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Unitary Multiperipheral Models. II*

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A new class of multipheripheral-like models is presented for which the full multiparticle S matrix is unitary. As in previous models of this type it is crucial for s-channel unitarity that one take into account the exchange of an arbitrary number of multiperipheral chains between the two leading particles. The new features of the present models are that multichain forces and some low subenergy effects are taken into account. As in earlier unitarity models, the elastic scattering amplitude has a dynamical branch cut in the angular momentum plane that plays a crucial role in enforcing the Froissart bound. It is shown that the qualitative features of these models will be present in a wide class of multiperipheral-like models.

I. INTRODUCTION

Recently a class of multiperipheral-like models was constructed for which the full multiparticle S matrix is unitary.[1,2] In this paper another class of such models is presented.

The basic idea of all of these models is the same. One starts with two very high energy particles which are assumed to propagate through the interaction region without making appreciable fractional changes in their energies or longitudinal momenta. These primary particles interact by exchanging multiperipheral-like chains from which secondary particles can be emitted or absorbed. It is crucial for s-channel unitarity that one take into account the exchange of an arbitrary number of chains.[1] The inclusion of multichain effects constitutes the main difference between these models and the standard multiperipheral model. Typical diagrams which contribute to the production amplitudes are shown in Fig. 1. Notice that a secondary emitted from one chain can either come off as a real particle or be reabsorbed on a second chain thereby giving rise to a force between the two chains.

The input in these models is the amplitude, W_n, for the production of n secondaries from a single chain. In the models discussed in I (MI), W_n was chosen so that only one particle was emitted from each vertex and so that the rapidity difference between any two particles on the same chain was large. A typical diagram contributing to W_n in MI is shown in Fig. 2(a). The wavy lines correspond to the exchange of either a fixed or moving pole in the angular momentum plane. In the present models (MII) vertices are considered from which pairs of particles can be emitted as shown in Fig. 2(b). The vertex functions will depend on the transverse momenta of the emitted particles and on their rapidity difference. For example, the vertex functions could correspond to the exchange of a secondary trajectory. The most important restriction on these functions is that the invariant mass of the pair be limited.

MII contains several features not present in MI. In Fig. 1(a) one sees that in MI only two chain forces arise from the emission and reabsorption of secondaries. On the other hand, from Fig. 1(b) it is clear that in MII a rather complicated class of multichain forces comes into play. Furthermore in MII one is making a start at taking into account low subenergy effects that were completely neglected in MI. In particular in MII it is possible for an arbitrary number of secondaries to be emitted with low relative subenergies.

The presence of multichain forces is of particular interest. In MI one of the most striking features of the elastic scattering amplitude is the existence of a dynamical branch cut in the angular momentum plane which is of an entirely different nature than the familiar Mandelstam cuts. This unitarity cut plays a crucial role in enforcing the Froissart bound. In MI the amplitude for the exchange of N chains has a pole in the angular momentum plane that moves to the right with N, as in $\frac{1}{2}N(N-1)$. This happens because for a set of N chains there are $\frac{1}{2}N(N-1)$ attractive two-chain potentials. There is no violation of the Froissart bound because all poles to the right of $l=1$ are on the unphysical sheet of the unitarity cut. In MII the presence of multichain forces drastically changes the dependence of the pole positions on N. Nevertheless, for most values of the input parameters there are still poles in the l plane arbitrarily far to the right of $l=1$. All such poles are again on the unphysical sheet of a dynamical branch cut. However, there is a restricted range of input parameters for which the multichain forces cancel the two-chain forces sufficiently so that no l-plane poles are to the right of $l=1$. In this case there is

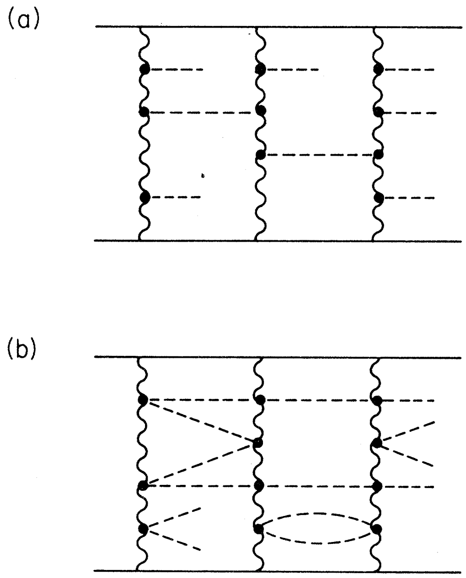

FIG. 1. (a) A typical diagram contributing to the production amplitude in MI. (b) A typical diagram contributing to the production amplitude in MII.

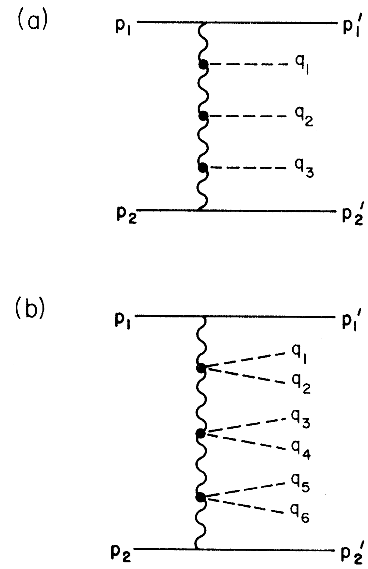

FIG. 2. (a) The amplitude W_3 in MI. (b) The amplitude W_6 in MII.

no dynamical branch cut, but there is also no pole in the neighborhood of $l=1$.

The outline of the paper is as follows. The model is constructed in Sec. II. In the solvable model considered in I, only a single mode of the secondary field was excited; however, in the present model an infinite number of modes come into play. Nevertheless it is still possible to diagonalize the S matrix. In Sec. III elastic scattering is studied in detail. For any choice of the input function, W_n, it is possible to construct a variety of unitary S matrices. These different S matrices correspond to different mechanisms for emission and absorption of the exchanged chains by the primary particles. It is shown that the position and nature of the l-plane singularities of the elastic amplitude do not depend on the details of the emission and absorption mechanism. In Sec. IV, production processes are discussed. It is found that some quantities, such as the average multiplicity, do depend on how the S operator is unitarized. The model makes no predictions for such quantities. Finally our results are discussed briefly in Sec. V. It is shown that a dynamical branch point in the angular momentum plane will exist in a far wider class of models than has been discussed here or in I.

II. CONSTRUCTION OF THE MODEL

A general prescription for constructing unitary models is given in I. In the present case two types of particles will be considered. Each state contains two primary particles whose momenta are labeled by p_1 and p_2. These leading particles cannot be created or destroyed and will be treated as distinguishable. The S matrix will be taken to be unity when acting on a state with other than two primaries. The secondary particles, whose momenta are labeled by q_i, can be created or destroyed and will be treated as identical particles. For simplicity, spin and internal quantum numbers will be neglected although there is no real difficulty in including them for either the primaries or the secondaries.

It is convenient to work in a coordinate system in which both primaries are initially moving along the z axis. A general four-vector, q, will then be written in terms of the transverse momentum, $\vec{q}$, which is a two-dimensional vector in the x-y plane, and the longitudinal rapidity, y, defined by

$$y=\tfrac{1}{2}\ln[(q_0+q_z)/(q_0-q_z)]. \qquad (1)$$

The input into the model is the amplitude, W_{2n}, for the production of $2n$ secondaries from a single chain. This amplitude is shown schematically in Fig. 2(b). W_{2n} will be constructed so that the primaries do not undergo significant fractional changes in their energies or longitudinal momenta. This eikonal approximation for the primaries is an oversimplification of the leading particle effect. Obviously one is leaving out important effects associated with the fragmentation of the primaries; however, it is expected that such effects will not play a significant role in determining the energy dependence of the amplitudes which will be the main subject of interest in this work. Because of the eikonal approximation for the primaries, the S matrix will be diagonal in their relative impact parameter, $\vec{B}$, and in their rapidity difference Y. $\vec{B}$ is just the transverse distance between the primaries. At high energies $Y \simeq \ln(s/m^2)$, where s is the square of the center-of-mass energy and m is the mass of the primaries.

By crossing symmetry, the amplitude W_{2n} also describes processes in which some or all of the secondaries are incoming. It is convenient to introduce a single operator which handles all processes described by W_{2n}. We first introduce creation and annihilation operators for the secondaries. In our normalization the commutation relations are

$$[a(\vec{q}, y), a^\dagger(\vec{q}\,', y')] = 2(2\pi)^3\delta^2(\vec{q} - \vec{q}\,')\delta(y - y'). \tag{2}$$

Since under crossing $\vec{q} \to -\vec{q}$ and $y \to y$, the required operator, $Z_n(Y, \vec{B})$ can be written in the form

$$Z_n(Y, \vec{B}) = \frac{1}{2s} \int \prod_{i=1}^{2n} dq_i\, W_{2n}(Y, \vec{B}; \vec{q}_1, y_1, \ldots, \vec{q}_{2n}, y_{2n}) \times : \prod_{i=1}^{2n} [a^\dagger(\vec{q}_i, y_i) + a(-\vec{q}_i, y_i)] : . \tag{3}$$

The invariant phase-space volume is given by $dq = d^2\vec{q}\,dy/2(2\pi)^3$. The creation and annihilation operators have been normal ordered so that secondaries are not absorbed on the same chain from which they are emitted. W_{2n} will be chosen so that Z_n is Hermitian. This merely requires that the two-dimensional Fourier transform of W_n with respect to $\vec{B}$ be real and invariant under a change of sign of all the transverse momenta.

The eikonal model suggests that the primary particles emit and absorb the chains independently. In this case the unitary S matrix is given by

$$S(Y, \vec{B}) = \exp[\,iZ(Y, \vec{B})], \tag{4}$$

where

$$Z(Y, \vec{B}) \equiv \sum_{n=0}^{\infty} Z_n(Y, \vec{B}). \tag{5}$$

More generally one can construct a unitary S matrix by writing

$$S(Y, \vec{B}) = \exp[\,if(Z)], \tag{6}$$

where f is any real function of Z. Different choices of f correspond to different mechanisms for emission and absorption of the chains by the primary particles. We shall concentrate on those aspects of the model that do not depend on the form of $f(Z)$. The more familiar momentum-space operators are obtained by taking the two-dimensional Fourier transform with respect to $\vec{B}$. The conjugate momentum is $\vec{\Delta} = \frac{1}{2}(\vec{p}_1' - \vec{p}_1) - \frac{1}{2}(\vec{p}_2' - \vec{p}_2)$.

Let us now consider the structure of the amplitude W_{2n}. At the ith vertex along the chain a pair of secondaries of momenta q_{2i-1} and q_{2i} is emitted. The rapidity of the pair, Y_i, is defined by

$$q_{2i-1} + q_{2i} = (M_i \cosh Y_i\,;\, M_i \sinh Y_i\,,\, \vec{q}_{2i-1} + \vec{q}_{2i}), \tag{7}$$

with

$$M_i^2 = (q_{2i-1} + q_{2i})^2 + (\vec{q}_{2i-1} + \vec{q}_{2i})^2. \tag{8}$$

A particularly simple class of models can be obtained by having fixed poles exchanged along the chains and by taking the rapidities of the pairs to be strongly ordered. In this case one can write W_{2n} in the form

$$\frac{1}{2s} W_{2n} = e^{-Y} f(\vec{B}) \prod_{i=0}^{n} e^{\alpha(Y_i - Y_{i+1})}\theta(Y_i - Y_{i+1}) \times \prod_{i=1}^{n} \tfrac{1}{2}G(\vec{q}_{2i-1}, \vec{q}_{2i}\,;\, y_{2i-1} - y_{2i}), \tag{9}$$

α is the spin of the input pole. In the center-of-mass system $Y_0 = -Y_{n+1} = \frac{1}{2}Y$. All correlations have been neglected except for those between secondaries emitted from the same vertex.

In order that the primaries not lose an appreciable fraction of their energy or longitudinal momenta, as has been assumed from the start, the rapidity variables must be confined to the range

$$-\tfrac{1}{2}Y \leq y_i \leq \tfrac{1}{2}Y \tag{10}$$

in the center-of-mass system.[3] W_{2n} will be taken to be zero when Eq. (10) does not hold.

Z_n can now be written in the form

$$Z_n(Y, \vec{B}) = \frac{1}{n!}\, e^{(\alpha-1)Y} f(\vec{B}) : V^n : , \tag{11}$$

where

$$V = \int dq_1 dq_2\, \tfrac{1}{2}G(\vec{q}_1, \vec{q}_2; y_1 - y_2) \times [\,a^\dagger(\vec{q}_1, y_1) + a(-\vec{q}_1, y_1)] \times [\,a^\dagger(\vec{q}_2, y_2) + a(-\vec{q}_2, y_2)], \tag{12}$$

so

$$Z(Y, \vec{B}) = e^{(\alpha-1)Y} f(B) : e^V : . \tag{13}$$

In high-energy hadron-hadron collisions the transverse momenta of the secondaries is sharply limited. As a result, correlations in transverse momenta may not play too strong a role in determining the energy dependence of the S matrix. In this spirit the vertex function, G, will be written in the separable form[4]

$$G(\vec{q}_1, \vec{q}_2; y_1 - y_2) = g(\vec{q}_1) g(\vec{q}_2) h(y_1 - y_2). \tag{14}$$

It is then convenient to introduce a new set of creation and annihilation operators

$$a(y) = (4\pi g^2)^{-1/2} \int \frac{d^2q}{(2\pi)^2} g(\vec{q}) a(\vec{q}, y), \tag{15}$$

where

$$g^2 = \int \frac{d^2q}{(2\pi)^2} |g(\vec{q})|^2, \tag{16}$$

has been chosen so that

$$[a(y), a^\dagger(y')] = \delta(y - y'). \tag{17}$$

Absorbing a factor of $g^2/4\pi$ into h, the operator V can be written in the form

$$V = \tfrac{1}{2} \int_{-Y/2}^{Y/2} dy_1 dy_2 h(y_1 - y_2) \times [a(y_1) + a^\dagger(y_1)][a(y_2) + a^\dagger(y_2)]. \tag{18}$$

The operator V, and therefore the S matrix, can be brought to diagonal form by expanding the creation and annihilation operators in a Fourier series on the interval $-\frac{1}{2}Y \leq y \leq \frac{1}{2}Y$. Defining

$$a_0^c = Y^{-1/2} \int_{-Y/2}^{Y/2} dy\, a(y), \tag{19}$$

$$a_n^c = (\tfrac{1}{2}Y)^{-1/2} \int_{-Y/2}^{Y/2} dy \cos(2\pi n y/Y)\, a(y), \quad n = 1, 2, \ldots . \tag{20}$$

$$a_n^s = (\tfrac{1}{2}Y)^{-1/2} \int_{-1/2}^{Y/2} dy \sin(2\pi n y/Y)\, a(y), \quad n = 1, 2, \ldots . \tag{21}$$

V can be written as

$$V = \tfrac{1}{2} h_0 (a_0^c + a_0^{c\dagger})^2 + \tfrac{1}{2} \sum_{n=1}^{\infty} h_n [(a_n^c + a_n^{c\dagger})^2 + (a_n^s + a_n^{s\dagger})^2], \tag{22}$$

with

$$h_n = \int_{-Y/2}^{Y/2} dy\, h(y) \cos(2\pi n y/Y). \tag{23}$$

In obtaining Eq. (22) use has been made of the fact that $h(y)$ is an even function of y. The notation can be simplified by writing

$$\begin{aligned} &\lambda_{2n} = h_n, \\ &b_{2n} = a_n^c, \quad n = 0, 1, 2, \ldots \\ &\lambda_{2n-1} = h_n, \\ &b_{2n-1} = a_n^s, \quad n = 1, 2, \ldots \end{aligned} \tag{24}$$

and

$$x_n = 2^{-1/2}(b_n + b_n^\dagger).$$

Then

$$V = \sum_{n=0}^{\infty} \lambda_n x_n^2. \tag{25}$$

Making use of the fact that

$$\begin{aligned} :\exp\tfrac{1}{2}\lambda(b + b^\dagger)^2: \; &= \pi^{-1/2} \int_{-\infty}^{\infty} dt\, e^{-t^2} : \exp[(2\lambda)^{1/2} t(b + b^\dagger)]: \\ &= \pi^{-1/2} \int_{-\infty}^{\infty} dt\, e^{-(1+\lambda)t^2} \exp[(2\lambda)^{1/2} t(b + b^\dagger)] \\ &= (1+\lambda)^{-1/2} \exp\tfrac{1}{2}[(b + b^\dagger)^2 \lambda/(1+\lambda)], \end{aligned} \tag{26}$$

one finds

$$:e^V: = D^{-1/2} \exp\left[\sum_{n=0}^{\infty} x_n^2 \lambda_n/(1+\lambda_n)\right], \tag{27}$$

with

$$D \equiv \prod_{n=0}^{\infty} (1 + \lambda_n) = \det(1 + h). \tag{28}$$

In the last step of Eq. (28) $h(y_1, y_2) = h(y_1 - y_2)$ is to be treated as an integral operator. Eqs. (4), (11), and (27) define a unitary S matrix which is diagonal in the coordinate representation of the creation and annihilation operators b_n and $b_n^\dagger$.

There is no problem in including vertices from which a single secondary is emitted or absorbed. If one takes such vertices to have the form $\eta' g(\vec{q})$ as in I, then one finds

$$:e^V: = D^{-1/2} \exp\left[\sum_{n=0}^{\infty} x_n^2 \lambda_n/(1+\lambda_n) + [\eta(2Y)^{1/2} x_0 - \tfrac{1}{2}\eta^2 Y]/(1+\lambda_0)\right], \tag{29}$$

with $\eta = \eta'(g^2/4\pi)^{1/2}$. Clearly single-particle vertices lead to the excitation of only one mode of the secondary field; however, the effective coupling constant for this mode is proportional to $Y^{1/2}$.

III. ELASTIC SCATTERING

In the present normalization the elastic scattering amplitude is given by

$$M(Y,\vec{\Delta}) = \int d^2B\, e^{-i\vec{\Delta}\cdot\vec{B}} M(Y,\vec{B})$$
$$= 2is \int d^2B\, e^{-i\vec{\Delta}\cdot\vec{B}}[1 - \langle 0|S(Y,\vec{B})|0\rangle]. \tag{30}$$

Here $\vec{\Delta} = \frac{1}{2}(\vec{p}_1' - \vec{p}_1) - \frac{1}{2}(\vec{p}_2' - \vec{p}_2)$ is the transverse momentum transfer of the primaries. At high energies $\vec{\Delta}^2 \simeq -t$. $|0\rangle$ is the state containing no secondaries. If the eigenstates of the coordinate operators x_n defined in Eq. (24) are denoted by $|x_0, x_1, x_2\rangle$, then

$$\langle x_0, x_1, x_2, \ldots |0\rangle = \prod_{n=0}^{\infty} \pi^{-1/4} e^{-x_n^2/2}. \tag{31}$$

The vertex function $h(y_1 - y_2)$ will be chosen so that the invariant mass of the emitted pair is limited. This means that h must fall off when $|y_1 - y_2|$ is larger than some correlation length L. At very high energies $Y \gg L$. One then sees from Eq. (23) that all h_n for which $n \ll Y/L$ are equal and that the h_n become negligible for $n \gg Y/L$. In order to orient our thinking let us first consider a particularly simple example which incorporates these features. Take

$$\lambda_n = \lambda, \quad n \leq 2Y/L$$
$$\lambda_n = 0, \quad n > 2Y/L, \tag{32}$$

which corresponds to

$$h(y_1 - y_2) \simeq \frac{\lambda}{\pi} \frac{\sin[2\pi(y_1 - y_2)/L]}{y_1 - y_2}. \tag{33}$$

Using Eq. (4) the elastic S-matrix element can be written in the form

$$\langle 0|S(Y,\vec{B})|0\rangle = \pi^{-M-1/2} \int_{-\infty}^{\infty} \prod_{n=0}^{2M} dx_n\, e^{-x_n^2} \exp\left[if(\vec{B})(1+\lambda)^{-1/2} e^{-cY} \exp\left(\Lambda \sum_0^{2M} x_n^2\right)\right]$$
$$= \frac{Y^{M+1/2}}{\Gamma(M+\frac{1}{2})} \int_0^{\infty} dt\, t^{M-1/2} e^{-Yt} \exp[if(\vec{B})(1+\lambda)^{-1/2} e^{-Y(c-\Lambda t)}], \tag{34}$$

where

$$c = 1 - \alpha + L^{-1}\ln(1+\lambda),$$
$$\Lambda = \lambda/(1+\lambda),$$
$$M = Y/L. \tag{35}$$

The amplitude for the exchange of N chains is

$$\frac{2is}{N!}\langle 0|Z^N(Y,\vec{B})|0\rangle = \frac{2is}{N!}[if(\vec{B})(1+\lambda)^{-1/2} e^{-Yc}]^N \frac{Y^{m+1/2}}{\Gamma(M+\frac{1}{2})} \int_0^{\infty} dt\, t^{M-1/2} e^{-Yt(1-N\Lambda)}$$
$$= \frac{2is}{N!}[if(\vec{B})(1+\lambda)^{-1/2}]^N (1+\lambda)[1-(N-1)\lambda]^{-1}$$
$$\times \exp(Y\{N(\alpha-1) - (N-1)L^{-1}\ln(1+\lambda) - L^{-1}\ln[1-(N-1)\lambda]\}). \tag{36}$$

Clearly the only l-plane singularity of this amplitude is a pole at

$$\alpha_N = 1 + N[\alpha - 1 - L^{-1}\ln(1+\lambda)] + L^{-1}\ln\{(1+\lambda)[1-(N-1)\lambda]^{-1}\}. \tag{37}$$

All of the poles in Eq. (37) are dynamical in origin except the input pole located at $\alpha_1 = \alpha$. Notice that

$$\alpha_N \underset{\lambda\to 0}{\sim} 1 + N(\alpha - 1) + \tfrac{1}{2}N(N-1)L^{-1}\lambda^2, \tag{38}$$

which aside from a redefinition of the coupling constant is just the result of the solvable model discussed in I. This is hardly surprising since in the weak coupling limit only two chain forces are important, so one has just replaced the exchange of a single secondary by the exchange of a pair.

For $\lambda > 0$ or $\lambda < -1 + e^{-L(1-\alpha)}$ there are poles in the angular momentum plane to the right of $l = 1$; however, none of these poles is on the physical sheet. To see this let us return to the integral representation for the S-matrix element given in Eq. (34) and start by considering the case $\lambda > 0$. It is convenient to break up the integral into two parts, one for $t \leq t_c = c/\Lambda$ and the other for $t \geq t_c$. For $t < t_c$

$$Z(t)\equiv f(\vec{B})(1+\lambda)^{-1/2}e^{-Y(c-\Lambda t)} \tag{39}$$

is small, so the integrand can be expanded in powers of $Z(t)$. For $t>t_c$, $Z(t)$ is large and no expansion is possible. Now the integrand of Eq. (36) is sharply peaked at $t_N=[L(1-\Lambda N)]^{-1}$. Any pole for which $t_N\le t_c$ will be present in the full S-matrix element; whereas a pole for which $t_N>t_c$ will be washed out by the rapid oscillation of the integrand of Eq. (34). More explicitly one can write

$$\begin{aligned}S_{\rm I}(Y,\vec{B})&=\frac{Y^{M+1/2}}{\Gamma(M+\frac{1}{2})}\int_{t_c}^{\infty}dt\,t^{M-1/2}e^{-Yt}e^{iZ(t)}\\&\simeq K_{\rm I}(\vec{B})\frac{Y^{M-1/2}}{\Gamma(M+\frac{1}{2})}t_c^{M-1/2}e^{-Yt_c}\\&\simeq(2\pi Lc/\Lambda)^{-1/2}K_{\rm I}(\vec{B})Y^{-1/2}\exp\{YL^{-1}[1+\ln(Lc/\Lambda)-Lc/\Lambda]\},\end{aligned} \tag{40}$$

and

$$\begin{aligned}S_{\rm II}(Y,\vec{B})&=\frac{Y^{M+1/2}}{\Gamma(M+\frac{1}{2})}\int_0^{t_c}dt\,t^{M-1/2}e^{-Yt}\sum_{N=0}^{\infty}\frac{[iZ(t)]^N}{N!}\\&\simeq\sum_{N=0}^{\infty}(1+\lambda)[1-(N-1)\lambda]^{-1}\frac{[if(\vec{B})(1+\lambda)^{-1/2}]^N}{N!}\exp[(\alpha_N-1)Y]\theta(t_c-t_N)\\&\quad+(2\pi Lc/\Lambda)^{-1/2}K_{\rm II}(\vec{B})Y^{-1/2}\exp\{YL^{-1}[1+\ln(Lc/\Lambda)-Lc/\Lambda]\}.\end{aligned} \tag{41}$$

Here $\alpha_0\equiv1$; $K_{\rm I}(\vec{B})$ and $K_{\rm II}(\vec{B})$ are functions of $\vec{B}$ and λ, but not of Y. Clearly the elastic scattering amplitude has poles in the angular momentum plane at $l=\alpha_N$, $N=1,2,\ldots,N_0$. N_0 is the largest integer such that $t_N\le t_c$, i.e., the largest integer such that $N\le\Lambda^{-1}-(cL)^{-1}$. In addition to the poles, the elastic scattering amplitude has a square-root branch point at $l=\alpha_c$,

$$\alpha_c=2+\ln(Lc/\Lambda)-Lc/\Lambda\,. \tag{42}$$

The poles at $l=\alpha_N$, $N>N_0$ are on the second sheet of the branch cut.

The l-plane structure of the elastic amplitude is illustrated in Fig. 3 for $\alpha<1$, $\lambda>0$. For small values of λ the branch point is far to the left in the l plane, and there are a large number of poles on the physical sheet. As λ is increased the dynamical poles move to the right but the branch point moves even faster. The Nth pole collides with the branch point when $t_N=t_c$. If λ is increased further the pole moves through the branch cut onto the unphysical sheet. α_c moves to the right as λ is increased until λ reaches λ_1, which is defined by the transcendental equation

$$\lambda_1L^{-1}=1-\alpha+L^{-1}\ln(1+\lambda_1)\,. \tag{43}$$

At $\lambda=\lambda_1$, $\alpha_c=\alpha$. If λ is increased further, α_c decreases. For $\lambda>\lambda_1$, the branch point is the only singularity on the physical sheet.

For $t_c>t_2$, the pole arising from the two chain diagrams is the leading dynamical singularity. In this case the total cross section has the asymptotic behavior

$$\sigma_T(Y)\underset{Y\to\infty}{\sim}K_2e^{(\alpha_2-1)Y}, \tag{44}$$

where the constant K_2 can be read off from Eq. (36). For $t_2>t_c$ the branch point is the leading dynamical singularity, and

$$\sigma_T(Y)\underset{Y\to\infty}{\sim}K_cY^{-1/2}e^{(\alpha_c-1)Y}. \tag{45}$$

Notice that for $\alpha\le1$, no l-plane singularity reaches one, so the total cross section always goes to zero at infinite energy.

Let us now imagine increasing α for a fixed, positive value of λ. The branch point reaches one when $Lc/\Lambda=1$, which means that

$$\alpha=1+L^{-1}[\ln(1+\lambda)-\Lambda^{-1}]>1\,. \tag{46}$$

If α is increased further then $c/\Lambda<L^{-1}$, and one

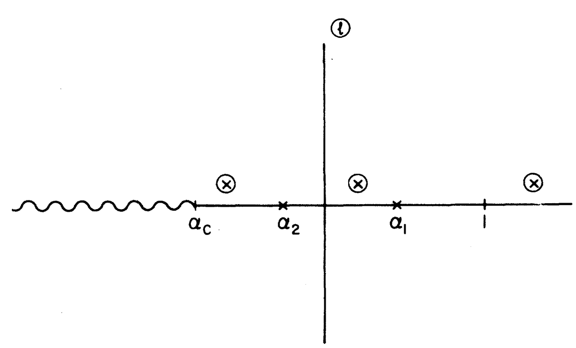

FIG. 3. The l-plane structure of the elastic amplitude for $\alpha<1$, $\lambda>0$. X denotes a pole on the physical sheet. ⊗ denotes a pole on the unphysical sheet.

sees from Eq. (34) that the S-matrix element goes to zero unless $f(\vec{B})$ is small. In particular taking $f(\vec{B})=e^{-B/R}$, one finds that

$$\langle 0|S(Y,\vec{B})|0\rangle \underset{Y\to\infty}{\longrightarrow} 0, \quad B<R_0$$
$$\langle 0|S(Y,\vec{B})|0\rangle \underset{Y\to\infty}{\longrightarrow} 1, \quad B>R_0, \tag{47}$$

with

$$R_0=RY(L^{-1}-c/\Lambda)\equiv R'Y, \tag{48}$$

which corresponds to scattering off a black disk of radius R_0. In the l plane the leading singularity of the elastic amplitude has the well-known form

$$M(l,\vec{\Delta})\simeq 2im^2R'^2[(l-1)^2+R'^2\vec{\Delta}^2]^{-3/2}. \tag{49}$$

The complex conjugate branch points enter the physical sheet through the square-root branch point when $\alpha_c=1$. In this case the total cross section has the asymptotic form

$$\sigma_T(Y) \underset{Y\to\infty}{\longrightarrow} 2\pi R_0^2. \tag{50}$$

If λ is now increased for fixed α, R_0 shrinks to zero and the complex conjugate branch points leave the physical sheet when cL/Λ again reaches one. Thus for sufficiently large λ, the branch point at $l=\alpha_c$ is the only singularity on the physical sheet and $\sigma_T(Y)$ always goes to zero at infinite energy. For $L_c/\Lambda\neq 1$, α_c is always less than one.

The behavior of the elastic scattering amplitude for $\lambda>0$ is qualitatively the same as in the solvable model discussed in I. However, the situation is different when λ becomes negative. Let us first consider the case $\alpha<1$. As λ approaches zero along the positive real axis the branch point, α_c, retreats to minus infinity. For small negative values of λ, $c>0$, and $\Lambda<0$. In this case the phase $Z(t)$ is small for all values of t. The integrand of Eq. (34) can now be expanded as a power series in $Z(t)$, and the integration performed term by term.[5] The elastic amplitude is given by the infinite series

$$M(Y,\vec{B})=2im^2\sum_{n=1}^{\infty}\frac{[if(\vec{B})(1+\lambda)^{-1/2}]^N}{N!}\times(1+\lambda)[1-(N-1)\lambda]^{-1}e^{\alpha_N Y}, \tag{51}$$

which is uniformly convergent in Y. The only l-plane singularities are the poles at $l=\alpha_N$, with α_N again given by Eq. (37). Notice that for $c<0$, α_N is a monotonically decreasing function of N, so the leading singularity is the input pole, $\alpha_1=\alpha$.

As λ decreases, c decreases, reaching zero when $\lambda=-1+e^{-L(1-\alpha)}$. As c decreases through zero the poles arising from the exchange of a large number of chains rapidly move from large negative values of l to large positive values of l. However, at $c=0$ the square-root branch point reenters the l plane at minus infinity. α_c is again given by Eq. (42) with c and Λ both negative. Once again all poles to the right of $l=1$ are on the unphysical sheet of the l plane. In the present case $Z(t)$ goes to zero for large t, and the troublesome poles are washed out by oscillations of the integrand for small values of t. In particular

$$S_{\rm I}(Y,\vec{B})=\frac{Y^{M+1/2}}{\Gamma(M+\frac{1}{2})}\int_{t_c}^{\infty}dt\,t^{M-1}e^{-Yt}e^{iZ(t)}$$
$$\simeq\sum_{N=0}^{\infty}(1+\lambda)[1-(N-1)\lambda]^{-1}\frac{[if(\vec{B})(1+\lambda)^{-1/2}]^N}{N!}e^{(\alpha_N-1)Y}\theta(t_N-t_c)+[2\pi Lc/\Lambda]^{-1/2}K_{\rm I}'(\vec{B})Y^{-1/2}e^{(\alpha_c-1)Y}, \tag{52}$$

and

$$S_{\rm II}(Y,\vec{B})=\frac{Y^{M+1/2}}{\Gamma(M+\frac{1}{2})}\int_0^{t_c}dt\,t^{M-1}e^{-Yt}e^{iZ(t)}$$
$$\simeq[2\pi Lc/\Lambda]^{-1/2}K_{\rm II}'(\vec{B})Y^{-1/2}e^{(\alpha_c-1)Y}. \tag{53}$$

Again $t_c=c/\Lambda$ and $t_N=[L(1-\Lambda N]^{-1}$.

For $c<0$ the behavior of the elastic amplitude as a function of λ and α is essentially the same as for the case $\lambda>0$, so it will not be cataloged. The important point is that there exists a range of input parameters ($c>0$, $\lambda<0$) for which the only singularities of the elastic amplitude are poles. For this range of parameters there is sufficient cancellation among the multichain forces so that α_N is a monotonically decreasing function of N. Notice that this "saturation" of forces occurs for only a limited range of input parameters, and when it does occur there are never any dynamical singularities near $l=1$.

The positions of the l-plane singularities listed above will be unchanged if one makes use of the more general definition of the S matrix given in Eq. (6). The only restrictions on $f(Z)$ are that it have a power-series expansion that converges inside the unit circle and that it have no singularities on the real axis. In that case for $\lambda > 0$ one can make a power-series expansion of $S_{II}(Y, \vec{B})$ analogous to the one given in Eq. (41). Clearly the positions of the poles will be unchanged although their residues will be altered. For $S_I(Y, \vec{B})$ one has an integral analogous to the first line of Eq. (40). If $f(Z)$ goes to infinity for large Z, then the integrand oscillates rapidly and one obtains a result of exactly the same form as Eq. (40). On the other hand if $f(Z)$ approaches a finite limit for large Z, the factor $\exp[if(\infty)]$ can be taken out of the integral and the Y dependence of S_I is again the same as in Eq. (40). As a result, the position and nature of the branch point, α_c, is unchanged. A similar result holds for the case of black-disk scattering.

Let us briefly consider more general forms for the vertex function, $h(y_1 - y_2)$. It is convenient to write the elastic S matrix element in the form

$$\langle 0|S(Y,\vec{B})|0\rangle = \int_{-\infty}^{\infty} dt\, e^{iZ(t)} F(t), \tag{54}$$

where

$$\begin{aligned} F(t) &= \int_{-\infty}^{\infty} \prod_{n=0} \pi^{-1/2} dx_n\, e^{-x_n^2} \delta\left(\sum_{n=0}^{\infty} x_n^2 \Lambda_n - t\right) \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dp\, e^{ipt} \prod_{n=0}^{\infty} [1 + ip\Lambda_n]^{-1/2} \end{aligned} \tag{55}$$

and

$$Z(t) = f(\vec{B}) e^{-cY+t}. \tag{56}$$

In the present case

$$c = 1 - \alpha + \frac{1}{2Y} \sum_{n=0}^{\infty} \ln(1 + \lambda_n). \tag{57}$$

c is a slowly varying function of Y, which goes to a constant as Y goes to infinity. This can be seen from the fact that

$$\begin{aligned} \sum_{n=0}^{\infty} \ln(1+\lambda_n) &= \ln[\det(1+h)] \\ &= \sum_{n=1}^{\infty} \frac{(-)^{N+1}}{N} \mathrm{tr}(h^N) \end{aligned} \tag{58}$$

and that

$$\begin{aligned} &\mathrm{tr}(h) = Yh(0), \\ &\mathrm{tr}(h^N) = Y\int_{-Y/2}^{Y/2} dy_1 \cdots dy_{N-1} h(y_1) h(y_1 - y_2) \cdots h(y_{N-1} - y_{N-1}) h(y_{N-1}). \end{aligned} \tag{59}$$

The amplitude for the exchange of N chains is now given by

$$2is\langle 0|Z(Y,\vec{B})^N|0\rangle = 2is[if(\vec{B})e^{-cY}]^N \prod_{n=0}^{\infty} [1 - N\Lambda_n]^{-1/2}, \tag{60}$$

where $\Lambda_n = \lambda_n/(1-\lambda_n)$. The leading l-plane singularity of this amplitude is a pole at $l = \alpha_N$.

$$\alpha_N = 1 + N(\alpha - 1) + \lim_{Y\to\infty}\left[(2Y)^{-1} \sum_{n=0}^{\infty} \ln\{(1+\lambda_n)[1-(N-1)\lambda_n]^{-1}\} - (2Y)^{-1}\sum_{n=0}^{\infty} \ln(1+\lambda_n)\right]. \tag{61}$$

That the limits in Eq. (61) are finite follows from the same reasoning as for c.

From Eq. (55) one sees that $F(t)$ vanishes for negative (positive) values of t if all of the Λ_n are positive (negative). As a result, if all of the λ_n are negative and small enough so that c is positive, the integrand of Eq. (54) can be expanded in powers of $Z(t)$ for all values of t. In this case the only l-plane singularities of any importance are the poles at $l = \alpha_N$. On the other hand if any of the λ_n are positive or if c is negative there will be branch cuts in the l plane associated with the rapid oscillation of the integrand of Eq. (54). The precise position of the branch points depends upon the details of the function $F(t)$; however, the l-plane structure of the S matrix is expected to be qualitatively the same as for the solvable model discussed above.[6] As long as $h(y_1 - y_2)$ is a smooth function which falls off for $|y_1 - y_2| > L$, its eigenvalues will be qualitatively the same as in the solvable model. Namely the λ_n will be approximately degenerate for $n < 2Y/L$ and will fall off rapidly for $n > 2Y/L$.

IV. PARTICLE PRODUCTION

The inclusive cross section for two incident primaries to produce one secondary plus anything else is given by

$$\frac{d\sigma}{d\vec{q}^{\,2}dy}=\frac{1}{2(2\pi)^3}\int d^2B\sum_{n=0}^{\infty}\int\prod_{i=1}^{n}dq_i\,\frac{1}{n!}\,|\langle\vec{q},y;\vec{q}_1,y_1;\cdots\vec{q}_n,y_n|S(Y,\vec{B})|0\rangle|^2$$
$$=\frac{1}{2(2\pi)^3}\int d^2B\,\langle 0|[S^\dagger(Y,\vec{B}),a^\dagger(\vec{q},y)][a(\vec{q},y),S(Y,\vec{B})]|0\rangle\,. \tag{62}$$

Using Eq. (6) for $S(Y,\vec{B})$ one has

$$[a(\vec{q},y),S(Y,\vec{B})]=if'(Z)S(Y,\vec{B})U\,, \tag{63}$$

where $f'(Z)$ means the derivative of $f(Z)$ with respect to its argument. U is given by

$$U=[a(\vec{q},y),Z]$$
$$=(4\pi/g^2)^{1/2}g(\vec{q})Z\left[Y^{-1/2}H_0(a_0^c+a_0^{c\dagger})+(\tfrac{1}{2}Y)^{-1/2}\sum_{n=1}^{\infty}H_n[(a_n^c+a_n^{c\dagger})\cos(2\pi ny/Y)+(a_n^s+a_n^{s\dagger})\sin(2\pi ny/Y)]\right]. \tag{64}$$

Here we have returned to the notation of Eqs. (19)–(23) and set $H_n=h_n/(1+h_n)$. Notice that U is independent of y, so the rapidity distribution is flat.

The inclusive cross section is particularly simple when $S(Y,\vec{B})=\exp[iZ]$. In this case $f'(z)=1$ and

$$\frac{d\sigma}{d^2q\,dy}=Kg^2(\vec{q})e^{(\alpha_2-1)Y}\,. \tag{65}$$

K is a slowly varying function of Y which goes to a constant as Y approaches infinity. In our solvable model

$$K=g^{-2}\lambda(1-\lambda)^{-2}(2L^{-1}+Y^{-1})(2\pi)^{-2}\int d^2B\,[f(\vec{B})]^2\,. \tag{66}$$

Equation (65) holds whether or not the elastic amplitude has the pole at $l=\alpha_2$ on the physical sheet. Equation (65) is just the result one would obtain in the multiperipheral model where all secondaries are produced from a single chain. In the present case the multichain effects have canceled. If the pole at $l=\alpha_2$ is the leading dynamical singularity of the elastic amplitude, then the total and inclusive cross sections have the same energy dependence and the average multiplicity $\bar{n}$, increases like $Y\simeq\ln(s/m^2)$. In this case the only important diagrams for particle production are those in which all the secondaries are come off the same chain. So, at high energies the model reproduces all of the results of the multiperipheral model for both inclusive and exclusive cross sections. If the square-root branch point is the leading dynamical singularity of the elastic amplitude then the average multiplicity grows like $Y^{3/2}e^{(\alpha_2-\alpha_c)Y}$.[7] $\bar{n}$ also increases like a power of s for the case of black-disk scattering. In these two cases multichain diagrams do contribute to the exclusive cross sections.

Multichain effects cancel out of the inclusive cross section only when one takes $S=\exp(iZ)$. For example, if one writes

$$S=(1+iZ)/(1-iZ)\,, \tag{67}$$

then

$$f'(Z)S=2/(1+Z^2)\,. \tag{68}$$

From Eqs. (62) and (63) one easily sees that the inclusive cross section now has the same energy dependence as the total cross section when either α_2 or α_c is the leading dynamical singularity of the elastic amplitude. In either case the average multiplicity grows like $\ln(s/m^2)$.

It is hardly surprising that the average multiplicity depends on the form of $f(z)$. Clearly

$$\bar{n}=\bar{n}_c\bar{N}\,, \tag{69}$$

where $\bar{n}_c$ is the average number of particles emitted by a single chain, and $\bar{N}$ is the average number of chains exchanged. Although $\bar{n}_c$ always grows like Y, $\bar{N}$ depends on the mechanism for emission and absorption of the chains by the primaries, in other words on $f(Z)$. Since there is no compelling reason for choosing any particle form for $f(Z)$ it seems best to concentrate on those quantities that do not depend on it, such as the positions of the l-plane singularities of the elastic amplitude. The model really does not make predictions for quantities such as $\bar{n}$ which depend strongly on $f(Z)$.

V. DISCUSSION

The point which should be emphasized is the simple mechanism by which the Froissart bound is enforced. This mechanism will come into play in a far wider class of models than has been discussed here or in I. For example, one might wish to build exact energy-momentum conservation into the operator $Z(Y, \vec{B})$ so that the primaries can be allowed to undergo finite fractional changes in their energies or longitudinal momenta; or one might wish to treat low-subenergy effects in a more sophisticated manner than has been done to date. In any case $Z(Y, \vec{B})$ will always be an Hermitian operator with a complete set of eigenvectors, $|a\rangle$

$$Z(Y,B)|a\rangle = a|a\rangle . \qquad (70)$$

If the elastic scattering amplitude has a pole in the angular momentum plane it must show up in one of the amplitudes $\langle 0|Z^N|0\rangle$, since these are the amplitudes that are given by a linear integral equation in the t channel. If the leading l-plane singularity of $\langle 0|Z^N|0\rangle$ is indeed a pole at $l=b$, then at high energies

$$\langle 0|Z^N|0\rangle = \int da\, a^N |\langle 0|a\rangle|^2 \sim s^b . \qquad (71)$$

There would appear to be a problem with the Froissart bound if b were greater than zero. However, if the S matrix is defined by Eq. (4), the full elastic S-matrix element is[8]

$$\langle 0|S|0\rangle = \int da\, e^{ia} |\langle 0|a\rangle|^2 . \qquad (72)$$

If b is positive, then for large s the important contribution to the integral in Eq. (71) must come from large values of a since $|\langle 0|a\rangle|^2$ is bounded by unity. However, there will be no important contributions to the full S-matrix element arising from large values of a due to the rapid oscillation of the integrand of Eq. (72). In the models studied to date, a branch point in the angular momentum plane is associated with the onset of this rapid oscillation. The troublesome poles show up on an unphysical sheet of this branch cut. There is no essential change if the more general definition of the S matrix given in Eq. (6) is used. Since $\exp[if(a)]$ is always bounded by one, the large region of integration can never give rise to an energy-increasing contribution to the elastic S-matrix element. In the simple models discussed here and in I, the position and nature of the l-plane singularities of the elastic amplitude does not depend on the function $f(a)$.

Notice that in order to construct unitary models it has been necessary to take into account the exchange of an arbitrary number of chains. When dynamical branch cuts are present, the expansion of the S matrix in powers of Z does not converge. As a result, models in which production takes place from only a finite number of chains can be quite misleading.

It is interesting that in the type of models that we have been discussing the elastic amplitude can never have an isolate pole at $l=1$. Since the position of a pole is a continuous function of the input parameters, an infinitesimal change in these parameters would move such a pole to the right of $l=1$ and therefore onto an unphysical sheet of the l plane. At best a pole at $l=1$ must be in the process of colliding with a dynamical branch point. So, if this type of model does give an adequate description of high-energy scattering processes, then multichain effects must play a crucial role in determining the nature of the Pomeranchuk singularity.

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[1]S. Auerbach, R. Aviv, R. Blankenbecler, and R. Sugar, Phys. Rev. Letters 29, 522 (1972); Phys. Rev. D 6, 2216 (1972). The second of these papers will be referred to as I.

[2]Different classes of unitary models have been discussed recently by G. Calucci, R. Jengo, and C. Rebbi, Nuovo Cimento 4A, 330 (1971); 6A, 601 (1971); R. Aviv, R. Blankenbecler, and R. Sugar, Phys. Rev. D 5, 3252 (1972); L. B. Redei, Nuovo Cimento 11A, 279 (1972).

[3]This point is discussed in detail in I. Strictly speaking the magnitude of the rapidity variable should be bounded by $\frac{1}{2}Y(1-\epsilon)$, but ϵ can be chosen to be arbitrarily small provided that the multiplicity does not grow like a power of the energy.

[4]In I models with and without correlations in the transverse momenta were studied. It was found that the main effect of taking W_n to be separable in the transverse momenta was to make the l-plane singularities of the elastic amplitude independent of the momentum transfer.

[5]Notice that if one attempts to do this for $\lambda > 0$, the resulting series for the elastic amplitude diverges. The difficulty is that one cannot interchange the order of summation and integration in the region $t_c \le t < \infty$, where the integrand oscillates rapidly.

[6]One difference is that in general there will be an infinite number of square-root branch points on the angular momentum plane.

[7]Energy conservation constrains $\alpha_2 - \alpha_c$ to be less than $\frac{1}{2}$. This point is discussed in detail in I.

[8]$\int da$ stands for an integral over all continuous eigenvalues of Z and a sum over all discrete ones.