with a small (perhaps vanishing) $c_3(\theta)$ term, can be achieved with an input which is chiral-SU(2)-symmetric up to a U-spin singlet octet term, the F_7 invariant direction of SU(3) breaking by electromagnetism.

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Dual Models with Global SU(2,2) Symmetry*

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The possibility of enlarging the gauge symmetry of the dual resonance models is considered by studying the structure of SU(2,2)- [or SO(4,2)] invariant dual models. n-point functions based on the *degenerate* representations of SU(2,2) are worked out in detail, and a condition under which these amplitudes are dual is specified. Dual models based on the nondegenerate representations are also discussed. Through a physical interpretation of the characteristics which emerge, a possible connection between the dimension $N-1$ of the hadronic matter and the gauge-symmetry group SO(N,2) is pointed out.

I. INTRODUCTION

Recent developments in the dual resonance models (DRM) have led to a better understanding of the attractive features as well as the limitations of these models. A description of these models in terms of quantized minimal surfaces in space-time has shown[1] that such models arise naturally from the dynamics of one-dimensionally extended objects. The relevance of the gauge conditions in these models was shown in I and II to be related to the coordinate-independent description of the minimal surfaces. Further arguments were given in these works that the well-known tachyon condition on the *external* masses, which comes about because of the requirement of gauge invariance, cannot be removed without drastic (and hitherto unknown) modifications of the dual vertices. In particular, no multiplicative modification of dual vertices is likely to succeed in completely removing the mass-shell condition without introducing ghosts or other unphysical features.[2] There are of course other shortcomings.

In view of the many attractive features possessed by DRM and of the limitations mentioned above, it would be of interest to consider possible departures from the underlying principles which might lead to more realistic models.

Interesting attempts in this direction have been taken by Domokos[3] and by Del Giudice *et al.*[4] Although the points of view adopted by these authors differ in detail, they have one common feature: They are both led to consider subgroups of the conformal group SU(2, 2) which leave a particular direction in space-time invariant and to construct composite states which are covariant under such groups. Since the generators of these subgroups contain dilatation and special conformal generators of SU(2, 2), if it is demanded, on physical grounds, that the resulting theory be Poincaré-invariant, one is led to require the full SU(2, 2) invariance of the kinematics. As is well known, the mass spectrum of such theories is either continuous or contains only zero-mass particles.

In the absence of a satisfactory symmetry-breaking scheme, and because we believe a reasonable mass spectrum is an important feature of any dual model,[5] we wish to consider SU(2, 2) *not* as a kinematical invariance group but as a *global*[6] *gauge group* in the same way that SU(1, 1) or SU(1, 1)⊗SU(1, 1) is employed in currently popular dual models. One motivation for such a gauge symmetry is its apparent connection with the "dimension" of the hadronic matter. It is clear from the formalism developed in I and II that one may regard the SU(1, 1)⊗SU(1, 1) or SO(2, 2) as the global gauge symmetry of a one-dimensional hadronic matter. If it turns out that the dimension of hadronic matter is not one but three, then it would be natural to regard SU(2, 2) which is the universal covering group of SO(4, 2) as the relevant global gauge group.[7] For brevity, we shall henceforth indicate the relation between a group and its covering group by the symbol ~.

In this paper we will consider the SU(2, 2) symmetry as an input and explore the possibility of constructing SU(2, 2)-invariant dual models. Since it is not *a priori* clear what type of operators are relevant in these models, we shall adopt a c-number approach and obtain the general form of the SU(2, 2)-invariant dual n-point functions. Our method of obtaining these is quite similar to the one used in obtaining SL(2, c)-invariant n-point functions in the Gel'fand-Naimark z basis, which are generalizations of Naimark's trilinear-invariant functional.[8] Of the two sets of SU(2, 2)-invariant anharmonic ratios, one is the generalization to Minkowski space of the cross ratios in the complex plane, and the other set involves local spinors. Because of the indefiniteness of the metric in the Minkowski space, the cross ratios of the former type are in general *not* dual to one another. They can, however, be made dual by an $i\epsilon$ prescription which is discussed in Sec. IV.

To make contact with the conventional DRM, we specialize to the simplest form of the dual amplitudes which can be obtained from the anharmonic ratios in the Minkowski space alone, and explicitly evaluate their 4-point function. We find, not surprisingly, that they are just 4-point functions of the Virasoro type.[9]

The plan of this paper is as follows: In Sec. II we derive the general form of the n-point functions based on the *degenerate* representations of SU(2, 2) ~SO(4, 2). We then demonstrate that the form of these n-point functions does *not* depend on the dimension of the projective space (four in this case), so that a class of n-point functions on SO(N, 2) can be constructed in the same way. This unified approach allows us to offer a physical interpretation of such amplitudes. In Sec. III we discuss a class of n-point functions based on *nondegenerate* representations, which are obtained by a method due to Domokos and Kövesi-Domokos.[10] In Sec. IV, we specialize to the simplest form of n-point functions on the degenerate representations and explicitly evaluate the corresponding 4-point function. Section V is devoted to a discussion of the results. We present in an appendix the relevant properties of the conformal group SU(2, 2) and its representations, which are needed for the evaluation of the n-point functions.

II. n-POINT FUNCTIONS ON THE DEGENERATE REPRESENTATIONS OF SO(4,2) AND THEIR GENERALIZATION TO SO(N,2)

The n-point functions which we want to construct are the generalizations of the usual bilinear invariants. As explained in the Appendix, the irreducible representations $\phi^{d\beta m}(x^\mu, z)$ of the conformal group SU(2, 2) are realized, generally, on the six-dimensional manifolds (x^μ, z). In addition, this group can also be realized on a manifold of lower dimension, namely, on the Minkowski space spanned by x^μ. These are called the *degenerate* representations of SU(2, 2). They are realized on the space of functions $\phi^d(x)$ where the action of the group is given by

$$T_g{}^d\,\phi^d(x) = |\Delta|^d(\mathrm{sgn}\Delta)^\epsilon\,\phi^d(x'), \tag{2.1}$$

where x' and x are related by (B10) and, from (B16), $\Delta=\det(h)=\det(XB+D)$. ϵ is the parity of the representation.

Consider the n-point functional

$$A_n = \int \prod_{i=1}^{n} d\mu(x_i)\phi^{d_i}(x_i)\,K(x_1, \ldots, x_n; \{d_i\}). \tag{2.2}$$

In a dual model the functions $\phi^d(x)$ may be interpreted as wave functions of external particles or parts of dual vertices. The invariant-volume element is given by

$$\prod_{i=1}^{n} d\mu(x_i) = \prod_{i=1}^{n} d^4x_i\left[\prod_{j=1}^{n}(x_j - x_{j+1})^2\right]^{-2},$$
$$x_{n+1} = x_1, \quad x^2 = x^\mu x_\mu\,. \tag{2.3}$$

The kernel $K(x_1, \ldots, x_n; \{d_i\})$ is a generalized Clebsch-Gordan coefficiént which must be determined from the requirement that the n-point function A_n be invariant under SU(2, 2).

As discussed in the Appendix, every SU(2, 2) transformation can be obtained from an appropriate combination of (a) Poincaré transformations, (b) scale transformations, and (c) inversions. We shall therefore determine the form of K by considering these transformations one at a time.

If $T_g{}^d$ is a transformation operator which represents the action of an element $g \in$ SU(2, 2) on the functions $\phi^d(x)$, then the action of SU(2, 2) on A_n is given as follows:

$$T_g : A_n = \int \prod_{i=1}^{n} d\mu(x_i)[T_g{}^{d_i}\phi^{d_i}(x_i)]$$
$$\times K(x_1, \ldots, x_n; \{d_i\}). \tag{2.4}$$

The requirement that A_n be an SU(2, 2)-invariant, i.e.,

$$T_g A_n = A_n, \quad g \in \mathrm{SU}(2,2), \tag{2.5}$$

imposes restrictions on the form of the kernel K.

We first consider Poincaré transformations. Under translations we have

$$x_i^\mu \to x_i^\mu + a^\mu,$$

so that to keep A_n invariant, K must have the form

$$K(x_1, \ldots, x_n) = K((x_i - x_j), \ldots);$$
$$i \neq j;\ i,j = 1, \cdots, n. \tag{2.6}$$

Consider next homogeneous Lorentz transformations. As is clear from (2.4)–(2.6), these transformations restrict the x dependence of K to the form

$$(x_i - x_j)^\mu(x_k - x_l)_\mu, \quad i \neq j; \quad k \neq l.$$

We now turn to scale transformations. Making use of (2.1), it is easy to see that the kernel K must be of the form

$$\prod_{\substack{i,j=1\\ i\neq j}}^{n}\ \prod_{\substack{k,l=1\\ k\neq l}}^{n} [(x_i - x_j)^\mu(x_k - x_l)_\mu]^{-d_{ij}}, \tag{2.7}$$

where

$$\sum_{\substack{j=1\\ j\neq i}}^{n} d_{ij} = d_i, \quad i = 1, \ldots, n. \tag{2.8}$$

Finally, consider the effect of inversions. These transformations further restrict the form of K to

$$K = C_n(x)\prod_{\substack{i,j=1\\ i\neq j}}^{n}[(x_i - x_j)^2]^{-d_{ij}}, \tag{2.9}$$

where d_{ij} are still given by (2.8), and $C_n(x)$ is an SU(2, 2)-invariant function of the x's. To see the nature of these invariant functions, consider $C_4(x)$. By imposing constraints due to SU(2, 2) transformations, it is easy to verify that, aside from a constant, $C_4(x)$ must have the form

$$C_4(x) = C_4(R_1(x), R_2(x)), \tag{2.10}$$

where, for example,

$$R_1(x) = \frac{(x_1 - x_2)^2(x_3 - x_4)^2}{(x_1 - x_3)^2(x_2 - x_4)^2},$$
$$R_2(x) = \frac{(x_1 - x_4)^2(x_2 - x_3)^2}{(x_1 - x_3)^2(x_2 - x_4)^2}. \tag{2.11}$$

These anharmonic ratios are the generalization to Minkowski space of the well-known cross ratios in the complex plane.

We thus find that the general n-point functions based on the degenerate representations of SU(2, 2) must have the form

$$A_n = \int\left[\prod_{i=1}^{n} d\mu(x_i)\,\phi^{d_i}(x_i)\right]$$
$$\times\left\{\prod_{\substack{i,j=1\\ i\neq j}}^{n}[(x_i - x_j)^2]^{-d_{ij}}\right\}C_n(x), \tag{2.12}$$

where the C_n's are functions of the SU(2, 2)-invariant cross ratios and d_{ij} are constrained by the condition (2.8).

Since SU(2, 2) is the universal covering group of SO(4, 2), the expression (2.12) is also an n-point function on SO(4, 2). Moreover, we note that in obtaining A_n, the dimension $N=4$ of the Minkowski-space coordinates x^μ has not played a crucial role. In fact, let $x^\mu = (x^0, x^1, \ldots, x^{N-1})$ be the Minkowski-space coordinates of the N-dimensional manifold with signature $(1, -1, -1, \ldots, -1)$. Such a space is related to the homogeneous space associated with SO(N, 2) by the usual stereographic projection.[11] Then it is easy to see that the cross ratios will still be of the form (2.11) and that the Poincaré, scale, and inversion transformations in such an N-dimensional projective space will determine the kernel to be of the form

$$(x_i - x_j)^\mu (x_i - x_j)_\mu, \quad i \neq j; \quad \mu = 0, 1, \ldots, N-1. \tag{2.13}$$

The invariant-volume element will now take the form

$$d^N v = \prod_{i=1}^{n} d^N x_i \left[\prod_{j=1}^{n} (x_j - x_{j+1})^2\right]^{-N/2}, \quad x_{n+1} = x_1, \quad x^2 = x^\mu x_\mu. \tag{2.14}$$

Thus for those irreducible representations of SO(N, 2) which can be constructed in the lower dimensional manifold x^μ, $\mu = 0, \ldots, N-1$, and which can be specified by a single Casimir operator d, the dimension, and the parity ϵ, one can write a class of n-point functions[12]:

$$A_n^N = \int \left[\prod_{i=1}^{n} d^N\mu(x_i)\phi^{d_i}(x_i)\right] \times \left\{\prod_{\substack{i,j=1\\ i\neq j}}^{n} [(x_i - x_j)^2]^{-d_{ij}}\right\} C_n^N(x). \tag{2.15}$$

For $N=1$, one obtains in this way the n-point functions on SO(2, 1) ~ SL(2R). For $N=2$, one obtains a class of n-point functions on the conformal group SO(2, 2) which is locally isomorphic to SL(2R)⊗SL(2R). For $N=4$, one of course recovers (2.12), and so on.

Although the *n-point functions A_n^N are independent of any particular interpretation of the underlying dynamics,* the unified description (2.15) makes it possible to give a physical interpretation to the quantities x^μ. Since the quantities (τ, ξ) in papers I and II, where the global gauge group is $\overline{\text{SU}(1,1)} \otimes \text{SU}(1,1) \sim \text{SO}(2,2)$,[13] may be associated with the "orbital" degrees of freedom of a one-dimensional object, one expects that when the global gauge group is changed from SO(2, 2) to SO(N, 2), the corresponding quantities x^μ, $\mu = 0, \ldots, N-1$, retain their significance and could be interpreted as the orbital degrees of freedom of N-dimensionally extended objects. Thus the space part of $x^\mu = (x^0, \vec{x})$ is in this way related to the "dimension" of the hadronic matter or to the number of internal orbital parameters necessary to describe a hadron.

III. n-POINT FUNCTIONS ON THE NONDEGENERATE REPRESENTATIONS OF SU(2,2)

We now consider a class of n-point functions which can be constructed on the manifold (x^μ, z) discussed in the Appendix. One again starts with the n-point functional

$$A_n = \int \left[\prod_{i=1}^{n} d\mu(x_i, z_i)\phi^{d_i\beta_i m_i}(x_i, z_i)\right] K(x_1, \ldots, x_n;\ z_1, \ldots, z_n,\ \bar{z}_1, \ldots, \bar{z}_n; \{d_i\beta_i m_i\}). \tag{3.1}$$

The functions

$$\phi^{d\beta m}(x, z) \equiv \phi^{d\beta m}(x^0, x^1, x^2, x^3, z, \bar{z}) \tag{3.2}$$

transform according to the irreducible representations $|d\beta m\rangle$ of SU(2, 2). An essential complication that one has to deal with in this case is the explicit dependence of the quantities z_i on the Minkowski coordinates x_i^μ. This is clear from (A35) and (A36). Thus, except under the subgroup of Weyl transformations, the differences of the form $z_i(x_i) - z_j(x_j)$ do not transform into differences of the same form. As can be seen from (A34) and (A35) the multipliers which arise from the x dependence and the z dependence of the functions $\phi^{d\beta m}(x, z)$ after an SU(2, 2) transformation are not of the same form. It then follows that the kernel of (3.1), which must compensate for such multipliers to keep the n-point function invariant, can be written in the form

$$K(x, z) = K_x(x_1, \ldots, x_n) \times K_z(z_1 x_1, \ldots, z_n x_n;\ \bar{z}_1 x_1, \ldots, \bar{z}_n x_n). \tag{3.3}$$

The factor K_x can be determined in exactly the same way as for the degenerate series and is, in fact, given by (2.9). To find the factor K_z, we follow the method of Domokos and Kövesi-Domokos.[10] For $m_i = 0$, the quantities which have the correct transformation properties to compensate for the multipliers arising from (A44) turn out to be of the form

$$F(z_i, x_i^\mu;\ z_j, x_j^\mu) = z_i(x_i^\mu)(X_i - X_j)z_j^\dagger(x_j^\mu) = z_i(x_i^\mu)\sigma^\mu(x_i^\mu - x_j^\mu)z_j^\dagger(x_j^\mu), \tag{3.4}$$

where $\sigma^\mu = (I, \vec{\sigma})$, and z and x^μ belong to the manifold (x^μ, z). Under SU(2, 2), F transforms as follows:

$$F(z_i', x_i'^\mu;\ z_j', x_j'^\mu) = [h_{12}(x)z_i + h_{22}(x)]^{-1}[\bar{h}_{12}(y)\bar{z}_j + h_{22}(y)]^{-1} F(z_i, x_i^\mu;\ z_j, x_j^\mu); \tag{3.5}$$

since these expressions are not symmetrized with respect to i and j, we consider instead

$$\Phi(z_i, x_i^\mu;\ z_j, x_j^\mu) = F(z_i, x_i^\mu;\ z_j, x_j^\mu)F(z_j, x_j^\mu;\ z_i, x_i^\mu), \tag{3.6}$$

which transform according to

$$\Phi(z_i', x_i'^\mu;\ z_j', x_j'^\mu) = |h_{12}(x_i)z_i(x_i)+h_{22}(x_i)|^{-2}\,|h_{12}(x_j)z_j(x_j)+h_{22}(x_j)|^{-2}\,\Phi(z_i, x_i^\mu;\ z_j, x_j^\mu). \tag{3.7}$$

These functions can now be used to define an invariant-volume element

$$\prod_{i=1}^{n} d\mu(x_i^\mu, z_i) = \left(\prod_{i=1}^{n} d^4x_i\, d^2z_i\right)\left[\prod_{j=1}^{n}(x_j - x_{j+1})^2\Phi(z_j, x_j^\mu;\ z_{j+1}, x_{j+1})\right]^{-1}. \tag{3.8}$$

It is now a straightforward matter to show that the n-point function (3.1) must have the form

$$A_n = \int\left[\prod_{i=1}^{n} d\mu(x_i, z_i)\,\phi^{d_i\beta_i m_i}(x_i, z_i)\right]\left\{\prod_{\substack{i,j=1\\ i\neq j}}^{n}[(x_i^\mu - x_j^\mu)]^{-d_{ij}}\right\}C_n(x)\left\{\prod_{\substack{i,j=1\\ i\neq j}}^{n}[\Phi(z_i, x_i^\mu;\ z_j, x_j^\mu)]^{-\beta_{ij}}\right\}C_n(z, x), \tag{3.9}$$

where $m_i = 0$ for all i, β_{ij} satisfy the constraint

$$\sum \beta_{ij} = \beta_i\ , \tag{3.10}$$

and $d\mu(x_i, z_i)$ is given by (3.8). The quantities $C_n(x)$ and $C_n(z, x)$ are SU(2, 2) invariants, $C_n(x)$ being given by (2.10) and (2.11) and $C_n(z, x)$ being a function of the cross ratios constructed from the quantities $\Phi(z_i, x_i^\mu;\ z_j, x_j^\mu)$. A prototype of such cross ratios is[10]

$$\frac{\Phi(z_1, x_1^\mu;\ z_2, x_2^\mu)\Phi(z_3, x_3^\mu;\ z_4, x_4^\mu)}{\Phi(z_1, x_1^\mu;\ z_3, x_3^\mu)\Phi(z_2, x_2^\mu;\ z_4, x_4^\mu)}\ . \tag{3.11}$$

Once again the n-point functions (3.9) are independent of any particular interpretation of the underlying dynamics. One may try, however, to give a physical interpretation to these amplitudes along the lines discussed in Sec. II. In particular, since $z(x)$ transforms as a local spinor under Lorentz transformations, it would be an attractive possibility to associate it with the "spin" degrees of freedom in the same way as x^μ is associated with orbital degrees of freedom. The n-point function (3.9) could then be thought to arise from a model of hadrons as three dimensionally extended objects in which the intrinsic spins of its constituents are nontrivially coupled to their orbital motion.[14] Such an interpretation can be made more concrete by using an operator formalism which will be discussed elsewhere.

IV. SIMPLEST POSSIBLE AMPLITUDES

We mentioned in Sec. II that the orbital amplitudes (2.15) are the analogs of the Bose-type SU(1, 1)⊗SU(1, 1)-invariant dual models in the present formalism. To make this analogy more explicit, we consider a *special case* of the n-point functions of the type (2.15) in which the wave functions $\phi^{d_i}(x_i)$ have $d_i = 0$, for all i, and thus transform as *scalars* under SO(N, 2). It follows that the factor $\prod_{i\neq j}[(x_i - x_j)^2]^{-d_{ij}}$ in (2.15) is also invariant under SO(N, 2). It must therefore be expressible in terms of the anharmonic ratios of the type (2.11). In particular, the 4-point function will have the form

$$B_4^N = \int d^N v\left[\prod_{i=1}^{4}\phi^0(x_i)\right]\left[\frac{(x_1 - x_2)^2(x_3 - x_4)^2}{(x_1 - x_3)^2(x_2 - x_4)^2}\right]^{-d_{12}}\left[\frac{(x_1 - x_4)^2(x_2 - x_3)^2}{(x_1 - x_3)^2(x_2 - x_4)^2}\right]^{-d_{23}} C_4^N(x)\ . \tag{4.1}$$

We now evaluate this expression for the case where $\phi^0(x_i)$ belong to the identity representation and are therefore constants. For $N = 1$ and 2 these are the Minkowski-space analogs of the well-known amplitudes.[15] Although evaluation for arbitrary N does not present any difficulty, we consider the case $N = 4$ for illustration. Since the group SU(2, 2) acts *transitively* on the Minkowski space, we can follow the Koba-Nielson prescription[15] to fix any three points in (4.1) and isolate a divergent but SU(2, 2)-invariant factor. This method of isolating divergent factors is of course not limited to the 4-point functions or the special case we are considering. More general cases can be dealt with in a similar manner.

It is convenient to fix the three points as follows: $x_1^\mu \to I^\mu$, $x_2^\mu \to \infty$, $x_3^\mu \to 0$. After separating the divergent factor and setting $C_4^4(x) = 1$, one is left with

$$B_4 \equiv B_4^4 = \int d^4x\,(x^2)^{-d_{12}-2}[(I^\mu - x)^2]^{-d_{23}-2}\ . \tag{4.2}$$

The quantity I^μ is a *non-null* but otherwise arbitrary unit 4-vector. Because of the SU(2, 2) invariance of B_4, it should not matter whether I^μ is timelike or spacelike. To ensure this and to make B_4

well defined, one must specify the manner in which the singularities in (4.2) are to be handled. To this end we shall make the replacements[16]

$$x^2 \to x^2 + i\epsilon\,,\qquad (I-x)^2 \to (I-x)^2 + i\epsilon\,. \tag{4.3}$$

With this prescription we can make use of the integral representation

$$(x^2+i\epsilon)^{-\alpha} = \frac{(-i)^{-\alpha}}{\Gamma(\alpha)}\int_0^\infty s^{\alpha-1}\, e^{+is(x+i\epsilon)}\,ds \tag{4.4}$$

to evaluate B_4. It is a straightforward matter to show that, aside from singularity-free multipliers, it has the form

$$B_4 = \frac{\Gamma(-d_{12})\Gamma(-d_{23})\Gamma(d_{12}+d_{23}+2)}{\Gamma(d_{12}+2)\Gamma(d_{23}+2)\Gamma(-d_{12}-d_{23})}\,. \tag{4.5}$$

If we make, among others, the identification

$$d_{12} = d(s) - 1\,,\qquad d_{23} = d(t) - 1\,, \tag{4.6}$$

we can write (4.5) in the form

$$B_4^{(1)} = \frac{\Gamma(1-d(s))\Gamma(1-d(t))\Gamma(d(s)+d(t))}{\Gamma(1+d(s))\Gamma(1+d(t))\Gamma(2-d(s)-d(t))}\,. \tag{4.7}$$

With the constraint

$$d(s)+d(t)+d(u) = 1\,, \tag{4.8}$$

(4.7) can be written in the symmetric form

$$B_4^{(1)} = \frac{\Gamma(1-d(s))\Gamma(1-d(t))\Gamma(1-d(u))}{\Gamma(1+d(s))\Gamma(1+d(t))\Gamma(1+d(u))}\,. \tag{4.9}$$

If instead of (4.6) we make the identification

$$d_{12} = \tfrac{1}{2}d(s)\,,\quad d_{23} = \tfrac{1}{2}d(t)\,, \tag{4.10}$$

the amplitude (4.5) will reduce to

$$B_4^{(2)} = \frac{\Gamma(-\frac{1}{2}d(s))\Gamma(\frac{1}{2}d(t))\Gamma(-\frac{1}{2}d(u))}{\Gamma(2+\frac{1}{2}d(s))\Gamma(2+\frac{1}{2}d(t))\Gamma(2+\frac{1}{2}d(u))}\,, \tag{4.11}$$

where

$$d(s)+d(t)+d(u) = -4\,. \tag{4.12}$$

These special cases of our orbital amplitudes are thus the analogs of the Virasoro-Shapiro amplitudes with arbitrary intercept.[8,9,17]

The expressions (4.11) and (4.12) agree with the results of Sommerfield[18] and of Brower and Goddard,[18] who start with an ansatz which could be obtained from (2.15) by setting $\phi^{d_i}(x_i) = C_n^N(x) = 1$ and going to the Euclidean region. This is understandable since our $i\epsilon$ prescription makes it possible to perform a Wick rotation where the amplitudes coincide. This agreement does not mean, however, that it is purely a matter of taste whether, e.g., one takes SO(2, 2) or SO(3, 1) as the underlying group structure. The dual models of interest from the theoretical point of view are those which are endowed with a Virasoro algebra.[19] It is well known that SO(2, 2) possesses *analytic* representations which allow for the additional (gauge) freedom of analytic mappings.[20] The generators of these transformations are precisely the Virasoro algebra.[1,21] The group SO(3, 1), on the other hand, does *not* possess analytic representations. It is only when one puts the SO(3, 1) wave functions $\phi^{\sigma M}(z,\bar{z})$,[8] the very causes of nonanalyticity, equal to constants that the remaining amplitude becomes analytic. This is because in this particular case the dependence on z and $\bar{z}$ completely decouple, and transformations can be applied to z ($\bar{z}$) without affecting $\bar{z}$ (z). The underlying group structure thus becomes effectively $SL(2R)\otimes SL(2R)$.

V. DISCUSSION OF THE RESULTS

The main purpose of this paper has been to study the possible forms of dual models with an underlying SU(2, 2) ~ SO(4, 2) global gauge symmetry. As a motivation we have noted that if, as seems to be the case, one considers SO(2, 2) as the global gauge symmetry of the one-dimensional hadronic matter, it would be natural to consider SO(4, 2) as the corresponding symmetry of a three-dimensional hadronic matter. The n-point functions discussed in Secs. II and III are generally dual once one specifies the manner in which the singularities in the Minkowski space are to be handled.

We have chosen to work within a c-number formalism mainly to find out the type of amplitudes one expects to obtain. A more consistent operator formalism is possible, however, and will be discussed elsewhere. Our c-number approach is clearly not without built-in limitations. For example, the dependence of the n-point functions on the external momenta is not determined by SU(2, 2) invariance and is introduced indirectly through the Casimir operators. Also the nature of the singularities is determined by the channel trajectories $d_{ij}(s)$ and $\beta_{ij}(s)$, the specific forms of which must be assumed separately. Some general features emerge, however, independently of such assumptions: Whatever the nature of the operator formalism, the resulting n-point functions would have the general characteristics of those discussed in Secs. II and III.

We have also discussed a class of n-point functions invariant under SO(N, 2). A special case of these, i.e., when $\phi^{d_i}(x_i)$ = constant and $C_n^N = 1$, are related to those of Ref. 18 by a Wick rotation. If it turns out that the intercept moves up with increasing N if one wishes to impose the Virasoro conditions, it is clear that an important role would have to be played by the functions $\phi^{d_i}(x_i)$ and $C_n^N(x)$ (which were left out in the special case considered) if one is to obtain models with more realistic intercepts. Should such attempts fail, it would certainly be tempting to regard this as favoring the one-dimensional nature of hadronic matter. One would then have to look elsewhere for the solution of the problems mentioned in the Introduction.

Finally, we note that the mass spectra in the models we have discussed manifest themselves in the d-s plane, where d is the eigenvalue of one of the Casimir operators of SU(2, 2), which specifies the dimension of a given representation. When restricted to Weyl group, d specifies the eigenvalues of the dilatation operator. An operator of this type also plays an essential role in field theories on hyperboloids.[22] It would be interesting to see whether there is a connection between such attempts and ours.

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APPENDIX: SUMMARY OF THE PROPERTIES OF THE GROUP SU(2,2) AND SOME OF ITS REPRESENTATIONS[24,25,26]

1. The Group SU(2,2) and Its Subgroups

The group SU(2, 2) is the group of unitary unimodular 4×4 matrices which leave the bilinear form

$$ZDZ^\dagger$$

invariant, where Z is the complex 4-vector

$$Z=[z_1, z_2, z_3, z_4]. \quad \text{(A1)}$$

The metric D is given by

$$D=\begin{bmatrix} 0 & iI \\ -iI & 0 \end{bmatrix}, \quad \text{(A2)}$$

where I is the unit 2×2 matrix. The invariance of the bilinear form implies that for every element $g \in$ SU(2, 2) we must have

$$Z'DZ'^\dagger = ZgDg^\dagger Z^\dagger \equiv ZDZ^\dagger$$

or

$$gDg^\dagger = D\,. \quad \text{(A3)}$$

We write g in the form

$$g=\begin{bmatrix} A & B \\ C & D \end{bmatrix}, \quad \text{(A4)}$$

where A, B, C, D are all 2×2 matrices. The constraint (A3) requires that

$$AC^\dagger - CA^\dagger = 0\,,\quad AD^\dagger - CB^\dagger = I\,,\quad BD^\dagger - DB^\dagger = 0\,. \quad \text{(A5)}$$

The submatrices are further constrained by the unimodularity condition

$$\det(g)=1. \quad \text{(A6)}$$

Next, consider a few subgroups of G:

(1) The subgroup

$$X \ni g_x = \begin{bmatrix} I & 0 \\ X & I \end{bmatrix} \quad \text{(A7)}$$

Clearly,

$$[g_x, g_{x'}] = 0 \text{ for every } x \text{ and } x'. \quad \text{(A8)}$$

From (A5)

$$X^\dagger = X\,, \quad \text{(A9)}$$

so that X is Hermitian:

$$X=\begin{bmatrix} x_{11} & x_{12} \\ x_{12}^\dagger & x_{22} \end{bmatrix}, \qquad \det(X) = \text{real}. \quad \text{(A10)}$$

(2) The subgroup

$$K \ni g_k = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix}. \quad \text{(A11)}$$

In this case the conditions (A5) and (A6) reduce to

$$A = D^{\dagger -1} = D^{-1\dagger}$$
$$BD^\dagger - DB^\dagger = 0 \quad \text{(A12)}$$
$$\det(g_k) = \det(AD) = 1\,.$$

Thus

$$g_k = \begin{bmatrix} d^{-1\dagger} & B \\ 0 & d \end{bmatrix}, \quad \text{(A13)}$$

where

$$d \in \mathrm{GL}(2, C)/\mathrm{U}(1) \equiv \text{Weyl group}. \quad \text{(A14)}$$

(3) The subgroup of diagonal matrices

$$D \ni g_d = \begin{bmatrix} d^{-1\dagger} & 0 \\ 0 & d \end{bmatrix}, \quad \text{(A15)}$$

where d is given by (A14).

(4) The subgroup

$$\Lambda \ni g_\Lambda = \begin{bmatrix} d_\lambda^{-1\dagger} & B_\lambda \\ 0 & d_\lambda \end{bmatrix}, \tag{A16}$$

where

$$d_\lambda = \begin{bmatrix} \lambda^{-1}\Delta & \mu \\ 0 & \lambda \end{bmatrix}. \tag{A17}$$

(5) The subgroup

$$Y \ni g_y = \begin{bmatrix} d_z^{-1\dagger} & 0 \\ x_z & d_z \end{bmatrix}, \tag{A18}$$

where

$$d_z = \begin{bmatrix} 1 & 0 \\ z & 1 \end{bmatrix} \tag{A19}$$

and $X_z = d_z X$, X being given by (A10). Thus

$$g_y = g_{d_z} g_x \ . \tag{A20}$$

(6) The subgroup

$$B \ni g_B = \begin{bmatrix} 1 & B \\ 0 & 1 \end{bmatrix}. \tag{A21}$$

Clearly

$$[g_{B'}, g_B] = 0 \quad \text{for every } B \text{ and } B'. \tag{A22}$$

We note that every element $g_B \in B$ can be written in the form

$$g_B = I g_x I \ , \tag{A23}$$

where I is the inversion matrix

$$I = \begin{bmatrix} 0 & i\sigma^2 \\ -i\sigma^2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}. \tag{A24}$$

2. The Decomposition of the Elements g SU(2,2) and the Poincaré Subgroup

It is easy to verify that any element $g \in$ SU(2, 2) can be written uniquely in the forms

$$g = g_k g_x = g_\Lambda g_y \ , \tag{A25}$$

where g_x, g_k, g_Λ, and g_y are given, respectively, by (A7), (A11), (A16), and (A18). Because of these decompositions, a transformation $g \in$ SU(2, 2) defines a transformation on the submanifolds X and Y. That is,

$$g\colon\ x \to x' \text{ such that } g_x g = g_k g_{x'}\ , \tag{A26}$$

and

$$g\colon\ y \to y' \text{ such that } g_y g = g_\Lambda g_{y'}\ . \tag{A27}$$

The transformation $x \to x'$ is a realization of the group SU(2, 2) in the coset space SU(2, 2)/K, and the transformation $y \to y'$ is a realization of the group SU(2, 2) on the coset space SU(2, 2)/Λ.

We now consider the Poincaré subgroup of SU(2, 2) in more detail. By (A8) the 4-parameter subgroup X is Abelian and can be put in 1-1 correspondence with the translation subgroup T_4 of the Poincaré group as follows:

$$\begin{aligned} x_{11} &= x^0 + x^3, \\ x_{12} &= x^1 - ix^2, \\ x_{21} &= x^1 + ix^2, \\ x_{22} &= x^0 - x^3, \end{aligned} \tag{A28}$$

where

$$x^\mu = (x^0, x^1, x^2, x^3) \tag{A29}$$

specifies a point in the Minkowski space. Thus

$$X = \begin{bmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{bmatrix} \tag{A30}$$

and

$$\det(X) = x^\mu x_\mu \ . \tag{A31}$$

Under the action of the block diagonal matrices g_d given by (A15), the manifold x transforms according to

$$X' = d^{-1} X d^{-1\dagger} \ . \tag{A32}$$

When the matrices d are restricted to SL(2, c), this is just the transformation law of a 4-vector x^μ under SL(2, c). It is thus clear that the Poincaré subgroup of SU(2, 2) can be realized by matrices g_x and g_d, $d \in$ SL(2, c). In this notation, the familiar product law of the Poincaré group may be written as

$$(d_1, X_1)\,(d_2, X_2) = (d_1 d_2, X_2 + d_2^{-1} X_1 d_2^{-1\dagger}) \ . \tag{A33}$$

The remaining transformations of the group SU(2, 2) are scale transformations which are a subgroup of the diagonal matrices $g_d \in D$, and special conformal transformations $g_B \in B$ given by (A21). By (A23) and (A24) any special conformal transformation can be obtained by an inversion followed by a translation and another inversion. Therefore, conformal transformations may be obtained by combining Poincaré transformations with inversions and scale transformations.

It is useful, for later reference, to have explicit expressions for the transformations $x \to x'$ and $z \to z'$ which follow from (A26) and (A27). Writing $g \in$ SU(2, 2) in the form given by (A4), one finds

$$X' = (XB + D)^{-1}(XA + C) , \tag{A34}$$

where x is given by (A10). "Symbolically"

$$X' = \frac{XA + C}{XB + D} .$$

It is also easy to show that

$$z' = \frac{h_{11}z + h_{21}}{h_{12}z + h_{22}} , \tag{A35}$$

where h_{ij} are elements of the matrix

$$h = XB + D \in \mathrm{GL}(2, c)/\mathrm{U}(1) . \tag{A36}$$

From the physical point of view, it is important to note that because of the dependence of h on x, the variable z is, in general, a function of x, i.e., $z = z(x)$. Under the subgroup of Weyl transformations, $h = D$, and there is no mixing of the x and z coordinates. Thus if z is independent of x to start with, it will not acquire x dependence under Weyl transformations.

The invariant volume elements for the subgroups X, Z, and Y are, respectively,

$$\begin{aligned} d\mu(x) &= d^4x \\ &= dx^0 dx^1 dx^2 dx^3 , \\ d\mu(z) &= d^2z , \\ d\mu(y) &= d\mu(x) d\mu(z) . \end{aligned} \tag{A37}$$

Under an SU(2, 2) transformation these measures transform as follows:

$$d\mu(x') = |\Delta|^{-4} d\mu(x) , \tag{A38}$$

$$d\mu(y') = |\Delta|^{-2} |h_{12}z + h_{22}|^{-4} d\mu(y) , \tag{A39}$$

$$\begin{aligned} \Delta &= \det(h) \\ &= \det(XB + D). \end{aligned} \tag{A40}$$

3. Principal Series of Unitary Irreducible Representations of SU(2,2) and Their Analytic Continuation

According to Graev, the representations of the principal series of SU(2, 2) are induced by the representations of a Cartan subgroup. Since SU(2, 2) has three distinct nonequivalent Cartan subgroups, there are three *nondegenerate* principal series of unitary irreducible representations of SU(2, 2), which he labels as d_0, d_1, and d_2. Of these, d_0 is discrete, and d_1 and d_2 are continuous. We summarize below the properties of the *nondegenerate* representations obtained by analytic continuation from d_2.

Let H_x be the space of all c^∞ functions of $f(g_x) \equiv f(x) = f(x^0, x^1, x^2, x^3)$, $g_x \in X$, square integrable with respect to the invariant measure $d\mu(x)$. Also let $\hat{H}_z$ be the space of functions $f(d_z) = f(z)$ upon which an irreducible representation $|d\beta m\rangle$ of the Weyl group $W = [\mathrm{SL}(2, c) \times \text{dilatations}]$ is realized by operators $\hat{T}_d$, where m is an integer or half integer, and d and β are in general complex numbers. For obvious reasons d is called the *dimension* of the representation. Then the representation $\phi^{d\beta m}(x,z) \equiv |d\beta m, x^\mu, z\rangle$ of SU(2, 2) is realized on the tensor product space:

$$\mathcal{H}_2 = \hat{H}_z \otimes H_x . \tag{A41}$$

The action of the operators T_g, $g \in$ SU(2, 2), on the functions $\phi^{d\beta m}(x,z)$ of $\mathcal{H}_2$ is as follows:

$$T_g : \phi^{d\beta m}(x,z) = |\Delta|^{-2} \hat{T}_d \, \phi^{d\beta m}(x',z) , \tag{A42}$$

where Δ is given by (A40) and x' by (A34). The action of $\hat{T}_d$ on functions belonging to $\hat{H}_z$ is given by

$$\begin{aligned} \hat{T}_d : f^{d\beta m}(z) = {} & |\Delta|^{d+2} (\operatorname{sgn}\Delta)^\epsilon \, (h_{12}z + h_{22})^{\beta+m} \\ & \times (\bar{h}_{12}\bar{z} + \bar{h}_{22})^{\beta-m} f^{d\beta m}(z') , \end{aligned} \tag{A43}$$

where ϵ determines the parity of the representation, and z' is given by (A35). It is easy to check that the tensor product of spaces $\hat{H}_z$ and H_x is identical to the space of functions $H_y = \mathrm{SU}(2, 2)/\Lambda \equiv Y$. The action of SU(2, 2) on $\mathcal{H}_2 \equiv H_y$ is given by

$$\begin{aligned} T_g : \phi^{d\beta m}(x,z) = {} & |\Delta|^d (\operatorname{sgn}\Delta)^\epsilon \, (h_{12}z + h_{22})^{\beta+m} \\ & \times (\bar{h}_{12}\bar{z} + \bar{h}_{22})^{\beta-m} \phi^{d\beta m}(x',z') \\ \equiv {} & |\Delta|^d (\operatorname{sgn}\Delta)^\epsilon \, \alpha(h,z) \, \phi^{d\beta m}(x',z) , \end{aligned} \tag{A44}$$

where again x' and z' are given by (A34) and (A35).

The specification of vectors transforming according to a given representation requires, in general, nine labels, three of which are the eigenvalues of the Casimir operators, i.e., d, β, and m. In the realization (A44), the remaining labels are taken to be the 4-vector x^μ and the complex number z. In particular, the unitary representations of the principal series d_2 are realized according to (A44), where m = half odd integer, β is purely imaginary, and $d = -1 + i\rho$, where ρ is a real number.

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