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A Theory of Spontaneous T Violation*

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A theory of spontaneous T violation is presented. The total Lagrangian is assumed to be invariant under the time reversal T and a gauge transformation (e.g., the hypercharge gauge), but the physical solutions are not. In addition to the spin-1 gauge field and the known matter fields, in its simplest form the theory consists of two complex spin-0 fields. Through the spontaneous symmetry-breaking mechanism of Goldstone and Higgs, the vacuum expectation values of these two spin-0 fields can be characterized by the shape of a triangle and their quantum fluctuations by its vibrational modes, just like a triangular molecule. T violations can be produced among the known particles through virtual excitations of the vibrational modes of the triangle which has a built-in T-violating phase angle. Examples of both Abelian and non-Abelian gauge groups are discussed. For renormalizable theories, all spontaneously T-violating effects are finite. It is found that at low energy, below the threshold of producing these vibrational quanta, T violation is always quite small.

I. INTRODUCTION

In this paper we discuss a theory of spontaneous T violation. To illustrate the theory, we shall first discuss a simple model in which the weak-interaction Lagrangian, as well as the strong- and electromagnetic-interaction Lagrangians, is assumed to be invariant under (1) the time reversal T and (2) a gauge transformation, e.g., that of the hypercharge Y. Yet the physical solutions are required to exhibit both T violation and Y nonconservation. In its construction, the model is similar to those gauge-group spontaneous symmetry-violating theories[1-4] that have been extensively discussed in the literature. The only difference is that one now has, in addition, the spontaneous violation of a discrete symmetry.[5] As we shall see, there exists a general class of such spontaneously T-violating theories. (*Note added in proof.* There actually exist two general classes of such spontaneously T-violating theories, depending on whether the spin-0 fields belong to a complex representation or a real representation of the gauge group. In this paper, except in Appendix D, only the class of complex representation is discussed. For real representation, such as that in the Georgi-Glashow model, one may follow the method developed in Appendix D. The details will be given in a separate paper.) The simple model serves only as a prototype which nevertheless embodies most of the essential features.

In addition to the known matter fields, the model consists of two independent spin-0 $Y=1$ complex fields ϕ_1, ϕ_2 and a neutral spin-1 gauge field B_μ. Under the hypercharge gauge transformation $e^{iY\Lambda}$, we have

$$\phi_k \to e^{i\Lambda}\phi_k$$

and (1)

$$B_\mu \to B_\mu + f^{-1}\frac{\partial\Lambda}{\partial x_\mu},$$

where f is the hypercharge coupling constant and the subscript $k=1$ and 2. As usual, T is assumed to commute with Y,

$$TYT^{-1}=Y. \tag{2}$$

This gives then a well-defined difference between T and either CT or CPT. Since T is an antiunitary operator, we can always choose the phase of ϕ_k such that

$$T\phi_k T^{-1}=\phi_k. \tag{3}$$

To avoid irrelevant complications, we assume the theory *not* to be symmetric under any linear transformation which mixes ϕ_1 and ϕ_2, so that the right-hand side of (3) must remain ϕ_k.

As will be discussed in the next section, the spontaneous T-violation mechanism can be introduced by assuming a T-invariant potential energy $V(\phi)$ between ϕ_1 and ϕ_2 which has a minimum at the c-number point

$$(\phi_1, \phi_2)=2^{-1/2}(\rho_1 e^{i\theta}, \rho_2), \tag{4}$$

where $\rho_1>0$, $\rho_2>0$, and $\theta\neq 0$ or π. This minimum point therefore defines a triangle where ρ_1 and ρ_2 form two sides and θ the angle in between. Because of quantum effects there must be fluctuations of ϕ_1 and ϕ_2 around their average values. These fluctuations can be shown to correspond to the vibrations of the triangle. The entire ϕ_1, ϕ_2 system can then be visualized as a triangular molecule which is defined by both its shape and its three vibrational modes of oscillation in the plane of the triangle. For convenience of nomenclature, we shall refer to this ϕ_1, ϕ_2 complex simply as the *triangle*.

In the absence of the gauge field B_μ, there would be a zero-mass boson, in accordance with the Goldstone theorem.[1] In the present case, this Goldstone boson corresponds simply to the rotational degree of freedom of the triangle. Because of the Higgs mechanism,[2] the presence of the gauge field B_μ eliminates the zero-mass boson. As a result, B_μ acquires a mass, and the would-be Goldstone boson becomes, as usual, the longitudinal mode of B_μ. The detailed description of the triangle and its interaction with the gauge field is given in Sec. II.

While the Lagrangian is assumed to be T-invariant, its solution, as characterized by the triangle, carries a phase angle $\theta \neq 0$ or π. Therefore, it has a built-in T violation, somewhat analogous to the two-component neutrino theory which carries a built-in screw direction. We recall that just on the basis of the two-component neutrino theory alone, but without any appropriate interaction, one cannot distinguish[6] between P and CP, and consequently there is no observable parity-violation effect. Here, one has a similar situation. Both the gauge field and the vibrational levels of this triangular molecule are of zero average hypercharge, $\langle Y \rangle = 0$. Thus, although these vibrational levels are not invariant under T, there is *no* violation of the reciprocity relations, since for states with $\langle Y \rangle = 0$ reciprocity relations can be derived by using CT invariance alone. In order that violations of the reciprocity relations be observed, there must be states with $\langle Y \rangle \neq 0$, such as K^0, $\overline{K}^0$, etc.

Once this triangle is allowed to interact with known particles with $\langle Y \rangle \neq 0$, T violation becomes a natural consequence. However, the existence of the triangle does not determine the exact form of its interactions (just as the interaction of a neutrino is not specified by the two-component theory). As a pure illustration, we consider in Sec. III a particularly simple form in which the usual T-invariant $\Delta Y = \pm 1$ weak-interaction Lagrangian $L_\pm$ is replaced by

$$\mathcal{L}_{\text{int}} = (g_1 \phi_1 + g_2 \phi_2) L_- + \text{H.c.} \tag{5}$$

Because of the transformation property (1), this new Lagrangian clearly conserves Y. It is also T-invariant, provided g_1 and g_2 are relatively real. Through the virtual emission and absorption of the triangle, violations of reciprocity relations can occur among the known particles. As we shall see, this can give rise to $K_L^0 \to 2\pi$, and (if we assume the threshold energy for producing these triangles is $\gtrsim$ a few GeV) the resulting CP violation in K_L^0-K_S^0 decays is of the superweak form.

As will also be discussed in the subsequent sections, in addition to the direct exchange of the triangle between the matter fields there is still another important mechanism which can violate the reciprocity relations via the coupling between the matter field and the gauge field. In this mechanism, the triangle propagates only in a loop diagram, and as a result one may have violations of the Furry theorem, i.e., the loop diagram connecting an odd number of the gauge-field quanta may now be nonzero. Such a loop diagram can in turn produce T violations among the matter fields.

In Sec. IV, we examine some generalizations of the model to other gauge groups, either Abelian or non-Abelian, but we restrict our discussion only to renormalizable theories. In all these cases, the general mechanism of T violation remains the same, and the basic structure of the triangle remains intact, though its interaction with the known matter fields can be quite different. Because in these cases the spontaneous T violation is tied to the spontaneous gauge-symmetry violations[7] of the weak and electromagnetic interactions, at low energy the magnitude of T violation among known particles always turns out to be very small, either milliweak or superweak. Furthermore, since such theories are renormalizable, all spontaneously T-violating effects are finite and computable, at least in principle. In any case, one feels that whatever the eventual gauge theory may be for the weak and electromagnetic interactions, it should contain T violation as an integral part. The mechanism of spontaneous T violation discussed in this paper may provide just such a needed possibility.

II. THE TRIANGLE AND THE GAUGE FIELD

In this section we consider the simple system of spontaneous T violation mentioned in the Introduction. The system consists of two complex spin-0 fields ϕ_1, ϕ_2 and a gauge field B_μ. The most general form of a gauge-invariant, T-invariant, and renormalizable Lagrangian density is

$$\mathcal{L}(B,\phi) = -\sum_{k=1,2}\left[\left(\frac{\partial}{\partial x_\mu} + i f B_\mu\right)\phi_k^\dagger\right] \times \left[\left(\frac{\partial}{\partial x_\mu} - i f B_\mu\right)\phi_k\right] - \frac{1}{4}\left(\frac{\partial}{\partial x_\mu}B_\nu - \frac{\partial}{\partial x_\nu}B_\mu\right)^2 - V(\phi), \tag{6}$$

where the dagger denotes the Hermitian conjugate, and the potential energy $V(\phi)$ is given by

$$V(\phi) = -\lambda_1 \phi_1^\dagger \phi_1 - \lambda_2 \phi_2^\dagger \phi_2 + A(\phi_1^\dagger \phi_1)^2 + B(\phi_2^\dagger \phi_2)^2 + C(\phi_1^\dagger \phi_1)(\phi_2^\dagger \phi_2) + \tfrac{1}{2}[(\phi_1^\dagger \phi_2)(D\phi_1^\dagger \phi_2 + E\phi_1^\dagger \phi_1 + F\phi_2^\dagger \phi_2) + \text{H.c.}], \quad (7)$$

where the eight constants $\lambda_1, \lambda_2, A, \ldots, F$ are all real so that T invariance holds.

In the spirit of renormalization, the renormalized values of these constants can be arbitrarily assigned. Following the standard treatment of spontaneous symmetry-breaking mechanism for the gauge group,[1] we assume

$$\lambda_1 \text{ and/or } \lambda_2 > 0 . \quad (8)$$

As we shall see, the spontaneous T violation can be induced by imposing

$$D > 0 . \quad (9)$$

In addition, in order for $V(\phi)$ to have a lower bound, we require

$$A - \frac{E^2}{8D} > 0 ,$$

$$B - \frac{F^2}{8D} > 0 ,$$

and

$$\left(A - \frac{E^2}{8D}\right)\left(B - \frac{F^2}{8D}\right) > \tfrac{1}{4}\left(C - D - \frac{EF}{4D}\right)^2 . \quad (10)$$

As usual, all the above conditions refer to the renormalized constants.

Let us first locate the minimum of the function $V(\phi)$ in its c-number form. In the tree approximation, this minimum determines the vacuum expectation values of ϕ_1 and ϕ_2:

$$\langle \phi_1 \rangle_{\text{vac}} = 2^{-1/2} \rho_1 e^{i\theta}$$

and

$$\langle \phi_2 \rangle_{\text{vac}} = 2^{-1/2} \rho_2 .$$

Because of (8) the minimum is not at the origin, and because of the gauge invariance of the Lagrangian we can always transform one of the vacuum expectation values, say $\langle \phi_2 \rangle_{\text{vac}}$, to be real and not negative. It is straightforward to obtain the necessary and sufficient condition for both $\rho_1 > 0$ and $\rho_2 > 0$. [See Appendix A for further details.] Similarly, one can readily verify that because of (9)

$$\cos\theta = -(4D\rho_1\rho_2)^{-1}(E\rho_1^2 + F\rho_2^2) , \quad (11)$$

in which the constants are chosen to satisfy $-1 < \cos\theta < 1$. Equation (11) has two solutions: θ and $-\theta$. By using (3), one sees that either solution is not invariant under T, and therefore one has a spontaneous T violation. The T invariance of the Lagrangian ensures that both solutions exist, and that they transform into each other under T.

The normal modes of this system can be derived by expanding the operators ϕ_1 and ϕ_2 around their vacuum expectation values. We write

$$\phi_1 = 2^{-1/2}(\rho_1 + R_1 + iI_1)e^{i\theta}$$

and (12)

$$\phi_2 = 2^{-1/2}(\rho_2 + R_2 + iI_2) ,$$

where ρ_1, ρ_2, and θ are, as before, c-numbers, but R_1, R_2, I_1, and I_2 are Hermitian fields. If the coupling constant f between the gauge field B_μ and ϕ_1, ϕ_2 were zero, then the Goldstone theorem would apply and there should be one normal mode, called the Goldstone boson G, that has a zero mass. It can be easily verified that in the tree approximation G is given by

$$G = (\rho_1^2 + \rho_2^2)^{-1/2}(\rho_1 I_1 + \rho_2 I_2) . \quad (13)$$

(This can also be established by using the geometrical considerations given below.) The remaining three normal modes, which will be referred to as t_1, t_2, and t_3, are linear combinations of the fields R_1, R_2 and

$$I \equiv (\rho_1^2 + \rho_2^2)^{-1/2}(\rho_2 I_1 - \rho_1 I_2) . \quad (14)$$

This linear relation may be written as

$$t = \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} = U \begin{pmatrix} R_1 \\ R_2 \\ I \end{pmatrix} , \quad (15)$$

where U is a (3×3) real orthogonal matrix.

As already mentioned in the Introduction, the description of the system can be characterized by a triangular molecule. For example, Fig. 1 gives a schematic picture of such a triangular molecule where the two sides are ρ_1 and ρ_2, respectively, and the angle in between is θ. In the plane of the triangle, a triangular molecule also has three normal modes of vibration, each of which is a linear combination of the displacements $\delta\rho_1 = R_1$, $\delta\rho_2 = R_2$, and $\delta\theta = \rho_1^{-1} I_1 - \rho_2^{-1} I_2$, i.e.,

$$\delta\theta = \frac{(\rho_1^2 + \rho_2^2)^{1/2}}{\rho_1\rho_2} I$$

(as illustrated in Fig. 1). Under the gauge transformation $e^{iY\alpha}$, the entire triangle rotates an angle α. Thus, the Goldstone boson G corresponds simply to the rotational degree of freedom of the triangle; this then leads to Eq. (13).

The configuration and vibration of a triangular molecule depend on nine real parameters: three for the shape of the triangle, three for the Eulerian angles that specify the real orthogonal matrix U, and three for the frequencies (or masses) of the normal modes. In the present case, the function $V(\phi)$ depends only on eight constants $\lambda_1, \lambda_2, A, \ldots, F$. This imposes a constraint

$$\sum_{a=1}^{3} (\rho_1 U_{a_1} + \rho_2 U_{a_2}) U_{a_3} m_a{}^2 = 0 , \tag{16}$$

where m_a is the mass of the normal mode t_a. Since the coupling constant $f \neq 0$, the zero-mass Goldstone boson is removed through the Higgs mechanism.[2] G now joins the two transverse components of B_μ to form a single massive neutral spin-1 boson B. In the tree approximation, the mass of B is

$$m_B{}^2 = f^2(\rho_1{}^2 + \rho_2{}^2) . \tag{17}$$

The Lagrangian (6) is constructed to be invariant under the gauge transformation (1). Therefore one has the current conservation

$$\frac{\partial j_\mu}{\partial x_\mu} = 0 ,$$

where

$$j_\mu = i \sum_{k=1,2} \left(\frac{\partial \phi_k^\dagger}{\partial x_\mu} \phi_k - \phi_k^\dagger \frac{\partial \phi_k}{\partial x_\mu} \right),$$

and the spatial integral of its time component is Y. The Lagrangian (6) is T-invariant; in addition, it is symmetric under the particle-antiparticle conjugation C and the space inversion P. The parity of B_μ is -1; the parity of ϕ_1 must be the same as that of ϕ_2, but it can be either $+1$ or -1, since the Lagrangian is an even function of ϕ_k. Under C, one has

$$C \phi_k C^\dagger = \phi_k^\dagger ,$$

$$C B_\mu C^\dagger = -B_\mu ,$$

and consequently

$$C Y C^\dagger = -Y .$$

[If one wishes, one may also set $C\phi_k C^{-1} = -\phi^\dagger$ for both $k=1$ and 2.]

The normal modes t_1, t_2, and t_3 are not eigenstates of C nor of T. As an example of C violation or T violation, we may consider diagrams for

$$n B_\mu \to m B_\mu , \tag{18}$$

where $n + m$ is an *odd* number. Because of loop diagrams in which the propagators are those of the triangles, the amplitudes for these C-violating processes can be nonzero. (However, see Appendix A for a list of special circumstances under which some of these amplitudes may happen to be zero.)

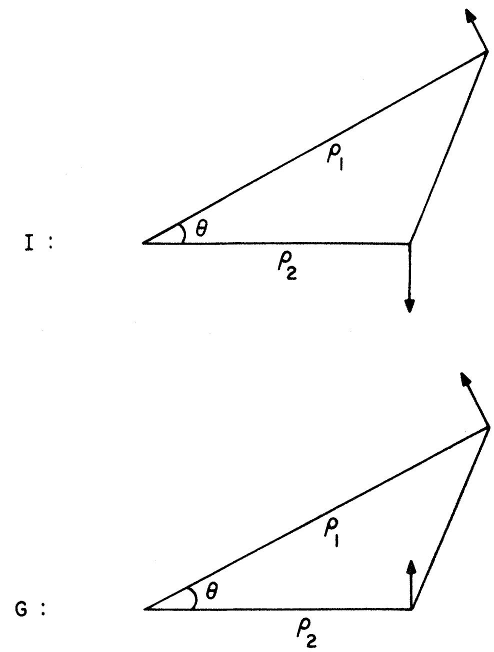

FIG. 1. A schematical drawing of the triangle; I represents one of its vibrational degrees of freedom defined by (14), and G represents its rotational degree of freedom defined by (13).

By using (1) and (12)–(15), one can readily verify that both the gauge field and the normal modes of the triangle are of zero average hypercharge; i.e., $\langle Y \rangle = 0$. Thus, for the system of the triangle and the gauge field alone, one can always introduce a new "time-reversal" operator T_n and a new "particle-antiparticle conjugation" operator C_n such that

$$T_n t_a T_n{}^{-1} = t_a ,$$

$$C_n t_a C_n{}^{-1} = t_a ,$$

$$T_n G T_n{}^{-1} = G ,$$

$$C_n G C_n{}^{-1} = G ,$$

$$T_n B_\mu T_n{}^{-1} = B_\mu ,$$

and

$$C_n B_\mu C_n{}^{-1} = B_\mu .$$

Since the Lagrangian $\mathcal{L}(B, \phi)$ can be written as a real function of these Hermitian fields, it must be invariant under the new C_n and T_n. Reaction (18) does not violate either T_n invariance or C_n invariance. The reciprocity relations are then maintained. However, under T_n one now has $\phi_1 \to \phi_1^\dagger$ and $\phi_2 \to \phi_2^\dagger$, and under C_n $\phi_1 \to \phi_1$ and $\phi_2 \to \phi_2$. Thus, the hypercharge Y neither commutes with T_n nor anticommutes with C_n. Nevertheless, this is totally acceptable, since in this simple system all the eigenstates are of $\langle Y \rangle = 0$. In order to observe violation of reciprocity relations, one should enlarge the system to include some known particles with $\langle Y \rangle$ nonzero.

III. VIOLATIONS OF RECIPROCITY RELATIONS

To illustrate how violations of reciprocity relations may occur, we discuss the example of a particular weak-interaction Lagrangian given by (5). [Other forms will be discussed in Sec. IV.] For clarity, let us consider first only the $P=-1$ part of the usual $\Delta Y = \pm 1$ nonleptonic weak interaction Lagrangian $L_\pm$. The operator $L_\pm$ is in general rather complicated, *not* a single canonical field; but so far as its transformation properties are concerned, $L_\pm$ is the same as the appropriate K^0 or $\bar{K}^0$ meson field. Thus we may write

$$L_- \sim K^0$$

and

$$L_+ \sim K^{0\dagger},$$

where ~ indicates that both sides have the same transformation properties. According to (5), with the inclusion of the triangle the corresponding weak interaction becomes

$$\mathcal{L}_{\text{int}} \sim (g_1\phi_1 + g_2\phi_2)K^0 + \text{H.c.}, \tag{19}$$

where g_1 and g_2 are both real so that T invariance holds. It is clear that (19) is also invariant under the hypercharge gauge transformation. By using (12), one may rewrite (19) in the form

$$\mathcal{L}_{\text{int}} \sim \Gamma K_1^0 + g(K_1^0\chi_1 + K_2^0\chi_2), \tag{20}$$

where Γ and g are both real and positive;

$$\Gamma^2 = g_1{}^2\rho_1{}^2 + g_2{}^2\rho_2{}^2 + 2g_1g_2\rho_1\rho_2\cos\theta \tag{21}$$

and $g=(g_1{}^2+g_2{}^2)^{1/2}$. The K_1^0 and K_2^0 meson fields are defined by

$$K_1^0 = 2^{-1/2}(K^0e^{i\alpha} + K^{0\dagger}e^{-i\alpha}),$$

$$K_2^0 = i2^{-1/2}(K^0e^{i\alpha} - K^{0\dagger}e^{-i\alpha})$$

and α is given by

$$\Gamma e^{i\alpha} = g_1\rho_1e^{i\theta} + g_2\rho_2. \tag{22}$$

The χ_1 and χ_2 fields are related to R_1, R_2 and I_1, I_2 by

$$\chi_1 = g^{-1}[g_1\cos(\theta-\alpha)R_1 - g_1\sin(\theta-\alpha)I_1 + g_2\cos\alpha R_2 + g_2\sin\alpha I_2]$$

and (23)

$$\chi_2 = g^{-1}[g_1\sin(\theta-\alpha)R_1 + g_1\cos(\theta-\alpha)I_1 - g_2\sin\alpha R_2 + g_2\cos\alpha I_2].$$

Under a hypercharge gauge transformation, the relative phase between K^0 and $\bar{K}^0$ meson states changes; therefore, we can always choose their relative phase so that K_1^0 represents the usual $CP=+1$ component, and K_2^0 the usual $CP=-1$ component. The first term ΓK_1^0 in (20) gives rise to the CP-conserving transition $K_1^0 \to 2\pi$. The second term in (20) leads to the CP-violating transition

$$K_1^0 \rightleftharpoons K_2^0;$$

its amplitude is determined by the Fourier transform of the contraction[8]

$$g^2\chi_1^\cdot(x)\chi_2^\cdot(0). \tag{24}$$

With this CP violation, there is automatically also a violation of the reciprocity relation. Similarly, one may phenomenologically include the $P=+1$ part of the usual $\Delta Y = \pm 1$ nonleptonic weak interaction and, if one wishes, also the usual $\Delta Y = \pm 1$ semileptonic weak interaction in the Lagrangian (5). The former gives, among other transitions, the CP-conserving reaction $K_2^0 \to 3\pi$, and the latter gives all the usual CP-conserving semileptonic $\Delta Y \neq 0$ transitions. It is easy to show that in both cases there is, in addition, a CP-violating transition amplitude which also depends linearly on (24).

The magnitude of the T-violating amplitude (24) depends on the detailed characteristics of the triangle: both its shape and its vibrational modes t_a. It is of interest to search for the maximum of (24). As will be shown in Appendix B, if the coupling constants f, Γ, g_1, g_2 and the masses m_B, m_1, m_2, m_3 are fixed, then by varying the shape and the vibrational modes of the triangle, under the constraint (16), the maximum value of the Fourier transform of (24) at zero 4-momentum transfer, and for $m_1 > m_2 > m_3$, is found to be

$$\tfrac{1}{2}g[g^2 - (f\Gamma/m_B)^2]^{1/2}(m_3{}^{-2} - m_1{}^{-2}). \tag{25}$$

The corresponding vibrational modes of the triangle are given by (B10) and (B11) in Appendix B, and the shape is determined by (17), (21), and (B15). As an illustration, we may mention the special case in which $g_1 = g_2$ and $m_2{}^2 = \frac{1}{2}(m_1{}^2 + m_3{}^2)$; then the maximum T-violating amplitude (25) can be realized if the triangle is isosceles. If in addition we assume a right-angle isosceles triangle, then according to Eq. (B20) in Appendix B the maximum T-violation amplitude is

$$\left(\frac{1}{2\sqrt{2}}\right)g^2(m_3{}^{-2} - m_1{}^{-2}). \tag{26}$$

In any case, (25) or (26) is proportional to g^2 and is therefore of second order in the weak interaction. For example, by using (17) and (26) one finds the magnitude of the CP-violating amplitude

$K_1^0 \rightleftarrows K_2^0$ to be of the order of

$$\sim\Gamma^2 f^2 m_B^{-2}(m_3^{-2} - m_1^{-2}).$$

Since Γ denotes the first-order weak interaction constant, one expects this CP-violating amplitude to be of the superweak strength.[9]

As discussed earlier, there is another mechanism through which T violations can be produced, and that is via the coupling between the matter fields and the gauge field. Such T-violation effects are at least proportional to f^6; furthermore, such a mechanism conserves Y. Thus, if f^2 is arbitrarily set to be $\sim\alpha$, the fine-structure constant, one expects it to generate a $\Delta Y = 0$ but C, T-violating weak (or milliweak) interaction among known particles. For K_L^0, K_S^0 decays, it may add to the above $\Delta Y = \pm 2$ $K_1^0 \rightleftarrows K_2^0$ transition an amplitude $\sim\Gamma^2\alpha^3$. Hence, the superweak character of CP violation in K decay remains the same.

This simple interaction Lagrangian (5) is not intended to be a realistic theory of weak interactions because it contains many defects. It leaves out all $\Delta Y = 0$ weak reactions, and, since (without introducing additional gauge fields) the usual weak interaction Lagrangian $L_\pm$ is not renormalizable, it is also nonrenormalizable. Nevertheless, this simple example does illustrate how, through virtual emissions and absorptions of the triangle, T violations, and consequently also reciprocity violations, can be observed among known particles.

IV. OTHER APPLICATIONS

The above theory of spontaneous T violation can be applied to a large class of interactions, which can be quite different from the simple model discussed in the previous section. To illustrate these possibilities, we consider the following two examples of renormalizable theories.

A. An Abelian Example

We may identify the transformation (1) not with the hypercharge, but with a different gauge, say $e^{iN\Lambda}$ where for the known particles N is the number of left-handed charged leptons. So far as the descriptions of the triangle and the gauge field are concerned, the discussion given in Sec. II remains intact, except that ϕ_1 and ϕ_2 are now considered to be of $N=1$ (instead of $Y=1$). Of course, the discussion given in Sec. III has to be modified.

To study the interaction with matter fields in this new case, let us introduce a left-handed charged lepton field $l_L(x)$ and a right-handed charged lepton field $l_R(x)$ which satisfy

$$\gamma_5 l_L(x) = l_L(x)$$

and (27)

$$\gamma_5 l_R(x) = -l_R(x).$$

Throughout the paper, all Dirac matrices $\gamma_1, \gamma_2, \ldots, \gamma_5$ are Hermitian. By definition, $l_L(x)$ is of $N=1$ and $l_R(x)$ of $N=0$. The total Lagrangian density of the system can be written as

$$\mathcal{L}(l,B) + \mathcal{L}(B,\phi) + \mathcal{L}(\phi,l), \tag{28}$$

where $\mathcal{L}(B,\phi)$ is given by (6),

$$\mathcal{L}(l,B) = -l_L^\dagger \gamma_4\gamma_\mu\left(\frac{\partial}{\partial x_\mu} - ifB_\mu\right) l_L - l_R^\dagger \gamma_4\gamma_\mu \frac{\partial}{\partial x_\mu} l_R,$$

and

$$\mathcal{L}(\phi, l) = -(g_1\phi_1 + g_2\phi_2) l_L^\dagger \gamma_4 l_R + \text{H.c.}, \tag{29}$$

where g_1 and g_2 are both real so that T invariance holds. The total Lagrangian (28) is also invariant under the gauge transformation $e^{iN\Lambda}$. In addition, it is invariant under a second gauge transformation,

$$l_L \to e^{i\xi} l_L$$

and

$$l_R \to e^{i\xi} l_R.$$

By using (12), we may rewrite (29) in the form

$$\mathcal{L}(\phi, l) = -m_l \psi_l^\dagger \gamma_4 \psi_l - (g/\sqrt{2})(\chi_1 \psi_l^\dagger \gamma_4\psi_l - i\chi_2 \psi_l^\dagger\gamma_4\gamma_5\psi_l), \tag{30}$$

where χ_1 and χ_2 are exactly of the same expressions given by (23) in the previous section, $g = (g_1^2 + g_2^2)^{1/2}$ as before,

$$\psi_l(x) = l_L(x) + e^{i\alpha} l_R(x), \tag{31}$$

$$m_l e^{i\alpha} = 2^{-1/2}(g_1\rho_1 e^{i\theta} + g_2\rho_2), \tag{32}$$

and therefore

$$m_l^2 = \tfrac{1}{2}(g_1^2\rho_1^2 + g_2^2\rho_2^2 + 2g_1g_2\rho_1\rho_2\cos\theta). \tag{33}$$

Since $\psi_l^\dagger\gamma_4\psi_l$ is of $P=1$, $C=1$, and $T=1$ while $i\psi_l^\dagger\gamma_4\gamma_5\psi_l$ is of $P=-1$, $C=1$, and $T=-1$, through the direct emission and absorption of the triangle, there is a P-, T-violating effect in the $(l+l)$ scattering amplitude. To lowest order, the amplitude is proportional to (24), exactly as before. According to (25), and after replacing Γ by $\sqrt{2}\, m_l$, one finds the maximum value of the Fourier transform of (24), at the zero 4-momentum transfer, to be

$$\frac{g[g^2 - 2(fm_l/m_B)^2]^{1/2} m_T \Delta_T}{(m_T^2 - \frac{1}{4}\Delta_T^2)^2}, \tag{34}$$

where m_T is the mean mass of the vibrational modes of the triangle

$$m_T=\tfrac{1}{2}(m_1+m_3)$$

and $\Delta_T=(m_1-m_3)$ is the corresponding difference. The other mass m_2 lies between $m_T+\frac{1}{2}\Delta_T$ and $m_T-\frac{1}{2}\Delta_T$. If in addition we assume the triangle to be a simple right-angle isosceles triangle and $g_1=g_2=2^{-1/2}g$, then (26) holds; the maximum T-violation amplitude (34) becomes

$$\frac{g^2 m_T\Delta_T}{\sqrt{2}\,(m_T{}^2-\frac{1}{4}\Delta_T{}^2)^2}\,. \tag{35}$$

Moreover, there can also be T-violating effects due to the direct coupling f between B_μ and l, just as before. We emphasize that although in this example both the gauge group and the interaction are quite different from those in Sec. III, the basic mechanism of T violation is identical.

B. A Non-Abelian Example

Let us first consider the Weinberg model of the leptons.[3] The group is $SU_2\times U_1$. There are four gauge fields $\vec{A}_\mu$ and B_μ. The usual l-neutrino field ν_l and the right- and left-handed charged lepton fields l_R and l_L form an SU_2 doublet and an SU_2 singlet:

$$L=\begin{pmatrix}\nu_l\\ l_L\end{pmatrix}$$

and (36)

$$R=l_R\,.$$

In order to have spontaneous T violation, we assume that there are two SU_2-doublet spin-0 fields

$$\phi_1=\begin{pmatrix}\phi_1^+\\ \phi_1^0\end{pmatrix}$$

and (37)

$$\phi_2=\begin{pmatrix}\phi_2^+\\ \phi_2^0\end{pmatrix},$$

where the superscript denotes the electric charge. Both ϕ_1 and ϕ_2 are assumed to transform like the product $R^\dagger L$ under the $SU_2\times U_1$ group; therefore, their coupling to the gauge fields is completely determined by the requirements of gauge invariance. The most general form of a renormalizable, gauge-invariant, and T-invariant potential energy $V(\phi)$ is now given by, instead of (7),

$$\begin{aligned}V(\phi)=&-\lambda_1\phi_1^\dagger\phi_1-\lambda_2\phi_2^\dagger\phi_2+A(\phi_1^\dagger\phi_1)^2+B(\phi_2^\dagger\phi_2)^2\\&+C(\phi_1^\dagger\phi_1)(\phi_2^\dagger\phi_2)+\overline{C}(\phi_1^\dagger\phi_2)(\phi_2^\dagger\phi_1)\\&+\tfrac{1}{2}[(\phi_1^\dagger\phi_2)(D\phi_1^\dagger\phi_2+E\phi_1^\dagger\phi_1+F\phi_2^\dagger\phi_2)+\text{H.c.}]\,,\end{aligned} \tag{38}$$

which contains nine constants; all these constants are assumed to be real so that T invariance holds. The only formal difference between (38) and (7) is the $\overline{C}$ term. We assume that both (8) and (9) are valid, and in addition

$$D>\overline{C}\,. \tag{39}$$

In the tree approximation, the minimum of the c-number function $V(\phi)$ determines the vacuum expectation values of ϕ_1 and ϕ_2. As will be shown in Appendix C, the additional condition (39) ensures that the minimum of $V(\phi)$ occurs at

$$\langle\phi_1^+\rangle_{\text{vac}}=\langle\phi_2^+\rangle_{\text{vac}}=0\,,$$
$$\langle\phi_1^0\rangle_{\text{vac}}=2^{-1/2}\rho_1 e^{i\theta}\,,$$

and (40)

$$\langle\phi_2^0\rangle_{\text{vac}}=2^{-1/2}\rho_2\,,$$

which again defines a triangle. Both ρ_1 and ρ_2 are assumed to be >0, and $\theta\neq 0$ or π. So far as the neutral (but complex) fields ϕ_1^0 and ϕ_2^0 are concerned, the discussion is exactly the same as previously given in Sec. II, except that the constant C in Sec. II is now replaced by $C+\overline{C}$.

We may expand, as before in (12),

$$\phi_1^0=2^{-1/2}e^{i\theta}(\rho_1+R_1+iI_1)$$

and (41)

$$\phi_2^0=2^{-1/2}(\rho_2+R_2+iI_2)\,.$$

The three vibrational modes of the triangle t_1, t_2, and t_3 are given by (15), and they correspond respectively to three neutral bosons of masses m_1, m_2, and m_3. There are now three Goldstone bosons; besides the neutral one

$$G^0=(\rho_1{}^2+\rho_2{}^2)^{-1/2}(\rho_1 I_1+\rho_2 I_2)\,, \tag{42}$$

which is identical to the G given by (13), there are two charged ones:

$$G^\pm=(\rho_1{}^2+\rho_2{}^2)^{-1/2}(\rho_1\phi_1^\pm e^{\mp i\theta}+\rho_2\phi_2^\pm)\,, \tag{43}$$

where $\phi_k^-=(\phi_k^+)^\dagger$ and $k=1$ or 2. Through the Higgs mechanism, these three Goldstone bosons (just as in the usual Weinberg model) join the gauge fields to form a set of three massive spin-1 intermediate bosons $W^\pm$ and W^0. In addition, there are also two massive charged spin-0 bosons

$$H^\pm=(\rho_1{}^2+\rho_2{}^2)^{-1/2}(\rho_2\phi_1^\pm e^{\mp i\theta}-\rho_1\phi_2^\pm)\,, \tag{44}$$

and their masses are

$$m_H{}^2=\tfrac{1}{2}(D-\overline{C})(\rho_1{}^2+\rho_2{}^2)\,.$$

For simplicity, we shall assume the mass m_H to be much greater than the masses m_1, m_2, and m_3 of the neutral spin-0 bosons. Therefore, all $H^\pm$-exchange processes will be ignored in the following discussions. So far as the mechanism of T violation is concerned, one has then exactly the same basic structure as before. The triangle is again characterized by its shape and its vibrational modes, and with the same constraint (16).

The interaction between the lepton fields and the gauge fields is determined by the requirement of gauge invariance; it is exactly the same as in the usual Weinberg model. The interaction between the spin-0 fields ϕ_1, ϕ_2 and the lepton fields is now given by

$$\mathcal{L}(l,\phi) = -(g_1\phi_1 + g_2\phi_2)L^\dagger\gamma_4 R + \text{H.c.}, \qquad (45)$$

where g_1 and g_2 are again assumed to be real so that T invariance holds. This Lagrangian is clearly also invariant under the $SU_2 \times U_1$ gauge transformation; it describes an interaction between the charged lepton $l^\pm$ and ϕ_k^0 which is exactly the same as (29) and which can again be rewritten as (30). Therefore there is a P-, T-violating amplitude in $l+l$ scattering that is proportional to (24).

The same P-, T-violating amplitude also leads to an electric dipole moment for the charged lepton. For definiteness, let us assume (35) holds; one then has a simple right-angle isosceles triangle. In this case, one finds

$$\rho_1{}^2 = \rho_2{}^2 = (2\sqrt{2}\,G_F)^{-1},$$
$$g_1{}^2 = g_2{}^2 = 2\sqrt{2}\,G_F m_l{}^2,$$

where $G_F \cong 10^{-5} m_N{}^{-2}$ is the Fermi constant. The P-, T-violating amplitude (35) becomes

$$\frac{4G_F m_l{}^2 m_T \Delta_T}{(m_T{}^2 - \frac{1}{4}\Delta_T{}^2)^2}. \qquad (46)$$

The electric dipole moment $eD(l)$ of $l^\pm$ can then be readily evaluated. By using (30), (24), and (46), we find

$$D(l) = \frac{G_F m_l{}^3}{8\pi^2}\left[m_3{}^{-2} J(\epsilon_3) - m_1{}^{-2} J(\epsilon_1)\right], \qquad (47)$$

where $m_1 = m_T + \frac{1}{2}\Delta_T$ and $m_3 = m_T - \frac{1}{2}\Delta_T$ denote respectively, as before, the largest and the smallest mass of the vibrational modes of the triangle, $\epsilon_1 = (m_l/m_1)^2$, $\epsilon_3 = (m_l/m_3)^2$, and

$$J(\epsilon) = \epsilon^{-1} + (2\epsilon^2)^{-1}\left\{\ln\epsilon + \frac{1-2\epsilon}{(1-4\epsilon)^{1/2}} \times \ln\left[\frac{1+(1-4\epsilon)^{1/2}}{1-(1-4\epsilon)^{1/2}}\right]\right\}.$$

For $\epsilon \ll 1$,

$$J(\epsilon) \cong \ln(1/\epsilon) - \tfrac{3}{2} + O(\epsilon\ln\epsilon).$$

If m_1 and m_3 are arbitrarily set to be ~15 GeV and ~10 GeV, respectively, then $D(\mu)$ is $\sim 1.3\times10^{-25}$ cm and $D(e)$ is $\sim 3.6\times10^{-32}$ cm. At present, both are too small to be detected.

The extension of the Weinberg model to hadrons is not without arbitrariness. The direct coupling between hadrons and the spin-0 fields ϕ_1^0 and ϕ_2^0 has the same form as that in (45), except that L and R now refer to the appropriate hadron fields. Such a coupling is usually assumed to conserve the isospin. Thus, similar to (46), in the hadron-hadron scattering there is a P-, T-violating, but $|\Delta\vec{I}| = 0$, amplitude which is given by

$$\frac{4G_F m_h{}^2 m_T \Delta_T}{(m_T{}^2 - \frac{1}{4}\Delta_T{}^2)^2} \qquad (48)$$

and which can lead to an electric dipole $eD(h)$ of the order of

$$D(h) \sim \frac{G_F m_h{}^3 \Delta_T}{4\pi^2 m_T{}^3}, \qquad (49)$$

where m_h denotes the corresponding hadronic mass.

The present experimental limit on the electric dipole moment of the neutron[10] is $D(n) < 10^{-23}$ cm. If we arbitrarily set $m_h \sim m_N$ (m_N is the nucleon mass), then (49) gives

$$\frac{m_N{}^2 \Delta_T}{m_T{}^3} \lesssim 2\times10^{-3}, \qquad (50)$$

which implies that the P-, T-violating amplitude (48) in a $|\Delta\vec{I}| = 0$ hadronic scattering process is

$$\lesssim 10^{-2} G_F. \qquad (51)$$

For the $\Delta Y = \pm1$ processes, the corresponding T-violating amplitude should be at least smaller by an additional factor $G_F m_W{}^2 \sim \alpha$, i.e.,

$$\lesssim 10^{-4} G_F. \qquad (52)$$

For the $\Delta Y = \pm 2$ processes, some special constructions must be introduced to make the usual T-conserving amplitude in the K_L^0, K_S^0 mass-difference calculation $\sim G_F m_N{}^2$ (not $\sim G_F m_W{}^2$) times smaller than the corresponding T-conserving $\Delta Y = \pm1$ amplitude. It seems reasonable to expect that relative to (52), a similar factor $\sim G_F m_N{}^2$ also applies for the corresponding T-violating $\Delta Y = \pm 2$ amplitude, and that would lead to a T-violating amplitude

$$\lesssim 10^{-9} G_F \qquad (53)$$

in the mass matrix of the K^0-$\bar{K}^0$ complex. Since (52) seems to be smaller than the milliweak strength, one may expect the CP-violating phenomena in the K decay to be dominated by (53); the result would be of the observed superweak character.[9]

For the $|\Delta\vec{I}| = 0$ processes, the T-violating amplitude can be of the milliweak strength, and this may have important experimental consequences. As discussed earlier (and also in Appendix A), there exist other T-violating diagrams in which the T violation is generated via the direct coupling between the matter fields and the spin-1 intermediate bosons. In addition, T violation can also occur due to exchanges of the charged spin-0 bosons $H^{\pm}$. However, a full investigation lies outside the scope of this paper.

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APPENDIX A

1. We first discuss the vacuum expectation values of ϕ_1 and ϕ_2:

$$\langle \phi_1 \rangle_{\text{vac}} \equiv 2^{-1/2}\rho_1 e^{i\theta}$$

and (A1)

$$\langle \phi_2 \rangle_{\text{vac}} \equiv 2^{-1/2}\rho_2 \,.$$

In the tree approximation, ρ_1, ρ_2, and θ can be determined by setting the minimum of the c-number function $V(\phi)$ at $(\phi_1, \phi_2) = 2^{-1/2}(\rho_1 e^{i\theta}, \rho_2)$. According to (7),

$$V = -\tfrac{1}{2}(\lambda_1\rho_1^2 + \lambda_2\rho_2^2) + \tfrac{1}{4}(p\rho_1^4 + q\rho_2^4 + 2r\rho_1^2\rho_2^2) + \tfrac{1}{2}D\rho_1^2\rho_2^2(\cos\theta - \delta)^2 \,, \tag{A2}$$

where

$$p = A - (8D)^{-1}E^2 \,,$$

$$q = B - (8D)^{-1}F^2 \,,$$

$$r = \tfrac{1}{2}\left(C - D - \frac{EF}{4D}\right),$$

and

$$\delta = -(4D\rho_1\rho_2)^{-1}(E\rho_1^2 + F\rho_2^2) \,.$$

The function V must have a lower bound, and therefore (10) holds; i.e.,

$$p > 0 \,,$$

$$q > 0 \,,$$

and (A3)

$$pq > r^2 \,.$$

Since $D > 0$, the minimum of V is at

$$\cos\theta = \delta \,, \tag{A4}$$

and since (λ_1 and/or λ_2) > 0, this minimum is not at the origin but at

$$\rho_1^2 = (pq - r^2)^{-1}(q\lambda_1 - r\lambda_2)$$

and (A5)

$$\rho_2^2 = (pq - r^2)^{-1}(p\lambda_2 - r\lambda_1) \,.$$

In order that $\theta \neq 0$ or π, and both ρ_1 and ρ_2 are >0, we require, in addition to (8) and (9),

$$|\delta| < 1 \,,$$

$$q\lambda_1 > r\lambda_2 \,,$$

and (A6)

$$p\lambda_2 > r\lambda_1 \,.$$

We note that if $r > 0$, then both λ_1 and λ_2 must be >0; but if $r < 0$, then at least one of them (either λ_1 or λ_2) must be >0, but the other one could be <0, provided (A6) is satisfied.

2. The expansion of ϕ_1 and ϕ_2 around their vacuum expectation values

$$\phi_1 = 2^{-1/2}(\rho_1 + R_1 + iI_1)e^{i\theta}$$

and (A7)

$$\phi_2 = 2^{-1/2}(\rho_2 + R_2 + iI_2)$$

leads to

$$V(\phi) = V_{c\text{-No.}} + V_{\text{quad}} + V_{\text{cub}} + V_{\text{quart}} \,, \tag{A8}$$

where the subscripts refer to, respectively, a c-number expression, a quadratic function of R_k, I_k, and corresponding cubic and quartic functions. (The linear function is absent because of the minimum condition.) To obtain the normal modes t_1, t_2, and t_3, we need only to diagonalize V_{quad}:

$$V_{\text{quad}} = [A\rho_1^2 + \tfrac{1}{2}\cos\theta(D\rho_2^2\cos\theta + E\rho_1\rho_2)]R_1^2 + [B\rho_2^2 + \tfrac{1}{2}\cos\theta(D\rho_1^2\cos\theta + F\rho_1\rho_2)]R_2^2 + [C - D(1 + \cos^2\theta)]\rho_1\rho_2 R_1 R_2 + \tfrac{1}{2}D(\rho_1^2 + \rho_2^2)\sin^2\theta\, I^2 - (4\rho_1\rho_2)^{-1}\sin\theta(E\rho_1^2 - F\rho_2^2)(\rho_1^2 + \rho_2^2)^{1/2}(\rho_2 R_1 - \rho_1 R_2)I \,, \tag{A9}$$

where ρ_1, ρ_2, and θ are given by (A4) and (A5). It is convenient to introduce I, G, R, and R', where G and I are defined by (13) and (14), respectively, and

$$R \equiv (\rho_1^2 + \rho_2^2)^{-1/2}(\rho_2 R_1 - \rho_1 R_2)$$

and (A10)

$$R' \equiv (\rho_1^2 + \rho_2^2)^{-1/2}(\rho_1 R_1 + \rho_2 R_2) \,.$$

The constraint (16) is derived by noting that the product $R'I$ is absent in V_{quad}, and the Goldstone boson (13) is determined by observing that G is absent in V_{quad}.

3. We shall now derive a set of conditions under which certain C-, T-violating diagrams must be zero if the system contains only ϕ_k and B_μ. (As explained before, even if such diagrams are not zero there is no violation of reciprocity relations without other fields.) We define

$$\phi_1' \equiv \rho^{-1}(\rho_2\phi_1 e^{-i\theta} - \rho_1\phi_2), \quad \phi_2' \equiv \rho^{-1}(\rho_1\phi_1 e^{-i\theta} + \rho_2\phi_2) \tag{A11}$$

where

$$\rho = (\rho_1{}^2 + \rho_2{}^2)^{1/2}.$$

From (A1) we find

$$\langle \phi_1' \rangle_{\text{vac}} = 0 \quad \text{and} \quad \langle \phi_2' \rangle_{\text{vac}} = 2^{-1/2}\rho, \tag{A12}$$

which are both real. In terms of ϕ_1' and ϕ_2', (A7) becomes

$$\phi_1' = 2^{-1/2}(R + iI) \quad \text{and} \quad \phi_2' = 2^{-1/2}(\rho + R' + iG). \tag{A13}$$

The function $V(\phi)$, defined by (7), can now be written as

$$V = -\lambda_1'\phi_1'^\dagger \phi_1' - \lambda_2'\phi_2'^\dagger\phi_2' - (\lambda_3'\phi_1'^\dagger\phi_2' + \text{H.c.}) + A'(\phi_1'^\dagger\phi_1')^2 + B'(\phi_2'^\dagger\phi_2)^2 + C'(\phi_1'^\dagger\phi_1')(\phi_2'^\dagger\phi_2') + \tfrac{1}{2}[(\phi_1'^\dagger\phi_2')(D'\phi_1'^\dagger\phi_2' + E'\phi_1'^\dagger\phi_1' + F'\phi_2'^\dagger\phi_2') + \text{H.c.}], \tag{A14}$$

where the new constants $\lambda_1', \lambda_2', \ldots, F'$ can be readily expressed in terms of the original eight real parameters $\lambda_1, \lambda_2, \ldots, F$; e.g.,

$$\lambda_1' = \rho^{-2}(\lambda_1\rho_2{}^2 + \lambda_2\rho_1{}^2),$$

$$\lambda_2' = \rho^{-2}(\lambda_1\rho_1^2 + \lambda_2\rho_2{}^2),$$

$$\lambda_3' = \rho^{-2}(\lambda_1 - \lambda_2)\rho_1\rho_2,$$

etc.

Because of Hermiticity only λ_3', D', E', and F' may have imaginary parts. By using (A4), (A5), and (A11), we find

$$\text{Im}\lambda_3' = 0, \quad \text{Im}F' = 0, \quad \text{Im}\,D' = -(2\rho_1\rho_2)^{-1}(E\rho_1{}^2 - F\rho_2{}^2)\sin\theta, \quad \text{and} \quad \text{Im}E' = 4\rho^{-2}\rho_1\rho_2 D\sin 2\theta. \tag{A15}$$

In addition, there are three equalities:

$$\rho_1\rho_2(\lambda_1' - \lambda_2') + (\rho_1{}^2 - \rho_2{}^2)\lambda_3' = 0, \quad \rho^2 B' = \lambda_2', \quad \text{and} \quad \rho^2 F' = 4\lambda_3'. \tag{A16}$$

These three equalities imply that among the eleven new real parameters λ_1', λ_2', λ_3', A', B', C', $\text{Re}D'$, $\text{Im}D'$, $\text{Re}E'$, $\text{Im}E'$, and F' only eight are independent. In terms of these new fields, (A9) becomes

$$V_{\text{quad}} = \tfrac{1}{2}[-\lambda_1' + \tfrac{1}{4}\rho^2(C' + \text{Re}\,D')]R^2 + \tfrac{1}{4}\lambda_2'R'^2 + \tfrac{1}{2}[-\lambda_1' + \tfrac{1}{4}\rho^2(C' - \text{Re}\,D')]I^2 + 2\lambda_3' RR' + \tfrac{1}{2}\rho^2(\text{Im}\,D')IR. \tag{A17}$$

The coupling between ϕ_k' and the gauge field B_μ has the same covariant form as that between ϕ_k and B_μ; e.g., the current operator j_μ remains given by

$$j_\mu = i\sum_{k=1,2}\left(\frac{\partial\phi_k'^\dagger}{\partial x_\mu}\phi_k' - \phi_k'^\dagger\frac{\partial\phi_k'}{\partial x_\mu}\right). \tag{A18}$$

The corresponding interaction Lagrangian is

$$-fB_\mu\left(G\frac{\partial R'}{\partial x_\mu} - R'\frac{\partial G}{\partial x_\mu} + I\frac{\partial R}{\partial x_\mu} - R\frac{\partial I}{\partial x_\mu}\right) - f m_B B_\mu{}^2 R' - \tfrac{1}{2}f^2 B_\mu{}^2(I^2 + R^2 + R'^2 + G^2). \tag{A19}$$

In addition to this interaction, we also have V_{cub} and V_{quart} in (A8). In a power-series expansion in f, we regard all masses to be of the zeroth order, and therefore

$$V_{\text{cub}} \sim O(f) \quad \text{and} \quad V_{\text{quart}} \sim O(f^2). \tag{A20}$$

We note that because of (A12) and (A15), in order to have C, T violations [e.g., $nB_\mu \rightleftarrows mB_\mu$ when $n + m$ is an odd number] one must have $\text{Im}\,D' \neq 0$ and/or $\text{Im}E' \neq 0$. The following theorems can then be easily established:

Theorem 1. If $\theta = \frac{1}{2}\pi$ and if I is a normal mode, then (without other fields) there is no C, T violation.

Proof. If I is a normal mode, then the coefficient of IR in (A17) must be zero. Hence $\text{Im}\,D' = 0$. If $\theta = \frac{1}{2}\pi$, then according to (A15) $\text{Im}E' = 0$. The theorem is then established. Thus, for example, the amplitude for $nB_\mu \rightleftarrows mB_\mu$ must be zero if $n + m$ is an odd number.

The same theorem can also be proved by noting that in this case, by using (A4) and (A9), one has $E = F = 0$. Hence, a new time-reversal operation may be defined, under which $\phi_1 \to -\phi_1$ and $\phi_2 \to \phi_2$, instead of (3). The vacuum expectation values $\langle \phi_1 \rangle_{vac} = i2^{-1/2}\rho_1$ and $\langle \phi_2 \rangle_{vac} = 2^{-1/2}\rho_2$ are compatible with this new time-reversal operation, and therefore without other matter fields there is no T violation.

Theorem 2. If the normal modes t_1, t_2, and t_3 are all degenerate (i.e., $m_1 = m_2 = m_3$), then there is also no C, T violation.

Proof. Because of the degeneracy, we may choose the normal modes to be $t_1 = R$, $t_2 = I$, and $t_3 = R'$. From (A17), one sees that the absence of RR' and IR coupling gives $\lambda_3' = 0$ and $\mathrm{Im}\, D' = 0$. The degeneracy $m_1 = m_2$ gives $\mathrm{Re}\, D' = 0$. These together with (A16) imply

$$\lambda_3' = F' = D' = 0 .$$

There is, therefore, only one term in (A14) that depends on the relative phase between ϕ_1' and ϕ_2', and it is proportional to E'. We may rotate $\phi_1' \to e^{i\beta}\phi_1'$, $\phi_2' \to \phi_2'$; this does not alter their vacuum expectation values, nor the coupling between ϕ_k' and B_μ, but it can transform E' to real. Once E' becomes real, one may define the time reversal to be the antiunitary operator under which $\phi_1' \to \phi_1'$ and $\phi_2' \to \phi_2'$. Since (A14) contains only real parameters and since $\langle \phi_1' \rangle_{vac}$ and $\langle \phi_2' \rangle_{vac}$ are also both real, the theorem is proved.

Remark. The condition of Theorem 2 can be weakened: We need only $t_1 = R$, $t_2 = I$, $t_3 = R'$, and $m_1 = m_2$, but m_3 can be different.

Theorem 3. If R' is a normal mode, then to order f^3 the amplitude of $B_\mu(k) \rightleftarrows B_\mu(p) + B_\mu(q)$ vanishes for arbitrary virtual momenta k, p, and q.

Proof. From (A17), it follows that if R' is a normal mode then $\lambda_3' = 0$, which implies $F' = 0$, on account of (A16). In (A14) there are only two terms, one proportional to D' and the other to E', that depend on the relative phase between ϕ_1' and ϕ_2'. Just as in the previous proof, we may rotate $\phi_1' \to e^{i\beta}\phi_1'$ and $\phi_2' \to \phi_2'$, but this time to make D' real. All C-, T-violating effects must then be proportional to $\mathrm{Im}E'$. It is easy to verify that in (A14) the $\mathrm{Im}E'$ term is of the form

$$\tfrac{1}{4}(\mathrm{Im}E')(R^2 + I^2)(I\rho + IR' - GR) .$$

Because of (A20), $V_{\rm quart}$ does not contribute to the lowest-order f^3 diagrams for $B_\mu(k) \rightleftarrows B_\mu(p) + B_\mu(q)$, but $V_{\rm cub}$ may. Since R' is assumed to be a normal mode, $V_{\rm cub}$ can contribute to such diagrams only if it contains at least one factor of R' [in which case the $V_{\rm cub}$ vertex can link with the $-f m_B B_\mu{}^2 R'$ vertex in (A19) through the R' propagator]. However, the cubic-function part of the above $\mathrm{Im}E'$ term does not contain any R' factor; hence, the theorem.

For a general triangle, the actual calculation of the transition $B_\mu(k) \rightleftarrows B_\mu(p) + B_\mu(q)$ is rather complicated, but we hope to give some of the details in a separate publication.

APPENDIX B

To establish (25), we first express χ_1 and χ_2, defined by (23), as linear functions of the normal modes t_1, t_2, t_3 and the Goldstone mode G:

$$\chi_1 = a_1 t_1 + a_2 t_2 + a_3 t_3 \tag{B1}$$

and

$$\chi_2 = b_1 t_1 + b_2 t_2 + b_3 t_3 + \gamma G \tag{B2}$$

where a_i, b_j, and γ are constants. From (22), it follows that

$$g_1\rho_1 \sin(\theta - \alpha) - g_2\rho_2 \sin\alpha = 0$$

and

$$g_1\rho_1 \cos(\theta - \alpha) + g_2\rho_2 \cos\alpha = \Gamma ;$$

these together with (23) require χ_1 to be independent of G and the constant γ in (B2) given by

$$\gamma = (g\rho)^{-1}\Gamma , \tag{B3}$$

where $g = (g_1{}^2 + g_2{}^2)^{1/2}$ and $\rho = (\rho_1{}^2 + \rho_2{}^2)^{1/2}$. Let us define

$$a = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

and (B4)

$$b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} .$$

By using (23), one derives

$$\tilde{a}a = 1 ,$$

$$\tilde{a}b = 0 ,$$

and (B5)

$$\tilde{b}b = 1 - \gamma^2 = 1 - (g\rho)^{-2}\Gamma^2 ,$$

where the tilde denotes the transpose.

The Fourier transform of (24) at the zero momentum is

$$g^2 \tilde{a} M^{-2} b \tag{B6}$$

where

$$M^2 = \begin{pmatrix} m_1{}^2 & 0 & 0 \\ 0 & m_2{}^2 & 0 \\ 0 & 0 & m_3{}^2 \end{pmatrix} . \tag{B7}$$

We may search for the maximum of (B6) by varying a and b, but keeping M^2 and the three orthonormal relations given in (B5) fixed. It is straightforward to show that for $m_1 > m_2 > m_3$,

$$|g^2\tilde{a}M^{-2}b| \leq \tfrac{1}{2}g^2[1-(g\rho)^{-2}\Gamma^2]^{1/2}(m_3^{-2}-m_1^{-2}), \tag{B8}$$

and the equal sign holds when

$$a_1 = a_3 = 2^{-1/2},$$
$$b_1 = -b_3 = -2^{-1/2}[1-(g\rho)^{-2}\Gamma^2]^{1/2}, \tag{B9}$$
and
$$a_2 = b_2 = 0.$$

Clearly, the maximum value for $|g^2\tilde{a}M^{-2}b|$ remains the same, if one changes $t_1 \to -t_1$, or $t_3 \to -t_3$, or $a \to -a$, or $b \to -b$.

For any given shape of the triangle ρ_1, ρ_2, and θ, one finds that the maximum in (B8) can be realized if the normal modes are given by

$$\left.\begin{matrix} t_1 \\ t_3 \end{matrix}\right\} = 2^{-1/2}\{-[\beta \pm (\alpha^2+\beta^2)^{-1/2}\alpha]I + [\alpha \mp (\alpha^2+\beta^2)^{-1/2}\beta]R + \gamma R'\} \tag{B10}$$

and

$$t_2 = (\alpha^2+\beta^2)^{-1/2}[\gamma\beta I - \gamma\alpha R + (\alpha^2+\beta^2)R'], \tag{B11}$$

where t_1 assumes the upper sign in (B10) and t_3 the lower sign, γ is given by (B3), I, R, and R' are defined by (14) and (A10),

$$\alpha = (\rho g\Gamma)^{-1}[(g_1^2-g_2^2)\rho_1\rho_2 - g_1g_2(\rho_1^2-\rho_2^2)\cos\theta], \tag{B12}$$

$$\beta = (g\Gamma)^{-1}\rho g_1 g_2 \sin\theta, \tag{B13}$$

and therefore

$$\alpha^2+\beta^2+\gamma^2 = 1. \tag{B14}$$

The only condition is that the constraint (16) should hold. Because of (B10) and (B11), this constraint can also be written as

$$\tfrac{1}{2}(\alpha^2+\beta^2)^{-1/2}\alpha(m_1^2-m_3^2) = \beta[m_2^2 - \tfrac{1}{2}(m_1^2+m_3^2)]. \tag{B15}$$

One can readily verify that the solutions (B10) and (B11) together with (B9) indeed lead to χ_1 and χ_2 defined by their original expression (23).

As an explicit example, we may consider the special case $g_1 = g_2 = 2^{-1/2}g$ and $\rho_1 = \rho_2 = 2^{-1/2}\rho$; i.e., an isosceles triangle. From (B12), one sees that $\alpha = 0$; therefore, (B15) implies

$$m_2^2 = \tfrac{1}{2}(m_1^2 + m_3^2). \tag{B16}$$

The angle θ and the side $2^{-1/2}\rho$ of the isosceles triangle are determined by (17) and (21), which can now be written as

$$\Gamma^2 = \tfrac{1}{2}g^2\rho^2(1+\cos\theta) \tag{B17}$$

and

$$m_B^2 = f^2\rho^2. \tag{B18}$$

If in addition we assume $\theta = \frac{1}{2}\pi$, i.e., a right-angle isosceles triangle, then $\Gamma = 2^{-1/2}g\rho$; hence, $\beta = \gamma = 2^{-1/2}$, the normal modes become

$$t_1 = \tfrac{1}{2}(-I - \sqrt{2}R + R'),$$
$$t_2 = \frac{1}{\sqrt{2}}(I + R'), \tag{B19}$$
$$t_3 = \tfrac{1}{2}(-I + \sqrt{2}R + R'),$$

and the corresponding maximum T-violation amplitude in (B8) is

$$2^{-3/2}g^2(m_3^{-2} - m_1^{-2}). \tag{B20}$$

As another example we may take the limiting case $\theta = 0$, or π. From (B13) it follows that $\beta = 0$; hence (B15) implies $m_1^2 = m_3^2$, and therefore according to (B8) the maximum T-violation amplitude is zero, as it should be.

In general, if the coupling constants f, g_1, g_2, and Γ and the masses m_1, m_2, m_3, and m_B are given, then in order to realize the maximum T-violation amplitude in (B8) the shape parameters ρ_1, ρ_2, and θ are determined by (17), (21), and (B15), and the vibrational modes t_1, t_2, and t_3 by (B10) and (B11).

APPENDIX C

In this appendix we discuss the minimum of the c-number potential energy $V(\phi)$ defined by (38); this minimum is assumed to be at

$$\phi_1 = 2^{-1/2}\begin{pmatrix} 0 \\ \rho_1 e^{i\theta} \end{pmatrix} \tag{C1}$$

and

$$\phi_2 = 2^{-1/2}\begin{pmatrix} \sigma \\ \rho_2 \end{pmatrix}, \tag{C2}$$

where θ is real, and σ, ρ_1, ρ_2 are all real and ≥ 0. Since V is invariant under the $SU_2 \times U_1$ gauge transformation, we can always transform the upper component of ϕ_1 to zero, and both components of ϕ_2 to real and non-negative. Equation (38) can then be written as

$$V = -\tfrac{1}{2}\lambda_1\rho_1^2 - \tfrac{1}{2}\lambda_2(\sigma^2+\rho_2^2) + \tfrac{1}{2}D\rho_1^2\rho_2^2[(\cos\theta-\Delta)^2-\Delta^2] + \tfrac{1}{4}[A\rho_1^4 + B(\sigma^2+\rho_2^2)^2 + C\rho_1^2(\sigma^2+\rho_2^2) + (\bar{C}-D)\rho_1^2\rho_2^2], \tag{C3}$$

where

$$\Delta = -(4D\rho_1\rho_2)^{-1}[E\rho_1^2 + F(\sigma^2+\rho_2^2)]. \tag{C4}$$

For $D > 0$, the minimum of V is at

$$\cos\theta = \Delta. \tag{C5}$$

Keeping (C5) satisfied, we find

$$\frac{\partial V}{\partial \sigma^2} = \frac{\partial V}{\partial \rho_2^2} + \tfrac{1}{4}(D-\overline{C})\rho_1^2 . \tag{C6}$$

Since $D > \overline{C}$, at $\partial V/\partial \rho_2^2 = 0$ the derivative $\partial V/\partial \sigma^2$ is always positive; hence to obtain the minimum of V we require

$$\sigma^2 = 0 . \tag{C7}$$

The function V then reduces to (A2) discussed in Appendix A provided the constant C in (A2) is replaced by $C+\overline{C}$.

APPENDIX D

It is possible to have a spontaneous T violation, but without a spontaneous gauge-symmetry violation. Let us consider a simple example which consists of a spin-$\frac{1}{2}$ field ψ and a single Hermitian spin-0 field ϕ. The Lagrangian density $\mathcal{L}$ is assumed to be renormalizable, and it is invariant under T, C, and P:

$$\mathcal{L} = -\frac{1}{2}\left(\frac{\partial \phi}{\partial x_\mu}\right)^2 - V(\phi) - \psi^\dagger \gamma_4\left(\gamma_\mu \frac{\partial}{\partial x_\mu} + m\right)\psi - ig\psi^\dagger\gamma_4\gamma_5\psi\phi , \tag{D1}$$

where the potential $V(\phi)$ is given by

$$V(\phi) = -\tfrac{1}{2}\lambda\phi^2 + \tfrac{1}{4}A\phi^4 . \tag{D2}$$

From Hermiticity, the parameters m, g, λ, and A must be real. It can be readily verified that the Lagrangian $\mathcal{L}$ is invariant under T, C, and P under which

$$T\phi(\vec{r},t)T^{-1} = -\phi(\vec{r},-t) , \tag{D3}$$

$$C\phi(\vec{r},t)C^{-1} = \phi(\vec{r},t) , \tag{D4}$$

and

$$P\phi(\vec{r},t)P^{-1} = -\phi(-\vec{r},t) ; \tag{D5}$$

the corresponding transformations of ψ are standard. In addition, $\mathcal{L}$ is invariant under the simple gauge transformation

$$\psi \to e^{i\beta}\psi \text{ and } \phi \to \phi . \tag{D6}$$

As we shall see, the solution may violate T invariance, but it remains gauge-invariant.

To generate a spontaneous T violation (but *without* a spontaneous gauge-symmetry violation), we assume the (renormalized) coupling constants to satisfy

$$\lambda > 0 \text{ and } A > 0 . \tag{D7}$$

Thus, the vacuum expectation value of ϕ is not zero;

$$\langle \phi \rangle_{\text{vac}} = \rho \neq 0 . \tag{D8}$$

Since according to (D3)–(D5) ϕ is of $P=-1$, $CP=-1$, and $T=-1$, such a nonzero vacuum expectation value implies spontaneous violations of P, CP, and T.

In the tree approximation, ρ is determined by the minimum of the c-number function V. Therefore,

$$\rho^2 = A^{-1}\lambda . \tag{D9}$$

We may write

$$\phi = \rho + \chi . \tag{D10}$$

The potential V becomes

$$V(\chi) = V_0 + \tfrac{1}{2}\mu^2\chi^2 + A\rho\chi^3 + \tfrac{1}{4}A\chi^4 , \tag{D11}$$

where $V_0 = -\frac{1}{4}A^{-1}\lambda^2$ and $\mu^2 = 2\lambda$. In order to render the quadratic expression $-\psi^\dagger\gamma_4(m+ig\rho\gamma_5)\psi$ in a more familiar form, we perform a unitary transformation U:

$$U\psi U^\dagger = [\exp(-\tfrac{1}{2}i\gamma_5\alpha)]\psi \tag{D12}$$

and

$$U\phi U^\dagger = \phi ,$$

where

$$\sin\alpha = M^{-1}g\rho , \quad \cos\alpha = M^{-1}m \tag{D13}$$

and

$$M = (m^2 + g^2\rho^2)^{1/2} . \tag{D14}$$

The Lagrangian $\mathcal{L}$ becomes

$$U\mathcal{L}U^\dagger = -\frac{1}{2}\left(\frac{\partial \chi}{\partial x_\mu}\right)^2 - V(\chi) - \psi^\dagger\gamma_4\left(\gamma_\mu\frac{\partial}{\partial x_\mu} + M\right)\psi - g\psi^\dagger\gamma_4(\sin\alpha + i\gamma_5\cos\alpha)\psi\chi . \tag{D15}$$

Since $\psi^\dagger\gamma_4\psi$ is of $P=1$, $C=1$, and $T=1$ while $i\psi^\dagger\gamma_4\gamma_5\psi$ is of $P=-1$, $C=1$, and $T=-1$, the Lagrangian (D15) satisfies spontaneous T, P, and CP violations, but the gauge invariance (D6) remains preserved.

From (D9) one sees that there are two solutions of $\langle\phi\rangle_{\text{vac}}$: ρ and $-\rho$. Either solution is not invariant under T, P, and CP. But since the Lagrangian is invariant under T, P, and CP both solutions must exist, and they should transform into each other under either T, or P, or CP.

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[6]For the free neutrino, the Weyl theory and the Majorana theory are clearly equivalent. For further discussions, see K. M. Case, Phys. Rev. 107, 307 (1957).

[7]The spontaneous T violation is logically an independent proposition; it does not have to be tied to any spontaneous gauge-symmetry violations. See Appendix D for a simple example.

[8]See the definition given by G. C. Wick, Phys. Rev. 80, 268 (1950).

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