PHYSICAL REVIEW D VOLUME 8, NUMBER 4 15 AUGUST 1973

Is there a Maximal Electrostatic Field Strength?

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(Received 22 May 1972)

Neglecting all nonelectromagnetic interactions, nonperturbative techniques are used based on an effective quantum-electrodynamic Lagrangian. A semiclassical argument leads to a finite "classical" electron radius $r_{\rm min}$ of the order of $\exp[-3\pi/(2\alpha)]$ times the Compton wavelength, and to a corresponding maximal field strength $E_{\rm max} \sim 1/r_{\rm min}{}^2$ (in natural units). The charge renormalization constant Z_3 computed with the cutoff thus suggested consequently has a minimum value of $\alpha/3\pi$, corresponding to a finite bare charge. The bare mass is estimated to be of order α^2 or possibly zero.

I. INTRODUCTION AND SUMMARY

The following considerations are highly speculative. However, we believe that the results are of sufficient interest to warrant wider distribution.

Within the framework of special relativity there is no natural limit below the velocity of light for the relative speed of two inertial systems. Consequently, there is no upper bound for the energy of radiation traveling in a particular direction; one only needs to view it from an appropriately moving Lorentz frame.

The situation is quite different for the Coulomb field (which can be separated invariantly from the radiation field). The quantum-field-theoretic modifications of the $1/r^2$ law have been explored little beyond the lowest orders of perturbation expansion. In particular, the question of the existence of a maximum electrostatic field strength is left open. While it is obvious that nonelectromagnetic interactions modify the emerging picture, it is nevertheless of considerable interest to explore this question for a theory which recognizes only electromagnetic interactions.

Such a study must clearly use nonperturbative techniques. These are rather difficult to come by and one must resort to semiclassical methods, with the understanding that the results must be viewed with suspicion and only as a possible indication of what might be the actual situation.

One of the very few nonperturbative results of quantum electrodynamics is the effective electromagnetic field Lagrangian first derived by Weisskopf.[1] It was later rederived in a different way by Schwinger.[2] This Lagrangian utilizes the *exact* solution of the Dirac equation for an electron in an external, almost constant field. The polarization of the vacuum due to electron pairs described by this solution is fully taken into account (to all orders of the external field, but excluding radiative corrections due to virtual photons) and the expansion in powers of α yields the usual well-known result.

If one wishes to restrict this Lagrangian to an electric field (which can be done invariantly) one finds that it is a complex-valued functional of the field, the imaginary part describing the probability of pair production by the field. In the following we shall concentrate our attention on the real part.

For large fields the Lagrangian is highly nonlinear, and it is far from trivial to determine whether it implies an upper bound on the field. At this point a method comes to mind which had been forgotten for a number of years: the classical model of Born and Infeld.[3] In this model a Lagrangian was invented with the express purpose of yielding a maximum field strength and with it a finite classical electron self-energy. Obviously, this Lagrangian was entirely *ad hoc*.

In contradistinction, the present Lagrangian is an exact consequence of quantum electrodynamics and is a nonperturbative result as far as the external field is concerned. Does this Lagrangian lead to similar qualitative features as the Born-Infeld model?

Before answering this question we note an essential difference between the Born-Infeld model (and similar models) and our case here: In the Born-Infeld model the electromagnetic field Lagrangian was intended to describe the electron as well, so that no separate matter field need to be introduced; the electron is stable as a purely electromagnetic particle derived from that nonlinear Lagrangian. The Weisskopf Lagrangian does not describe a closed system but only the pair-production effects of an external field.

It is therefore not surprising that the present Lagrangian by itself does not describe a stable electron. The matter terms are essential for a closed electromagnetic system, in that the "external" field has a source and becomes part of the system and its dynamics.

But if we restrict quantum electrodynamics to the static limit of a single electron, which limit is essentially a classical system, then we find a surprising result: *The vacuum polarization effects modify the Coulomb field to such an extent that there is a minimum electrostatic electron radius* $r_{\min}$ *and an associated maximum electrostatic field strength* $E_{\max}$.

The physical picture that brings this about is not well understood. At very large distance from a static charge e the Coulomb field $D=e/(4\pi^2r^2)$ coincides with the field strength E. At low fields (i.e., not too close to a static charge) we know from perturbation expansion that a test charge would see a field strength E which is less than D, i.e., less than what would be present without the polarizability of the vacuum; thus, at these low field strengths the vacuum acts like a dielectric with a dielectric "constant" $\epsilon \equiv D/E > 1$. This was known to the earliest authors.[1]

However, as the test charge approaches further the semiclassical picture tells us that the effective polarization diminishes again until ϵ reaches $\epsilon = 1$; then the polarization changes sign, yielding an $\epsilon < 1$. The smallest value ϵ reaches is $\alpha/(3\pi)$, at which point D as a function of E has a maximum. This maximum is responsible for the fact that the test charge cannot come closer than a certain minimum distance $r_{\min}$. Mathematically, Maxwell's equations derived from the Weisskopf Lagrangian have no solutions for values of r less than $r_{\min}$. The double-valuedness of E as a function of D (and therefore r) permits us to accept only one branch (the lower one) as physically meaningful. Thus, there is no D larger than $D_{\max}$.

Alternatively, this situation can be viewed in terms of a bare charge e_0 surrounded by vacuum polarization which diminishes its effectiveness, as measured by E, i.e., by the force it exerts on a test charge. At very large distance one sees the physical charge $e<e_0$ $(Z_3<1)$. As one approaches, one sees at each distance a different effective charge which is at first *less* than e $(\epsilon > 1)$ and then, at smaller distances, greater than e. The effective charge increases until the minimum radius is reached.

If we accept this result we find a number of interesting consequences:

(a) The field-theoretic modifications of the Coulomb potential energy between two static point charges are such that this energy reaches a maximum corresponding to a minimum distance between the charges. The Coulomb behavior persists in shape but is scaled (by a factor 2) since the (larger) bare charge is more effective at this distance.

(b) If one uses the above minimum distance (maximum momentum) as a cutoff on divergent integrals, the charge renormalization constant Z_3 so computed cannot reach zero, but reaches a finite lower limit.

(c) The bare electron mass is of order α and could be zero, corresponding to the fact that all or almost all of the physical electron mass is electromagnetic.

In Sec. II we shall derive the maximum field strength, followed by a discussion in Sec. III. Then in Sec. IV we shall explore its consequences for quantum electrodynamics. Section V contains concluding remarks.

II. DERIVATION OF THE MAXIMUM FIELD STRENGTH

As a preliminary which will also introduce our notation we summarize the pertinent results previously obtained.

The Weisskopf Lagrangian is a functional of the two invariants

$$\mathcal{F} \equiv \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} = \tfrac{1}{2}(B^2 - E^2)$$

and

$$\mathcal{G} \equiv \tfrac{1}{4}F_{\mu\nu}\,{}^{\star}F^{\mu\nu} = B\cdot E,$$

where the star indicates the dual. It does not depend on the derivatives of these invariants; these are assumed to be negligible.[4] By means of the complex variable X defined by

$$X^2 = (B+iE)^2 = 2(\mathcal{F}+i\mathcal{G})$$

Schwinger[2] has written this Lagrangian in the form

$$\mathcal{L} = -\mathcal{F} - \frac{1}{8\pi^2}\int_0^\infty \frac{ds}{s^3}\, e^{-m^2 s}\left[(es)^2\mathcal{G}\,\frac{\operatorname{Re}\cosh(esX)}{\operatorname{Im}\cosh(esX)} - 1 - \tfrac{2}{3}(es)^2\mathcal{F}\right].$$

The invariant electric field limit $\mathcal{G}\to 0$, $\mathcal{F}<0$ leads to a complex $\mathcal{L}$. Schwinger showed that $\mathcal{L}$ can be separated into its real and imaginary parts by integrating along $s = x + i\epsilon$, just above the real axis. One finds an imaginary part $\mathcal{L}_2$ which describes the probability of pair creation per unit space-time volume; this part is of no concern to us at the moment. The real part becomes[2] with $B=0$

$$\mathcal{L}_1(E) = \tfrac{1}{2}E^2 + \frac{\alpha E^2}{2\pi}\int_0^\infty \frac{dt}{t^3}\cos\beta t\,(t\coth t - 1 - \tfrac{1}{3}t^2), \tag{2.1}$$

where $\alpha \equiv e^2/4\pi \approx 1/137$ is the renormalized fine-structure constant and $\beta \equiv m^2/eE$ is dimensionless.

We have investigated this integral in some detail, but only the limiting cases are of interest here. When E is sufficiently small $(\beta \gg 1)$ the integral in (2.1) yields the well-known perturbation-expansion result

$$\mathcal{L}_1(E)=\tfrac{1}{2}E^2\left(1+\frac{4\alpha^2}{45m^4}E^2\right)\quad (E \ll m^2). \tag{2.2}$$

But when E is large ($\beta \ll 1$) the situation is completely different. In that case the dominant contribution to the integral comes from the last term in parentheses, yielding

$$-\tfrac{1}{3}\int_{\pi}^{\infty}\frac{dt}{t}\cos\beta t=\tfrac{1}{3}\ln(\gamma\beta\pi)+O(\beta^2\pi^2).$$

The contribution of the remaining integral is $O(1)$. Thus

$$\mathcal{L}_1(E)=\tfrac{1}{2}E^2\left(1+\frac{\alpha}{3\pi}\ln(\gamma\beta\pi)+O(\alpha)\right).$$

This expression can be written more conveniently as

$$\mathcal{L}_1(E)=\tfrac{1}{2}E^2\left(1-\frac{\alpha}{3\pi}\ln\frac{\kappa E}{m^2}\right)\quad (E \gg m^2), \tag{2.3}$$

where κ is a constant of order 1 which takes into account all terms of order α.

The Maxwell equations which derive from the Lagrangian (2.1) for a source-free region are

$$\begin{aligned}&\dot{\vec{D}}=0,\\ &\nabla\cdot\vec{D}=0,\\ &D_k\equiv\frac{\partial\mathcal{L}_1}{\partial E_k}\quad (k=1,2,3).\end{aligned} \tag{2.4}$$

For a point singularity at the origin (point charge e) the spherically symmetric solution which vanishes at infinity is uniquely

$$\vec{D}=e\vec{r}/4\pi r^3$$

Here e is the renormalized charge, as seen by a test charge at large distance. Since we have spherical symmetry, $D_r=|\vec{D}|\equiv D$ and $E_r=|\vec{E}|\equiv E$. The last relation (2.4) then gives D as a function of E, i.e., it gives the constitutive equation of the vacuum,

$$D=\frac{\partial\mathcal{L}_1}{\partial E}. \tag{2.5}$$

For small fields this becomes with (2.2) the well-known

$$D=E\left(1+\frac{8\alpha^2}{45m^4}E^2\right), \tag{2.6}$$

while for large fields (2.3) gives

$$\begin{aligned}\frac{e}{4\pi r^2}&=D\\ &=E\left(1-\frac{\alpha}{6\pi}-\frac{\alpha}{3\pi}\ln\frac{\kappa E}{m^2}\right)\quad (E\gg m^2).\end{aligned} \tag{2.7}$$

If ϵ is defined as D/E we see that $\epsilon>1$ for small fields and $\epsilon<1$ for large fields.

The result (2.7) shows that, as r decreases, D and E both increase monotonically up to the point where D has a maximum as a function of E; there,

$$D_{\max}=\frac{\alpha}{3\pi}E_{\max}, \tag{2.8}$$

where $E_{\max}$ is that value of E for which D attains its maximum,

$$E_{\max}=\frac{m^2}{\kappa}e^{3\pi/\alpha-3/2}. \tag{2.9}$$

This occurs at a radius

$$r_{\min}=\frac{1}{m}\left(\frac{9\pi\kappa^2e^3}{4\alpha}\right)^{1/4}e^{-3\pi/2\alpha} \tag{2.10}$$

as follows by an easy calculation. (Note that e here is 2.718..., while the electron's charge occurs only in α.)

Thus, a minimum radius emerges from the double-valuedness of $E=E(r)$ at very small r, because this function has a domain of $r\geq r_{\min}$, and only the lower branch can have physical meaning.

The numerical value of $r_{\min}$ is much smaller than any length of physical interest. Its meaning lies of course in the field-theoretic interpretation of $r_{\min}$ as a cutoff length corresponding to a momentum cutoff

$$k_{\max}\sim\frac{1}{r_{\min}}=\sqrt{C}\,me^{3\pi/2\alpha}. \tag{2.11}$$

This cutoff refers to static interaction only.

III. DISCUSSION

An obvious objection to the results of the preceding section is that they were obtained under assumptions for which the Weisskopf-Schwinger Lagrangian is not applicable. This Lagrangian was derived for almost constant fields. Near the Coulomb singularity such an assumption is clearly violated.[4] $\nabla\cdot E$ does not vanish and for small distances can easily exceed $m|E|$. In fact at $E_{\max}$ one finds that $\nabla\cdot E$ diverges.

Since this argument seems very convincing let us look at a physical system for which the Weisskopf Lagrangian certainly does hold. Consider a parallel-plate capacitor of infinite plate size producing a constant homogeneous electrostatic field. If the voltage difference is large enough the expression (2.3) for $\mathcal{L}_1$ will be valid, and so will be the field equations (2.4) as well as the relation between D and E, the second equality (2.7). Thus, all these equations hold quite independent of the boundary conditions for which the field equations are solved.

This implies that $D(E)$ will have a maximum also for the constant-field case and Eqs. (2.8) and (2.4) still hold. The difference becomes apparent only when we ask how D is determined by the external conditions. In the parallel-plate capacitor we have $D=\sigma$, the charge density on the plates, instead of Coulomb's law. As we increase σ, i.e., D, we see that E will also increase. But this continues only up to $D_{\max}$. The equations do not allow for an increase of the charge density σ beyond $\sigma_{\max}=D_{\max}$. Thus the only liberty we have taken in the preceding section was to apply this rigorous consequence of the Weisskopf-Schwinger Lagrangian to the Coulomb case.

Another question that may be raised is whether the spin of the electron (and the associated Fermi statistics) may have something to do with this impossibility to exceed a maximum field.

It is not difficult to show that for charged particles of *spin zero* the real part of the Weisskopf Lagrangian, the analog of (2.1), is

$$\mathfrak{L}_1(E)=\tfrac{1}{2}E^2-\frac{\alpha E^2}{4\pi}\int_0^\infty\frac{dt}{t^3}\cos\beta t\left(\frac{t}{\sinh t}-1+\tfrac{1}{6}t^2\right). \tag{3.1}$$

The low-field limit now becomes

$$\mathfrak{L}_1(E)=\tfrac{1}{2}E^2\left(1+\frac{7\alpha^2}{180m^4}E^2\right)\quad(E\ll m^2), \tag{3.2}$$

which is a known result. The high-field limit is

$$\mathfrak{L}_1(E)=\tfrac{1}{2}E^2\left(1-\frac{\alpha}{12\pi}\ln\frac{\kappa' E}{m^2}\right)\quad(E\gg m^2), \tag{3.3}$$

with κ' having the same significance as κ.

Thus, all high-field results of spin $\frac{1}{2}$ remain valid for spin 0 if only one replaces α by $\frac{1}{4}\alpha$ and κ by κ'.

The ability of the electric field to permit the production of *real* pairs in addition to virtual pair production is of course responsible for the complex nature of the Weisskopf Lagrangian. And since we ignored the imaginary part, it can well be argued that our result is meaningless, being associated with a highly unstable system in which a more careful calculation would be necessary to be convincing.

For this reason it is of interest to consider the limit $E\to 0$ in $\mathfrak{L}$. The case of a purely magnetic constant field leads to a *real* Lagrangian for which the above objection cannot be raised: A constant homogeneous magnetic field cannot produce real pairs.

One now takes the limit $\mathfrak{G}\to 0$, $\mathfrak{F}\to\frac{1}{2}B^2$ and finds for spin $\frac{1}{2}$

$$\mathfrak{L}=-\tfrac{1}{2}B^2-\frac{(eB)^2}{8\pi^2}\int_0^\infty\frac{dt}{t^3}e^{-\beta t}(t\coth t-1-\tfrac{1}{3}t^2), \tag{3.4}$$

where

$$\beta=\frac{m^2}{eB}. \tag{3.5}$$

For small B this leads to

$$\mathfrak{L}=-\tfrac{1}{2}B^2\left(1-\frac{4}{45m^4}\alpha^2B^2\right), \tag{3.6}$$

so that

$$H\equiv-\frac{\partial\mathfrak{L}}{\partial B}=B\left(1-\frac{8\alpha^2}{45m^2}B^2\right). \tag{3.7}$$

This is completely analogous to the electrostatic case (2.2) and (2.6), yielding

$$\mu\equiv\frac{B}{H}=1+\frac{8\alpha^2}{45m^2}B^2>1$$

in this approximation.

For large magnetic fields one finds

$$\mathfrak{L}=-\tfrac{1}{2}B^2\left(1-\frac{\alpha}{3\pi}\ln\frac{\kappa B}{m^2}\right). \tag{3.8}$$

Thus,

$$H=B\left(1-\frac{\alpha}{6\pi}-\frac{\alpha}{3\pi}\ln\frac{\kappa B}{m^2}\right) \tag{3.9}$$

and we see that the purely magnetic case is completely analogous to (the real part of) the purely electric case also for very large fields: One finds an $H_{\max}$ despite the fact that no real pairs can be produced in a constant magnetic field. The maximal fields are equal: $H_{\max}=D_{\max}$, $B_{\max}=E_{\max}$.

Finally, the approximate nature of the Weisskopf Lagrangian should be recalled: It neglects radiative corrections to electron propagators. However, it is difficult to believe that this approximation would cause a qualitative change in the results presented here.

As an illustration how this physically reasonable assertion may emerge mathematically one can consider the following example. It is plausible that radiative corrections modify (2.7) as follows:

$$D=E\left[1-\frac{\alpha}{6\pi}-\frac{\alpha}{3\pi}\ln\frac{\kappa E}{m^2}-\sum_{n=2}^{\infty}c_n\left(\ln\frac{\kappa E}{m^2}\right)^n\right]. \tag{3.10}$$

In order to see what this may do let us assume that

$$c_n=-\frac{\alpha}{3\pi}\frac{a^{n-1}}{(n-1)!}, \tag{3.11}$$

where a must clearly be of order α. Then the sum can be carried out and one obtains

$$D=E\left[1-\frac{\alpha}{6\pi}-\frac{\alpha}{3\pi}\left(\frac{\kappa E}{m^2}\right)^a \ln\frac{\kappa E}{m^2}\right]. \tag{3.12}$$

A simple calculation shows that the factor $(\kappa E/m^2)^a$ has little effect and that the maximum obtained from (3.12) is qualitatively the same as the one obtained from (2.7). We shall return to this question at the end of the following section.

IV. CONSEQUENCES

If we take seriously the high-momentum cutoff suggested by (2.11) then all calculations of radiative corrections must take this cutoff into account. In the following we shall do so for the estimate of radiative corrections to the photon propagator, and we shall obtain a corresponding potential energy between two static charges. We shall also apply it to the estimate of the radiative corrections to the charge renormalization constant Z_3 and to the electron self-energy. These results will of course be speculative, depending on (2.11).

We define our free causal photon propagator as $D_c(k^2)=(4\pi^2k^2)^{-1}$, with the usual Feynman contour. We also have the well-known relation between the renormalized propagator D'_{c1} and the unrenormalized one D'_c,

$$D'_{c1}=Z_3^{-1}D'_c, \tag{4.1}$$

where for very large k^2, $D'_c\cong D_c$. The one-photon exchange contribution to the potential energy between two static point charges then becomes[5]

$$V(r)=2\alpha\int D'_{c1}(k^2)\,e^{ik\cdot x}\,d^4k, \tag{4.2}$$

where α is the renormalized fine-structure constant, $\alpha\approx 1/137$. For small α this yields asymptotically[6]

$$V(r)=\frac{\alpha}{r}\left[1+\frac{2\alpha}{3\pi}\left(\ln\frac{1}{mr}-\tfrac{5}{6}-\ln\gamma\right)+O(\alpha^2)\right]. \tag{4.3}$$

With our result (2.10) the maximum value is reached at $r=r_{\min}$, viz.,

$$V(r_{\min})=\frac{2\alpha}{r_{\min}}\,[1+O(\alpha)]. \tag{4.4}$$

This is to be compared with $V(r)=\alpha/r$ at large distances. Vacuum polarization therefore increases the Coulomb potential energy between two charges at small distances until it reaches exactly twice its classical value.

The renormalization constant Z_3 can be estimated[7] by a partial summation of diagrams which, one can argue,[8] give the leading contributions,

$$Z_3^{-1}=\left(\frac{\alpha}{3\pi}\right)^{-1}\left(\frac{1}{x}+\frac{1}{1-e^x}\right), \tag{4.5}$$

where

$$x=\frac{3\pi}{\alpha}-\ln\frac{|k_m^2|}{m^2}$$

and k_m^2 is the cutoff on the divergent integrals. If we substitute

$$k_m^2=m^2Ce^{3\pi/\alpha},\quad C=\frac{2}{3\kappa}\left(\frac{\alpha}{\pi e^3}\right)^{1/2} \tag{4.6}$$

we find $x=-\ln C$ and

$$Z_3=\frac{\alpha}{3\pi}\,\frac{(C-1)\ln C}{1+C(\ln C-1)}. \tag{4.7}$$

This function is monotonically decreasing and reaches $Z_3=\alpha/3\pi$ asymptotically for $C\to\infty$. It has the value $Z_3=2\alpha/3\pi$ for $C=1$. One finds that $Z_3<1$ for all values of C greater than $\exp(-3\pi/\alpha)$. The latter value is of course unreasonably small, since it would mean $r_{\min}\sim 1/m$. Thus we are led to the result

$$\frac{\alpha}{3\pi}<Z_3<1, \tag{4.8}$$

with the most likely value near the lower limit. The estimate (4.6) for C yields less than 10^{-2} when $\kappa=1$, a small enough value for Z_3 to be well approximated by $(\alpha/3\pi)(\ln C)$.

The best closed expression for the electron self-energy known to us is the one derived by Landau *et al.*[8] which gives the bare mass m_0 in terms of the physical mass m as

$$m_0=m\left(1-\frac{\alpha}{3\pi}\ln\frac{\Lambda^2}{m^2}\right)^{9/4},$$

where Λ is the upper cutoff. Using $k_{\max}$ for Λ, (2.9) yields

$$m_0=m\left(\frac{\alpha}{3\pi}\ln\frac{1}{C}\right)^{9/4},$$

i.e., suggests $C\leqslant 1$ and a very small and possibly vanishing bare electron mass.

Conversely, if we require $m_0=0$ we obtain a cutoff

$$\Lambda=me^{3\pi/2\alpha}$$

which is very close to the value $k_{\max}$ of (2.11), and which gives $Z_3=\alpha/3\pi$ from (4.5).

These considerations are related to the work of Gell-Mann and Low[6] in two respects. First, it is noted that Z_3 and m_0/m are here found without any explicit reference to the bare charge. The high-momentum cutoff which emerges from the maximal field strength considerations is provided by the theory itself and is not put in *ad hoc*. This intrinsic cutoff depends only on the renormalized charge.

Gell-Mann and Low also found that the renormalized photon propagator is independent of the electron mass and its asymptotic form can be obtained from the limit $m \to 0$. Here we find that the same intrinsic cutoff which makes $m_0 = 0$ (or of order α^2) also determines Z_3 and thus fixes the asymptotic form of the photon propagator.

A vanishing bare mass dominates the electrodynamics of Johnson, Baker, and Willey.[9] Their work would correspond to $C = 1$ in the above equations.

It must be mentioned here that Gell-Mann and Low[6] and Johnson, Baker, and Willey[9] presented arguments in favor of a finite Z_3 such that the coefficient of the logarithmically divergent parts of this renormalization constant would vanish. This suggests, but does not imply, that the finite renormalized vacuum polarization contains no logarithm for large fields.[10] While there is no rigorous proof for such speculations, one may arrive in this way at the opinion that the higher radiative corrections modify the results presented here in a qualitative way and do not permit a $k_{\max}$.

On the other hand, if one assumes that the exact solution of the Dirac equation is a qualitative guide to the exact renormalized solution of quantum electrodynamics, one is led to the result that vacuum polarization provides a natural cutoff of the Coulomb field so that the charged particles are not point singularities, but have an effective finite size even in the absence of other (e.g., weak) interactions.[11]

Finally, the stability question of the quantum field-theoretic electron is not affected by these considerations. The electron is stable (the self-stress vanishes) simply as a consequence of having only one fundamental length in the theory[12] (viz., $1/m$). The finiteness (i.e., *any* cutoff) of the self-energy suffices to yield a vanishing self-stress.

V. DISCUSSION

The double-valuedness of E as a function of r, Eq. (2.7), corresponds to a field distribution remindful of a situation in general relativity: Two (outer) Schwarzschild solutions can be joined to form a single (double-sheeted) surface which, in suitable coordinates, is free from singularities[13] (Einstein-Rosen bridge). Thus, in vacuum polarization as in the Schwarzschild solution the double-valuedness of a function leads to a description in terms of two sheets, only one of which is physical; these sheets are joined by a "neck" which in some sense is the radius of the object.

The prevention of the point singularity in electrodynamics as a consequence of the highly nonlinear behavior of the theory for very high fields is a most desirable feature. As was pointed out repeatedly by Einstein, a consistent field theory should not describe its sources as point singularities.

We have shown that in order to obtain a singularity-free static field in quantum electrodynamics reminiscent of the Einstein-Rosen bridge, one can proceed in two steps. First one shows that for a constant external field the Weisskopf Lagrangian predicts a maximum field. Then one assumes[14] that these results can be extended to the Coulomb field where the same $E_{\max}$ leads to a minimum radius.

If one accepts these arguments then one concludes that at least for static fields quantum electrodynamics provides its own cutoff, replacing the point charges by finite-size objects.

We conclude with a statement about the nature of this paper. We are presenting a *conjecture*, i.e., something that cannot at the present time be either proven or disproven. The arguments which we give in favor of this conjecture are not proofs and are not meant to be proofs. Their strengths depends on one's individual prejudices. But an honest and objective statement can only be "we do not know" when confronted with unproven assertions. In order to save time and energy to the readers we summarize here the main open questions relevant to our conjecture.

(1) We use a Lagrangian for almost constant fields for a situation in which the fields are not constant. Whether this leads to *qualitatively* incorrect results is not known.

(2) We neglect radiative corrections. Whether this leads to qualitatively incorrect results is not known. Some arguments pro and con are presented in footnote 10.

Despite all this we felt that the conjecture of the existence of maximal field strengths as a consequence of the (so far hardly explored) nonlinear nature of quantum electrodynamics is of sufficient interest to have others think about it.

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*Some of the results presented here are contained in a thesis submitted by M. Greenman in partial fulfillment for the requirement of the M.A. degree.

†Work supported in part by a grant from the National Science Foundation.

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[3]M. Born, Proc. R. Soc. A143, 410 (1934); M. Born and L. Infeld, *ibid.* A144, 425 (1934).

[4]The precise validity condition (Ref. 2), $|2F^{\mu\nu}(p_\nu - eA_\nu)| >> |\partial_\nu F^{\mu\nu}|$, reduces for $A=0$ in the static limit to $2|mE| >> |\nabla \cdot E|$.

[5]See, e.g., S. S. Schweber, *An Introduction to Relativistic Quantum Field Theory* (Harper and Row, New York, 1961).

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[8]L. D. Landau, A. Abrikosov, and L. Halatnikov, Nuovo Cimento Suppl. 3, 80 (1956) and earlier work by these authors quoted therein.

[9]K. Johnson, M. Baker, and R. Willey, Phys. Rev. 136, B1111 (1964) and later papers; M. Baker and K. Johnson, Phys. Rev. 183, 1292 (1969).

[10]The argument is as follows: If Z_3 is to be finite one can impose an eigenvalue condition on the coupling constant which leads to a vanishing of the logarithmically divergent counterterms of charge renormalization. Tracing the origin of the terms (Ref. 2) in (2.1), one finds that the $-\frac{1}{3}t^2$ term should be absent. But this may lead to a divergent result for the low-field case ($\beta >> 1$) instead of (2.2) and may contradict perturbation theory, which is known to be valid in that limit. However, it is still possible to maintain that the $-\frac{1}{3}t^2$ term in (2.1) is absent in the high-field limit ($\beta << 1$). In that case no logarithm would be obtained in (2.3). But such a radical modification of the $-\frac{1}{3}t^2$ term in (2.1) will undoubtedly be concomitant with severe modifications of the other terms in (2.1), so that the absence of a logarithm in (2.3) can only be a conjecture. (The conjecture of a finite Z_3 runs contrary to recent results of constructive quantum field theory. Models which can be treated rigorously confirm the existence of the *divergent* renormalization constants exactly as predicted by perturbation expansion and suggest that the same would also be true in quantum electrodynamics.)

[11]It must be emphasized that the above applications of $k_{\max}$ to various divergent expressions such as (4.3), (4.5), etc. should not be taken too seriously and in any case are not meant to justify the existence of a $k_{\max}$. The reason is that the divergent expressions used above are all derived from perturbation expansion. This is just the approximation we have been trying to avoid. For example, the photon propagator is found from perturbation expansion to contain the factor

$$d^{-1} = 1 - \frac{\alpha}{3\pi} \ln \left| \frac{k^2}{m^2} \right| ,$$

which, with our $k_{\max}$, is of order α. But "the question of the true behavior of the photon Green's function in the region where $\alpha d << 1$ does not hold cannot be answered on the basis of results of perturbation theory" is an almost verbatim quote from N. N. Bogoliubov and D. D. Shirkov, *Introduction to the Theory of Quantized Fields* (Interscience, New York, 1959), p. 528. In contradistinction, our logarithmic term, (2.7), was obtained without the use of perturbation expansion.

[12]See, e.g., J. M. Jauch and F. Rohrlich, *Theory of Photons and Electrons* (Addison-Wesley, Reading, Mass., 1955), especially Sec. 16-4.

[13]A. Einstein and N. Rosen, Phys. Rev. 48, 73 (1935); M. D. Kruskal, *ibid.* 119, 1743 (1960).

[14]In order to establish these results in a more acceptable way a better calculation is necessary for example along the lines of E. Wichmann and N. M. Kroll, Phys. Rev. 101, 843 (1956).