Slope of the Forward Peak: An s-Channel Analysis*

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Starting from the exact multiparticle unitarity relation, we derive the derivative extension of the optical theorem. The slope of the forward peak is then related in a model-independent way to the rotational properties of production amplitudes. A sequential representation is developed for treating an n-particle state. The formalism is then applied to the multiperipheral and diffractive models. The analysis makes clear the mechanisms in which the forward slope is built up in those models. In the case of the multiperipheral model the slope is proportional to the number of links in the multiperipheral chain. Quantitatively, the slope of the Pomeranchuk trajectory at $t=0$ is too large. In an oversimplified version of the diffractive model, the predicted slope of the elastic forward peak is one-half the mean slope, the average being taken over all two-cluster production processes; consequently, it is too small. The realistic high-energy model is somewhere between the two extremes considered.

I. INTRODUCTION

Various models have been suggested for the description of multiparticle production processes at very high energies. Among them the multiperipheral[1] and the diffractive[2] models have almost opposite points of views, and yet their predictions on measurable quantities are so similar that to this date no experiment has been accurate enough to rule out either one of the two basic views. Besides, both models can be and have been modified to accommodate new empirical features, so an experimental selection of the realistic model is likely to be a long and indecisive procedure.

There is, however, one feature on which the two models make drastically different claims. That is the slope of the forward peak. In the multiperipheral model (MPM) the slope increases at least as fast as $\ln s$, whereas in the diffractive excitation model (DEM) it is limiting, i.e., approaches a constant as $s\to\infty$. Experimentally,[3] the derivative of the slope with respect to $\ln s$ is becoming smaller and smaller as s increases, although the experimental errors again do not preclude either possibility. Nevertheless, the divergence of the theoretical assertions on this matter makes it a worthwhile subject for a close examination. It should be possible to calculate either the slope b or its derivative $db(s)/d\ln s$ from the parameters of the models. As we shall show in this paper, the calculations in two simple versions of the two models yield results that are on two opposite sides of what may be regarded as acceptable. Thus, we foresee the slope calculation as an effective guidepost for the modification and improvement of high-energy models.

The slope of the forward peak has been investigated in the past mainly in the t channel. In the MPM the slope of the Pomeranchuk trajectory at $t=0$ is related to the slope parameter of the average multiplicity $\langle n\rangle$ and the eigenfunction of the multiperipheral integral equation.[4] In the DEM the slope of the diffraction peak is calculated in the t channel assuming a weak, fixed branch point for the Pomeranchukon,[5] and the results are reasonable for a variety of processes.[6] But none of these calculations provide a good insight on what really builds up the slope, a question which can be satisfactorily answered only if we analyze directly in the s channel. An s-channel approach has recently been undertaken by Tiktopoulos and Treiman,[7] but they used the Schwarz inequality and therefore obtained only a lower bound.

In this paper we start from the complete s-channel unitarity relation. We derive a "derivative optical theorem" which relates the slope of the forward peak to "matrix elements" of a product of rotation operators. To illustrate how this relation can be used as a diagnostic tool for examining high-energy models, we apply it to two simple versions of the MPM and DEM. It is found that in the case of the MPM the slope of the forward peak shrinks too fast, and that in the case of the DEM the limiting slope is too small. Evidently, this result reflects the extremities of the two views. The realistic situation must lie somewhere in between.

II. DERIVATIVE OPTICAL THEOREM

The relation to be proven follows only from unitarity. Define the S matrix in the usual way

$$S_{fi}=\delta_{fi}+i(2\pi)^4\delta^4(K_f-K_i)T_{fi}\ , \tag{1}$$

where K_i and K_f are the total four-momenta of the initial and final states; we shall use K to denote both. The imaginary part of the elastic amplitude is given by

$$A(s,\theta)\equiv(2i)^{-1}(T_{fi}-T_{fi}^*)$$
$$=\tfrac{1}{2}\sum_n\int\left[\prod_{j=1}^{n}(dp_j)\right](2\pi)^4$$
$$\times\delta^4\left(\sum_{j=1}^{n}p_j-K\right)T_{in}^*T_{fn}\,, \qquad (2)$$

where $\sum_n$ represents the sum over all intermediate states, each of which has n particles. For simplicity, we consider only scalar particles here. In (2) we have used the abbreviation

$$(dp_j)\equiv[(2\pi)^3 2p_j^0]^{-1}d^3p_j\,. \qquad (3)$$

The normalization of T is such that the optical theorem is

$$A(s,0)=2k\sqrt{s}\;\sigma_T\,, \qquad (4)$$

where k is the incident c.m. momentum, and that the differential cross section for two-body reaction is

$$\frac{d\sigma}{d\Omega}=\frac{p}{k}(64\pi^2 s)^{-1}|T|^2\,, \qquad (5)$$

where p is the final c.m. momentum.

A. Two-Body States

We want to take the derivative of (2) with respect to $\cos\theta$. Let us first consider the contribution from any two-body intermediate state and denote it by $A_2(s,\theta)$. With p designating the intermediate c.m. momentum, and $\epsilon=\frac{1}{2}$ (or 1) if the intermediate particles are identical (or not), we have

$$A_2(s,\theta)=\frac{p\epsilon}{32\pi^2\sqrt{s}}\int d\Omega_p\,T^*(\vec{k}_i,\vec{p})T(\vec{k}_f,\vec{p})\,, \qquad (6)$$

where

$$\vec{k}_f=R_y(\theta)\vec{k}_i\,. \qquad (7)$$

Here $R_y(\theta)$ represents a rotation about the y axis by an angle θ, and $\vec{k}_i$ is along the z axis. The derivative of (6) with respect to $z\equiv\cos\theta$ can be carried inside the integral and acts only on the second T, giving rise to the quantity $\partial T(\vec{k}_f,\vec{p})/\partial z$. To evaluate this quantity in the limit $z\to1$, let us use (ψ,ϕ) to denote the polar and azimuthal angles of $\vec{p}$, and χ the angle between $\vec{k}_f$ and $\vec{p}$. Then

$$\cos\chi=\cos\theta\cos\psi+\sin\theta\sin\psi\cos\phi\,, \qquad (8)$$

hence,

$$\frac{\partial T(\cos\chi)}{\partial z}=\frac{\partial T(\cos\chi)}{\partial\cos\chi}(\cos\psi-\cot\theta\sin\psi\cos\phi)\,. \qquad (9)$$

The $\cot\theta$ term in (9) diverges as $\theta\to0$, so we must expand $\partial T/\partial\cos\chi$ to first order in $\sin\theta$:

$$\frac{\partial T(\chi)}{\partial\cos\chi}=\frac{\partial T(\psi)}{\partial\cos\psi}+\sin\theta\sin\psi\cos\phi\frac{\partial^2T(\psi)}{\partial(\cos\psi)^2}+\cdots\,. \qquad (10)$$

Upon substitution of (10) into (9) the remaining $\cot\theta$ term does not contribute after integration over ϕ, so we keep only

$$\lim_{z\to1}\frac{\partial T(\cos\chi)}{\partial z}=\cos\psi\frac{\partial T(\cos\psi)}{\partial\cos\psi}-\sin^2\psi\cos^2\phi\frac{\partial^2T(\cos\psi)}{\partial(\cos\psi)^2}\,. \qquad (11)$$

Using the symbol $\zeta=\cos\psi$, we obtain for the derivative of (6)

$$A_2'(s,0)=\frac{p\epsilon}{16\pi\sqrt{s}}\int_{-1}^{1}d\zeta\,T^*(\zeta)\times\left[\zeta\frac{\partial T(\zeta)}{\partial\zeta}-\tfrac{1}{2}(1-\zeta^2)\frac{\partial^2T(\zeta)}{\partial\zeta^2}\right]\,. \qquad (12)$$

B. Multiparticle States: $\partial A/\partial\theta$

For the multiparticle intermediate states, let us first consider the simpler problem of the derivative with respect to θ, not $\cos\theta$. The result is trivial if we make use of the fact that the eigenfunctions of the rotation operator for a two-body state are the spherical harmonics, which in the spinless case are functions only of $\cos\theta$. Thus, $A(s,\theta)$ contains only even powers of θ, and $\partial A/\partial\theta$ at $\theta=0$ must necessarily vanish.

The question that we address ourselves to here is whether that implies any constraint on the production amplitudes through unitarity. In answering this question we set up the formalism useful in Sec. II C. If we move the differential operator $\partial/\partial\theta$ through the unitarity integral, it acts on $T(\vec{k}_f,\vec{p}_1\cdots\vec{p}_n)$. By rotational invariance we express $T(\vec{k}_f,\vec{p}_j)$ in terms of $\vec{k}_i$ and $\vec{q}_j$ where

$$\vec{q}_j=R_y^{-1}(\theta)\vec{p}_j\,,\quad j=1,\ldots,n\,. \qquad (13)$$

We then obtain

$$\frac{\partial}{\partial\theta}T(\vec{k}_f,\vec{p}_1\cdots\vec{p}_n)=\sum_{j=1}^{m}\frac{\partial}{\partial\vec{q}_j}T(\vec{k}_i,\vec{q}_1\cdots\vec{q}_n)\left(\frac{\partial\vec{q}_j}{\partial\theta}\right)_{p_j}, \qquad (14)$$

which implies in the $\theta\to0$ limit

$$\lim_{\theta\to 0}\frac{\partial}{\partial\theta}A_{n>2}(s,\theta)=\tfrac{1}{2}\sum_n\int d\Phi_n\sum_j\left[T^*_{in}\frac{\partial}{\partial\vec{p}_j}T_{in}\right]\times\lim_{\theta\to 0}\left(\frac{\partial\vec{q}_j}{\partial\theta}\right)_{p_j}, \tag{15}$$

where $d\Phi_n$ is the phase-space density in (2). Since $A(s,\theta)$ is real, the quantity in the square bracket in (15) is equivalent to its complex conjugate, and consequently also to

$$\frac{1}{2}\frac{\partial}{\partial\vec{p}_j}|T_{in}(\vec{k}_i,\vec{p}_1\cdots\vec{p}_n)|^2 . \tag{16}$$

From (13) we have

$$\vec{q}_j=\hat{u}_x(p_x^j\cos\theta-p_z^j\sin\theta)+\hat{u}_y p_y^j+\hat{u}_z(p_x^j\sin\theta+p_z^j\cos\theta), \tag{17}$$

so

$$\lim_{\theta\to 0}\frac{\partial}{\partial\theta}A_{n>2}(s,\theta)=-\tfrac{1}{4}\sum_n\int d\Phi_n\sum_j iJ_y^j|T_{in}|^2, \tag{18}$$

where

$$J_y^j\equiv -i\left(p_z^j\frac{\partial}{\partial p_x^j}-p_x^j\frac{\partial}{\partial p_z^j}\right). \tag{19}$$

The right-hand side of (18) can be expressed in terms of the inclusive cross section, as we now show.

Let us first be a little more explicit in the description of the phase-space integration:

$$\sum_n\int d\Phi_n=\sum_t\sum_{n_t}\int\prod_t\left[\frac{1}{n_t!}\prod_{l=1}^{n_t}(dp_l)\right](2\pi)^4\times\delta^4\left(\sum_{j=1}^n p_j-K\right), \tag{20}$$

where t denotes the type of particle produced, n_t the number of particles of type t, and $n=\sum_t n_t$. The total production cross section is

$$\sigma_P=\sum_t\sum_{n_t}\sigma_{n_1\cdots n_t\cdots}\equiv\sum_t\sum_{n_t}\sigma_{\{n_t\}}, \tag{21}$$

where the summand is the cross section for producing n_t particles of type t with t ranging over all types, i.e.,

$$\sigma_{\{n_t\}}=(4k\sqrt{s})^{-1}\int\prod_t\left[\frac{1}{n_t!}\prod_{l=1}^{n_t}(dp_l)\right](2\pi)^4\times\delta^4(\cdots)|T_{i\{n_t\}}|^2 . \tag{22}$$

If in the above integral there is no integration over (dp_j), where j labels a particle of type t_j, we get a cross section which we denote by $\sigma_{\{n_t\}}(dp_j)$. Let us further define

$$\sigma_{n(t_j)}(p_j)=\sum_{\substack{t\\ t\neq t_j}}\sum_{n_t}\sigma_{\{n_t\}}(p_j). \tag{23}$$

Then clearly we have

$$\sigma_P=\sum_{n_t}\int(dp)\sigma_{n_t}(p), \tag{24}$$

$$f_t(p)=\sum_{n_t}n_t\sigma_{n_t}(p), \tag{25}$$

where $f_t(p)$ is the inclusive cross section for detecting a particle of type t at $\vec{p}$ in the phase volume (dp). We now see that the right-hand side of (18) can be written as

$$\begin{aligned}-ik\sqrt{s}\sum_t\sum_{n_t}\sum_j\int(dp_j)J_y^i\sigma_{\{n_t\}}(p_j)&=-ik\sqrt{s}\sum_{t_j}\sum_{n(t_j)}n(t_j)\int(dp_j)J_y^i\sigma_{n(t_j)}(p_j)\\&=-ik\sqrt{s}\sum_t\int(dp)J_y f_t(p).\end{aligned} \tag{26}$$

The last expression vanishes upon integration over ϕ because $f_t(p)$ can have no azimuthal dependence in a problem involving only scalar particles. Note that this azimuthal independence applies only to the inclusive cross section but not to the amplitude, which involves other momenta p_l, $l\neq j$, that are held fixed in (15) or (18). Hence, we conclude that the vanishing of $\partial A/\partial\theta$ at $\theta=0$ puts no constraint on the production amplitudes. It is interesting to see the intricate consistency among rotational symmetry, multiparticle unitarity, and the analyticity of $A(s,t)$ at $t=0$.

C. Multiparticle States: $\partial A/\partial z$

Let us now consider the derivative of (2) with respect to $\cos\theta$. Instead of (14) we have

$$\frac{\partial}{\partial z}T(\vec{k}_f,\vec{p}_1\cdots\vec{p}_n)=\sum_{j=1}^n\frac{\partial}{\partial\vec{q}_j}T(\vec{k}_i,\vec{q}_1\cdots\vec{q}_n)\left(\frac{\partial\vec{q}_j}{\partial z}\right)_{p_j}, \tag{27}$$

where

$$\left(\frac{\partial\vec{q}}{\partial z}\right)_p=\hat{u}_x(p_x+p_z\cot\theta)+\hat{u}_z(p_z-p_x\cot\theta). \tag{28}$$

The singularity at $\theta=0$ due to the $\cot\theta$ terms requires an expansion of $\partial T/\partial\vec{q}$ in $\sin\theta$, just as in the two-body case. We have to the first order in $\sin\theta$

$$\frac{\partial T(\cdots\vec{q}_j\cdots)}{\partial\vec{q}_j}=\frac{\partial T(\cdots\vec{p}_j\cdots)}{\partial\vec{p}_j}+\sum_{k=1}^{n}(\vec{q}_k-\vec{p}_k)\cdot\frac{\partial^2}{\partial\vec{p}_k\,\partial\vec{p}_j}\times T(\cdots\vec{p}_j,\vec{p}_k\cdots)\ . \tag{29}$$

It is straightforward to establish for small θ

$$(\vec{q}_k-\vec{p}_k)\cdot\frac{\partial}{\partial\vec{p}_k}=\sin\theta\left(p_x^k\frac{\partial}{\partial p_z^k}-p_z^k\frac{\partial}{\partial p_x^k}\right)\ . \tag{30}$$

We thus obtain using the definition in (19)

$$\frac{\partial T(\vec{k}_f,\vec{p})}{\partial z}=\sum_j i\cot\theta J_y^j T(\vec{k}_i,\vec{p})+\sum_{j,k}\cos\theta J_y^k J_y^j T(\vec{k}_i,\vec{p})\ . \tag{31}$$

The first term on the right-hand side is the same as the right-hand side of (14), apart from the $\cot\theta$ factor. Since we perform the integration over the phase space first, and then let $\theta\to 0$, that term does not contribute to $\partial A/\partial z$ at $z=1$ for the same reason that $\partial A/\partial\theta=0$ at $\theta=0$. We are therefore left with the regular term in (31), which yields the final result

$$A'(s,0)=\lim_{z\to 1}\frac{\partial}{\partial z}A(s,z)=\tfrac{1}{2}\sum_n\sum_{j,k}\int d\Phi_n\,T_{in}^*J_y^jJ_y^kT_{in}\ . \tag{32}$$

It is possible to verify (as we shall indicate later) that when $n=2$, (32) reduces to our earlier result (12). Equation (32) may be regarded as the derivative optical theorem.

D. A Formal Derivation

The result obtained above suggests an alternative method for its derivation. We have used rotational invariance to write

$$T(\vec{k}_f,\{\vec{p}\})=T(\vec{k}_i,\{\vec{q}\})\ , \tag{33}$$

where $\{\vec{p}\}\equiv\vec{p}_1\cdots\vec{p}_n$, and $\vec{q}$ is defined in (13). To generalize this to a problem that involves spinning particles is a standard procedure which we will not consider here. Since $\vec{k}_i$ points along the z axis, let us omit the dependence of T on it, and write

$$T_{in}(\vec{k}_i,\{\vec{p}\})=T_n(\{\vec{p}\})\equiv|T_0\rangle\ , \tag{34}$$

$$T_{fn}(\vec{k}_i,\{\vec{q}\})=T_n(\{\vec{q}\})\equiv|T_\theta\rangle\ . \tag{35}$$

Now, let $|T_\theta\rangle$ be vectors in a linear vector space in which the scalar product is defined as follows:

$$\langle T_0|T_\theta\rangle=\tfrac{1}{2}\sum_n\int d\Phi_n\,T_n^*(\{\vec{p}\})T_n(\{\vec{q}\})=A(s,\theta)\ . \tag{36}$$

The norm of these vectors is (for any θ)

$$\langle T_\theta|T_\theta\rangle=A(s,0)\ , \tag{37}$$

which exists for every finite s by virtue of the optical theorem (4). We can define a representation of the rotation operator in this linear vector space by the homomorphism

$$R(\vec{n}\theta)\to U(R)=\exp(-i\theta\vec{n}\cdot\vec{J})\ , \tag{38}$$

where

$$\vec{J}=\sum_{j=1}^{n}\vec{J}^j\ ,\qquad J_x^j=-i\left(p_y^j\frac{\partial}{\partial p_z^j}-p_z^j\frac{\partial}{\partial p_y^j}\right)\ , \tag{39}$$

and cyclic permutations. We thus have

$$T_n(\{\vec{q}\})=T_n(\{R_y^{-1}(\theta)\vec{p}\})=\exp(-i\theta J_y)T_n(\{\vec{p}\})\ . \tag{40}$$

From (36) we then obtain

$$A(s,\theta)=\langle T_0|\exp(-i\theta J_y)|T_0\rangle=\langle T_0|T_0\rangle-i\theta\langle T_0|J_y|T_0\rangle-\tfrac{1}{2}\theta^2\langle T_0|J_y^2|T_0\rangle+\cdots\ . \tag{41}$$

The second term on the right-hand side corresponds to (18) and has been shown to vanish. Evidently, we get

$$A'(s,0)=\langle T_0|J_y^2|T_0\rangle\ , \tag{42}$$

which agrees with (32) because of (39). Combining this expression with (37) enables us to relate the logarithmic derivative of A to the forward expectation value of J_y^2:

$$\left\{\frac{d}{dz}[\ln A(s,z)]\right\}_{z=1}=\frac{\langle T_0|J_y^2|T_0\rangle}{\langle T_0|T_0\rangle}\ . \tag{43}$$

The operator J_y is not *a priori* Hermitian; such properties are determined by the nature of the inner-product space. It follows directly from

$$\langle T_\theta|T_\theta\rangle=\langle T_0|T_0\rangle \tag{44}$$

and (40) that J_y is indeed Hermitian. Thus, the slope as defined by (43) must be positive. Moreover, the Hermiticity of J_y implies that $\langle T_0|J_y|T_0\rangle$ must be real and therefore must vanish in order to maintain the reality of $A(s,\theta)$ in (41). The same argument applies to all higher odd-power terms of J_y in (41). We thus obtain

$$\left\{\frac{\partial^m}{\partial(\theta^2)^m}[\ln A(s,\theta)]\right\}_{\theta=0}=\frac{(-1)^m m!}{(2m)!}\frac{\langle T_0|(J_y)^{2m}|T_0\rangle}{\langle T_0|T_0\rangle}\ . \tag{45}$$

Since the left-hand side is measurable, this relation may be utilized to constrain the possible forms of $T_n(\{\vec{p}\})$ in the construction of realistic models.

III. A SEQUENTIAL REPRESENTATION

The n vectors in the set $\{\vec{p}\}$ describe the momenta of the n particles in the intermediate state; they are constrained by the energy-momentum conservation: $\delta^4(\sum_i p_i - K)$. When the operator J_y^j acts on $T_n(\{\vec{p}\})$, all the other momenta $\vec{p}_i$, $i\neq j$, are held fixed. This is as required by the chain rule in the differentiation in (27) in spite of the energy-momentum conservation, which is guaranteed by the δ function in $d\Phi_n$. If in a model this representation in terms of $\{\vec{p}\}$ is cumbersome because of the conservation constraint, we suggest a different representation which has the constraint already built in. We call it the sequential representation; it seems particularly suitable for the MPM.

Let us order the momenta in $\{\vec{p}\}$ in some way and define

$$P_j^\mu=\sum_{i=1}^{j} p_i^\mu\ . \tag{46}$$

Then, clearly $P_n^\mu=K^\mu$. We can define the invariants

$$s=K^2,\quad s_j={P_j}^2,\quad t_j=(k_a^\mu-P_j^\mu)^2\ , \tag{47}$$

where k_a^μ is the 4-momentum of one of the initial particles. The remaining set of independent variables may be taken to be the azimuthal angles of $\vec{P}_j$. Instead of the set $\{\vec{p}\}$, let us first choose to express T_{in} in terms of the set

$$\{\vec{P}\}=\vec{P}_1\cdots\vec{P}_{n-1}\ . \tag{48}$$

By momentum conservation $\vec{P}_n$ vanishes in the c.m. frame. Energy conservation fixes the over-all normalization of these vectors, which is unaffected by rotation. Thus, insofar as the rotation operator is concerned, these momenta may be regarded as independent. In this representation we have

$$\vec{J}=\sum_{j=1}^{n-1}\vec{J}^j\ , \tag{49}$$

$$J_y^j=-i\left(P_z^j\frac{\partial}{\partial P_x^j}-P_x^j\frac{\partial}{\partial P_z^j}\right)\ . \tag{50}$$

This expression will be used in Sec. IV in an application to the MPM.

Formally, the representation in terms of $\{\vec{P}\}$ is unsatisfactory because the phase-space density $d\Phi_n$, when expressed in terms of $\prod_j d^3P_j$, involves some awkward kinematical factors. It is more convenient to replace the momenta magnitudes by s_j as independent variables, and to use the set

$$\hat{P}_1, s_2, \hat{P}_2\cdots s_{n-1}, \hat{P}_{n-1}\ , \tag{51}$$

where $\hat{P}_j$ is the unit vector, specified by two angles. This set $\{s_j,\hat{P}_j\}$ has $3n-4$ variables, the required independent number for an n-particle state. Let us consider one fixed Lorentz frame for all j; for definiteness, let it be the over-all s-channel c.m. system. Then we have for the invariant mass of the $(j+1)$th particle

$$\begin{aligned}{m_{j+1}}^2&=(P_{j+1}^\mu-P_j^\mu)^2\\&=s_{j+1}+s_j-2P_{j+1}^0P_j^0+2\vec{P}_{j+1}\cdot\vec{P}_j\ .\end{aligned} \tag{52}$$

From this we see that P_j (the magnitude of the 3-momentum), is determined by s_j, s_{j+1}, $\hat{P}_j\cdot\hat{P}_{j+1}$, and P_{j+1}, where by recursion P_{j+1} depends on all s_l and $\hat{P}_l$ for $l>j$. It is useful to introduce the variable s_j', defined by

$$\begin{aligned}s_j'&=(P_j'^\mu)^2\\&=\left(\sum_{l=j+1}^{n}p_l^\mu\right)^2\\&=(K^\mu-P_j^\mu)^2\ .\end{aligned} \tag{53}$$

Its value is determined by s, s_j, and P_j.

We now establish the expression of $d\Phi_n$ in this representation. Define

$$\begin{aligned}d\phi_n&=\left(\frac{1}{2\pi}\prod_t n_t!\right)d\Phi_n\\&=\left[\prod_{i=1}^{n}(dp_i)\right](2\pi)^3\delta^4\left(\sum_{i=1}^{n}p_i^\mu-K^\mu\right)\ .\end{aligned} \tag{54}$$

Multiplying $d\phi_n$ by

$$d^4P_{n-1}ds_{n-1}\delta^4\left(P_{n-1}^\mu-\sum_{i=1}^{n-1}p_i^\mu\right)\delta({P_{n-1}}^2-s_{n-1})\theta({P_{n-1}}^2)\ , \tag{55}$$

and rearranging, we obtain

$$d\phi_n=d\phi_{n-1}ds_{n-1}d\rho_{n-1}\ , \tag{56}$$

where

$$\begin{aligned}d\rho_{n-1}&=(dP_{n-1})(dp_n)(2\pi)^3\delta^4(P_{n-1}^\mu+p_n^\mu-K^\mu)\\&=\frac{P_{n-1}}{16\pi^2\sqrt{s}}d^2\hat{P}_{n-1}\ .\end{aligned} \tag{57}$$

Evidently, $d\rho_{n-1}$ is the phase-space density for a two-body state involving the nth particle and a "quasiparticle" of mass $(s_{n-1})^{1/2}$. More generally, we have

$$d\phi_{j+1}=\left[\prod_{i=1}^{j}(dp_i)\right](dp_{j+1})(2\pi)^3\delta^4\left(\sum_{i=1}^{j}p_i^\mu+p_{j+1}^\mu-P_{j+1}^\mu\right)d^4P_j\,ds_j\,\delta^4\left(P_j^\mu-\sum_{i=1}^{j}p_i^\mu\right)$$

$$\times\delta(P_j^2-s_j)\theta(P_j^2)d^4P_j'ds_j'\delta^4\left(P_j'^\mu-\sum_{l=j+1}^{n}p_l^\mu\right)\delta(P_j'^2-s_j')\theta(P_j'^2)$$

$$=d\phi_j\,d^4p_{j+1}ds_j(dP_j)(dP_j')(2\pi)^3\delta^4(P_j^\mu+p_{j+1}^\mu-P_{j+1}^\mu)\delta^4(P_j'^\mu-p_{j+1}^\mu-P_{j+1}'^\mu)$$

$$=d\phi_j\,ds_j\,d\rho_j\,, \tag{58}$$

where

$$d\rho_j=(dP_j)(dP_j')(2\pi)^3\delta^4(P_j^\mu+P_j'^\mu-K^\mu) \tag{59}$$

$$=\frac{P_j}{16\pi^2\sqrt{s}}\,d^2\hat{P}_j\,. \tag{59a}$$

Since by the same procedure one can establish that

$$d\phi_2=d\rho_1\,, \tag{60}$$

we have by recursive application of (58)

$$d\phi_n=d\rho_1ds_2d\rho_2ds_3\cdots d\rho_{n-2}ds_{n-1}d\rho_{n-1}\,. \tag{61}$$

This is just what we need for the sequential set (51) by virtue of (59a).

In this representation all the $\hat{P}_j$ are independent, so the rotation operator is particularly simple. If we choose the y axis to be the polar axis, with reference to which the polar and azimuthal angles of $\hat{P}_j$ are denoted by ξ_j and η_j, respectively, then (50) becomes

$$J_y^j=-i\frac{\partial}{\partial\eta_j}\,. \tag{62}$$

Thus, using this in (49) and (43) we have formally a very simple expression for the derivative optical theorem.

Because of the arrangement of p_i in a sequence, this representation obviously should have useful application in the MPM. It also can be applied to the single-diffraction-dissociation processes if p_n is identified with the undissociated particle. For double diffraction dissociation, $a+b\to A+B$, we should use two sequences, i.e.,

$$d\phi_n=ds_Ads_Bd\rho_{AB}d\phi_Ad\phi_B\,, \tag{63}$$

where

$$s_A=P_l^2=\left(\sum_{i=1}^{l}p_i^\mu\right)^2,$$

$$s_B=P_l'^2=\left(\sum_{i=l+1}^{n}p_i^\mu\right)^2, \tag{64}$$

$$d\rho_{AB}=d\rho_l\,.$$

In (63) $d\phi_A$ and $d\phi_B$ are the phase-space densities of the two clusters; they should separately be expressed in two sequential representations, one starting from $i=1$, the other starting from $i=n$, both working towards the middle.

IV. APPLICATIONS

The foregoing consideration has been rather formal; its implications can best be made clear by some examples. We give below two simple examples, which are chosen because they represent two extreme views about high-energy collisions. Clearly, in order to calculate the slope, what has to be supplied is the production amplitude T_n. Among high-energy models the MPM is most explicit in the specification of the amplitude; it will be considered first. The DEM, on the other hand, is not as specific, but the simplicity of the model permits an estimate of the result without detailed information.

A. The Multiperipheral Model

We consider an extremely simple version of this model. We assume

$$T_n=h(s,s_{i,i+1})\prod_{j=1}^{n-1}\exp(\lambda t_j)\,, \tag{65}$$

where $s_{i,i+1}$ are the subenergies $(p_i+p_{i+1})^2$; t_j as defined in (47) are the momentum-transfer variables, j running over all the links of the multiperipheral chain; and h is some unspecified function of the energy variables. The parameter λ is chosen such that the average transverse momentum of the produced particles agrees with the observed value of about 350 MeV/c. Thus, roughly it is given by

$$2\lambda\equiv t_0^{-1}\sim\langle p_\perp^2\rangle^{-1}\sim8\ (\mathrm{GeV}/c)^{-2}\,. \tag{66}$$

The simplicity of this model lies in the factorizability of the energy and momentum-transfer dependences, and in the fixed exponential form for the latter. A more elaborate MPM might deviate quantitatively from (65), but not qualitatively, so the formula should represent the spirit of all versions of the MPM, viz., short-range correlation and rapid $p_\perp$ damping.

It is straightforward to apply the derivative optical theorem to this problem. Since the function h is invariant under rotation, the operator J_y acts only on $\prod_j \exp(\lambda t_j)$. Identifying k_a^μ in (47) with the incident particle that is connected to the multiperipheral chain at the terminal labeled $i=1$, and letting its 3-vector point in the z direction, we have

$$t_j = m_a{}^2 + s_j - 2k_a^0 P_j^0 + 2k_a P_j^z \;. \tag{67}$$

From (50) we then obtain a form of J_y^j appropriate for operating on a function of t_j only:

$$J_y^j = i2kP_x^j \frac{\partial}{\partial t_j} \;, \tag{68}$$

where $k=k_a$. Using (50) to act from the left again yields

$$J_y^j J_y^k = 2k\left[\delta_{jk} P_z^j \frac{\partial}{\partial t_j} - 2kP_x^j P_x^k \frac{\partial^2}{\partial t_j \partial t_k}\right] . \tag{69}$$

Since T_n has no dependence on any azimuthal angles, the off-diagonal terms in (32) vanish upon integration because of (69). Thus, we need only consider

$$(J_y^j)^2 T_n = 2k\left[P_z^j \frac{\partial}{\partial t_j} - k(P_\perp^j)^2 \frac{\partial^2}{\partial^2 t_j}\right] T_n \;, \tag{70}$$

where $(P_x^j)^2$ has been replaced by $(P_\perp^j)^2/2$ in anticipation of the ϕ integration to come. The suffix j is supposed to run through all the links of the multiperipheral chain. In the two-body case there is only one link, and it is easy to show that (70) leads to the result in (12), so that this serves as a consistency check.

Substituting (65) in (70), and then in (32), we have

$$A'(s,0) = \tfrac{1}{2}\sum_n \sum_j \int d\Phi_n \, T_n^* 2\lambda k[P_z^j - \lambda k(P_\perp^j)^2] T_n \;. \tag{71}$$

On the other hand, by the Hermiticity of J_y we can write (42) in the form

$$A'(s,0) = \langle J_y T_0 | J_y T_0 \rangle \;, \tag{72}$$

so that upon the application of (65) and (68) we get

$$A'(s,0) = \tfrac{1}{2}\sum_n \sum_j \int d\Phi_n \, 2\lambda^2 k^2 (P_\perp^j)^2 T_n^* T_n \;. \tag{73}$$

Comparing (71) and (73) yields

$$A'(s,0) = \tfrac{1}{2}\lambda k \sum_n \sum_j \int d\Phi_n \, P_z^j T_n^* T_n \;. \tag{74}$$

The sum in j extends over all links of the multiperipheral chain from $j=1$ to $n-1$. Assuming symmetry of the chain, the sum is equivalent to twice the sum over half the chain. At the jth link the final state can be partitioned into two clusters of mass-squared s_j and s_j', which satisfy the inequality

$$s_j s_j' < t_0 s \;, \tag{75}$$

where t_0 is approximately given in (66). Thus we have

$$s_j < (t_0 s)^{1/2}, \quad \text{for } j < n/2 \;. \tag{76}$$

Now to determine P_z^j we first note that (67) may be rewritten as

$$t_j = \tau_j + s_j - 2k(P_0^j - P_z^j) \;, \tag{77}$$

where

$$\begin{aligned} \tau_j &= m_a{}^2 - 2(k^0 - k)P_0^j \\ &\simeq m_a{}^2\left(1 - \frac{P_0^j}{k}\right) \\ &\simeq \frac{m_a{}^2(s_j' - s_j)}{4k^2} \;. \end{aligned} \tag{78}$$

From (77) we then obtain

$$\begin{aligned} P_z^j &= k - \frac{1}{4k}(s_j + s_j' + 2\tau_j - 2t_j) \\ &> k\left[1 - \frac{s_j}{s} - \frac{t_0}{s_j} - \frac{2}{s}(\tau_j - t_j)\right] , \end{aligned} \tag{79}$$

where (75) has been used in arriving at the lower bound. Since $t_0 \ll s_j$ except possibly at $j=1$ (which can be ignored in the sum over j when n is large), we therefore may approximate $P_z^j \approx k$ for all $j<n/2$ because of (76). By symmetry of the problem, this approximation must be valid for all j. It then follows from (74) that

$$\begin{aligned} A'(s,0) &= \tfrac{1}{2}\lambda k^2 \sum_n (n-1) \int d\Phi_n \, T_n^* T_n \\ &= \lambda k^2 \sum_n (n-1) 2k\sqrt{s}\, \sigma_n \;. \end{aligned} \tag{80}$$

As usual we define the average multiplicity of the pions (assuming only pions are produced in this model) as follows:

$$\langle n \rangle = \sum_n n\sigma_n \Big/ \sum_n \sigma_n \;; \tag{81}$$

then we obtain

$$A'(s,0) = 2\lambda k^3 \sqrt{s}\, \sigma_T(\langle n \rangle - 1) \;. \tag{82}$$

To relate this result to the slope of the forward peak, we parametrize the t dependence of the differential cross section by

$$\frac{d\sigma}{dt}=\exp[a(s)+b(s)t+c(s)t^2+\cdots]\ . \quad (83)$$

Assuming that the elastic amplitude in the forward direction becomes purely imaginary, we have

$$b(s)=[k^2A(s,0)]^{-1}A'(s,0)$$
$$=(2k^3\sqrt{s}\ \sigma_T)^{-1}A'(s,0)\ . \quad (84)$$

Hence, we conclude from (82) that in the MPM

$$b(s)=\lambda(\langle n\rangle-1)\ . \quad (85)$$

This result is based on the assumption that there is only one particle in each rung of the MP ladder. If a model specifies that there are on the average n_0 particles in each rung, then (85) should be modified to read

$$b(s)=\lambda[\langle n\rangle/n_0-1]\ . \quad (86)$$

Parametrizing $\langle n\rangle$ by

$$\langle n\rangle=c_0+c_1\ln s\ , \quad (87)$$

we obtain for the slope of the Pomeranchuk trajectory at $t=0$

$$\alpha'=\lambda c_1/2n_0\ . \quad (88)$$

This result agrees in spirit with that of Goldberger,[4] who obtained a similar relation between α' and c_1 in a t-channel consideration. Instead of λ he has a mean inverse (mass)^2 that depends on the eigenfunction solution of the multiperipheral integral equation.

Numerically, the average multiplicity of charged particles is given by[8]

$$\langle n_{ch}\rangle\approx c_2+c_3\ln s\ ,$$
$$c_2=-4.02\pm0.22\ ,\quad c_3=1.99\pm0.03\ , \quad (89)$$

in a fit for momentum from 69 to 10^4 GeV/c. We therefore take

$$c_1\approx3\ . \quad (90)$$

Using $\lambda\approx4$ from (66), and setting $n_0=2$ as is customary, we obtain

$$\alpha'\approx3\ (\text{GeV}/c)^{-2}\ . \quad (91)$$

We note that instead of estimating λ as in (66), one could use (65) with λ as a free parameter to fit the transverse-momentum distribution. Amann[9] has done this using a model with single emission at each vertex, i.e., $n_0=1$, and he obtained a good fit with $\lambda=1$. That leads to the result $\alpha'\approx1.5$ (GeV/c)$^{-2}$. The experimental value of α' as determined by the CERN Intersecting Storage Rings (ISR) data is <0.3 (GeV/c)$^{-2}$. Evidently, the simple MPM considered here predicts a value of α' which is an order of magnitude too large.

B. The Diffractive Excitation Model

In the DEM the production process is viewed as one in which the two incident particles are raised diffractively to two excited states which subsequently decay into two clusters of particles. Thus, in the unitarity relation if we integrate out the internal variables of the two clusters, we have essentially a quasi-two-body problem, the kinematics of which is similar to the elastic case. Let us therefore consider the elastic problem first.

Assuming the elastic amplitude T to be purely imaginary near the forward direction, we write

$$T=iA_0\exp(\tfrac{1}{2}bt)$$
$$=iA_0\exp[-bk^2(1-\zeta)]\ , \quad (92)$$

where $A_0=A(s,0)$ and ζ is as used in (12). Approximating $1-\zeta^2$ in (12) by $-t/k^2$ for ζ near $+1$, we obtain

$$A_{el}'(s,0)=\frac{k}{16\pi\sqrt{s}}\int_{-\infty}^{0}dt\,\tfrac{1}{2}b(1+\tfrac{1}{2}bt)T^*T$$
$$=\frac{kA_0^2}{64\pi\sqrt{s}}$$
$$=(16\pi)^{-1}k^3\sqrt{s}\ \sigma_T^2\ . \quad (93)$$

It then follows from (84) that the contribution to the slope b from the elastic unitarity is

$$b_{el}=\sigma_T/32\pi\ . \quad (94)$$

It is interesting to note that this result is independent of the input (true) value of b. If the parametrization of T is generalized to include a $ct^2/2$ term in the exponent in (92), the result would only differ by a term of order c/b^2, which is empirically very small. We also note that by relating A_0^2 to $b\sigma_{el}$ we obtain another expression,

$$b_{el}=b\sigma_{el}/2\sigma_T\ . \quad (95)$$

If one ignores the inelastic cross section, (95) leads to the familiar result that the elastic unitarity reduces the slope of the forward peak by half.

Numerically, if we take σ_T to be 38 mb for pp scattering, then (94) yields $b_{el}=0.97$ (GeV/c)$^{-2}$. This is only 7.5% of the observed value of b at the ISR energies. If the DEM is meaningful, the observed value of b sould be limiting, and the balance must be made up by two-cluster contributions.

In the multiparticle production case let us for simplicity consider an extremely naive version of the DEM. Our objective here is not on constructing a realistic model, but on seeing what builds up the slope. Suppose that the diffractive excitation processes raise only the masses of the particles, but not the spins, so that all the excited states are spinless. This is a tremendous simplification since the decay distributions of the clusters are then independent of the initial momenta. Thus, the only relevant momentum in the final state of T_n that is sensitive to the rotation operator is $\vec{P}_l$ as defined in (64). It is just the c.m. momentum of the two clusters. For every pair of fixed values of the cluster mass-squares, s_A and s_B, we have a quasi-two-body reaction which we label by r. Intergration over all s_A and s_B implies a summation over all reaction channels r. Thus, for a given r the calculation is just as in the elastic case; we have

$$A_r{}'(s,0)=p_r k^2\sqrt{s}\ \frac{d\sigma_r}{dt}\ \text{(forward)}\,, \qquad (96)$$

where $p_r \equiv |\vec{P}_l|$, the c.m. momentum of the clusters. For ease of comprehending the relative importance of the contributions from different channels, let us assume that $d\sigma_r/dt$ can be approximated by an exponential form for each r

$$\frac{d\sigma_r}{dt}=\left[\frac{d\sigma_r}{dt}\right]_{t=t_r}\exp[b_r(t-t_r)]\,, \qquad (97)$$

where t_r is the value of t in the forward direction of channel r. Denoting the integral of (97) over all physical values of t by σ_r, we get

$$A_r{}'(s,0)=p_r k^2\sqrt{s}\ b_r\sigma_r\ . \qquad (98)$$

Using (84) again, and including the elastic channel in the sum over r, we obtain

$$b=\frac{\sum_r(p_r/k)b_r\sigma_r}{2\sum_r\sigma_r}\ . \qquad (99)$$

Since the diffractive process is significant only if $s_As_B < t_0 s$, just as in (75), we can use the same argument as the one following (79) to obtain $p_r \approx k$ for all r. We therefore have

$$b=\frac{\sum_r b_r\sigma_r}{2\sum_r\sigma_r}\equiv\frac{\langle b\rangle}{2}\ , \qquad (100)$$

where $\langle b\rangle$ is the mean slope averaged over all two-cluster diffractive processes. A more elaborate derivation of this result with explicit integration over $d\Phi_n$ can be given following the formalism of Ref. 10, but it will be omitted here since it sheds no further light on the issue at hand. Because each channel has only one momentum-transfer link, the sum over r does not increase the value of b, but effects only an averaging.

The slopes of the forward peaks of diffractive excitation processes are known experimentally to be smaller than that of the elastic peak.[11] They generally decrease with increasing cluster mass. This is physically reasonable because for a given amount of momentum transfer it should be more likely to find a struck particle excited than being left in an unperturbed state. Of course, an excitation in mass is usually accompanied by an excitation in spin, so there is no experimental information on the slope parameters of our simplified model. It is, however, reasonable to take $\langle b\rangle$ to be no greater than the value of b for elastic scattering. Then (100) is a contradiction. The implication is, of course, that the naive version of the DEM is too unrealistic. The spin complication of the excited states must be taken into account, and by virtue of (43) it necessarily gives rise to an additional positive contribution.

V. CONCLUSION

We have developed a model-independent formalism for analyzing the slope of the forward peak. It takes into account the multiparticle states exactly. When applied to specific models of high-energy collisions, it enables one to see how the slope is built up in an s-channel view. It therefore can serve as a theoretical detector, useful in diagnosing the unrealistic features of high-energy models.

We have seen that in a simple version of the MPM considered, the slope of the Pomeranchuk trajectory at $t=0$ is too large – by almost an order of magnitude. This discrepancy could possibly be reduced to a certain extent by an appropriate modification of the details of the model. However, the qualitative feature of the model brought out into the open by our calculation is that the slope b of the forward peak is proportional to the number of links in the multiperipheral chain [cf. (86)]. That is why in the MPM the slope increases with $\ln s$, assuming that the basic hypothesis common to all MPM's is that the number n_0 of particles emitted at each vertex is fixed. Evidently, by increasing n_0 the value of α' can be reduced. Now, we note that whereas the average multiplicity $\langle n\rangle$ increases somewhat faster than $\ln s$ as the beam energy is increased from 20 to 1500 GeV, i.e., it is concave upward in the semilog plot, the slope parameter b decreases with $\ln s$ dramatically over the same energy range,[3] i.e., it is concave downward. Thus n_0 must increase continuously with s. That is a rather serious modification of the spirit of the MPM. In any case the slope consideration suggests the direction in which the model can be improved.

In the DEM our example illustrates the possibility in the other extreme. By ignoring the spins of the excited states, we have imposed essentially only one momentum-transfer link per channel. The predicted value of b naturally turns out to be too small. Obviously, to be realistic the angular momentum states of the clusters must be taken into account. It seems that the observed value of $b \approx 12$ (GeV/c)^2 is quite within reach of the DEM.

Shortcomings of the models aside, it is clear that the slope analyses by the method suggested here are very useful not only in testing high-energy models, but also as sum rules for constraining parameters in a variety of problems at any energy.

Note added in proof. After this paper was submitted, it was pointed out to me that parts of this work have already been considered by others. See, in particular, L. Van Hove, Nuovo Cimento 28, 798 (1963) and Z. Koba and M. Namiki, Nucl. Phys. B8, 413 (1968).

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