper bound for $d(t)$ determined from a respective inequality in Sec. I. The bottom curve represents a common lower bound for $d(t)$. The symbol ▲ represents a predicted value due to the soft-pion theorem.

Let us further assume that $f(t)$ is semimonotonic; it is then easy to derive an inequality

$$0 \leq \epsilon \frac{t_0-\Delta}{t_0-t}\,\frac{f(\Delta)-f(0)}{\Delta} \leq \epsilon \frac{f(t)-f(0)}{t} \leq \epsilon \frac{f(\Delta)-f(0)}{\Delta} \qquad (2.4)$$

for values of t satisfying

$$0 \leq t < \Delta\ . \qquad (2.5)$$

When we identify $f(t)=d(t)$, then we must have $\epsilon=+1$ in view of (1.12) and (1.13), and we immediately recover (1.24).

Before going into further details, we note that the inequality (2.4) will be valid even for a slightly more general case when either $f(t)$ or $-f(t)$ is a Herglotz function[11] of t. The Herglotz function $\epsilon f(t)$ ($\epsilon=\pm 1$) now satisfies a representation of the form[11]

$$\epsilon f(t) = at + \epsilon f(0) + \frac{t}{\pi}\int_{t_0}^{\infty} dt'\, \frac{\rho(t')}{t'(t'-t)}\ , \qquad (2.6)$$

with $a \geq 0$ and $\rho(t') \geq 0$. It is now easy to derive (2.4) from (2.6). Consequently, we note that if $\epsilon f(t)$ is a Herglotz function and if $f(t)$ has no zero point, then[11] the inverse function

$$g(t) = -\frac{\epsilon}{f(t)}$$

is also a Herglotz function without CDD (Castillejo-Dalitz-Dyson) poles. Hence, applying the inequality (2.4) to this function, we find

$$\epsilon \frac{(t_0-\Delta)f(0)[f(\Delta)-f(0)]}{t_0(\Delta-t)f(\Delta)+t(t_0-\Delta)f(0)} \leq \epsilon \frac{f(t)-f(0)}{t} \leq \epsilon \frac{f(0)[f(\Delta)-f(0)]}{(\Delta-t)f(\Delta)+tf(0)}\ , \qquad (2.7)$$

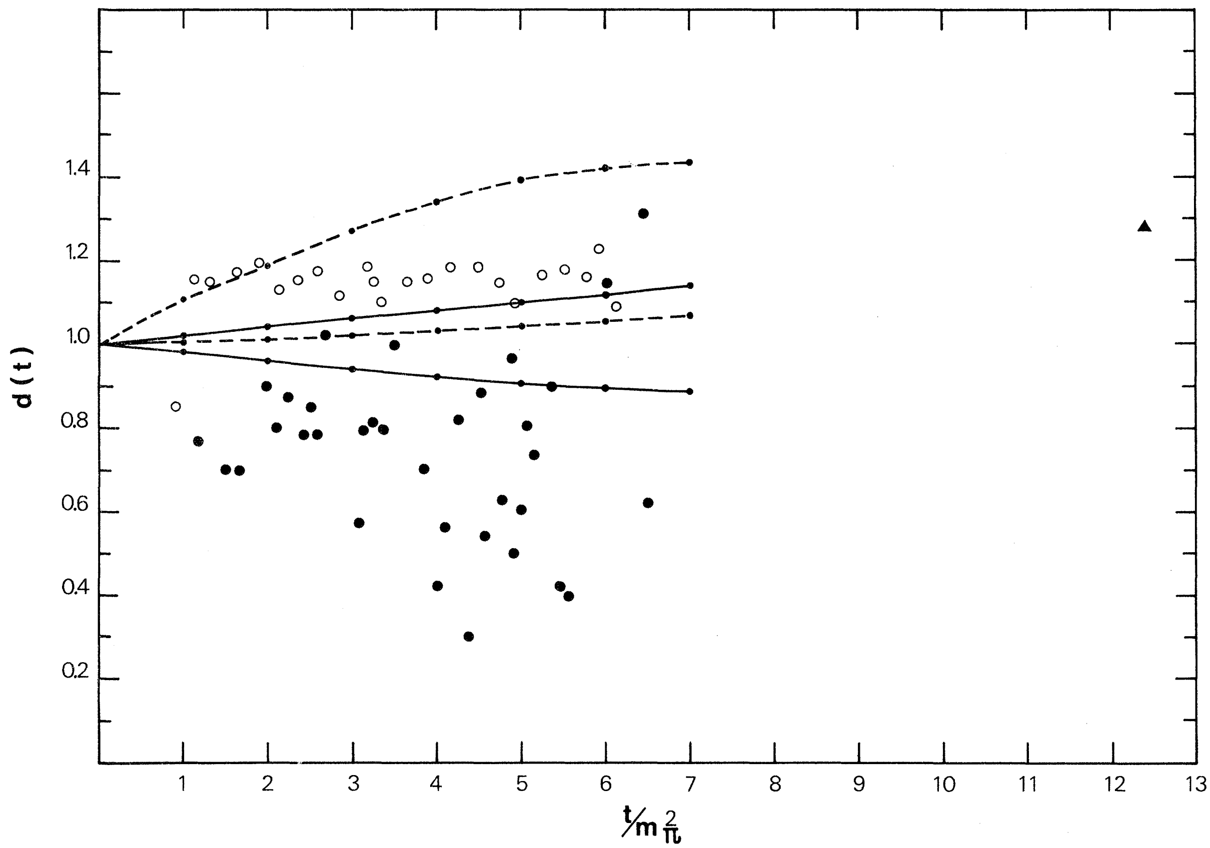

FIG. 2. Theoretical bounds for $d(t)$ corresponding to (1.27) and (1.29). Dashed lines: bounds for $d(t)$ determined from (1.27). Solid lines: bounds for $d(t)$ determined from (1.29). Symbol ▲ represents a predicted value due to the soft-pion theorem. Experimental data are plotted for comparison. Closed circles represent data compiled in Ref. 2. Open circles represent preliminary data of Ref. 4.

where for simplicity we have assumed

$$f(0)f(\Delta)>0\ . \tag{2.8}$$

Next, let us suppose that $f(t)$ satisfies USDR (unsubtracted dispersion relation):

$$f(t)=\frac{1}{\pi}\int_{t_0}^{\infty}dt'\,\frac{1}{t'-t}\,\mathrm{Im}f(t'+i0)\ . \tag{2.9}$$

Then, it also satisfies OSDR (2.3), so that we must have (2.4) automatically, if $f(t)$ is semimonotonic. Moreover, from (2.9) and from the semimonotonicity condition (2.2), $f(t)$ has no zero point in the cut plane. Hence, it must satisfy inequality (2.7), also. In addition, a further inspection of (2.9) leads to

$$0\leqslant\epsilon\frac{t_0-t}{t_0}f(t)\leqslant\epsilon f(0)\leqslant\epsilon f(t)\leqslant\epsilon f(\Delta)\ . \tag{2.10}$$

Note that the condition (2.8) is automatically satisfied because of (2.10). All inequalities (2.4), (2.7), and (2.10) for the unsubtracted case are now combined into a single inequality

$$0\leqslant\epsilon\frac{t_0-\Delta}{t_0-t}\,\frac{1}{\Delta}[f(\Delta)-f(0)]$$

$$\leqslant\epsilon\frac{1}{t}[f(t)-f(0)]$$

$$\leqslant\mathrm{Min}\{M_1(t),M_2(t)\}\ , \tag{2.11a}$$

$$M_1(t)=\epsilon\frac{f(0)[f(\Delta)-f(0)]}{(\Delta-t)f(\Delta)+tf(0)}, \tag{2.11b}$$

$$M_2(t)=\epsilon\frac{f(0)}{t_0-t}\ . \tag{2.11c}$$

When we identify $f(t)=d(t)$, then we must have $\epsilon=+1$ again and Eq. (2.11) gives (1.15) with (1.17).

If $d(t)$ satisfies USDR and if $\mathrm{Im}d(t'+i0)$ changes its sign only once at $t'=t_1>t_0$, then the function defined by

$$f(t)=(t_1-t)\,d(t)$$

is easily seen to be semimonotonic. Moreover, it obviously satisfies OSDR. Hence, applying (2.4), we obtain

$$0 \leqslant \epsilon \frac{t_0-\Delta}{t_0-t}\frac{1}{\Delta}\left[(t_1-\Delta)\,d(\Delta)-t_1\right] \leqslant \epsilon\frac{1}{t}\left[(t_1-t)\,d(t)-t_1\right] \leqslant \epsilon\frac{1}{\Delta}\left[(t_1-\Delta)\,d(\Delta)-t_1\right] . \tag{2.12}$$

Then, setting

$$\beta = \Delta\frac{d(\Delta)}{d(\Delta)-1} \tag{2.13}$$

and noting that t_1 is larger than t_0, we find (2.12) leads to the following inequalities according to the two cases:

(i) $\epsilon=+1$; $t_1 \geqslant \mathrm{Max}(\beta, t_0)$.

$$\frac{1}{\beta-\Delta} \geqslant \frac{1}{t}\left[d(t)-1\right] \geqslant \mathrm{Min}\{A_1(t), B_1(t)\} , \tag{2.14}$$

(ii) $\epsilon=-1$; $\beta \geqslant t_1 \geqslant t_0$.

$$B_1(t) \geqslant \frac{1}{t}\left[d(t)-1\right] \geqslant A_1(t) \geqslant 0 , \tag{2.15}$$

where $A_1(t)$ and $B_1(t)$ are given by Eq. (1.17a) and (1.19), respectively. It is obvious that we have

$$B_1(t) \gtrless A_1(t)$$

according to whether we have $\beta \gtrless t_0$. Moreover, we find

$$B_1(t) \geqslant \frac{1}{\beta-\Delta} \geqslant A_1(t)$$

if β is larger than $2t_0$. Assuming the exact soft-pion value of $d(\Delta)=1.28$, we compute $\beta = 2.6\,t_0 > 2t_0$, and hence it is sufficient to consider only the second alternative (2.15), which is nothing but the inequality (1.18).

Next, suppose that $d(t)$ satisfies a superconvergent dispersion relation, i.e., we have

$$\lim_{t\to\infty} t\,d(t) = 0 .$$

In that case, we must have a super-convergent sum rule

$$\frac{1}{\pi}\int_{t_0}^{\infty} dt'\,\mathrm{Im}d(t'+i0) = 0 .$$

As a result, $d(t)$ cannot be semimonotonic. Now allowing $\mathrm{Im}d(t'+i0)$ to change its sign twice on the cut at $t'=t_1$ and $t'=t_2$, then the function

$$f(t) = (t_1-t)(t_2-t)\,d(t)$$

will satisfy OSDR and it is, moreover, semimonotonic. Thus, the inequality (2.4) gives

$$\epsilon\frac{t_0-\Delta}{t_0-t}\frac{t}{\Delta}\frac{1}{(t_1-t)(t_2-t)}\left[(t_1-\Delta)(t_2-\Delta)d(\Delta)+\frac{t_0(\Delta-t)}{t(t_0-\Delta)}t_1t_2\right] \leqslant \epsilon\,d(t) \leqslant \epsilon\frac{t}{\Delta}\frac{1}{(t_1-t)(t_2-t)}\left[(t_1-\Delta)(t_2-\Delta)d(\Delta)+\frac{(\Delta-t)}{t}\,t_1t_2\right].$$

We have to optimize both sides of this inequality with respect to t_1 and t_2 under the condition $t_1 \geqslant t_0$ and $t_2 \geqslant t_0$ for two cases $\epsilon=\pm1$. Although the calculation is elementary, it is a bit tedious. We discover the result (1.21) with (1.22) after some calculations, assuming

$$\frac{t_0}{t_0-\Delta} \geqslant d(\Delta) \geqslant 1 .$$

If $d(t)$ has a real zero point at $t=\lambda$, we can say more. Suppose first that it lies below the threshold. Then, if $d(t)$ satisfies USDR and if $\mathrm{Im}d(t'+i0)$ changes its sign once at $t'=t_1 \geqslant t_0$, the function

$$f(t) = \frac{t_1-t}{t_1-\lambda}\,d(t)$$

also satisfies USDR and it is again semimonotonic. Hence, we can apply our inequalities (2.11a) for this function, from which we can prove that if $\lambda<\Delta$, then it must satisfy a stronger condition

$$\lambda \leqslant -\frac{\Delta}{d(\Delta)-1} \approx -2t_0 .$$

On the other hand, if $d(t)$ satisfies OSDR instead of USDR, then we can only apply a weaker inequality (2.4) for $f(t)$. Nevertheless, we can still prove that for negative λ, we must have (1.11) against the experimental result (1.4).

The case in which $d(t)$ has a zero point at $t=t_1 > t_0$ on the cut itself is interesting. Assuming that $\mathrm{Im}d(t'+i0)$ changes its sign only once exactly on that zero point and that the dispersion relation for $d(t)$ requires subtractions up to twice, the function defined by

$$f(t) = \frac{1}{t_1-t}\,d(t)$$

is semimonotonic and satisfies OSDR. Applying (2.4) for this function, we get

$$0 \leq \epsilon \frac{t_0 - \Delta}{t_0 - t} \frac{1}{\Delta}\left[\frac{d(\Delta)}{t_1 - \Delta} - \frac{1}{t_1}\right]$$
$$\leq \epsilon \frac{1}{t}\left[\frac{d(t)}{t_1 - t} - \frac{1}{t_1}\right]$$
$$\leq \epsilon \left[\frac{d(\Delta)}{t_1 - \Delta} - \frac{1}{t_1}\right]\frac{1}{\Delta} .$$

Using $t_1 \geq t_0$, and considering two cases $\epsilon = \pm 1$ separately, we obtain (1.27) and (1.28) after some calculations.

Another interesting case is that $d(t)$ has no zero point and that $d(t)$ requires more than one subtraction for its dispersion relation. More specifically, let us assume the asymptotic form

$$d(t) \simeq \text{const} \times t^{\alpha} \quad (t \to \infty) \tag{2.16}$$

at infinity. If we have $\alpha > 0$, then $d(t)$ needs at least one subtraction. Further, if $d(t)$ has no zero point in the entire cut plane, then the inverse function $1/d(t)$ will satisfy USDR for this case. Especially, the inequality (2.10) gives

$$0 \leq \epsilon \frac{1}{d(0)} \leq \epsilon \frac{1}{d(\Delta)} ,$$

which in turn leads to $d(\Delta) < 1$ in contradiction to our ansatz (1.12). Therefore, we have to conclude that either $d(t)$ has at least one zero point or $d(t)$ is not semimonotonic. Suppose now that $\operatorname{Im} d(t' + i0)$ changes its sign only once at $t' = t_1 > t_0$. Then under the same conditions that $d(t)$ has no zero point with $\alpha > 0$, the function

$$f(t) = \frac{t_1 - t}{d(t)}$$

is semimonotonic and satisfies OSDR. Applying again the inequality (2.4) for this function, we discover (1.29) with (1.30).

The total number of zero points of $d(t)$ is intimately related to the phase representation of $d(t)$. In general, suppose that $d(t)$ satisfies at most an n-times subtracted dispersion relation, and that $\operatorname{Im} d(t' + i0)$ changes its sign exactly m times on the cut. Then, the total number of zero points N of $d(t)$ must be restricted by

$$N \leq n + m \tag{2.17}$$

if $d(t)$ has no zero points on the cut. The proof for this statement is essentially analogous to that proved by Creutz[12] and we will not repeat it here since we shall rederive it by another method in Sec. III. Finally, if $d'(0)$ is known in addition to $d(\Delta)$, further inequalities can be derived. This will be briefly discussed in an appendix.

III. PHASE REPRESENTATION

Let $\delta(t)$ be the phase of $d(t)$ on the cut, i.e.,

$$\delta(t) = \operatorname{Arg} d(t + i\,0), \quad t \geq t_0 . \tag{3.1}$$

Then, assuming that $\delta(\infty)$ is bounded and that $d(t)$ is polynomially bounded at infinity, we can write[13]

$$d(t) = P(t) \exp\left[\frac{t}{\pi}\int_{t_0}^{\infty} dt' \frac{\delta(t')}{t'(t'-t)}\right] , \tag{3.2}$$

where $P(t)$ is a polynomial of t with real coefficients. We have assumed that $d(t)$ has no poles superimposed on the cut. The familiar final-state interaction theorem asserts[1] that $\delta(t)$ coincides with the $I = \frac{1}{2}$, S-wave pion-kaon scattering phase shift in the elastic interval

$$t_0 = (m_K + m_\pi)^2 \leq t \leq (m_K + 3m_\pi)^2 = \bar{t}_0 , \tag{3.3}$$

if we assume the so-called $\Delta I = \frac{1}{2}$ rule for K_{l3} decay. Especially, near the threshold, it will behave as

$$\delta(t) \simeq \text{const}\,(t - t_0)^{1/2} \quad (t \to t_0) , \tag{3.4}$$

so that we have

$$\delta(t_0) = 0 . \tag{3.5}$$

Now, assuming $\delta(t)$ to be continuous with Lipschitz condition

$$|\,\delta(t) - \delta(t')| \leq \text{const}\,|\,t - t'\,|^{\gamma} \qquad (\gamma > 0) ,$$

then the exponential function

$$G(t) = \exp\left[\frac{t}{\pi}\int_{t_0}^{\infty} dt' \frac{\delta(t')}{t'(t'-t)}\right] \tag{3.6}$$

is continuous even on the cut without any zero point. Thus, the total number N of the zero points of $D(t)$ is precisely the degree of the polynomial $P(t)$. Moreover, letting $t \to \infty$ in (3.2) and comparing the result[13] with the Regge-like asymptotic behavior (2.16), we find

$$N = \alpha + \frac{1}{\pi}[\,\delta(\infty) - \delta(t_0)] , \tag{3.7}$$

which is essentially an analog of the familiar Levinson's theorem. Supposing that $\sin \delta(t)$ changes its sign $\bar{m}$ times on the cut, we find

$$-(\bar{m} + 1) \leq \frac{1}{\pi}[\,\delta(\infty) - \delta(t_0)] \leq \bar{m} + 1$$

so that (3.7) leads to

$$\alpha-\bar{m}-1 \leqslant N \leqslant \alpha+\bar{m}+1 \ . \tag{3.8}$$

Further, if $P(t)$ has m_0 zero points of odd degree on the cut, then $\text{Im}d(t'+i0)$ can change its sign only on the points where either it passes through one of such zeros or $\sin\delta(t')$ changes its sign, as we see from (3.2). Therefore, we obtain

$$m=m_0+\bar{m} \geqslant \bar{m} \ , \tag{3.9}$$

barring an accidental situation that a zero point of $P(t)$ on the cut coincides with the point at which $\sin\delta(t)$ changes its sign. Excluding such a case, (3.8) and (3.9) give

$$\alpha-m-1 \leqslant N \leqslant \alpha+m+1 \ . \tag{3.10}$$

If α is negative or nonintegral, then we must have $\alpha<n$ so that we find

$$N \leqslant n+m \ . \tag{3.11}$$

If α is a non-negative integer, then $\alpha+1=n$ follows and again we find (3.11). This is again (2.17) of Sec. II.

Next, let us evaluate $D(t)$ on the basis of the phase representation (3.2). In order to compute the integral, it is necessary to know values of $\delta(t)$. Hereafter, we assume that contributions for the phase $\delta(t)$ from inelastic channels are negligible and that we can identify $\delta(t)$ with the $I=\frac{1}{2}$, S-wave pion-kaon scattering phase shift on the entire energy range $t \geqslant t_0$. So far, experimentally,[14] $\delta(t)$ is measured up to 1 GeV^2. Also, it is positive for that range with its magnitude less than π. Thus, $\text{Im}d(t'+i0)$ does not change its sign up to 1 GeV^2, unless $P(t)$ has a zero point on the real axis at that interval. Unfortunately, there is no information on $\delta(t)$ above 1 GeV^2 available. However, the precise behavior of $\delta(t)$ for $t>1$ GeV^2 is practically immaterial, if it remains reasonably small. Suppose that $\delta(t)$ satisfies

$$a \leqslant \frac{1}{\pi}\,\delta(t) \leqslant b, \quad t \geqslant t_2 \geqslant 1 \text{ GeV}^2 \tag{3.12}$$

for some constants a, b, and t_2. Then, setting

$$\bar{G}(t)=\exp\left[\frac{t}{\pi}\int_{t_0}^{t_2} dt' \frac{\delta(t')}{t'(t'-t)}\right], \tag{3.13}$$

it is easy to prove

$$\bar{G}(t)\left(\frac{t_2}{t_2-t}\right)^a \leqslant G(t) \leqslant \bar{G}(t)\left(\frac{t_2}{t_2-t}\right)^b \ . \tag{3.14}$$

Thus, for instance, if we can choose $b=-a=\frac{1}{2}$, with $t_2=1$ GeV^2, the error induced by neglecting contributions from $t'>1$ GeV^2 for $G(t)$ is very small, of the order of at most 7% for physical values of t. The choice $b=-a=\frac{1}{2}$ is perhaps justifiable for the so-called down solution for the phase shift. Even for the up solution where $\delta(t)$ crosses a resonant point $\frac{1}{2}\pi$ near $t\approx 0.75$ GeV^2, an explicit numerical calculation indicates essentially the same numerical values for $\bar{G}(t)$. The remaining task is to determine the polynomial $P(t)$. This is of course not *a priori* known. However, for the case $N=1$, the soft-pion theorem enables us to give

$$P(t)=1+\frac{t}{\Delta}\left[\frac{d(\Delta)}{G(\Delta)}-1\right] \ . \tag{3.15}$$

The case $N=0$ can be obtained by requiring

$$d(\Delta)=G(\Delta) \ . \tag{3.16}$$

Our numerical calculation indicated above gives

$$G(\Delta)\approx 1.06 \ , \tag{3.17}$$

which is a bit smaller than the left-hand side 1.28 in (3.16). This suggests[2,15] that indeed $P(t)$ cannot be a constant.

If we take into account the error suggested by (3.12), then (3.2) and (3.15) give

$$M_1(t) \leqslant d(t) \leqslant M_2(t) \,, \tag{3.18}$$

where $M_j(t)$ ($j=1,2$) are given by

$$M_1(t)=\left[\frac{\Delta-t}{\Delta}\left(\frac{t_2}{t_2-t}\right)^a+\frac{t}{\Delta}\left(\frac{t_2-\Delta}{t_2-t}\right)^b\frac{d(\Delta)}{\bar{G}(\Delta)}\right]\bar{G}(t) \ ,$$

$$M_2(t)=\left[\frac{\Delta-t}{\Delta}\left(\frac{t_2}{t_2-t}\right)^b+\frac{t}{\Delta}\left(\frac{t_2-\Delta}{t_2-t}\right)^a\frac{d(\Delta)}{\bar{G}(\Delta)}\right]\bar{G}(t) \ .$$

For the choice $t_2=1$ GeV^2 with $b=-a=\frac{1}{2}$, the bound (3.18) is plotted in Fig. 3.

If we choose $b=-a=1$, with $t_2=1$ GeV^2 instead, the lower bound becomes close to unity for the entire range of t and the upper bound increases only slightly compared with the previous case. We see again that we have $d(t)\geqslant 1$.

Thus, perhaps we have to consider the possibility of $P(t)$ being quadratic:

$$P(t)=1+C_1 t+C_2 t^2 \ . \tag{3.19}$$

Since the soft-pion theorem gives one constraint

$$1+C_1\Delta+C_2\Delta^2=\frac{d(\Delta)}{G(\Delta)} \quad , \tag{3.20}$$

(3.19) can be expressed in the form

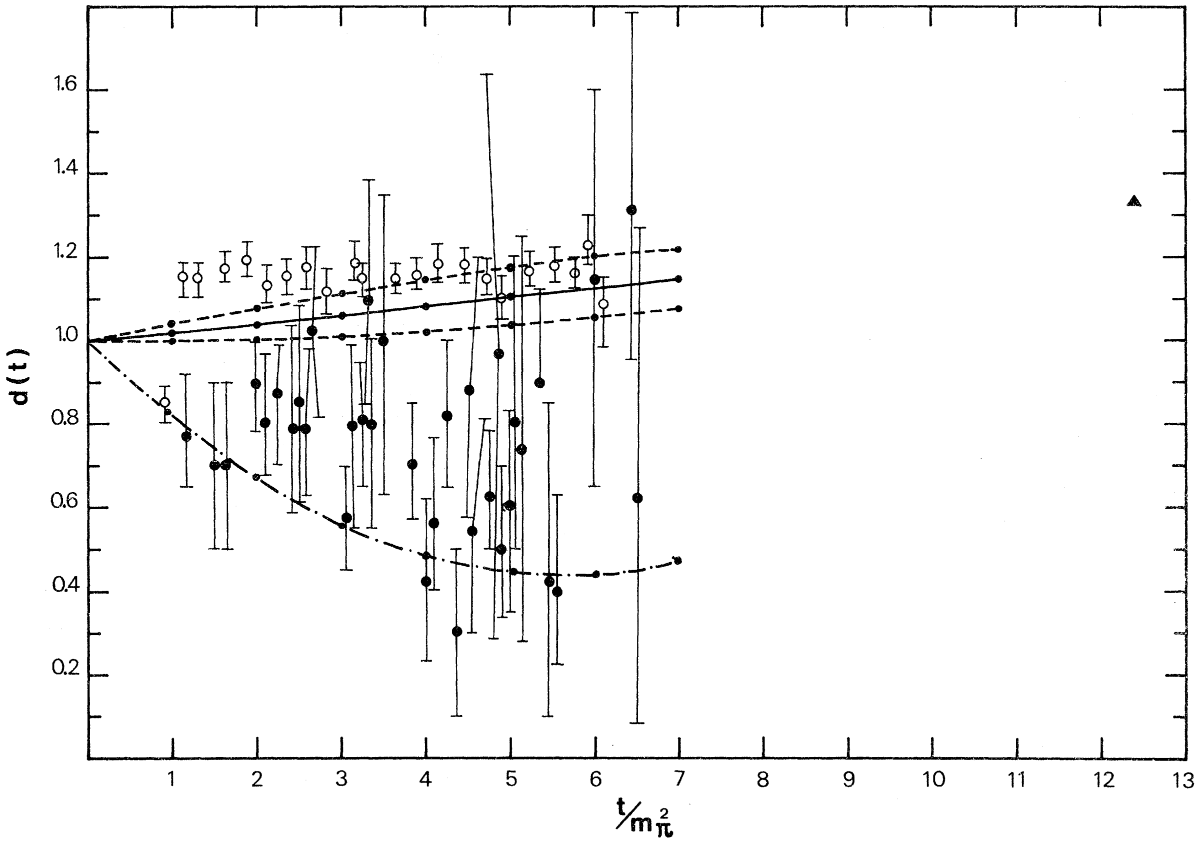

FIG. 3. A theoretical estimate of $d(t)$ based on the phase representation. (——): A theoretical estimate for $d(t) = P(t)\bar{G}(t)$. The cutoff energy t_2 is chosen to be 1 GeV^2. [See (3.13) and (3.15).] (---): A better theoretical estimate for $d(t)$, taking also into account the contributions beyond the cutoff energy. [See (3.18) and (3.19).] (–·–): A fit adjusting the free parameter C_2 in (3.21) suitably. Experimental data are again plotted for convenience. See the caption of Fig. 2 about symbols.

$$P(t) = 1 + \frac{1}{\Delta}\left[\frac{d(\Delta)}{G(\Delta)} - 1\right] t + C_2 t(t-\Delta) . \qquad (3.21)$$

If we choose C_2 to be around 0.018, it is possible to fit the present experimental data, as is seen in Fig. 3.

Lastly, let us briefly consider the general case where $P(t)$ has n zero points at $t = \lambda_j (j = 1, 2, \ldots, n)$;

$$P(t) = \prod_{j=1}^{n} \left(1 - \frac{t}{\lambda_j}\right) . \qquad (3.22)$$

If λ_j is complex, then $t = \lambda_j^*$ is also a zero point of $P(t)$ because of the reality condition.

Suppose now that all zero points lie in the left half plane, i.e., all λ_j satisfy

$$\mathrm{Re}\,\lambda_j \le 0 \quad (j = 1, 2, \ldots, n) . \qquad (3.23)$$

Then it is easy to prove that for values of t satisfying $0 \le t < \Delta$, we have

$$1 \le P(t) \le P(\Delta) . \qquad (3.24)$$

This in turn leads to the inequality

$$G(t) \le d(t) \le \frac{d(\Delta)}{G(\Delta)} G(t) . \qquad (3.25)$$

Again, if we use the numerical estimate of $G(t)$, this gives a result in contradiction with the experimental result (1.4), and we conclude that at least one zero point of $d(t)$ must lie in the right half plane. This conclusion is valid even if we do not assume the exact soft-pion value for $d(\Delta)$.

IV. DISCUSSION

In the previous sections, we have obtained various exact inequalities for $d(t)$ under several assumptions. Especially, we have seen that they contradict the present experimental data indicated by (1.4). As we have emphasized in the Introduction, the main difficulty is due to the soft-pion theorem, when implemented by the analyticity of $d(t)$. Suppose that $d(t)$ satisfies once-subtracted dispersion relation (1.23). Since the physical values of t lies quite far from the threshold t_0 of the cut, we may approximate the integral in (1.23) to be constant, unless the weight factor $\mathrm{Im}d(t'+i0)$ changes quite rapidly with t'. Then, $d(t)$ can be roughly approximated by a linear function of t. This fact together with the soft-pion theorem always leads to (1.11), i.e., $d(t)\geq 1$. Many conditions assumed in the previous sections are needed only to justify this simple picture more rigorously mathematically. Under this light, the result of Refs. 7 and 8 can be perhaps understood also. Conditions needed for results of these papers automatically require $d(t)$ to satisfy an unsubtracted dispersion relation, as Li and Pagels[16] have originally pointed out.

At any rate, a likely alternative is to assume that $d(t)$ requires at least two subtractions. For example, let us consider the twice-subtracted dispersion relation,

$$d(t)=1+at+\frac{t^2}{\pi}\int_{t_0}^{\infty}dt'\,\frac{1}{(t')^2(t'-t)}\mathrm{Im}d(t'+i0)\quad .$$

By a similar reasoning, we can approximate the integrand by a constant and this leads to a quadratic expression

$$d(t)\simeq 1+at+bt^2$$

for $d(t)$. Then, we can easily accommodate both the soft-pion theorem and $d(t)\leq 1$ (see Ref. 2). Analogously, in the phase-representation (3.2), the polynomial $P(t)$ must be at least quadratic. Hence, $d(t)$ must have at least two zero points. Perhaps, this fact also explains failures of some of our inequalities when $d(t)$ requires two subtractions but has no zero point.

Summarizing, we have to assume that either $d(t)$ requires two subtractions or the soft-pion theorem must be abandoned, if the present experiment with (1.4) is correct. Both are difficult to accept lightly. We may remark that $D(t)$ is obtained as a matrix element of

$$\langle\pi^0(p')|\partial_\mu V_\mu^{(4-i5)}(0)|K^+(p)\rangle=\frac{i}{\sqrt{2}}D(t)\,(4p_0p_0'V^2)^{-1/2},$$

$$t=-(p-p')^2\ .$$

The success of the ordinary PCAC (partial conservation of axial-vector currents) might indicate that $d(t)$ is a smooth function of t. Moreover, the electromagnetic form factor of the nucleon decreases in a dipole-like fashion for large t.

One possibility, as has been suggested[17] elsewhere, is to modify the ordinary Cabibbo theory so that the leptonic current responsible to the K_{l3} decay is now replaced for example by

$$\tilde{V}_\mu(x)=V_\mu^{(4-i5)}(x)+\frac{\partial}{\partial x_\mu}S^{(4-i5)}(x)$$

for some scalar density $S^{(4-i5)}(x)$. In that case, the soft-pion theorem is no longer applicable. Also, since we have

$$\partial_\mu\tilde{V}_\mu(x)=\partial_\mu V_\mu^{(4-i5)}(x)+\Box S^{(4-i5)}(x)\ ,$$

it is likely that $d(t)$ may require some subtraction for its dispersion relation. Another advantage of this modification is that the new term does not affect electron-decay made of hadrons, say, $A\to B+e+\nu$. Its effect will be noticeable only for muon decay such as $K^+\to\pi^0+\mu^++\nu$, $\Lambda\to p+\mu^-+\nu$, and $\Sigma^-\to n+\mu^-+\nu$.

Another possibility is to assume that the Cabibbo angles θ_V and θ_A for vector and axial-vector currents are very different. In that case, what we measure experimentally from $K_{\mu 2}$ and K_{l3} decay are

$$\frac{f_K}{f_\pi f_+(0)}=(1.30\pm 0.03)\frac{\sin\theta_V}{\sin\theta_A}\cos\theta_A\ .$$

Therefore, if θ_V is much smaller than θ_A, then we could have $d(\Delta)<1$ as we see from Eq. (1.6).

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APPENDIX

If $d'(0)$ is known in addition to $d(\Delta)$, then we can derive more inequalities. As an example, we shall consider the case in which $d(t)$ satisfies OSDR and $\mathrm{Im}d(t'+i0)$ changes its sign only once at $t'=t_1>t_0$ on the cut. Then setting

$$f(t)=(t_1-t)\,d(t),$$

$f(t)$ satisfies a twice-subtracted dispersion relation

$$g(t)\equiv\frac{f(t)-f(0)-tf'(0)}{t^2}$$

$$=\frac{1}{\pi}\int_{t_0}^{\infty}dt'\,\frac{1}{(t')^2(t'-t)}\ \mathrm{Im}f(t'+i0)\ .$$

Moreover, it is semimonotonic so that we have

$$0 \leq \epsilon \left(\frac{t_0-\Delta}{t_0-t}\right) g(\Delta) \leq \epsilon\, g(t) \leq \epsilon\, g(\Delta)$$

for values of t with $t \leq \Delta < t_0$. Considering two cases $\epsilon = \pm 1$ separately and optimizing the result with respect to t_1 under the condition $t_1 \geq t_0$, we find

$$\operatorname{Min}\{\gamma, C_1(t), C_2(t)\} \leq \frac{1}{t^2}[d(t)-1-t\,d'(0)] \leq \operatorname{Max}\{\gamma, C_2(t)\} ,$$

where

$$\gamma = \frac{1}{\Delta^2}[d(\Delta)-1-\Delta\, d'(0)] ,$$

$$C_1(t) = \gamma\,\frac{t_0-\Delta}{t_0-t} ,$$

$$C_2(t) = \gamma\left(\frac{t_0-\Delta}{t_0-t}\right)^2 + \frac{\Delta-t}{(t_0-t)^2}\, d'(0) .$$

Especially if we set $t=0$, this leads to

$$\operatorname{Min}\{\lambda, C_1(0), C_2(0)\} \leq \tfrac{1}{2} d''(0) \leq \operatorname{Max}\{\gamma, C_2(0)\} .$$

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*On leave of absence from Department of Physics and Astronomy, University of Rochester, Rochester, New York, 14627.

†Nordita Guest Professor from January to May 1973.

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