Some Speculations on High-Energy Quantum Electrodynamics*

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Recent work on quantum electrodynamics is reviewed, some speculations about the theory are made, and some conceivable future experimental implications are discussed.

I. INTRODUCTION

In this note we will first summarize the reasoning which leads to the possibility of a finite quantum electrodynamics. The arguments presented will in essence be those already discussed in previous publications.[1-7] However, we hope to be able to present the argument in a form which makes clear the beautiful simplicity of quantum electrodynamics. We will then discuss the implications of a finite theory for high-energy electrodynamic experiments.

In Sec. II we will show that a self-consistent finite solution of the quantum electrodynamics of zero-*physical*-mass electrons exists if the square[8] of the coupling constant x is chosen to be a positive root x_0 of the equation

$$f(x_0)=0, \tag{1.1}$$

where $(x/2\pi)f(x)$ is the sum of the coefficients of the logarithmically divergent integrals for the vacuum polarization in massless electrodynamics with coupling constant x.

In Sec. III we assume the existence of a root x_0 of Eq. (1.1) and study the properties of massless electrodynamics with coupling constant x_0, which is a self-consistent finite theory. We show that in this theory scattering amplitudes involving only external photons vanish and we obtain the following simplified equation for x_0:

$$f_1(x_0)=0, \tag{1.2}$$

where $f_1(x)$ is the contribution to $f(x)$ arising from diagrams containing a single closed fermion loop. Examples of diagrams which contribute to $f_1(x)$ are depicted in Fig. 1.

In Sec. IV we show that the quantum electrodynamics of electrons with physical mass $m\neq 0$ can be finite under two possible circumstances:

(i) The bare fine-structure constant α_0 is taken equal to x_0 [α_0 is determined from the high-momentum behavior of the photon propagator $D_{\mu\nu}(k)$].

(ii) The physical fine-structure constant α is taken equal to x_0 [α is determined from the behavior of $D_{\mu\nu}(k)$ for k^2 near zero].

If $m=0$, since no other scale must be introduced into a finite theory, the exact photon propagator $D(k)$ is proportional to the free propagator and hence the bare charge and the physical charge are equal. Alternatives (i) and (ii) are then equivalent and reduce to the result of Secs. II and III for massless electrodynamics. In the real world where $m\neq 0$ the distinct possibilities (i) and (ii) arise from two different orders of summing the series for the contributions to the vacuum polarization which depend upon the mass m. The order of summation which yields alternative (ii) was pointed out in the important work[7] of Adler. It yields the physically attractive possibility of determining theoretically the observed fine-structure constant α.

In Sec. V we review what is presently known about the fundamental function $f_1(x)$ and discuss the possibility of distinguishing between alternatives (i) and (ii) from high-energy experiments.

II. $m=0$ ELECTRODYNAMICS

The photon propagator $D_{\mu\nu}(k)$ can be written as

$$D_{\mu\nu}(k)=\left(g_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right)D(k^2)+b\,\frac{k_\mu k_\nu}{k^2}, \tag{2.1}$$

where b is a free parameter which determines the gauge. $D(k^2)$ is then determined in terms of the vacuum-polarization function $\Pi_{\mu\nu}(k)$ by the equations

$$D^{-1}(k^2)=k^2[1+\Pi(k^2)], \tag{2.2}$$

$$\Pi_{\mu\nu}(k)=(k^2 g_{\mu\nu}-k_\mu k_\nu)\Pi(k^2). \tag{2.3}$$

The electron propagator $S(p)$ is determined in terms of the electron self-energy function $\Sigma(p)$ by the equation

$$S^{-1}(\gamma\cdot p)=\gamma\cdot p+\Sigma(p). \tag{2.4}$$

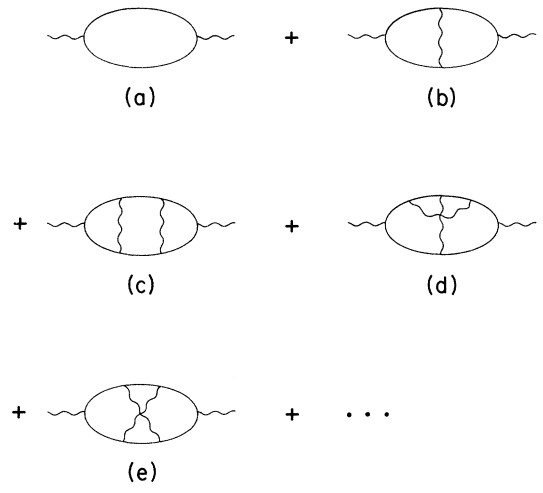

FIG. 1. Examples of diagrams contributing to $f_1(x)$.

$\Sigma(p)$ and $\Pi_{\mu\nu}(k)$ are given by a sum over all Feynman graphs, examples of which are depicted in Figs. 2 and 3. In these diagrams the solid lines stand for the electron propagator $S(p)$ and the wavy lines for the photon propagator $D_{\mu\nu}(k)$. If a finite, unique solution to these equations exists, then by scale invariance

$$D(k) = 1/k^2, \tag{2.5}$$

$$S(p) = 1/\gamma \cdot p. \tag{2.6}$$

For simplicity of presentation we have omitted from Eqs. (2.5) and (2.6) nonessential finite constant factors which rescale the coupling constants appearing at each of the vertices in Figs. 2 and 3. Let us call the square[8] of this rescaled coupling constant x.

We now insert the trial solutions (2.5) and (2.6) for S and D into the expansions for Σ and Π depicted in Figs. 2 and 3. This substitution gives rise to all the perturbation-theory integrals for Π and Σ in massless electrodynamics except for those containing internal electron or photon self-

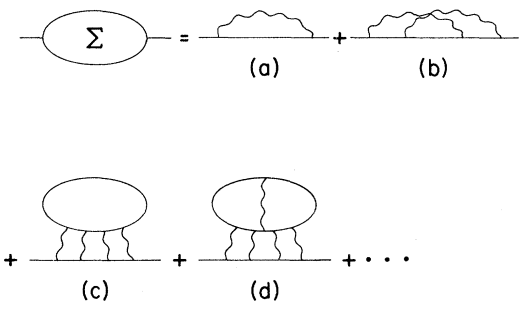

FIG. 2. Examples of graphs for $\Sigma(p)$.

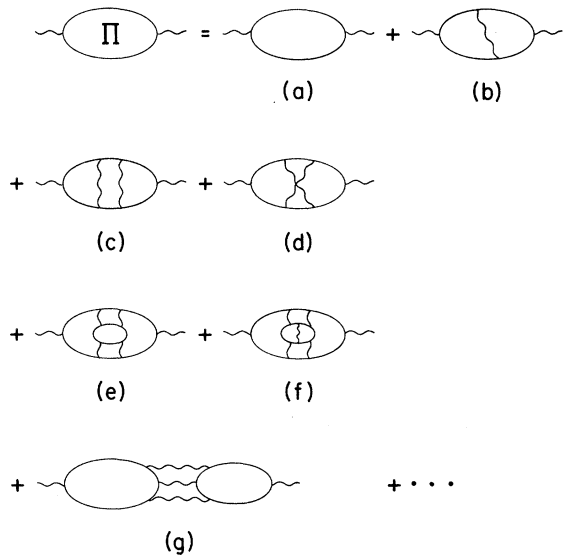

FIG. 3. Examples of graphs for $\Pi(k^2)$.

energy corrections. Because $m = 0$, the integrals not only possess the usual ultraviolet divergences, but they might also diverge in the "infrared" region where some subset of the momenta p_i of the internal lines become small. However, in the Appendixes of Ref. 3 it was shown by elementary power-counting arguments that no such "infrared divergence" arises when one or more of the p_i are held fixed. Furthermore, as long as the external electron and photon momenta are nonvanishing, there are no divergences arising from the small-p_i integration region when all the p_i are integrated over. This insensitivity of the relevant integrals of quantum electrodynamics to the electron mass m when m is set equal to zero is the essential feature of quantum electrodynamics which leads to the possibility of a consistent finite theory.

The situations concerning ultraviolet divergences in the above expressions for Σ and Π are quite distinct. For Σ the usual ultraviolet divergences in the perturbation-theory integrals can be isolated in terms of two infinite constants, the wave-function renormalization constant Z_2 and the electron self-mass δm. However, since δm is proportional to m and since m has been set equal to zero, the δm divergence is not present. The expansion of Fig. 2 then contains only the Z_2 divergence. However, this divergence depends upon the value of the gauge parameter b and can be eliminated in every order of perturbation theory with a suitable choice of b. The proof of this fact makes essential use of the properties of Feynman integrals for small as well as large values of the momenta of the external lines. In a suitably chosen gauge the

integrals of Fig. 2 for Σ possess neither infrared[9] nor ultraviolet divergences. By dimensional arguments it follows that Σ is proportional to $\gamma \cdot p$ multiplied by a power series in x with coefficients which are finite to every order in perturbation theory. Thus (2.5) and (2.6) are self-consistent solutions of Eq. (2.4). Equation (2.4) then simply determines the constant factor we have omitted from the expression (2.6) for S in terms of the coupling constant x.

We now investigate the ultraviolet divergences in Π to see if (2.5) and (2.6) can also be self-consistent solutions of Eq. (2.2). There is an over-all superficial quadratic divergence in the perturbation-theory integrals of Fig. 3. However, because of the factor $k^2 g_{\mu\nu} - k_\mu k_\nu$ which appears in the expression (2.3) for $\Pi_{\mu\nu}$, this quadratic divergence is reduced to the usual logarithmic ultraviolet divergence for Π. This over-all logarithmic divergence arises from the final integration over the momentum p of the electron line which is coupled to the external photon line on the left of the diagrams of Fig. 3. If p is held fixed, the integrations over the remaining internal lines in these diagrams are finite in the ultraviolet region. This follows from the following facts: (a) The diagrams of Fig. 3 contain neither electron nor photon self-energy insertions; (b) the only other kind of insertions which lead to ultraviolet-divergent subintegrations are vertex insertions as in Fig. 3(d), which, because of Ward's identity, are finite in the gauge in which Σ is finite.[10]

Using scale invariance we thus conclude that the integrals for $\Pi(k)$ can be written in the form

$$\Pi(k^2, x) = \frac{x}{2\pi} \int_0^\infty \frac{dp^2}{p^2} f(p^2/k^2, x), \tag{2.7}$$

where $f(p^2/k^2, x)$ is a finite function of p^2/k^2 and x which arises from carrying out all the integrations in the diagrams of Fig. 3 except the final integration over the magnitude of p. Equation (2.7) follows from the fact that the integrals for f are both infrared- and ultraviolet-finite. The ultraviolet divergence in Π is obtained by calculating the large-p limit of f. But by scale invariance this is equivalent to letting k approach zero. Now, since, as stated above, all integrals for f remain infrared-finite when k is set equal to zero, we conclude

$$\lim_{p^2 \to \infty} f(p^2/k^2, x) = f(x), \tag{2.8}$$

where $f(x)$ is the finite function of the coupling constant x which is obtained by setting $k = 0$ in the integrals for $f(p^2/k^2, x)$.[11] We can then write

$$\Pi(k^2, x) = \frac{x}{2\pi} \int_0^{k^2} \frac{dp^2}{p^2} f(p^2/k^2, x) + \frac{x}{2\pi} \int_{k^2}^\infty \frac{dp^2}{p^2} [f(p^2/k^2, x) - f(\infty, x)] + \frac{x}{2\pi} f(x) \int_{k^2}^\infty \frac{dp^2}{p^2}. \tag{2.9}$$

We see that the Π divergence is like a single power of the logarithm of an ultraviolet cutoff on the momentum p^2, and therefore the assumption of the existence of a finite solution to mass-zero electrodynamics is in general inconsistent. However, if the coupling constant x is chosen to be a root x_0 of the equation

$$f(x) = 0, \tag{2.10}$$

then the coefficient of this logarithmic divergence vanishes.[12] $\Pi(k^2, x_0)$ is then finite and by scale invariance [or by explicitly letting $p^2 \to k^2 z$ in the finite terms in (2.9)] is a constant $\Pi(x_0)$ independent of k^2. Thus, for coupling constant $x = x_0$, (2.5) and (2.6) are also self-consistent solutions of Eq. (2.2). Equation (2.2) then simply determines the constant factor we have omitted from Eq. (2.5) for D in terms of the finite constant $\Pi(x_0)$. We thus conclude that mass-zero quantum electrodynamics with coupling constant x_0 determined from Eq. (2.10) is a finite theory having the exact electron and photon propagators which are proportional to the free propagators.

The possibility of obtaining a finite mass-zero theory by imposing a single condition upon x depended upon the fact that the integrals of Fig. 3 for Π diverged only like a single power of a logarithm. The essential reason for this single logarithm was that all integrations except the final integration over p were finite in both the ultraviolet and infrared regions. If the integrals defining $f(p^2/k^2)$ of Eq. (2.7) possessed infrared divergences when $k = 0$, then f could have contained terms like $[\ln(p^2/k^2)]^n n \geq 1$ and the above argument leading to a finite theory could not have been carried out.

The arguments of this section leading to the possibility of a finite mass-zero electrodynamics depended only upon the explicit properties of perturbation-theory integrals. In Sec. III we will examine the properties of this finite theory using some results which follow from the operator representations of scattering amplitudes in field theory. Such results are not explicitly visible from the perturbation solution for these amplitudes and hence the theoretical basis for the simplifications to be obtained in Sec. III is fundamentally different from that for the results obtained above.

III. PROPERTIES OF MASS-ZERO ELECTRODYNAMICS

We now assume there exists a positive root x_0 of Eq. (2.10). Then the quantum electrodynamics of photons interacting with massless electrons with coupling constant x_0 is a finite relativistic quantum field theory. In this theory the exact photon propagator $D(k)$ is proportional to the free propagator, and hence the absorptive part of $D(k)$ vanishes. However, the absorptive part of $D(k)$ is determined by the Fourier transform of $\langle 0|j_\mu(x)j_\nu(y)|0\rangle$, where $j_\mu(x)$ is the electromagnetic current operator. We then conclude that the current operator in mass-zero electrodynamics satisfies the equation

$$\langle 0|j_\mu(x)j_\nu(y)|0\rangle = 0. \tag{3.1}$$

But we now apply the theorem[13]

$$\langle 0|j_\mu(x)j_\nu(y)|0\rangle = 0 \text{ implies}$$

$$\langle 0|j_{\mu_1}(x_1)\cdots j_{\mu_n}(x_n)|0\rangle = 0. \tag{3.2}$$

Thus all amplitudes involving only external photon lines (real or virtual) vanish. The result (3.2), unlike the previous ones of our discussion, cannot be understood from an examination of the properties of perturbation theory. That is, in perturbation theory an n-photon amplitude is expressed as a sum of all closed-loop diagrams with n external photon lines. Of course, to a given order in perturbation theory these diagrams do not vanish. The mechanism by which the sum over all diagrams vanishes is not at all clear. However, this vanishing must occur if one accepts the general principles upon which it is based.

If this theorem[13] could be extended to include processes where there are external electron lines, then all transition amplitudes would vanish. This would mean that the finite quantum electrodynamics of zero-mass electrons would be equivalent to a free-field theory. We would thus conclude that in the absence of an electron mass there are no interactions, i.e., interactions are a consequence of the nonvanishing electron mass. The difficulty of this extension is caused by the absence of a positive metric in the portion of the Hilbert space associated with external charged lines. However, in this paper we will not make any use of this speculation about the zero-mass theory.

We now show that the vanishing Eq. (3.2) of the n-photon amplitudes implies that x_0 satisfies the simpler Eq. (1.2). The vanishing of the four-photon amplitude means that the sum of all graphs of the type depicted in Fig. 4 which involve an internal fully interacting four-photon amplitude must vanish. Likewise the sum of all diagrams of the class of Fig. 5 vanishes because the multiphoton

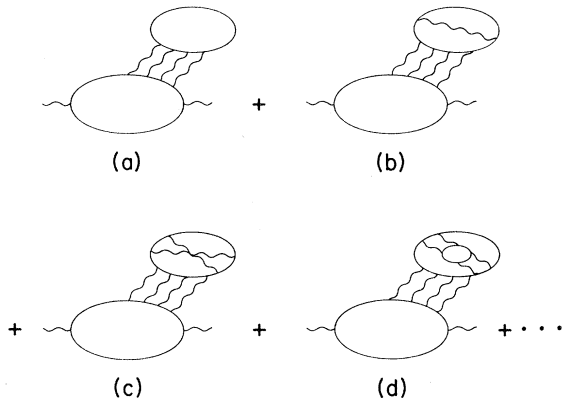

FIG. 4. Examples of contributions to $f(x)$ from graphs involving interacting photons.

interaction on the right vanishes when all the diagrams for that interaction are included. In general the diagrams for $f(x)$ break up into subclasses, each of which involves some subset of photons fully interacting with each other. Hence each of these subclasses must vanish separately. There remains the class of diagrams which involve no photon-photon interactions, namely those diagrams involving a single closed fermion loop, depicted in Fig. 1. These graphs give the contribution $f_1(x)$ to $f(x)$. Since all other contributions to $f(x)$ vanish at $x = x_0$, we see that $f_1(x)$ must also vanish at $x = x_0$.

If we apply the above argument to the n-photon amplitude, we immediately conclude that the single-closed-fermion-loop contribution to the n-photon amplitude vanishes by itself.

Adler[7] has made the important observation that the vanishing [(3.2)] of the n-photon amplitudes implies that all derivatives of $f(x)$ and $f_1(x)$ vanish at $x = x_0$. Another way of obtaining this result is to use the fact that the nth derivative of $f(x)$ is related to the forward n-photon→ n-photon amplitude. The latter can be seen by differentiating Eq. (2.9) with respect to x. This yields the relation[3]

$$\frac{d}{dx}f(x) = q^{2\frac{1}{2}}[K(0;q)], \tag{3.3}$$

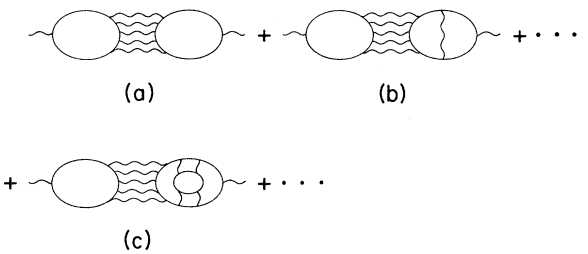

FIG. 5. Further examples of contributions to $f(x)$ from graphs containing a subgroup of interacting photons.

where $K(k; q)$ is the asymptotic Bethe-Salpeter kernel for the forward scattering of photons of momenta q and k. However, since when $x = x_0$ the scattering amplitude T for this process vanishes, so must the kernel K, since T and K are related by the usual linear scattering integral equation. Similarly differentiation of Eq. (3.3) with respect to x relates d^2f/dx^2 to the kernel for the forward $3 \to 3$ photon amplitude. Likewise differentiating Eq. (2.9) n times with respect to x relates $d^n f(x)/dx^n$ to the kernel for the forward $n \to n$ photon amplitude which vanishes at $x = x_0$. We then conclude

$$\frac{d^n f(x)}{dx^n} = 0 \quad \text{at } x = x_0 . \tag{3.4}$$

The above reasoning also relates the nth derivative of the single-closed-loop function $f_1(x)$ to the single-closed-loop contribution to the forward n-photon $\to$ n-photon amplitude which vanishes at $x = x_0$. As a consequence, we obtain Adler's result,

$$\frac{d^n}{dx^n} f_1(x) = 0 \quad \text{at } x = x_0 . \tag{3.5}$$

Thus both the functions $f(x)$ and $f_1(x)$ have essential singularities at $x = x_0$. Hence in order to test for the existence of a finite theory of mass-zero quantum electrodynamics, we must calculate the single-closed-loop function $f_1(x)$ and look for a value of the coupling for which $f_1(x)$ and all its derivatives vanish.

We would like to emphasize that the simplified eigenvalue Eq. (1.2) and the essential singularities of $f(x)$ and $f_1(x)$ are basically consequences of the theorem[13] of Eq. (3.2).

IV. QUANTUM ELECTRODYNAMICS OF FINITE-MASS ELECTRONS

We now turn to the theory of physical interest, ordinary quantum electrodynamics of electrons with mass $m \neq 0$. This theory will be consistent if the renormalized electron propagator $\bar{S}(p)$ and the renormalized photon propagator $\bar{D}(k)$ behave like free propagators at high energy, i.e.,

$$\lim_{p \to \infty} \bar{S}(p) \sim \text{const}/\gamma \cdot p , \tag{4.1}$$

$$\lim_{k^2 \to \infty} \alpha \bar{D}(k) \sim \alpha_0 / k^2 , \tag{4.2}$$

where α is the fine-structure constant and α_0 by definition is the bare fine-structure constant. We will see that the high-energy behavior of this theory is determined completely by the properties of mass-zero electrodynamics.

We first study the perturbation expansion for $\bar{D}(k)$ which is generated from the vacuum-polarization function $\bar{\Pi}(k)$ by the equations

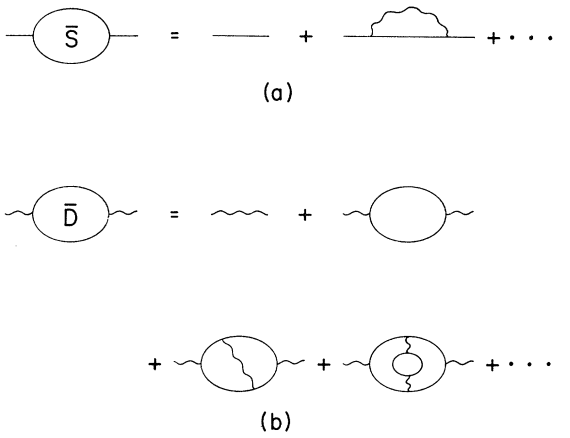

FIG. 6. Graphs representing the perturbation expansions of $\bar{S}(p)$ and $\bar{D}(k)$.

$$\bar{D}^{-1}(k) = k^2[\,1 + \alpha \bar{\Pi}_R(k)\,], \tag{4.3}$$

where

$$\bar{\Pi}_R(k) = \bar{\Pi}(k) - \bar{\Pi}(0) . \tag{4.4}$$

For convenience we have removed from the definition of $\bar{\Pi}$ the factor α which may be associated with the coupling to the external photon lines. The perturbation expansion for $\bar{\Pi}(k)$ can be obtained from the sum of all diagrams of the type of Fig. 3 by replacing all electron and photon lines in each diagram by the renormalized electron and photon propagators $\bar{S}(p)$ and $\bar{D}(k)$ of Fig. 6, and by associating with each internal vertex the renormalized charge e. The electron lines on the right-hand side of Fig. 6 of course stand for $1/(\gamma \cdot p + m)$, where m is the physical mass of the electron. The resulting expression for $\bar{\Pi}_R(k)$, Eq. (4.4), is the usual series of convergent renormalized perturbation-theory integrals, which for $k^2/m^2 \gg 1$ takes on the form

$$\bar{\Pi}_R(k) = c_0(\alpha) + c_1(\alpha) \ln \frac{k^2}{m^2} + \sum_{n=2}^{\infty} c_n(\alpha) \left(\ln \frac{k^2}{m^2} \right)^n + O\left(\frac{m^2}{k^2} \right) . \tag{4.5}$$

The coefficients $c_n(\alpha)$ are power series in the fine-structure constant α, obtained by evaluating the asymptotic form of the perturbation-theory integrals. The requirement (4.2) means that $\bar{\Pi}_R(k)$ must remain finite as $k^2 \to \infty$. This condition is equivalent to the requirement that the unrenormalized theory is finite or, equivalently, that Z_3 is finite.

We must thus evaluate the high-k^2 limit of $\bar{\Pi}_R(k)$. We first do this according to the procedure of Ref. 3. We make the following assumption (later to be checked by self-consistency):

(A) We can replace $\bar{\alpha}D(k)$ by its limiting form α_0/k^2 in the expression for $\bar{\Pi}(k)$ in order to calculate the dominant large-k behavior of $\bar{\Pi}_R(k)$. That is, we assume that the leading corrections to the asymptotic limit (4.2) fall off sufficiently rapidly that they yield contributions to $\bar{\Pi}(k)$ which will vanish as $k \to \infty$. Furthermore, if we choose the gauge in which Z_2 is finite, the electron self-energy corrections are unimportant. Thus with assumption (A) the large-k behavior of $\bar{\Pi}(k)$ is obtained from the sum of the diagrams of Fig. 3 by replacing each internal photon line by $1/k^2$, each vertex by $\sqrt{\alpha_0}$, and each internal electron line by $1/(\gamma \cdot p + m)$. We are then left with a series for the asymptotic behavior of $\bar{\Pi}(k)$ which is the same as the series for $\Pi(k)$ in the mass-zero theory except for the presence of the electron mass m in the denominators.

Since $\bar{\Pi}_R(k)$ is a function of k^2/m^2, we can calculate the $\lim_{k\to\infty}\bar{\Pi}(k)$ by taking the limit $m \to 0$. By our previous discussion of Eqs. (2.7)–(2.9), we know that all integrals for $\bar{\Pi}_R(k)$ are finite when m is set equal to zero except for the final integration over p, which with $m=0$ would be infrared-divergent because of $\bar{\Pi}(0)$ in Eq. (4.4). The presence of the electron mass in this final integration cuts off this divergence, and the subtraction in (4.4) makes the final integration over p ultraviolet-convergent. Thus we can set $m=0$ everywhere except in the final integration over p. This effectively replaces the p^2 in the denominators of the integrals in Eqs. (2.7) and (2.9) by p^2+m^2. Then, using (2.9) and (2.4), we conclude that

$$\lim_{k\to\infty}\bar{\Pi}_R\left(\frac{k^2}{m^2}\right)=\lim_{m\to 0}\bar{\Pi}_R\left(\frac{k^2}{m^2}\right) \sim -\frac{1}{2\pi}f(\alpha_0)\ln\frac{k^2}{m^2}+\text{constant}. \tag{4.6}$$

We can understand the relation between Eqs. (4.5) and (4.6) by first noting that from the previous discussion it follows that the higher powers of $\ln(k^2/m^2)$ in Eq. (4.5) come only from diagrams which include the vacuum-polarization corrections of Fig. 6(b) in the internal photon lines of Fig. 3. Thus, for example, when the internal photon line in Fig. 3(b) is replaced in the diagrams of Fig. 6(b) for $\bar{D}$, we obtain the diagrams depicted in Fig. 7, which individually behave asymptotically like high powers of $\ln(k^2/m^2)$. However, if we first sum over all diagrams of Fig. 7, then from (A) it follows that the high-k^2 limit of these diagrams is the same as that of Fig. 3(b) with α replaced by α_0. Thus if we take the $\lim k\to\infty$ of the individual terms in Fig. 7 we generate a series of the structure (4.5), while if we first sum over all diagrams

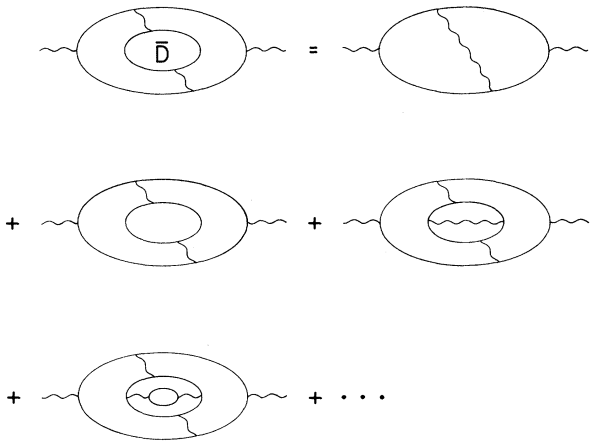

FIG. 7. Examples of contributions to $\bar{\Pi}_R(k)$ from diagrams having vacuum-polarization insertions in internal photon lines.

and then go to the limit $k\to\infty$ we obtain a result proportional to $f(\alpha_0)\ln(k^2/m^2)$. There is a similar relation between the full series (4.5) and the result (4.6).

From (4.6) we conclude that under assumption (A) Eq. (4.2) can be self-consistent only if the bare charge α_0 is chosen to be a root x_0 of Eq. (2.10). If we choose $\alpha_0=x_0$, we find

$$\lim_{k^2\to\infty}\bar{\Pi}_R(k^2)=\text{constant} \tag{4.7}$$

for $\alpha_0=x_0$, and Eqs. (4.2) and (4.3) are consistent. It remains to verify (A), namely, to show that (4.7) is still valid when the corrections to the limit (4.2) are included in the calculation of $\bar{\Pi}_R(k)$. This was done in Ref. 3 using the incorrect assumption that $f'(x_0)\neq 0$. However, Adler[7] showed that this conclusion is still valid by using the fact [Eq. (3.2)] that the n-photon amplitude vanishes. We hence conclude that there exists a self-consistent solution for the photon propagator which behaves for large k^2 like the free propagator provided the bare coupling constant α_0 is chosen to be a root of Eq. (2.10). Furthermore, it can be readily seen[1,3] that the physical fine-structure constant α is left undetermined except that it must be less than $\alpha_0=x_0$. This latter fact is more easily seen from the Gell-Mann–Low equation,[1] which we will use in Sec. V of this paper in order to determine the corrections to Eq. (4.7) at finite values of k^2. Finally, we note that $\alpha_0=x_0$ is of course also a root of the simpler equation (1.2).

Adler[7] has pointed out that a different order of performing the summation over the diagrams for $\bar{\Pi}_R(k)$ and taking the limit $k^2\to\infty$ can lead to the conclusion that a consistent solution for $\bar{D}$ exists when the physical fine-structure constant α is cho-

sen equal to x_0. The first step in Adler's procedure is to sum all diagrams in the renormalized perturbation expansion for $\overline{\Pi}_R(k)$ which contain a single fermion closed loop. There is no prior summation of vacuum-polarization insertions in internal photon lines. This series can thus be represented by the graphs of Fig. 1 where each vertex corresponds to the renormalized charge e and each internal photon line corresponds to free photon propagator $1/k^2$. From our previous discussion, it is clear that this series of diagrams gives a contribution to $\overline{\Pi}_R(k)$ which for $k^2 \to \infty$ behaves like

$$\overline{\Pi}_R(k) \underset{k^2\to\infty}{\sim} -\frac{1}{2\pi} f_1(\alpha) \ln\frac{k^2}{m^2} + \text{constant} + O\left(\frac{m^2}{k^2}\right). \tag{4.8}$$

Thus this subset of diagrams yields an asymptotically finite contribution to $\overline{\Pi}_R(k)$ if we choose $\alpha = x_0$, i.e., if α satisfies the equation

$$f_1(\alpha) = 0. \tag{4.9}$$

Adler assumes that $\alpha = x_0$ and considers contributions to $\overline{\Pi}_R(k)$ from diagrams containing two fermion closed loops, examples of which are depicted in Fig. 8. Since the individual diagrams of Fig. 8 contain vacuum-polarization insertions, they behave like higher powers of $\ln(k^2/m^2)$ for large k^2. However, the sum of all diagrams of the type in Fig. 8 contains an inner closed loop with all possible photon exchanges. Because of the eigenvalue condition [(4.9)], this gives a contribution to internal vacuum polarization in Fig. 8 which is asymptotically finite. The sum of the diagrams of Fig. 8 is thus equivalent to the diagram of Fig. 9. The crosses at the internal vertices of Fig. 9 indicate that the effective constant at these vertices differs from e because of the contribution of the finite part of the vacuum-polarization insertions in Fig. 8. By our previous argument, Fig. 9, and hence the sum of the diagrams of Fig. 8, then behave like a single power of $\ln(k^2/m^2)$.

Now consider the sum of diagrams of Fig. 10(a)

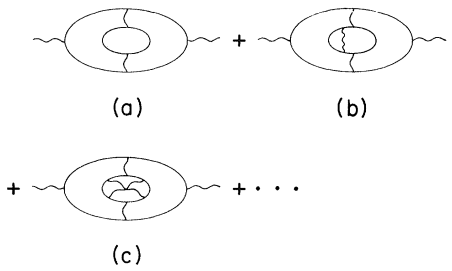

FIG. 8. Examples of diagrams for $\overline{\Pi}_R(k)$ which contain two fermion closed loops.

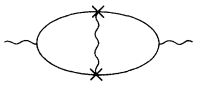

FIG. 9. Graph representing contribution of the sum of the graphs of Fig. 8.

where the outer closed loop contains an additional internal photon line. The sum of all diagrams of Fig. 10(a) is then equivalent to the diagram of Fig. 10(b). Likewise the sum over all diagrams which contain any number of internal photons in either closed fermion loop is equivalent to the sum of diagrams depicted in Fig. 11. From our previous discussion we see that each diagram of Fig. 11 behaves for large k^2 like a single power of $\ln(k^2/m^2)$ with a coefficient which is determined by setting $m = 0$ and performing all integrations except one. Let us then carry out all integrations in each diagram of Fig. 11 except for the integration over the photon line which joins the two crosses. These integrations generate diagrams for the single-loop contribution to photon-photon scattering.[14] (See Fig. 12.) (The external lines for this photon-photon amplitude are the two external photons of the diagrams of Fig. 11 and the two crosses.) Thus the coefficient of $\ln(k^2/m^2)$ in the sum of the diagrams of Fig. 11 is proportional to the single-closed-loop contribution to photon-photon scattering in mass-zero electrodynamics with coupling constant α. However, since α satisfied Eq. (4.9), it follows from Eq. (3.2) that this photon-photon amplitude vanishes. Hence the coefficient of $\ln(k^2/m^2)$ in the sum of the diagrams of Fig. 11 vanishes. Thus the sum of the two-fermion-closed-loop diagrams represented by Fig. 11 gives

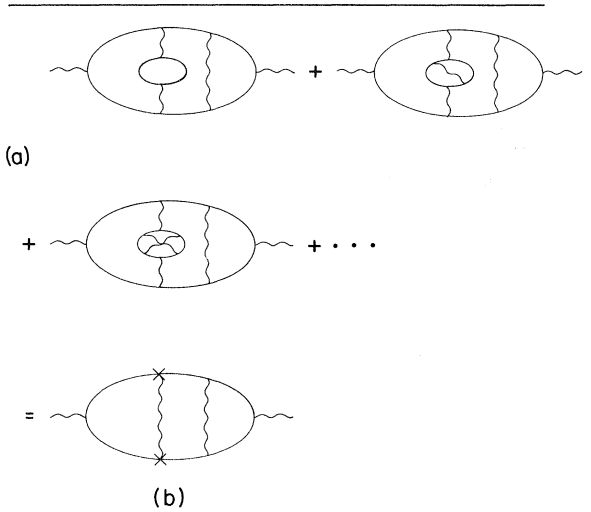

FIG. 10. Further examples of graphs containing two fermion closed loops.

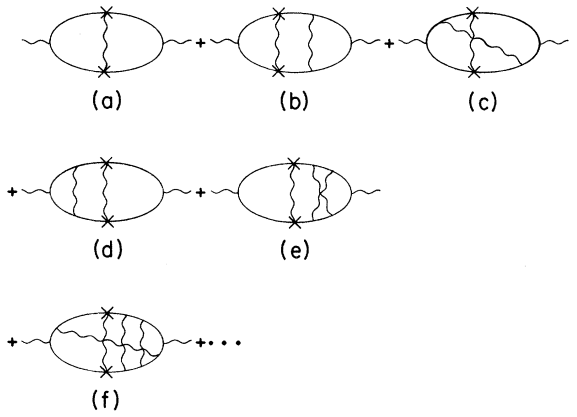

FIG. 11. Graphs representing sum of diagrams with two closed loops, each of which contains any number of internal photons.

a contribution to the vacuum polarization $\overline{\Pi}_R(k^2)$ which for $k^2/m^2 \gg 1$ has the behavior

$$\overline{\Pi}_R(k) \underset{k^2\to\infty}{\longrightarrow} \text{constant} + O\left(\frac{m^2}{k^2}\right). \tag{4.10}$$

Therefore if α satisfies (4.9), the sum over all diagrams in which the inner and outer fermion loops of Fig. 8(a) contain any number of internal lines gives a finite asymptotic contribution to the vacuum polarization.

The basic ingredients in the above argument are the following:

(a) diagrams for $\overline{\Pi}_R(k)$ without vacuum-polarization insertions behave like a single power of $\ln(k^2/m^2)$ for large k^2, and

(b) amplitudes with n external photons in mass-zero electrodynamics and coupling α vanish if $\alpha = x_0$.

Using (a) and (b) it is easy to see that if we sum over all diagrams containing a fixed number of closed fermion loops then $\Pi_R(k^2)$ will have the behavior (4.10) if α is chosen equal to x_0. In our previous vacuum-polarization insertion summation procedure based upon assumption (A), we summed over diagrams containing an infinite number of closed loops at the first stage when we replaced $\overline{D}$ by α_0/k^2 as in the example of Fig. 7. Thus in order to go from this procedure which leads to (4.10) and an eigenvalue equation for α, we must interchange the limit $k\to\infty$ with the summation over all diagrams.

If Adler's loopwise summation procedure for calculating $\overline{\Pi}(k)$ yields the physically correct solution of quantum electrodynamics, then the fine-structure constant α is fixed and equal to x_0. Now it follows from general arguments that the bare fine-structure constant α_0 is greater than α. Since

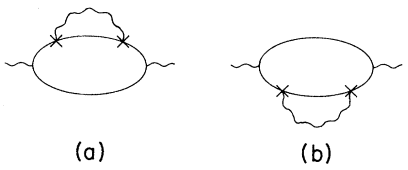

FIG. 12. Graphs representing two-closed-loop diagrams which contain an electron self-energy insertion.

the function $f_1(x)$ has an essential singularity at $x = x_0$, and $\alpha_0 > \alpha = x_0$, the point α_0 lies outside the radius of convergence of the function $f_1(x)$. Thus the rearrangement of diagrams and the interchange of the limit $k\to\infty$ with the summation over all diagrams which is necessary to convert the "loopwise" summation procedure into the vacuum-polarization insertion procedure [assumption (A)] is not justified. Hence one cannot deduce Eq. (4.6) and the eigenvalue condition for α_0. Conversely, suppose that the vacuum-polarization insertion assumption (A) yields the physically correct solution for $\overline{\Pi}_R(k)$; then $\alpha_0 = x_0$ and α is left undetermined except for the requirement that it be less than α_0. In this case the loopwise summation procedure will not work since $f_1(\alpha)$ does not vanish and hence the single-closed-loop diagram is not finite as $k^2 \to \infty$.

We now turn to the equation for the renormalized electron propagator $\overline{S}(p)$. We assume

(1) that there exists a consistent finite solution for $\overline{D}$ which has the behavior (4.2), and

(2) that we can replace $\alpha\overline{D}$ by its limiting form α_0/k^2 in the expression for the electron self-energy function $\overline{\Sigma}(p)$ in order to calculate the dominant high-energy behavior of $\overline{S}(p)$.

We know that assumption (1) is justified if there exists a positive root of the equation $f(x_0) = 0$. In the case $\alpha = x_0$, the asymptotic corrections to $\overline{\Pi}_R(k)$ [Eq. (4.10)] are of order m^2/k^2 and so assumption (2) is justified. In the case $\alpha_0 = x_0$, we will see in Sec. V that the asymptotic corrections to $\overline{\Pi}_R(k)$ vanish more slowly as $k^2 = 0$. In this case Adler has noted that assumption (2) will still be valid if the theorem of Eq. (3.2) can be extended to amplitudes containing external fermion lines.

Using (1) and (2) the Schwinger-Dyson equation for $\overline{S}$ then becomes an integral equation whose kernel is the Bethe-Salpeter kernel for electron-positron scattering. This equation was treated in detail in Ref. 5 and we will only state the results of that analysis here, since our main concern in this paper is the photon propagator. It was shown there that the solution of this equation for $\overline{S}(p)$ for $p^2 \gg m^2$ depended crucially upon the behavior of the

kernel for $m=0$. Moreover, the kernel is sufficiently well behaved so that in a suitably chosen gauge the solution for $\overline{S}(p)$ takes on the form

$$\overline{S}(p) \xrightarrow[p^2/m^2 \gg 1]{} \text{constant} \times \left[\frac{1}{\gamma \cdot p} + \frac{m}{p^2} a \left(\frac{m^2}{p^2} \right)^{\epsilon} \right], \tag{4.11}$$

where a and ϵ are constants. The parameter ϵ is determined from the expansion of the kernel for the scattering of zero-mass electrons and positrons with coupling constant α_0. To order α_0^2, ϵ is given by[5]

$$\epsilon = \frac{3}{2} \frac{\alpha_0}{2\pi} + \frac{3}{8} \left(\frac{\alpha_0}{2\pi} \right)^2 + \cdots . \tag{4.12}$$

We see that Eq. (4.11) is consistent with (4.1) if $\epsilon > -\frac{1}{2}$. If $\epsilon < -\frac{1}{2}$, the mass term dominates the $1/\gamma \cdot p$ term and there are no consistent finite solutions for electron propagators.

If we introduce a cutoff Λ, then the bare mass $m_0(\Lambda)$ is given in terms of the physical mass m by the relation[5]

$$m_0(\Lambda) = a m (m^2/\Lambda^2)^{\epsilon}. \tag{4.13}$$

Thus if $\epsilon > 0$, then $\delta m = m - m_0(\Lambda)$ is finite as the cutoff $\Lambda \to \infty$. The usual perturbation-theory divergence for δm arises from putting the second term in Eq. (4.11) with $\epsilon = 0$ into the equation for $\overline{\Sigma}$. This "divergence" is thus clearly not intrinsic to quantum electrodynamics but is the result of the inapplicability of perturbation theory.

Although ϵ is positive to order α_0^2, we have no general proof of positivity. We cannot rule out the possibility that ϵ might lie in the range $-\frac{1}{2} < \epsilon < 0$. In this case Eq. (4.11) would still be a valid asymptotic solution of the renormalized equation for $\overline{S}$, although from Eq. (4.13) we would find $\delta m = \infty$.[15]

V. EXPERIMENTAL IMPLICATIONS OF ALTERNATIVES (i) AND (ii) OF SEC. I

We have thus shown that quantum electrodynamics can be a consistent finite theory with electron and photon propagators which behave like free propagators at high energy, provided there exists a positive root x_0 of the equation $f(x_0) = 0$. $(x/2\pi) \times f(x)$ is the coefficient of the divergent logarithm in the sum of the $m=0$ diagrams depicted in Fig. 3. If $f(x_0) = 0$, then $f_1(x_0) = 0$, where $(x/2\pi) f_1(x)$ is the coefficient of the divergent logarithm in the sum of the simpler set of diagrams of Fig. 1. Furthermore $(d^n/dx^n) f(x) = (d^n/dx^n) f_1(x) = 0$ at $x = x_0$. Conversely, if we calculate $f_1(x)$ and find a positive value of x for which $f_1(x)$ and all its derivatives vanish, we can be fairly sure that $f(x)$ also vanishes at this point. However, this has not yet been proven.

There are then the following alternatives:

(i) A positive root x_0 exists and $\alpha_0 = x_0$.

(ii) A positive root x_0 exists and $\alpha = x_0$.

(iii) No positive root x_0 exists (in this case we have found no finite consistent solution of electrodynamics).

The best way to decide among the above alternatives is to calculate $f_1(x)$. We have been trying to find ways to calculate $f_1(x)$ exactly for the past five years, but no real progress has been made since Rosner's[16] sixth-order calculation which includes all the diagrams of Fig. 1. There has not even been a satisfactory explanation of the simplicity of Rosner's result[16]

$$f_1^{(6)}(x) = \frac{2}{3} + \left(\frac{x}{2\pi} \right) - \frac{1}{4} \left(\frac{x}{2\pi} \right)^2 . \tag{5.1}$$

Rosner carried out his calculation in the gauge in which Z_2 is finite. The sixth calculation of $f_1(x)$ in the Feynman gauge has been carried out by Brandt.[17] The importance of Brandt's calculation, aside from explicitly verifying the gauge invariance of $f_1(x)$, was to show that the calculation in the Feynman gauge possessed the following great simplicity: The sum of the nontrivial graphs (d) and (e) of Fig. 1, which involve crossed photon lines, gives no contribution to $f_1(x)$ when calculated in this gauge. Thus the sixth-order result (5.1) for $f_1(x)$ is obtained by evaluating in the Feynman gauge the uncrossed 2-photon diagram 1(c) and the internal electron self-energy corrections, both of which are trivial to calculate. However, the reason why Figs. 1(d) plus 1(e) do not contribute in the Feynman gauge has not been understood, and there has been no successful use of this simplicity to calculate the higher orders of $f_1(x)$.

Another class of attempts to calculate $f_1(x)$ have made use of coordinate-space methods.[18] Since $f_1(x)$ is calculated from a theory in which the electron mass m is equal to zero, the integrals for f_1 are all conformally invariant. We have attempted to exploit the conformal invariance in order to calculate f_1 to all orders. Although a little progress has been made, the method has been plagued with certain subtleties.[18] We will report on this work in a separate publication, hopefully to stimulate further attempts to overcome our present difficulties.

The fact that $f_1(x)$ has an essential singularity suggests a third approach to calculating $f_1(x)$. We note that in the power-series expansion of $f_1(x)$

$$f_1(x) = \sum_{j=0}^{\infty} a_j x^j \tag{5.2}$$

only the coefficients a_j for large j are relevant for

calculating f_1 near the point where it has a singularity. Before discussing our conjecture for calculating a_j for large j, we will decompose $\overline{\Pi}_R(k)$ into a sum of terms $\Pi_n(k,\alpha)$ having the n electron-positron pair threshold. This decomposition will be useful for our discussion of the experimental implications of alternatives (i) and (ii) of Sec. I as well as for our treatment of the coefficient a_j for large j.

We thus write

$$\overline{\Pi}_R(k) = \sum_{n=0}^{\infty} \Pi_n(k,\alpha), \tag{5.3}$$

where $\mathrm{Im}\Pi_n(k,\alpha)$ includes all terms in the sum

$$\mathrm{Im}\overline{\Pi}_{\mu\nu}(k) \sim \sum_N \langle 0|j_\mu(0)|N\rangle\langle N|j_\nu(0)|0\rangle, \tag{5.4}$$

for which the state $|N\rangle$ contains n electron-positron pairs and any number of photons. Clearly

$$\mathrm{Im}\Pi_n(k,\alpha) \underset{k^2<4m^2n^2}{=} 0, \tag{5.5}$$

while

$$\Pi_n(k,\alpha) \underset{k^2\gg 4m^2n^2}{\approx} \Pi_n^0(\alpha) + \Pi_n^1(\alpha)\ln\frac{k^2}{m^2} + \Pi_n^2(\alpha)\left(\ln\frac{k^2}{m^2}\right)^2 + \cdots. \tag{5.6}$$

Equations (5.3) and (5.6) combine to yield a reordered version of the asymptotic expression (4.5) for $\overline{\Pi}_R(k)$. Equation (5.6) is more precise than (4.5) in that it makes explicit the fact that as n increases one must go to larger and larger values of k in order to be in the asymptotic region.

The contribution to $\Pi_n(k,\alpha)$ which is of lowest order in α comes from the terms in the sum (5.4) for which the state $|N\rangle$ contains n electron positron pairs and no photons. The lowest-order perturbation value of $\langle 0|j^\mu|N\rangle$ for these states is proportional to α^{n-1}. This gives a contribution to $\Pi_n(k,\alpha)$ proportional to α^{2n-2}. For example, the lowest-order contribution to $\Pi_3(k,\alpha)$ comes from tree-graph contributions to $\langle 0|j_\mu|3e,3p\rangle$ like those depicted in Fig. 13. Higher-order contributions to $\Pi_3(k,\alpha)$ come from radiative corrections to

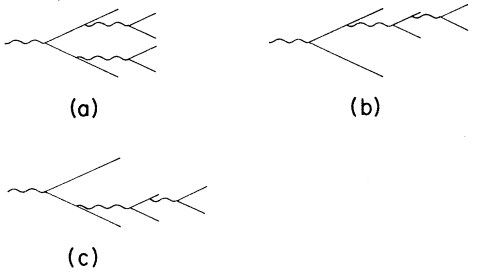

FIG. 13. Tree graphs for $\langle 0|j_\mu|3e,3p\rangle$.

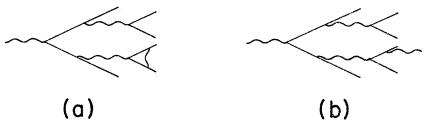

FIG. 14. Graphs for real and virtual radiative corrections to the graph of Fig. 13(a).

$\langle 0|j_\mu|3e,3p\rangle$ such as that depicted in Fig. 14(a) as well as from lowest-order matrix elements $\langle 0|j_\mu|3e,3p,1\text{ photon}\rangle$, depicted in Fig. 14(b). The amplitude of Fig. 14(a) gives a contribution to $\Pi_3(k,\alpha)$ containing infrared divergences which cancel the infrared divergences arising from the contribution to $\Pi_3(k,\alpha)$ of the amplitude of Fig. 14(b).

The single-closed-fermion-loop contribution to $\Pi_n(k,\alpha)$ for $k^2 \gg (2mn)^2$ behaves like a single power of $\ln(k^2/m^2)$ with a coefficient $\alpha^{2n-2}b_n(\alpha)$ which is the n-pair-state contribution to $f_1(\alpha)$. To calculate $b_n(\alpha)$ and hence $f_1(\alpha)$, we must evaluate the single-closed-loop contribution to the sum (5.4) with m set equal to zero.[19] Clearly states $|N\rangle$ which contain only photons do not contain any single-closed-loop contribution, i.e., $b_0(\alpha)=0$. Furthermore the lowest-order n-pair contribution $\alpha^{2n-2}b_n(0)$ to $f_1(\alpha)$ arises from tree diagrams such as those depicted in Fig. 13. We can thus write $f_1(x)$ as a sum over n-pair contributions:

$$f_1(x) = \sum_{n=1}^{\infty} x^{2n-2}b_n(x). \tag{5.7}$$

Comparison of (5.7) with the power-series expansion (5.2) shows that the n-pair state contributes to all terms in the series (5.2) for which $j \geq 2n-2$. That is, every time j increases by 2, the coefficient a_j is increased by the contribution from a state with one additional pair.

We now conjecture that the dominant contribution to the coefficients a_j for large $j = 2n-2$ arises from the n-pair state in the sum (5.7), that is, for large n we can write

$$a_{2n-2} \sim b_n(0). \tag{5.8}$$

Of course a_{2n-2} also receives contributions from radiative corrections from states having fewer than n pairs via the diagrams depicted in Fig. 14. The assumption is that these radiative corrections do not contribute substantially to the growth of a_j for large j. We know that the contribution of these corrections in the infrared region cancels, and we see no physical reason why they should be important for large j. Thus our assumption (5.8) is that the origin of the essential singularity in $f_1(x)$ is the rapid growth of the lowest-order n-pair contribution $b_n(0)$ as n, the number of pairs, increases. That is, the essential singularity in $f_1(x)$ is due to the presence of the infinite number

of multiparticle thresholds which is characteristic of a relativistic theory.

The above paragraph is of course pure speculation and is an attempt to give physical motivation for calculating the coefficients $b_n(0)$ for large n. To calculate $b_n(0)$ we must calculate the tree-graph contribution to the matrix element $\langle 0|j_\mu|n \text{ pairs}\rangle$, project the single-closed-loop contribution out of the product $\langle 0|j_\mu|n \text{ pairs}\rangle \times\langle n \text{ pairs}|j_\mu|0\rangle$, and finally integrate over $2n$-particle phase space. In carrying out this procedure, we, of course, use mass-zero electrons and positrons. We have shown that states in which each electron-positron pair is produced with zero total momentum contributes nothing to $b_n(0)$. The calculation of the contribution to $b_n(0)$ from more complicated kinematic configurations of n electron-positron pairs may be feasible, but we have not progressed very far with this calculation.

This concludes a summary of our current meager theoretical knowledge of $f_1(x)$ and we now speculate briefly on the implications of this discussion for high-energy experiments.

The question we pose in confronting alternatives (i) and (ii) of Sec. I with at least conceivable future experiment is the following: The renormalized perturbation expansion for $\overline{\Pi}_R$ contains terms which behave asymptotically like $\alpha\ln(k^2/m^2)$. Such terms modify the photon propagator in a way that can be experimentally detected in sufficiently accurate high-energy electron-electron or electron-positron scattering or electron-positron annihilation. On the other hand we know that if quantum electrodynamics is a finite theory via either mechanism (i) or (ii), then $\overline{\Pi}_R(k)\to$ constant, i.e., $\alpha\overline{D}\sim\alpha_0/k^2$ as $k^2\to\infty$. Hence, in such a case the $\alpha\ln(k^2/m^2)$ terms must not be present for sufficiently high energy. We then ask at what energy these logarithmic terms disappear. Our answer is the following: If the physically correct solution corresponds to alternative (i) with $\alpha_0^{-1}=x_0^{-1}\ll\alpha^{-1}\sim 137$ (e.g., if $\alpha_0=\frac{1}{4}$), then $\alpha\overline{D}$ will not approach its asymptotic limit α_0/k^2 until we reach experimentally unattainable superhigh energies of the order me^{137}. Thus we should expect no deviations from the renormalized perturbation-theory predictions due to electrodynamic effects in any foreseeable experiment. However, in alternative (ii), $\alpha=x_0$, we cannot rule out the possibility that $\alpha\overline{D}$ attains its asymptotic limit at an energy which may be experimentally accessible. Very roughly we might expect this energy to be of the order of magnitude mn_0 where states with n_0 electron-positron pairs give important contributions to $f_1(\alpha)$. (Perhaps $n_0\sim 137$.) However, in lieu of a calculation of f_1, we have no indication of what this energy is. However, if perturbation-theory logarithms disappear in accurate high-energy experiments, then this might be interpreted as information in favor of alternative (ii) and might at the same time give us some information about $f_1(\alpha)$.

The basic reason for the above distinction between (i) and (ii) is that the mechanism by which $\overline{\Pi}_R(k)$ is made finite as $k^2\to\infty$ is completely different in the two cases. In alternative (ii) the asymptotic expansion (4.5) gives a finite result for $\overline{\Pi}_R(k)$ as $k^2\to\infty$ because the coefficients $c_j(\alpha)$ of $[\ln(k^2/m^2)]^j$ all vanish when $\alpha=x_0$. In alternative (i) no conclusion can be drawn directly from (4.5) because one must sum the infinite set of diagrams corresponding to vacuum-polarization insertions in internal photon lines before taking the limit $k^2\to\infty$. When this is done, the different powers of $\ln(k^2/m^2)$ in Eq. (4.5) combine to form a new expression for Π in terms of α_0, which, for $\alpha_0=x_0$, is asymptotically finite. In this case the simplest way to find the rate at which $\overline{\Pi}_R(k^2)$ approaches its limiting finite value is to use the Gell-Mann–Low equation, which we will now briefly describe.

Making certain plausible assumptions about the behavior of perturbation theory integrals for the photon propagator when $m=0$,[4] Gell-Mann and Low[1] derived the following equation for the asymptotic behavior of the photon propagator $\overline{D}(k^2)$:

$$\ln\frac{k^2}{m^2}=\int_{q(\alpha)}^{k^2\alpha\overline{D}(k^2)}\frac{dx}{x^2\psi(x)}. \tag{5.9}$$

In Eq. (5.9), $q(\alpha)$ is the constant in the asymptotic expansion of $k^2\overline{D}(k^2)$. $q(\alpha)=\alpha+O(\alpha^2)$. The function $\psi(x)$ is a function which vanishes at root x_0 of the equation $f(x)=0$. In case (i) as $k^2\to\infty$, $k^2\alpha\overline{D}\to\alpha_0=x_0$, and the integral in (5.9) diverges at the upper limit since $\psi(x_0)=0$. For small values of x, $\psi(x)=\frac{1}{2}\pi(\frac{2}{3}+x/2\pi+\cdots)$. For concreteness let us assume $x_0=\frac{1}{4}$. Let us then ask, "How large does k^2 have to become before $k^2\alpha\overline{D}$ reaches a value which is greater than $\frac{1}{8}$?" We rewrite Eq. (5.9) in the form

$$\ln\frac{k^2}{m^2}=\int_{q(\alpha)}^{1/8}\frac{dx}{x^2\psi(x)}+\int_{1/8}^{k^2\alpha\overline{D}}\frac{dx}{x^2\psi(x)}. \tag{5.10}$$

Since $\psi(x)$ is positive for $0<x<x_0=\frac{1}{4}$, we conclude from Eq. (5.10) that

$$\ln\frac{k^2}{m^2}\underset{k^2\alpha\overline{D}>1/8}{>}\int_{q(\alpha)}^{1/8}\frac{dx}{x^2\psi(x)}\approx\frac{1}{\psi(0)}\left[\frac{1}{q(\alpha)}-8\right], \tag{5.11}$$

that is, if $\alpha_0=x_0=\frac{1}{4}$, the photon propagator will not come within a factor 2 of its asymptotic limit

until

$$\frac{k^2}{m^2} \geqslant \exp\left[\frac{1}{\psi(0)}\left(\frac{1}{q(\alpha)} - 8\right)\right] \sim e^{137}. \tag{5.12}$$

Thus we conclude that if $\alpha_0 = x_0$ is not very small, then the photon propagator does not start approaching its asymptotic limit until superhigh unattainable energies. Thus in alternative (i) we do not expect deviations from renormalized perturbation behavior at any experimentally attainable energy. Of course if x_0 is small, i.e., only slightly greater than α, one cannot arrive at the above conclusion. This is because the factor 8 on the right-hand side of Eq. (5.18) would have to be replaced by a factor of order $1/x_0 \sim 137$ and hence could not be neglected in comparison with that factor $1/q(\alpha) \sim 1/\alpha \sim 137$.

Now let us consider the alternative (ii), $\alpha = x_0$. In this case we can use directly the high-energy expansion (4.5) of perturbation theory. However, we must first make the decomposition (5.3) of $\overline{\Pi}_R(k)$, since as the thresholds become higher, one must go to higher values of k to reach the asymptotic region (5.6). If $\alpha = x_0$, then from (5.3), (5.6), and (4.5)

$$\sum_{n=1}^{\infty} \Pi_n^j(\alpha) = c_j(\alpha) = 0, \quad j = 1, 2, \ldots . \tag{5.13}$$

(5.13) is the condition that the coefficient $c_j(\alpha)$ of $[\ln(k^2/m^2)]^j$ in the high-k expansion of $\overline{\Pi}_R(k)$ vanishes. The coefficients (5.13) vanish as a consequence of (a) the condition $f_1(\alpha) = 0$ and (b) the vanishing of the n-photon amplitudes in $m = 0$ electrodynamics. Let us rewrite Eq. (5.3) in the form

$$\overline{\Pi}_R(k) = \sum_{n=1}^{n_0} \Pi_n(k, \alpha) + \sum_{n=n_0+1}^{\infty} \Pi_n(k, \alpha), \tag{5.14}$$

where n_0 is the number of pairs which is important in the expansion of $f_1(\alpha)$. Then for $k^2 \gg (2mn_0)^2$ we have, using (5.6),

$$\overline{\Pi}_R(k) \underset{k^2 >> (2mn_0)^2}{\approx} \sum_{n=1}^{n_0}\left[\Pi_n^0(\alpha) + \Pi_n^1(\alpha)\ln\frac{k^2}{m^2} + \cdots\right] + \sum_{n=n_0+1}^{\infty} \Pi_n(k, \alpha). \tag{5.15}$$

Then if we use Eq. (5.13), (5.15) becomes

$$\overline{\Pi}_R(k) \underset{k^2 >> (2mn_0)^2}{\approx} \sum_{n=1}^{\infty} \Pi_n^0(\alpha) + \sum_{n=n_0+1}^{\infty}\left[\Pi_n(k, \alpha) - \Pi_n^0(\alpha) - \Pi_n^1(\alpha)\ln\frac{k^2}{m^2} \cdots\right]. \tag{5.16}$$

The first summation on the right-hand side of Eq. (5.16) gives the asymptotic value $\overline{\Pi}_R(\infty)$. For each value of n the sum of the terms in the bracket under the second summation in Eq. (5.16) vanishes as $k^2 \to \infty$. Denoting this sum by $s_n(k, \alpha)$, we can write (5.16) as

$$\overline{\Pi}(\alpha) \underset{k^2 >> (2mn_0)^2}{\approx} \overline{\Pi}_R(\infty) + \sum_{n=n_0+1}^{\infty} s_n(k, \alpha), \tag{5.17}$$

where $s_n(k, \alpha) \to 0$ for k fixed as $n \to \infty$ and for n fixed as $k \to \infty$. $s_n(k, \alpha)$ is the n-pair contribution to the asymptotically vanishing part of the vacuum polarization. If we assume that the dominant contribution to $s_n(k, \alpha)$ comes from the same pair states that are important for $f_1(\alpha)$, we conclude that

$$\sum_{n=n_0+1}^{\infty} s_n(k, \alpha) < \epsilon \quad \text{independent of } k^2, \tag{5.18}$$

where ϵ is of the order of magnitude of $\sum_{n=n_0+1}^{\infty} \alpha^{2n-2} b_n(\alpha)$, which is small by our choice of n_0. In writing (5.18) we have assumed that the convergence is uniform, i.e., ϵ does not depend upon k^2.

Thus if we carry out an experiment at an energy which probes values of $k^2 > (2mn_0)^2$, the logarithmic behavior of perturbation theory disappears and

$$\Pi_R(k^2, \alpha) \underset{k^2 >> (2mn_0)^2}{\approx} \overline{\Pi}_R(k^2, \alpha) + \epsilon. \tag{5.19}$$

As of now we have no way of estimating n_0. It also may be that the value of n_0 that enters into Eq. (5.19) is not the value of n_0 that is important for $f_1(\alpha)$.

In any case it is not impossible that the transition from the perturbation-theory behavior (4.5) to the behavior (5.19) could show up at experimentally accessible energies. If in fact the behavior (5.19) were observed, we could rule out alternative (i) with α_0 not close to α, and it would be likely that alternative (ii) is the physical solution of quantum electrodynamics.

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[8]In the remainder of this paper we will usually call x simply the coupling constant. x is normalized to include the same factors of π as the fine-structure constant.
[9]Everywhere in this paper the photon mass is set equal to zero from the start. The usual photon-mass infrared divergence arises from the second subtraction in the equation for the renormalized electron propagator. This subtraction never has to be carried out if one works in the gauge in which Z_2 is ultraviolet-finite.
[10]Since π is gauge-invariant, in any other gauge the divergences arising from the vertex insertions cancel those arising from the electron self-energy insertions which must be included if Π is calculated in a gauge in which S is not finite.
[11]For $k = 0$ the integral (2.7) diverges at the lower limit. For $k \neq 0$ there is no such divergence since $f(p^2/k^2) \to 0$ as $p^2 \to 0$. This is obvious since the Feynman integrals for Π contain a factor $1/(p+k)^2$ rather than the $1/p^2$ factor which was introduced for convenience in Eq. (2.7).
[12]Setting $x = x_0$ in Eq. (2.7) clearly also eliminates the divergence in $\Pi(0)$ for small p mentioned in Ref. 10.
[13]P. G. Federbush and K. Johnson, Phys. Rev. 120, 1296 (1960).
[14]To obtain all the single-loop diagrams we should also include the self-energy corrections to the electron and positron on the outer loop. Such corrections yield diagrams which are equivalent to those depicted in Fig. 12. The sum of Figs. 11(a) and 12(a) and 12(b) then generate the lowest-order contribution to photon-photon scattering. In the gauge in which Z_2 is finite, we can omit the diagrams of Figs. 12(a) and 12(b).
[15]If the exact value of ϵ turned out to be close to $-\frac{1}{2}$, then there could be an electrodynamic enhancement of a small, weak-interaction-induced electron-muon mass difference. That is, since as $p^2 \to \infty$ electrodynamics behaves like a free-field theory, at some distance Λ the weak interaction becomes the predominant interaction of the electron and muon. Let us assume that the effect of the weak interaction is to cut off the integrals of electrodynamics when the momentum p becomes greater than Λ^{-1}. In this case the cutoff Λ in Eq. (4.13) is a physical cutoff produced by the weak interactions. If we let m_{0e} and $m_{0\mu}$ be the masses of the electron and muon in the absence of electromagnetism, then from Eq. (4.13) m_{0e} and $m_{0\mu}$ are given in terms of the physical masses m_e and m_μ of the electron and muon by the expressions

$$m_{0e} = a m_e \left[\frac{m_e^2}{\Lambda^2}\right]^{\epsilon},$$

$$m_0 = a m_\mu \left[\frac{m_\mu^2}{\Lambda^2}\right]^{\epsilon},$$

or

$$\frac{m_\mu}{m_e} = \left(\frac{m_{0\mu}}{m_{0e}}\right)^{1/(1+2\epsilon)}$$

Thus a small deviation from unity of $m_{0\mu}/m_{0e}$ produced by the weak interactions could be enhanced by electrodynamics into the large observed ratio m_μ/m_e if ϵ is close to $-\frac{1}{2}$.
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[19]Because we are evaluating only the single-closed-loop part of Eq. (5.4), this formula imposes no positivity requirement on $f_1(\alpha)$.