PHYSICAL REVIEW D VOLUME 8, NUMBER 5 1 SEPTEMBER 1973

Transverse Momenta and Overlap Functions in Multiperipheral Models*

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(Received 29 March 1973)

The multi-Regge model is studied, with particular emphasis on the predicted slope of the elastic diffraction peak, and the average transverse momentum of the produced particles. The slope of the diffraction peak is proportional to the mean-square impact parameter, and in the multi-Regge model the total impact parameter is built up in a random walk, with each link in the multiperipheral chain corresponding to a step in the walk. A simple Chew-Pignotti model is incapable of fitting the inclusive production data at high energies: It predicts too slow an increase in the multiplicity of produced pions, and too rapid a shrinkage of the diffraction peak. To cure this problem, one must introduce "clustering" effects such as to reduce the over-all spread in impact parameter, and increase the density of produced pions in longitudinal-momentum space. Such effects were to be expected, in fact, because of resonance formation. A multiperipheral cluster model is introduced, in which the decay of the clusters is described via the statistical bootstrap model of Hagedorn and Frautschi. A crude fit to the high-energy data, in which all the clusters are given a common mass, shows that the average cluster mass is at least a couple of GeV, a surprisingly large figure. This provides some *a posteriori* justification for Hagedorn's thermodynamic model. The calculations are carried out using both approximate analytical methods and a Monte Carlo numerical program.

I. INTRODUCTION

It is well-known that the multi-Regge model[1] of high-energy production processes, as it is usually formulated, incorporates the idea that at high energies the average transverse momentum of the produced particles will approach a finite limit. It also predicts that the width of the diffraction peak will shrink logarithmically with energy. These two results are related in a quite simple way, and it is our object to explore this relationship in quantitative terms. Comparisons can then be made with experimental data.

Roughly speaking, the model predicts that at high energies the production amplitude factorizes into a product of functions, each depending on the transverse momentum of a single exchanged Reggeon. As is shown in Sec. II, each of these factors makes an independent, additive contribution to the slope of the diffraction peak, i.e., to the mean-square absorption radius. This can be interpreted as the mean-square impact parameter corresponding to the exchange of that particular Reggeon. Thus the total mean-square impact parameter separating the initial particles is proportional to the average number of exchanged Reggeons. Since this number is growing logarithmically, the width of the diffraction peak will shrink in the same way.

As an order-of-magnitude estimate, one finds in fact,

$$\langle n\rangle \underset{s\to\infty}{\simeq} 2\alpha_P'\langle \vec{q}_\perp{}^2\rangle \ln s \,, \tag{1}$$

where $\langle n\rangle$ is the average number of exchanged Reggeons, α_P' is the "Pomeron slope," $\langle \vec{q}_\perp{}^2\rangle$ is the mean-square transverse momentum of the produced particles, and s, as usual, is the center-of-mass energy squared.

Experimentally, one finds that the diffraction peak is shrinking rather slowly.[2] This means that the average number of factorizable Reggeon links is rising slowly with energy. The multiplicity of produced pions, on the other hand, is increasing relatively rapidly with energy.[3] Therefore several pions must be produced per exchanged Reggeon. And this, in turn, implies that the final-state pions emerge from "clusters" (i.e., localized groups in configuration space, resulting for instance from resonance formation), and that single-Reggeon exchange takes place only between clusters. The average number of pions per cluster needs to be quite large.

The rest of this paper is devoted to putting these arguments on a quantitative footing. In Sec. II we outline the connection between multiparticle amplitudes and the diffraction peak, in a general multiperipheral model. In Sec. III we analytically investigate a simple multi-Regge model in the "strong ordering" approximation[4] where one evaluates the partial cross sections $\sigma_n(s)$ and the overlap functions $F_n(p,\theta)$ using approximations valid in the limit $s\to\infty$. In Sec. IV we carry out the sum over n, to find the inclusive behavior of the theory; this summation is only valid in a "weak coupling" approximation. The conclusions outlined above can then be explicitly demonstrated.

Since the "weak coupling" approximation is unlikely to be valid in the real world, we check the analytic results at each stage of the work against more exact Monte Carlo calculations. In Sec. IV we also discuss the likely effect of including more sophisticated and correct vertex functions. It is argued that modifications of this sort will not change our conclusions.

Finally, in Sec. V we discuss the physical effects which lead to clustering, and endeavor to make rough estimates of the parameters required in a multiperipheral cluster model in order to fit the experimental data. Our results and conclusions are summarized in Sec. VI.

Recently Hwa[5] has carried out a calculation which is similar in spirit to our Secs. II and III. His explicit multi-Regge amplitude is slightly different from ours and he does not carry out numerical calculations, but his conclusions are consistent with those reached here.

II. THE OVERLAP FUNCTION

The overlap function and its relation to elastic diffraction scattering has been extensively discussed.[6] We give here a derivation of some known results in a more transparent form particularly suitable for the discussion of multiperipheral models.

We define the overlap function for the elastic scattering of two spinless particles a and b as

$$F(p,\theta)=\sum_c \int T(p_a, p_b ; c)\, T^*(p_a', p_b'; c)\, d\Omega_c \ , \qquad (2)$$

where c is any possible inelastic channel for interactions of a and b; p_a, p_b and p_a', p_b' are the initial and final 4-momenta of particles a and b; p, θ are the center-of-mass momentum and scattering angle corresponding to this particular elastic scattering; $T(p_a, p_b; c)$ is the matrix element for the inelastic reaction $a+b\to c$, while $d\Omega_c$ is the phase-space volume element associated with the final state c. Unitarity then implies that the imaginary part of the elastic scattering amplitude is given by

$$\operatorname{Im}T_{\rm el}(p,\theta)=\int T_{\rm el}(p_a, p_b; p_a'', p_b'')\, T_{\rm el}^*(p_a', p_b'; p_a'', p_b'') \times d\Omega_2(p_a'', p_b'') + F(p,\theta) \ . \qquad (3)$$

Let us fix our attention on a particular state c consisting of n particles, i.e., we consider the contribution coming from the reaction

$$a+b\to c_1+c_2+\cdots+c_n \ . \qquad (4)$$

The kinematics of this process are usually specified by giving, in addition to p_a and p_b, the 4-momenta $p_1, p_2, \ldots, p_n$ of particles $c_1, c_2, \ldots, c_n$. The phase-space volume element then includes the n mass-shell constraints and 4 over-all energy-momentum conservation constraints. An alternative procedure is to specify the $(n-1)$ 4-momenta $q_1, \ldots, q_{n-1}$ defined by

$$q_i=q_{i-1}-p_i\ , \quad i=1,2,\ldots,n-1$$
$$q_0\equiv p_a\ , \qquad (5)$$
$$q_n\equiv -p_b\ .$$

The reaction specified by p_a', p_b' will have a different set of q's, $q_0', \ldots, q_n'$, satisfying the requirement

$$q_i'-q_i=p_a'-p_a\ . \qquad (6)$$

The phase-space volume element depends only on differences between the q_i and is therefore invariant under such a uniform translation. The kinematics are illustrated in Fig. 1.

If we write $f\{q\}$ for $f(q_0, \ldots, q_n)$, then we can write the contribution to the overlap function as

$$F_n^c(p,\theta)=\operatorname{Re}\int T\{q\}\, T^*\{q'\}\, d\Omega_n\{q\}\ , \qquad (7)$$

where we have introduced the real part since time-reversal and rotational invariance require $F(p,\theta)$ to be real and so only the real parts of the individual terms need be considered. Now we choose a coordinate system in the center-of-mass frame so that the momentum transfer in the elastic scattering lies along the x axis, and $\vec{p}_a, \vec{p}_a'$ lie in the x-z plane [Fig. 2].

Then we can write

$$q_i=\bar{q}_i+\tfrac{1}{2}\Delta\hat{x}-\epsilon\hat{z}\ ,$$
$$q_i'=\bar{q}_i-\tfrac{1}{2}\Delta\hat{x}-\epsilon\hat{z}\ , \qquad (8)$$

where $\hat{x}, \hat{z}$ are unit spacelike vectors along the x and z axes, $\Delta=2p\sin\frac{1}{2}\theta$, $\epsilon=p(1-\cos\frac{1}{2}\theta)$, and the $\bar{q}_i$ are independent of θ. For small Δ, ϵ we can expand about the origin:

$$T\{q\}=T\{\bar{q}\}+\frac{\Delta}{2}\sum_{i=0}^{n}\frac{\partial T}{\partial q_{ix}}+\frac{\Delta^2}{8}\left(\sum_{i=0}^{n}\frac{\partial}{\partial q_{ix}}\right)^2 T - \epsilon\sum_{i=0}^{n}\frac{\partial T}{\partial q_{iz}}+\cdots \qquad (9)$$

through terms second order in θ. Making the corresponding expansion of $T^*\{q'\}$ and using the fact that $d\Omega_n\{q\}=d\Omega_n\{\bar{q}\}$, we have

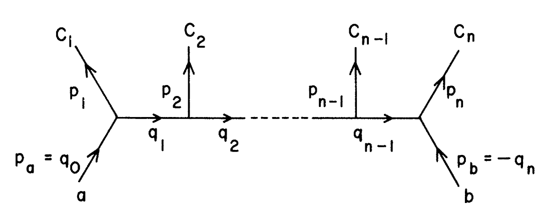

FIG. 1. Diagram illustrating the kinematic variables used. Note that variables can be defined in this way regardless of whether the production process is multiperipheral.

$$F_n^c(p,\theta)=\int\left(TT^*-\frac{\Delta^2}{4}\left(\sum_i\frac{\partial T}{\partial q_{ix}}\right)\left(\sum_j\frac{\partial T^*}{\partial q_{jx}}\right)+\frac{\Delta^2}{8}\left\{T\left(\sum_i\frac{\partial}{\partial q_{ix}}\right)^2T^*+\left[\left(\sum_i\frac{\partial}{\partial q_{ix}}\right)^2T\right]T^*\right\}\right.$$
$$\left.-2\epsilon\,\mathrm{Re}\left(T^*\sum_i\frac{\partial T}{\partial q_{iz}}\right)\right)d\Omega_n(\bar{q})+O(\theta^4) \tag{10}$$

$$\equiv\int\left[TT^*+\frac{\Delta^2}{8}T\left(\sum_i\frac{\overleftrightarrow{\partial}}{\partial q_{ix}}\right)^2T^*-2\epsilon\,\mathrm{Re}\left(T^*\sum_i\frac{\partial T}{\partial q_{iz}}\right)\right]d\Omega_n(\bar{q})+O(\theta^4)\ , \tag{11}$$

where T, T^* and their derivatives are all evaluated at $q_i=\bar{q}_i$ or $\theta=0$. Now writing $\Delta\simeq p\theta$, $\epsilon\simeq\frac{1}{8}p\theta^2$, we find

$$F_n^c(p,\theta)=\int\left[|T|^2+\frac{p^2\theta^2}{8}\,T\left(\sum_i\frac{\overleftrightarrow{\partial}}{\partial q_{ix}}\right)^2T^*-\frac{p\theta^2}{4}\,\mathrm{Re}T\sum_i\frac{\partial}{\partial q_{iz}}T^*\right]d\Omega_n\{q\}+O(\theta^4) \tag{12}$$

where we have written q for $\bar{q}$.

Thus

$$F_n^c(p,\theta)=F_n^c(p,0)(1-\tfrac{1}{2}p^2\theta^2\,\Gamma_n^c)+O(\theta^4)\ , \tag{13}$$

where

$$\Gamma_n^c=-\frac{1}{4}\left\langle\frac{T(\sum_i\overleftrightarrow{\partial}/\partial q_{ix})^2T^*}{|T|^2}\right\rangle+\frac{1}{2p}\left\langle\sum_i\frac{\partial\ln|T|}{\partial q_{iz}}\right\rangle \tag{14}$$

(the diagonal braces indicating averages over phase space, with weight function equal to the matrix element squared).

The full overlap function is therefore given by

$$F(p,\theta)=F(p,0)(1-\tfrac{1}{2}p^2\theta^2\,\Gamma),\qquad \Gamma=\frac{\sum_{c,n}\sigma_n^c\Gamma_n^c}{\sum_{c,n}\sigma_n^c} \tag{15}$$

where σ_n^c is the total cross section for the n-body inelastic channel c.

Now the second term in Eq. (14) turns out to be small in multiperipheral models, and we shall neglect it for the moment. The remaining term can be given a direct physical interpretation in terms of impact parameters. Suppressing all but the transverse momentum variables, and carrying out a two-dimensional Fourier transform of the reaction amplitude, one obtains

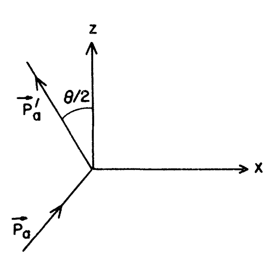

FIG. 2. Diagram illustrating the center-of-mass frame used for discussion of the overlap function.

$$h\{\vec{b}\}=\int\exp\left(i\sum_{i=0}^{n+1}\vec{b}_i\cdot\vec{p}_{i\perp}\right)T\{\vec{p}_\perp\}\times\prod_{i=0}^{n+1}\left(\frac{d^2p_{i\perp}}{2\pi}\right)2\pi\,\delta^2\left(\sum_{i=0}^{n+1}\vec{p}_{i\perp}\right), \tag{16}$$

defining $\vec{p}_{0\perp}=-\vec{p}_{a\perp}$, $\vec{p}_{n+1,\perp}=-\vec{p}_{b\perp}$. Using (5), this can be written in terms of the $\{\vec{q}_\perp\}$:

$$h\{\vec{b}\}=\int\exp\left[i\sum_{i=0}^{n}\vec{q}_{i\perp}\cdot(\vec{b}_{i+1}-\vec{b}_i)\right]T\{\vec{q}_\perp\}\prod_{i=0}^{n}\frac{d^2q_{i\perp}}{2\pi}\ . \tag{17}$$

A similar transformation can be written for the final-state amplitude:

$$h\{\vec{b}'\}=\int\exp\left[i\sum_{i=0}^{n}\vec{q}_{i\perp}\cdot(\vec{b}'_{i+1}-\vec{b}'_i)\right]T\{\vec{q}_\perp\}\prod_{i=0}^{n}\frac{d^2q_{i\perp}}{2\pi}. \tag{17$'$}$$

Now under the Fourier transformation the operator $\sum_{i=0}^{n}\overleftrightarrow{\partial}/\partial q_{ix}$ transforms as follows:

$$\sum_{i=0}^{n}\frac{\overleftrightarrow{\partial}}{\partial q_{ix}}\to i\sum_{i=0}^{n}\left[(b_{ix}-b_{i+1,x})+(b'_{ix}-b'_{i+1,x})\right]$$
$$=i\left[(b_{0x}-b_{n+1,x})+(b'_{0x}-b'_{n+1,x})\right]\ .$$

In most models of interest the transverse impact parameters will come out equal for the initial and final states at high energies, so that

$$\sum_{i=0}^{n}\frac{\overleftrightarrow{\partial}}{\partial q_{ix}}\to 2i\,(b_{0x}-b_{n+1,x})\ ,$$

and the first term in Eq. (14) will give

$$\Gamma_n^c\simeq-\frac{1}{4}\left\langle\frac{T(\sum_i\overleftrightarrow{\partial}/\partial q_{ix})^2T^*}{|T|^2}\right\rangle$$
$$=\langle(b_{0x}-b_{n+1,x})^2\rangle$$
$$=\tfrac{1}{2}\langle(\vec{b}_0-\vec{b}_{n+1})^2\rangle\ , \tag{18}$$

that is, the slope Γ_n^c of the overlap function is simply one-half the mean-square impact parameter between the incoming particles. The second term of Eq. (14) can be given a similar interpretation in terms of the longitudinal collision coordinates, but is of negligible importance for our present purposes, as previously remarked.

Let us now consider a multiperipheral-type model. In this case, if one orders the produced particles according to their longitudinal momenta, takes the limit $s\to\infty$ ("strong-ordering approximation"[4]), and ignores correlations between the transverse momenta of the exchanged particles, then the reaction amplitude at fixed s can be written in a factorized form:

$$T\{\vec{q}_\perp\} = \prod_{i=1}^{n-1} \phi_i(q_{i\perp}^2) \tag{19}$$

[see Eq. (42), below]. Its Fourier transform then must factorize also; from Eqs. (17) and (19)

$$h\{\vec{b}\} = \delta(\vec{b}_0 - \vec{b}_1) \prod_{i=1}^{n-1} \tilde{\phi}_i((\vec{b}_i - \vec{b}_{i+1})^2)\, \delta(\vec{b}_n - \vec{b}_{n+1}) \,. \tag{20}$$

This equation implies that the impact parameter between the incoming particles is the same as that between the outermost produced particles in the multiperipheral chain, and that each link in the multiperipheral chain makes an independent contribution to the over-all mean-square impact parameter, after the fashion of a random walk. This fact seems to have been first noticed by Gribov.[7]

We turn now to a more detailed treatment of the multi-Regge model.

III. THE "ASYMPTOTIC" THEORY FOR EXCLUSIVE STATES

An exact analytic solution of the multi-Regge model has not yet been derived. But useful results can be obtained by means of various approximations when the subenergy of every pair of produced particles is large, as we shall demonstrate below. This treatment is patterned after one given by DeTar.[7a]

Consider again the process shown in Fig. 1, where n particles are produced in the final state. For simplicity we shall ignore all spins and internal quantum numbers, and assume that there is only one type of external particle with mass m_0. Consider a multi-Regge exchange reaction with each exchanged Reggeon having the same (linear) trajectory

$$\alpha(t) = \alpha_0 + \alpha' t \,. \tag{21}$$

The amplitude for this process can be taken as

$$T\{q\} = g_0^{\,n-2} \prod_{i=1}^{n-1} (s_i^{\,\alpha(t_i)}\, e^{bt_i}\, e^{i\pi\alpha(t_i)/2}) \,, \tag{22}$$

where

$$s_i = (q_{i+1} - q_{i-1})^2 \,, \quad t_i = q_i^{\,2} \,. \tag{23}$$

The cross section is then

$$\sigma_n(s) = \frac{(g_0^{\,2})^{n-2}}{s} \int \prod_{i=1}^{n} \frac{dp_{iL}}{E_i}\, d^2 p_{i\perp} \delta^4\Big(\sum_{i=1}^{n} p_i - p_a - p_b\Big) \times \prod_{i=1}^{n} (s_i^{\,2\alpha(t_i)}\, e^{2bt_i}). \tag{24}$$

The factor $1/s$ is a flux factor, s being the total squared center-of-mass energy, g is an internal vertex coupling constant (the external vertex constants are arbitrarily chosen to be unity), s_i, t_i are the ith-pair subenergy squared and momentum transfer squared, and p_{iL}, $\vec{p}_{i\perp}$, and E_i are the longitudinal momentum, transverse momentum, and energy of the ith particle. Equation (22) is the simplest possible multi-Regge expression for the amplitude; more realistic expressions will be briefly discussed in Sec. IV.

Let us now introduce the "transverse mass" and "rapidity" variables

$$\mu_i = (m_0^{\,2} + \vec{p}_{i\perp}^{\,2})^{1/2} \,, \quad y_i = \sinh^{-1}(p_{iL}/\mu_i) \,. \tag{25}$$

In the lab frame, in which the target a is at rest and the projectile b moves with large rapidity Y, we have

$$s \simeq m_0^{\,2} e^{Y} \,. \tag{26}$$

We can then apply the "strong ordering approximation,"[4] appropriate when all subenergies are large:

$$E_{i+1} \gg E_i \gg m_0 \,. \tag{27}$$

This allows the replacements

$$s_1 s_2 \cdots s_{n-1} \simeq \mu_1^{\,2}\mu_2^{\,2} \cdots \mu_{n-1}^{\,2} s \,,$$

$$\delta\Big(\sum_i p_{iL} - p_{bL}\Big)\, \delta\Big(\sum_i E_i - E_a - E_b\Big) \simeq \frac{2}{s}\, \delta(y_1 - \ln(\mu_1/m_0))\, \delta(Y - y_n - \ln(\mu_n/m_0)) \,. \tag{28}$$

The cross section then becomes

$$\sigma_n \simeq 2 g_0^{\,2(n-2)} s^{2\alpha_0 - 2} \int \prod_{i=2}^{n} dy_i\, \theta(y_i - y_{i-1})\, dy_1\, \delta(y_1 - \ln(\mu_1/m_0))\, \delta(Y - y_n - \ln(\mu_n/m_0)) \int \prod_{i=1}^{n} d^2 p_\perp\, \delta^2\Big(\sum_{i=1}^{n} \vec{p}_{i\perp}\Big) \times \prod_{i=2}^{n-1} (\mu_i^{\,2})^{\alpha_0} \exp\Big[\sum_{i=1}^{n-1} 2t_i\,(b + \alpha' \ln s_i)\Big] \,. \tag{29}$$

Now since $\vec{p}_i = \vec{q}_i - \vec{q}_{i-1}$, and $\vec{q}_{0\perp} = 0$, then

$$\prod_{i=1}^{n} d^2 p_{i\perp}\, \delta^2\left(\sum_{i=1}^{n} \vec{p}_{i\perp}\right) = \prod_{i=1}^{n-1} d^2 q_{i\perp} \,. \tag{30}$$

We also note that in the strong-ordering approximation

$$t_i = q_{i0}{}^2 - q_{iL}{}^2 - \vec{q}_{i\perp}{}^2$$
$$\simeq -\vec{q}_{i\perp}{}^2 \,. \tag{31}$$

To proceed further we replace $\ln\mu_i{}^2$, $\ln s_i$ by their average values:

$$\ln\mu_i{}^2 \to \frac{1}{n-2}\left\langle \sum_{i=2}^{n-1} \ln\mu_i{}^2 \right\rangle$$
$$\equiv \ln\mu^2 \,,$$
$$b + \alpha' \ln s_i \to \frac{1}{n-1}\left\langle \sum_{i=1}^{n-1} (b + \alpha' \ln s_i)\right\rangle$$
$$= b + \alpha'\left[\ln\mu^2 + \frac{\ln(s/\mu^2)}{(n-1)}\right]$$
$$\equiv c_n(s) \,. \tag{32}$$

The replacements (32) remain approximate even in the limit $s \to \infty$, n fixed. A simple approximate expression is then obtained for σ_n:

$$\sigma_n = 2g_0{}^{2(n-2)}\, s^{2\alpha_0 - 2} \frac{[Y - \ln(\mu_1\mu_n/m_0{}^2)]^{n-2}}{(n-2)!} (\mu^2)^{2\alpha_0(n-2)}$$
$$\times \prod_{i=1}^{n-1} \int d^2 q_{i\perp}\, e^{-2c_n(s)\vec{q}_{i\perp}{}^2} \tag{33}$$

$$= \frac{\pi}{c_n(s)} s^{2\alpha_0 - 2} \frac{[g_n{}^2(s)\ln(s/\mu_1\mu_n)]^{n-2}}{(n-2)!} \,, \tag{34}$$

where

$$g_n{}^2(s) = g_0{}^2 \frac{\mu^{4\alpha_0}\pi}{2c_n(s)} \,. \tag{35}$$

Now it remains to find the average transverse mass. To deduce this we note that in the approximation of Eq. (33), the average transverse momentum squared of each of the exchanged Reggeons is

$$\langle \vec{q}_{i\perp}{}^2 \rangle = \frac{1}{2c_n(s)} \,. \tag{36}$$

It follows immediately that for the outermost particles on the chain

$$\langle \mu_1{}^2 \rangle = \langle \mu_n{}^2 \rangle$$
$$= m_0{}^2 + \frac{1}{2c_n(s)} \,. \tag{37}$$

For the internally produced particles, however, their transverse momentum is the difference between those of two Reggeons:

$$\vec{p}_{i\perp} = \vec{q}_{i\perp} - \vec{q}_{i-1,\perp} \,. \tag{38}$$

If there are no correlations between the transverse momenta of the Reggeons, then

$$\langle \vec{p}_{i\perp}{}^2 \rangle = 1/c_n(s), \quad i = 2, 3, \ldots, (n-1) \tag{39}$$

and

$$\mu^2 = m_0{}^2 + \frac{1}{c_n(s)} \,. \tag{40}$$

We shall take these as the results of the asymptotic theory. In general, however, there *are* correlations between the transverse momenta: they may enter through the factors $\mu_i{}^2$ in Eq. (29), for instance, or through the internal vertex functions (see Sec. IV). The resulting effects will have to be calculated numerically, and they are unlikely to affect the average transverse momentum squared by more than a factor of two, so we shall ignore them for the moment.

To calculate the effect of this multi-Regge contribution to the diffraction scattering, we can directly apply the results of Sec. III. Rewriting (22) using (21), we have

$$T\{q\} = g_0{}^{n-2} \prod_{i=1}^{n-1} \exp\{[\alpha'(\tfrac{1}{2}i\pi + \ln s_i) + b]\, t_i\}$$
$$\times s_i{}^{\alpha_0} e^{ni\pi\alpha_0/2} \,. \tag{41}$$

Using Eqs. (28), (31), and (32), this equation becomes

$$T\{q\} \simeq (\mu^2 g_0)^{n-2} \prod_{i=1}^{n-1} \exp\{-[c_n(s) + \tfrac{1}{2}i\pi\alpha']\vec{q}_{i\perp}{}^2\}$$
$$\times s^{\alpha_0} e^{ni\pi\alpha_0/2} . \tag{42}$$

Hence

$$\left\langle \frac{T(\sum_{i=0}^{n} \partial/\partial q_{ix})^2 T^*}{|T|^2} \right\rangle$$
$$= -4\left\{ (n-1)\, c_n(s) + \pi^2\alpha'^2 \left\langle \left(\sum_{i=1}^{n-1} q_{ix}\right)^2 \right\rangle \right\}, \tag{43}$$

while $\langle \sum_i \partial \ln|T|/\partial q_{iz} \rangle$ vanishes in the limit $s \to \infty$. Now

$$\left\langle \left(\sum_{i=1}^{n-1} q_{ix}\right)^2 \right\rangle = \left\langle \sum_{i=1}^{n-1} q_{ix}{}^2 \right\rangle$$
$$= (n-1)\frac{1}{4c_n(s)} \,, \tag{44}$$

if we ignore correlations among the Reggeon transverse momenta, and use Eq. (36).

Therefore, from Eqs. (13), (14), (43), and (44) one obtains

$$F_n(p,\theta)=F_n(p,0)\left[1-\tfrac{1}{2}p^2\theta^2\,\Gamma_n(s)\right]+O(\theta^4)\,, \quad (45)$$

where

$$\Gamma_n(s)=(n-1)\left[c_n(s)+\frac{\pi^2\alpha'^2}{4c_n(s)}\right]. \quad (46)$$

To test the validity of the asymptotic theory outlined so far, we have done some numerical calculations. The Monte Carlo procedure used was an adaptation of one previously developed to calculate transverse damped phase space.[8] For a detailed discussion, see Chen and Peierls.[9] Here we outline the procedure. First, using Van Hove's method,[8] a set of n transverse momenta was generated, distributed Gaussianly with sum zero. Then a rapidity was generated for each particle, Gaussianly distributed about $-\frac{1}{2}Y+(j-1)Y/(n-1)$ for the jth particle (we work in the center-of-mass system). The Gaussian widths were adjustable in both cases. The resulting events, which did not, in general, satisfy the longitudinal momentum or energy constraints, were Lorentz-transformed and rescaled so as to satisfy these constraints, and reweighted according to the transformation Jacobian. By adjusting the Gaussian widths and appropriately choosing the relative weights for different scale factors, good convergence is obtained. For the final runs, the number of events generated was about 1000 per final-state particle for each value of s.

We have considered two cases:

Example 1. $\alpha_0=0$, $\alpha'=0$, $b=5\ (\mathrm{GeV}/c)^{-2}$, $g_0^{\,2}=8b/\pi$, $m_0^{\,2}=m_\pi^{\,2}$.

For these values of the parameters, Eqs. (34), (39), and (46) lead to the following results: For the cross section,

$$\sigma_n=\frac{\pi}{b}s^{-2}\left[\frac{g_0^{\,2}\pi}{2b}\ln\left(\frac{s}{m_0^{\,2}+1/2b}\right)\right]^{n-2}\frac{1}{(n-2)!}\,, \quad (47)$$

the average transverse momentum squared of all the produced particles is

$$\langle \vec{p}_\perp^{\,2}\rangle_n=\left(\frac{n-1}{n}\right)\frac{1}{b} \quad (48)$$

and the slope of the overlap function is

$$\Gamma_n(s)=(n-1)\,b\,. \quad (49)$$

Furthermore, no correlations among the transverse momenta are present in this special case, so that these predictions should hold exactly in the limit of high energies.

The numerically calculated values for these quantities, for $n=2$, 3 and 4, are displayed in Figs. 3(a), 3(b), and 3(c), along with the predictions of the asymptotic theory.[10] It can be seen that the Monte Carlo results do approach those of the asymptotic theory as the center-of-mass energy E becomes large, but the rate of approach is rather slow. Note that the asymptotic limits are all approached from below.

Example 2. $\alpha_0=\frac{1}{2}$, $\alpha'=1\ (\mathrm{GeV}/c)^{-2}$, $b=5\ (\mathrm{GeV}/c)^{-2}$, $g_0^{\,2}=32.5$, $m_0^{\,2}=m_\pi^{\,2}$.

The numerical calculations are compared with the asymptotic theory for this case in Figs. 3(d), 3(e), and 3(f). Here, there *are* correlations among the transverse momenta of the exchanged Reggeons, which lead to values of $\langle\vec{p}_\perp^{\,2}\rangle$ about 50% above those predicted by our asymptotic theory at energies of order 10^3 GeV.[11] At very much higher energies still, when $\langle\vec{p}_\perp^{\,2}\rangle \ll m_0^{\,2}$, the Monte Carlo results should approach those of the asymptotic theory again, because the correlations will become unimportant. But such huge energies are clearly not of practical interest.

IV. INCLUSIVE BEHAVIOR

In this section, we shall consider the results obtained by summing over n, the number of produced particles. We shall first discuss the predictions of the asymptotic theory, and then make comparisons with results calculated numerically.

It is well-known that multiperipheral models embody Feynman scaling,[12] and lead to a logarithmically rising number of produced particles[13]:

$$\langle n\rangle \underset{s\to\infty}{\simeq} \mathrm{const}\times\ln s \quad (50)$$

so that for the quantities defined by Eqs. (32) and (35) of Sec. III, one finds

$$\begin{aligned} &c_{\langle n\rangle}(s)\underset{s\to\infty}{\simeq} c, \quad \text{a constant}\\ &g_{\langle n\rangle}^{\,2}(s)\underset{s\to\infty}{\simeq} g^2, \quad \text{a constant.}\end{aligned} \quad (51)$$

Since the multiplicity distribution is approximately Poisson, and sharply peaked at $n\simeq\langle n\rangle$, $c_n(s)$ and $g_n^{\,2}(s)$ can be replaced by c and g^2 in the formulas of Sec. III, for the purpose of summing over n.

So, assuming that the asymptotic theory can be used for $n\simeq\langle n\rangle$ (which is equivalent to the assumption that the coupling is weak), it follows that

$$\sigma_n(s)\underset{s\to\infty}{\simeq}\mathrm{const}\times s^{2\alpha_0-2}\frac{[g^2\ln(s/s_0)]^{n-2}}{(n-2)!} \quad (52)$$

$$\sigma_{\mathrm{inel}}(s)=\sum_n\sigma_n(s)$$

$$\underset{s\to\infty}{\simeq}\mathrm{const}\times s^{2\alpha_0-2+g^2}, \quad (53)$$

$$\langle n\rangle\underset{s\to\infty}{\simeq} g^2\ln s\,. \quad (54)$$

These results are all familiar from the work of Chew and Pignotti.[14] Also, one finds that

$$\langle\vec{p}_\perp^{\,2}\rangle\underset{s\to\infty}{\simeq}\frac{1}{c} \quad (55)$$

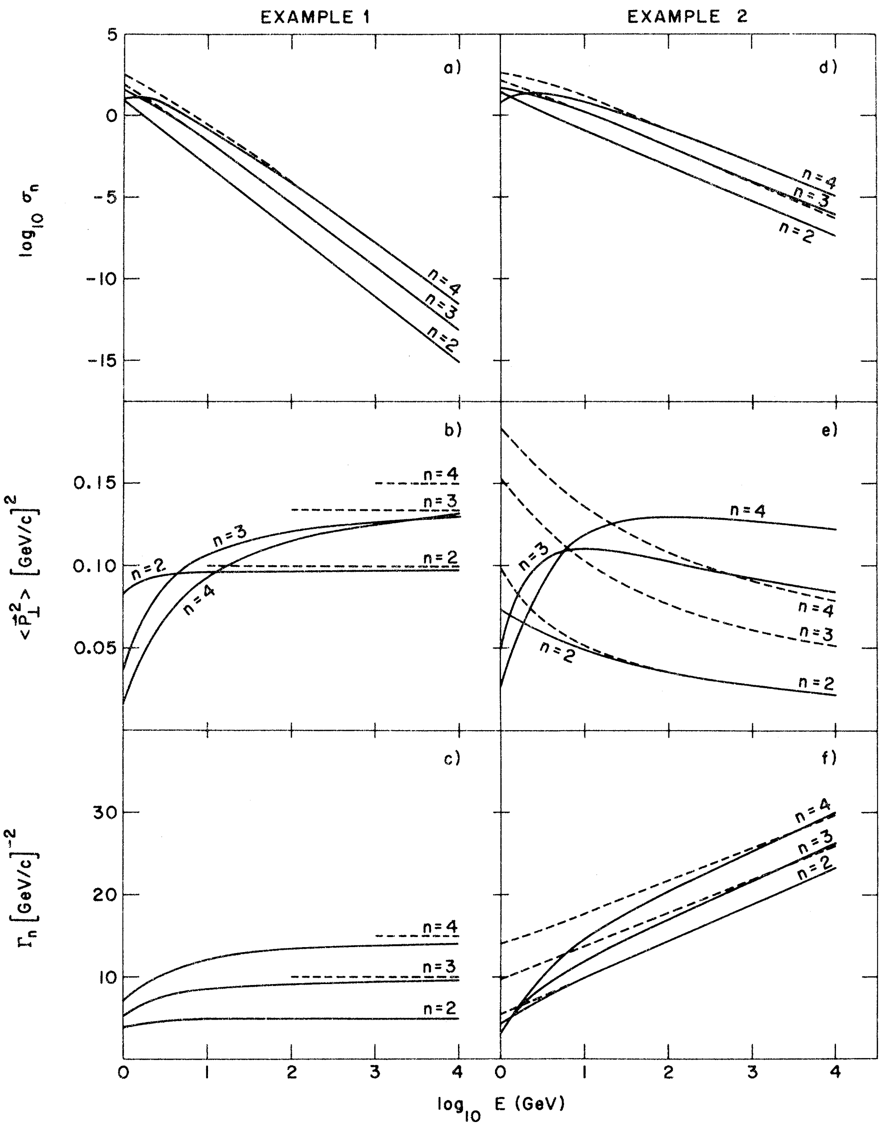

FIG. 3. Comparison of Monte Carlo results (solid lines) with the asymptotic theory (dashed lines) for the examples discussed in Sec. III. Figs. 3(a)–3(c) show $\log_{10}\sigma_n$, $\langle \vec{p}_\perp^{\,2}\rangle$, and Γ_n versus $\log_{10} E$ for Example 1; Figures 3(d)–3(f) show the same quantities for Example 2.[10]

and

$$\Gamma(s) = \langle \Gamma_n(s)\rangle \underset{s\to\infty}{\sim} g^2 \ln s\left(c + \frac{\pi^2 \alpha'^2}{4c}\right) . \tag{56}$$

The fact that the average transverse momentum squared is predicted to approach a constant in the high-energy limit is, of course, another well-known result of multiperipheral models.[1,13,15]

Note, finally, that the quantity $\Gamma(s)$ is predicted to rise logarithmically with s. Now assuming that the unitarity sum (3) is dominated by the inelastic intermediate states,[16] and that near the forward direction the elastic scattering amplitude is mainly imaginary, then $\Gamma(s)$ determines the rate of shrinkage of the diffraction peak, measured by the "Pomeron slope" α_P':

$$\left.\frac{d(\ln\sigma_{el})}{dt}\right|_{t=0} \underset{s\to\infty}{\sim} 2\alpha_P' \ln s \underset{s\to\infty}{\sim} \Gamma(s) . \tag{57}$$

These arguments, leading to a logarithmic shrinkage of the diffraction peak, are also familiar from multiperipheral bootstrap models.[14,16]

This behavior can be given a simple physical interpretation, as outlined in Sec. II. Each link in the multiperipheral chain corresponds to a step in impact-parameter space, whose mean-square length goes to a constant at high energy:

$$\langle \Delta \vec{b}_i{}^2 \rangle \underset{s\to\infty}{\simeq} 2\left(c+\frac{\pi^2\alpha'^2}{4c}\right), \quad i=1,2,\ldots,(n-1) \tag{58}$$

and these steps add in random walk fashion to give an over-all impact parameter

$$\begin{aligned}\langle \vec{b}_{\text{tot}}{}^2 \rangle &\underset{s\to\infty}{\simeq} 2\langle n-1\rangle\left(c+\frac{\pi^2\alpha'^2}{4c}\right)\\ &\underset{s\to\infty}{\simeq} 2g^2(\ln s)\left(c+\frac{\pi^2\alpha'^2}{4c}\right).\end{aligned} \tag{59}$$

This logarithmically-increasing impact parameter then implies a logarithmically-shrinking diffraction peak.

Now let us confront this model with experiment, identifying the produced particles with the experimentally observed pions. According to recent observations, it seems that the total inelastic cross section is roughly constant as a function of energy; the average multiplicity rises like[3]

$$\langle n_\pi \rangle \underset{s\to\infty}{\simeq} 2.3 \ln s\ , \tag{60}$$

the average transverse momentum squared is about[17]

$$\langle \vec{p}_\perp{}^2 \rangle \underset{s\to\infty}{\simeq} 0.16\ (\text{GeV}/c)^2\ , \tag{61}$$

and the rate of shrinkage of the diffraction peak is given by[2]

$$\alpha_P' \simeq 0.28\ (\text{GeV}/c)^{-2}\ . \tag{62}$$

The predictions of the theory are compared with experiment in Table I.

A very little calculation shows that our naive model is incapable of fitting the data. Even supposing that $\alpha'=0$, for instance, to satisfy relations (1), (3), and (4) of Table I simultaneously would require

$$\begin{aligned}&c \simeq [0.16\ (\text{GeV}/c)^2]^{-1},\\ &g^2 \simeq 0.09\ ,\\ &\alpha_0 \simeq 0.95\ .\end{aligned} \tag{63}$$

That is, the low value of α_P' implies that the number of links in the multiperipheral chain can rise only very slowly with $\ln s$, so that the "input" intercept α_0 must be close to 1. But then relation (2) of Table I cannot be satisfied: The number of produced pions is predicted to rise much too slowly.

The cure for this problem is to introduce a new ingredient into the model, namely "clustering". If we suppose that the objects produced at the vertices of the multiperipheral chain each give rise

TABLE I. The predictions of a multi-Regge model (computed in the weak coupling approximation) compared with experimental data on inclusive production of pions at high energies.

	Asymptotic theory ($s\to\infty$)	Experiment[18]
(1) σ_{inel}	$\propto s^{2\alpha_0-2+g^2}$	~const
(2) $\langle n_\pi\rangle/\ln s$	g^2	2.3
(3) $\langle \vec{p}_\perp{}^2\rangle$ [$(\text{GeV}/c)^2$]	$1/c$	0.16
(4) α_P' [$(\text{GeV}/c)^{-2}$]	$\frac{1}{2}g^2\left(c+\frac{\pi^2\alpha'^2}{4c}\right)$	0.28

to *several* final-state pions, on the average, then relation (2) of Table I can also be satisfied along with the others. This idea will be discussed more fully in Sec. V.

Before accepting this conclusion, one must ask whether the asymptotic theory is at all realistic. For the remainder of this section we shall address ourselves to this question.

First let us consider how the asymptotic theory compares with numerical calculations of the inelastic cross sections. To do this, we have taken Examples 1 and 2 of Sec. III, and adjusted the free parameters g_0^2 and b so that a constant total production cross section was obtained, and the mean-square transverse momentum of the produced particles was the same as that given by Eq. (61). The Monte Carlo results are compared with those of the asymptotic theory in Table II.[18]

It can be seen that the asymptotic theory is grossly inaccurate, in quantitative terms, at predicting the inclusive behavior. The reason for this is obvious (but we shall state it anyway): The underlying "strong ordering" assumption, or equivalently, the weak-coupling approximation, is inapplicable to the cases in question, and at a given energy the individual cross sections with $n \simeq \langle n \rangle$ have not yet approached their asymptotic forms.

Nevertheless, the qualitative conclusions obtained using the asymptotic theory still apply. In both the examples considered, the numerical results give too high a value for α_P', and too low a multiplicity. The only way in which both these defects can be simultaneously corrected is to introduce "clustering."

Finally, it remains to discuss how the naive model given by Eq. (22) compares with more general multi-Regge forms. It is clear that our results depend to some extent on the analytic forms assumed for the dependence of the amplitude on s_i and t_i, but it is hard to think of sensible alternative forms which might change our qualitative conclusions. In particular, let us look at more realistic vertex functions. Reggeon-particle–Reggeon vertex functions have been treated by

TABLE II. Analytic results obtained via the weak-coupling approximation compared with numerical results obtained via a Monte Carlo program, for the inclusive behavior of the two examples discussed in Sec. IV. All parameters in units of GeV/c.

	Example 1. ($\alpha_0=0$, $\alpha'=0$)		Example 2. ($\alpha_0=\frac{1}{2}$, $\alpha'=1$)		
	Asymptotic theory ($g^2=2$, $c=6.3$)	Monte Carlo results ($g_0^2=6.5$, $b=1.5$)	Asymptotic theory ($g^2=1$, $c=6.3$)	Monte Carlo results ($g_0^2=13.5$, $b=2.6$)	Experiment
(1) σ_{inel}	const	const	const	const	const
(2) $\langle n_\pi\rangle/\ln s$	2.0	1.0	1.0	0.7	2.3
(3) $\langle \vec{p}_\perp^{\,2}\rangle$	0.16	0.16	0.16	0.16	0.16
(4) α_P'	6.3	0.8	3.3	1.3	0.28

several authors over the past few years.[19] They concluded that the $2\to 3$ particle amplitude, for instance, can be written

$$A(s_1, s_2, t_1, t_2, \kappa) \underset{s\to\infty}{\simeq} s_1^{\alpha_1} s_2^{\alpha_2} \xi_1 \xi_2 \tilde{\beta}(t_1) \times R(t_1, t_2, \kappa)\tilde{\beta}(t_2) \,. \tag{64}$$

Here the t dependence of the trajectories α_1, α_2 and the signature factors ξ_1, ξ_2 has been suppressed. The variable $\kappa = s_1 s_2/s$ is linearly related[20] to $\cos\omega$, the Toller angle at the internal vertex, and is approximately equal to $\mu_{2\perp}^2$, the transverse mass squared of the produced particle. The vertex function can then be written

$$R(t_1, t_2, \kappa) = V(t_1, t_2, \kappa + i\epsilon) - \frac{\tau_1\tau_2}{\xi_1\xi_2}\text{Disc}_\kappa V(t_1, t_2, \kappa) \,, \tag{65}$$

using the Regge-pole signatures τ_1 and τ_2, and a function V which has a cut in the variable κ:

$$V(t_1, t_2, \kappa) = (-\kappa)^{-\alpha_1} V_1(t_1, t_2, \kappa) + (-\kappa)^{-\alpha_2} V_2(t_1, t_2, \kappa) \,. \tag{66}$$

The functions V_1 and V_2 are now real, with no cuts in κ, and tend to constant values as $\kappa \to 0$. They obey certain symmetry relations which we shall not write down.

Comparing these forms with the naive model of Eq. (22) where the vertex functions were taken to be constants × exponentials in the momentum transfers t_i, it can be seen that the net effect is two fold:

(i) Realistic vertex functions will introduce an extra dependence on the transverse masses μ_i^2. The factors $(-\kappa)^{-\alpha}$ in Eq. (66) will, in fact, tend to cancel the factors $(\mu_i^2)^\alpha$ in Eq. (29), and "damp out" any transverse momentum correlations. It is unlikely that there is any strong effect beyond this, because comparisons of the multi-Regge model with experiment[21] have not hitherto turned up any strong dependence on the Toller angles.

(ii) The realistic vertex functions will introduce additional phase factors, so that the phase of the amplitude will not simply be determined by the product of the signature factors of the exchanged Reggeons. It appears that the vertex functions tend to *cancel* the phase introduced by the signature factors. It is possible to separate the slope $\Gamma_n(s)$ given by Eq. (46) into two terms:

$$\Gamma_n^{(1)}(s) = (n-1)\, c_n(s) \,, \qquad \Gamma_n^{(2)}(s) = (n-1)\frac{\pi^2\alpha'^2}{4c_n(s)} \,, \tag{67}$$

the first of which depends on the variation in magnitude of the inelastic amplitude as the scattering angle changes, and the second on the variation in phase (see Koba and Namiki[6]). So the effect of a realistic vertex function is likely to be a reduction in the phase-dependent term $\Gamma_n^{(2)}(s)$. But this term is of minor importance in any case, in the examples considered so far.

Neither of these effects is likely to have much impact on the qualitative results given earlier in this section. So we believe that the resulting conclusions should hold in any realistic multi-Regge model. Some "clustering" effects will have to be included before any agreement with experiment is possible.

V. THE MULTIPERIPHERAL CLUSTER MODEL

In Secs. III and IV, it has been demonstrated that a naive model in which pions are produced singly along a multi-Regge chain is incapable of fitting the experimental data on inclusive pion production at high energies. Some clustering effects must be incorporated into the model if it is to achieve phenomenological success. A more precise definition of a cluster is given in the Appendix.

Such effects have been discussed many times in the past.[22] In fact, the classic ABFST (Amati, Bertocchi, Fubini, Stanghellini, and Tonin) multiperipheral model[13] already contained a primitive form of clustering, in that the final-state pions were supposed to emerge in pairs from each vertex.

It can be seen that these effects must be important, simply from the fact[23] that the average subenergies of adjacent pairs of the produced pions in rapidity space remain fixed and small (less than 1 GeV) as the total energy increases. At such small subenergies, the approximation of dropping all but the multi-Regge ladder diagrams involving a single dominant exchanged Reggeon must inevitably fail. Terms which are neglected, incorrectly, in such an approximation include:

(i) more complicated Regge exchange diagrams involving lower-lying trajectories, multiple s-channel exchanges, "crossed" ladders, etc.,[22,24] and

(ii) resonance terms. The scattering amplitude in any (nonexotic) two-particle exclusive reaction is generally found to be dominated by resonances at low energies. The same is presumably true of many-particle production amplitudes when the energy of any subset of those particles is low.

If one regards effects such as those mentioned in (i) as contributing to the formation of resonances, then the clustering effects can be said to be due entirely to resonances. For simplicity, we shall adopt this attitude for the remainder of this paper.

How can one include these resonance terms in the model? In the past, the hypothesis was put forward[25] that "duality"[26] might enable one to describe the *average* behavior in the resonance region in terms of the leading Regge pole exchange terms. This would lead one back to the original multi-Regge model of Chew and Pignotti,[27] for instance, where the pions are emitted singly after all and resonances are otherwise ignored. But it was recently pointed out[28] that duality applies to *amplitudes*, not to *cross sections*, and that the production cross sections are grossly underestimated at low subenergies by such a prescription. A more sensible strategy, in fact, on both phenomenological and aesthetic grounds, is to treat the production and decay of all particles on an equal footing, whether they be stable final-state particles such as pions, or unstable resonances. This can be done without any double counting[28] if one separates the resonance cross sections into a piece due to the dominant Regge-pole exchange ("coherent" term), and a remainder ("incoherent" term).

At low energies where the resonances are separable, they should really be dealt with individually. At intermediate and high energies, however, where they are densely spaced and strongly overlapping, a collective description is desirable. There are then alternative attitudes one may take to the "incoherent" terms defined above:

(i) One may regard them as being largely determined by a very small number of dynamical parameters, as in a dual resonance model or a quark model, perhaps. Such terms would then be calculable dynamically. The label "incoherent" is inappropriate in such a case.

(ii) One may regard them as being determined by a large number of approximately independent dynamical variables, and thus more amenable to a statistical treatment. In Ref. 28, for instance, it was proposed to treat these terms via the statistical bootstrap model of Hagedorn and Frautschi.[29] The "incoherent" terms would then be treated analogously to the compound nucleus cross sections[30] and Ericson fluctuations[31] found in nuclear physics. This viewpoint will again be adopted in the present paper.

We are thus led to construct a multiperipheral cluster model,[32] in which the usual multi-Regge chain of exchanges occurs, but where particles both stable and unstable are produced at each vertex with a probability depending on their mass in some unknown fashion, and subsequently decay statistically into "clusters" of final-state particles. This will result in the following improvements over the naive model of Sec. III:

(i) enhancements in the cross sections which occur whenever a group of final state particles emerge nearby each other in phase space are *included*, if the model is correct, in diagrams where the members of the group form a single cluster.

(ii) If the final-state pions are grouped into clusters in this way, then the average rapidity gap between clusters is larger than that between individual pions, and so both the "strong ordering" approximation[4] and the ladder approximation should have more success.[33]

For the remainder of this section we shall endeavor to make rough estimates of the parameters required in such a model in order to fit the data. The clusters will be treated as if they were elementary particles, and they will be given a unique "average" mass m_0, so that the same numerical program as in Secs. III and IV can be employed. We shall also suppose that the decay of each cluster gives rise to an average number of pions equal to

$$\bar{n}_\pi(m_0) = 0.30\left(\frac{m_0}{m_\pi}\right) + 0.6 \,. \tag{68}$$

This formula is in accordance with the results of the statistical bootstrap model, and also describes the decay patterns of the observed low-mass resonances reasonably well.[34] The corresponding value for the mean-square transverse momentum of the produced pions, relative to the center of mass of the cluster, can be deduced from the results of Frautschi and Hamer.[35] We shall mainly be interested in the asymptotic value as m_0 tends to infinity:

$$\langle \vec{p}_\perp{}^2\rangle_\infty = \lim_{m_0\to\infty}\langle \vec{p}_\perp{}^2(m_0)\rangle = \frac{2}{3}\frac{\int d^3p\, p^2 \exp[-(m_\pi{}^2+p^2)^{1/2}/T_{\rm eff}]}{\int d^3p\, \exp[-(m_\pi{}^2+p^2)^{1/2}/T_{\rm eff}]}\,. \tag{69}$$

If we put $T_{\rm eff}=128$ MeV, which is consistent with the parameters of Eq. (68),[34] then

$$\langle \vec{p}_\perp{}^2\rangle_\infty = 0.145\ ({\rm GeV}/c)^2\,. \tag{70}$$

Now the mean-square transverse momentum per produced pion which is contributed by the exchanged Reggeons (whose correlations are again neglected) is asymptotically equal to $1/c\bar{n}_\pi{}^2(m_0)$. Convoluting the momentum distribution due to the exchanged Reggeons with that resulting from the decay of the cluster, it follows that the *total* mean-square transverse momentum per pion is simply the sum:

$$\langle \vec{p}_\perp{}^2\rangle_\pi \underset{s\to\infty}{\sim} \langle \vec{p}_\perp{}^2(m_0)\rangle + \frac{1}{c\bar{n}_\pi{}^2(m_0)}\,, \tag{71}$$

since the two distributions are symmetric and uncorrelated.

The asymptotic theory for this crude form of multiperipheral cluster model therefore leads to the relations listed in Table III.

A few trials are enough to show that, according to the asymptotic theory, only Pomeranchuk exchange ($\alpha_0 \lesssim 1$) will suffice to fit the data. In this case $\bar{n}_\pi(m_0)$ must be large; in fact, m_0 may even diverge logarithmically as $s\to\infty$, as in the diffractive excitation models of Hwa, and Jacob and Slansky.[36] This would also imply that

$$\langle \vec{p}_\perp{}^2\rangle_\infty = \langle \vec{p}_\perp{}^2\rangle_\pi\,, \tag{72}$$

i.e., the observed average transverse momentum comes purely from the decay of the cluster. According to Eq. (70) $\langle \vec{p}^2\rangle_\infty = 0.145$ $({\rm GeV}/c)^2$ whereas $\langle \vec{p}_\perp{}^2\rangle_\pi = 0.16$ $({\rm GeV}/c)^2$. Allowing for a certain degree of error, these figures are not incompatible with Eq. (72).

It was found in Sec. IV, however, that the asymptotic theory may be grossly inaccurate in predicting inclusive properties of the produced pion spectrum. So we must check numerically whether the exchange of lower-lying trajectories is really excluded. In order to do this, we have chosen to consider the example with $\alpha_0=\frac{1}{2}$, $\alpha'=1$ $({\rm GeV}/c)^{-2}$ (ρ-like trajectory). The parameters $g_0{}^2$, b, and m_0 were adjusted by hand in an attempt to get a rough fit to the experimental behaviors of $\sigma_{\rm incl}$, $\langle n_\pi\rangle$, and $\alpha_P{}'$, as given in Table III; the corresponding value of $\langle \vec{p}_\perp{}^2\rangle_\pi$ can then be compared with data afterwards. With the parameters $g_0{}^2=22$, $b=-4.2$ $({\rm GeV}/c)^{-2}$, and $m_0=4.6$ GeV, for instance, we find

$$\begin{aligned}\sigma_{\rm incl} &\simeq {\rm const}\,,\\ \langle n_\pi\rangle/\ln s &\simeq 2.3\,,\\ \alpha_P{}' &= \frac{\Gamma}{2\ln s}\\ &\simeq 0.5\ ({\rm GeV}/c)^{-2}\,.\end{aligned} \tag{73}$$

These values are reasonably close to the experimental numbers given in Table III. The result for $\alpha_P{}'$ is somewhat too high, and it is difficult to get it any lower; but 30% of this number is contributed by the phase-dependent term $\Gamma^{(2)}$, and this fraction should perhaps be thrown away (see Sec. IV). If $\Gamma^{(2)}$ is neglected, a fit to the data is certainly possible.

The corresponding transverse momentum squared per cluster averaged approximately 0.4 $({\rm GeV}/c)^2$. But since the number of pions emitted

TABLE III. The predictions of a multiperipheral cluster model (MCM), derived in a weak-coupling approximation, compared with experimental data on inclusive production of pions at high energies.

	MCM asymptotic theory	Experiment
(1) $\sigma_{\rm incl}$	$s^{2\alpha_0-2+g^2}$	const
(2) $\langle n_\pi\rangle/\ln s$	$g^2\bar{n}_\pi(m_0)$	2.3
(3) $\langle \vec{p}_\perp{}^2\rangle_\pi$ $[({\rm GeV}/c)^2]$	$\langle \vec{p}_\perp{}^2(m_0)\rangle + \dfrac{1}{c\bar{n}_\pi{}^2(m_0)}$	0.16
(4) $\alpha_P{}'$ $[({\rm GeV}/c)^{-2}]$	$\frac{1}{2}g^2\left(c+\dfrac{\pi^2\alpha'^2}{4c}\right)$	0.28

from each cluster is high,

$$\bar{n}_\pi(m_0) = 10.5\,, \tag{74}$$

it follows that the average transverse momentum squared per *pion* which is contributed by the exchanged Reggeons is only 0.004 $(\mathrm{GeV}/c)^2$. Thus relation (3) of Table III can again be satisfied provided Eq. (72) is approximately true.

Similar conclusions hold for the exchange of pion-like Reggeons. The value of m_0 needed in this case is about 2.1 GeV.

VI. SUMMARY AND CONCLUSIONS

In this paper, we have used the present experimental data concerning the gross features of inclusive production in high-energy pp collisions, and compared it with the predictions of multi-Regge models. The calculations were made both by approximate analytical methods, and more exact numerical ones.

It was found that a naive model in which the final state pions are produced singly at each vertex of the multiperipheral chain cannot fit the data. The rate of increase of the multiplicity comes out too low, and the rate of shrinkage of the diffraction peak comes out too high (except for Pomeron exchange). There is no possibility of remedying both these defects together by adjusting parameters, because the two quantities are proportional to each other, with a scale set by the transverse momentum of the produced particles. A physical reason for this was discussed in terms of impact parameters in Sec. II.

The model can only be salvaged if each vertex in the multiperipheral chain gives rise to a *cluster* of final-state pions. As shown in the Appendix, even the simple multi-Regge model itself can formally be treated as a multicluster model. Compared with this case, however, we need clusters whose particles are more closely spaced in longitudinal momentum, to raise the multiplicity, and more closely spaced in the transverse direction, so as to reduce the shrinkage of the diffraction peak. Such clusters are in fact a very natural occurrence, and will result from enhancements of the production cross sections whenever several of the final-state particles emerge close to each other in phase space. One expects such enhancements to occur via resonance formation, in particular, and also due to nonplanar Regge exchange diagrams, etc.

A particular example was discussed[28] in which the decay of a cluster was treated according to the statistical bootstrap model of Hagedorn and Frautschi.[29] Study of this example leads one to the following conclusions:

(a) The inclusive-production data given in Table III are in principle sufficient to determine α_0, the intercept of the input trajectory, as well as the other major parameters involved in a multiperipheral cluster model. The results of the weak-coupling approximation, for instance, lead one to expect that Pomeranchuk exchange must dominate at high energies ($\alpha_0 \simeq 1$). But in practice, this conclusion cannot be maintained. In realistic numerical calculations, models with α_0 equal to $\frac{1}{2}$ or 0 (ρ, ω, ... or pion exchange) could be made to fit the data also. The basic reason for this ambiguity is that the two contributions to the transverse momentum of the produced pions (namely, that from the decay of a cluster, and that from Reggeon exchange) cannot be separated in practice, and relation (3) of Table III cannot be made into a useful restriction. Thus the question of which exchanges are dominant must be settled by other means.

(b) Regardless of which exchange is dominant, the average number of exchanged Reggeons grows much more slowly than one would expect from the weak coupling approximation. So in order to explain the observed pion multiplicity, each cluster must give rise to a surprisingly large number of pions, of order five to ten on the average. The average cluster mass must be correspondingly high, of order a few GeV at least. These results may depend to some extent on the specific parametrization of the model we have adopted. But they are in accord with the indications from longitudinal phase-space analyses of various exclusive final states,[37] where only small numbers of multi-Regge events are found.

(c) It follows from (b) that the mean-square transverse momentum of the produced pions comes mainly from the decay of the clusters, and the contribution of the exchanged Reggeons is of order a few percent only.[38] Under our assumption that these decays follow statistical laws, it therefore follows that the transverse-momentum distribution of the pions produced in the central region of rapidity space ("pionization region") should be predominantly a statistical one, as in the thermodynamic model.[39] This provides some *a posteriori* justification for Hagedorn's model.

APPENDIX: DEFINITION OF A CLUSTER

Consider the general n-particle production amplitude discussed in Sec. II, and illustrated in Fig. 4. Consider some subset of l final state particles and choose a kinematic ordering so that they are adjacent, i.e., their four-momenta are p_{r+1}, $p_{r+2}, \ldots, p_{r+l}$. Suppose also that the T matrix can be factorized as follows:

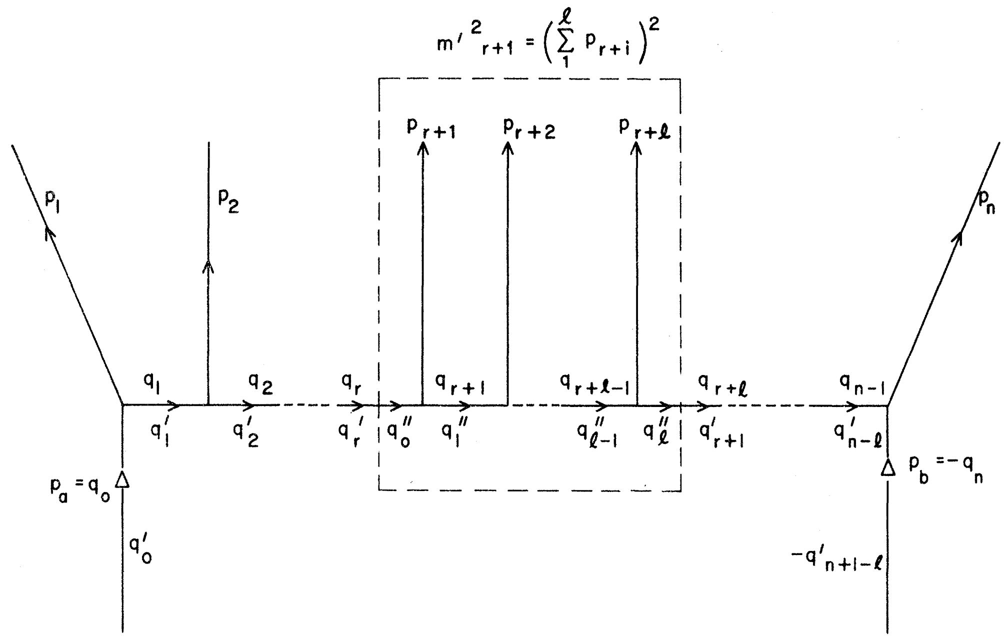

FIG. 4. Choice of kinematic variables for discussion of clusters.

$$T_n\{q\} = \tilde{T}(q_0, \ldots, q_r, q_{r+l}, \ldots, q_n)$$
$$\times D(q_r, q_{r+1}, \ldots, q_{r+l}). \qquad \text{(A1)}$$

Then we can describe these l-particles as a cluster. Let us label the q's as follows:

$$q_i' = q_i, \quad i = 0, 1, \ldots, r$$
$$q_i' = q_{i-l+1}, \quad i = r + l, \ldots, n \qquad \text{(A2)}$$
$$q_i'' = q_{i-r}, \quad i = r, r+1, \ldots, r+l$$

(so that $q_r' = q_0''$, $q_{r+1}' = q_l''$). Then

$$T_n\{q\} = \tilde{T}(q_0', \ldots, q_{n+1-l}')D(q_0'', \ldots, q_l'') \qquad \text{(A3)}$$

and the phase-space volume element

$$d\Omega_n = \delta((q_n - q_{n-1})^2 - m_n^2)$$
$$\times \prod_{i=1}^{n} d^4q_i\, \delta((q_i - q_{i-1})^2 - m_i^2) \qquad \text{(A4)}$$

can be written

$$d\Omega_n = \int d\Omega_{n+1-l}'\, d\Omega_l''\, dm_{r+1}'^{\,2}, \qquad \text{(A5)}$$

where the masses m', m'' are defined corresponding to (A2). The kinematics is illustrated in Fig. 4.

Now applying the arguments of Sec. II, we can easily see that

$$\Gamma_n^c = \tfrac{1}{2}\langle(\vec{b}_0' - \vec{b}_{n-l}')^2\rangle + \tfrac{1}{2}\langle(\vec{b}_0'' - \vec{b}_{l-1}'')^2\rangle, \qquad \text{(A6)}$$

where each average now also includes an average over $m_{r+1}'^{\,2} = (\sum_{i=1}^{l} p_{r+i})^2$. The generalization to several clusters is obvious.

A multiperipheral cluster model is one in which $\tilde{T}$ is itself a multiperipheral amplitude: the contribution of the first term becomes proportional to $n - l$ for a single cluster or $n - 1 - \sum (l - 1)$ for several clusters. The contribution of the second term can in principle be anything. Two special cases are simple to treat.

First, the pure multiperipheral model of Sec. III is a trivial example. In this case the amplitude D is itself multiperipheral and thus we recover the results of Sec. II.

To reduce Γ we need clusters which do not spread out in impact parameter to this extent. The other extreme comes when we set

$$D(\vec{q}_0'', \ldots, \vec{q}_l'') = D((\vec{q}_0'' - \vec{q}_l'')^2). \qquad \text{(A7)}$$

In this case the Fourier transform becomes $\delta^{(2)}(\vec{b}_0'' - \vec{b}_{l-1}'')$ and the second term contributes nothing. This corresponds to the case when all l particles in the cluster emerge with the same impact parameter, and is the basis for the calculations of Sec. V. A realistic cluster model would lie between these extremes.

*Work performed under the auspices of the U.S. Atomic Energy Commission.

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