and A. H. Taub, Zh. Eksp. Teor. Fiz. 55, 233 (1968) [Sov. Phys.–JETP 28, 122 (1969)].

[13]The wave equation in this form assumes minimal coupling. The more general form admitting conformal coupling contains an additional term $\frac{1}{6}R\Phi$, where R is the scalar curvature of spacetime. [The coefficient is given for a n-dimensional space as $(n-2)/4(n-1)$.] The equation becomes conformally invariant in the case $\mu=0$. See R. Penrose, in *Relativity Groups and Topology*, edited by C. DeWitt and B. DeWitt (Gordon and Breach, New York, 1964).

[14]P. Winternitz, I. Lukac, and Ya. A. Smorodinskii, Yad. Fiz. 7, 192 (1968) [Sov. J. Nucl. Phys. 7, 139 (1968)]; Ya. A. Smorodinskii and I. I. Tugov, Zh. Eksp. Teor. Fiz. 50, 653 (1966) [Sov. Phys.–JETP 23, 434 (1966)].

[15]Our presentations in Secs. IV and V follow G. W. King, R. M. Hainer, and P. C. Cross [J. Chem. Phys. 11, 27 (1943)], and C. Van Winter, [Physica (Utr.) 20, 274 (1954)]. The analogy of the present problem with the nuclear collective model was kindly called to the attention of the author by J. A. Wheeler. See, e.g., A. Bohr and B. R. Mottleson, K. Dan. Vidensk. Selsk. Mat.-Fys. Medd. 27, No. 16 (1953); D. L. Hill and J. A. Wheeler, Phys. Rev. 89, 1102 (1953).

[16]R. L. Kronig and I. I. Rabi, Phys. Rev. 29, 262 (1929); F. Reiche and H. Rademacher, Z. Phys. 39, 444 (1926).

[17]See, e.g., A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton Univ. Press, Princeton, New Jersey, 1957); E. P. Wigner, *Group Theory and Its Application to the Quantum Mechanics of Atomic Spectra* (Academic, New York, 1959).

[18]S. C. Wang, Phys. Rev. 34, 243 (1929). w_{KM}^{J} is an eigenfunction of the quantum-mechanical Hamiltonian for a symmetric top. As the group manifold of the mixmaster space (three-sphere) is in two-to-one correspondence with the configuration space of the top, half-integral J's appear in the mixmaster problem and not for the top.

[19]B. S. Ray, Z. Physik 78, 74 (1932).

[20]R. S. Mulliken, Phys. Rev. 59, 873 (1941).

[21]D. M. Dennison, Rev. Mod. Phys. 3, 280 (1931).

[22]G. W. King, J. Chem. Phys. 15, 820 (1947).

[23]For a detailed treatment of the asymmetric top problem using Lamé functions, see H. A. Kramer and G. P. Ittmann, Z. Phys. 53, 553 (1929); 58, 217 (1929); 61, 663 (1930).

[24]A comprehensive exposition of the theory of Lamé functions can be found in F. M. Arscott, *Periodic Differential Equations* (MacMillan, New York, 1964); see also E. W. Hobson, *The Theory of Spherical and Ellipsoidal Harmonics* (Chelsea, New York, 1955).

[25]I. Lukac and Ya. A. Smorodinskii, Zh. Eksp. Teor. Fiz. 57, 1342 (1969) [Sov. Phys.–JETP 30, 728 (1970)].

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One-Loop Corrections to Tree Diagrams

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Motivated by quantum gravity (where multiloop and in particular one-loop corrections pose large problems) we study the zero-space dimension ψ^4 model. A nonperturbative solution in the one-loop approximation for the relative vacuum amplitude is obtained. This amplitude is found to be nonanalytic in λ. This has possible implications for the gravity-modified quantum electrodynamics as proposed by Salam. The gravity Green's functions (at least the two-point functions) have a $\log\lambda$-type behavior. We discuss why it is plausible that summation over all orders in perturbation theory may change this $\log\lambda$ dependence. We use an approximate connection between the Schwarzschild radius of the electron and the behavior in λ to suggest plausible effects of quantum-loop corrections on the Schwarzschild solution.

I. INTRODUCTION

Ever since field theory was formulated it has been a major problem to go beyond perturbation theory. There are innumerable questions of interest which require a nonperturbative treatment for their solutions. However, we have been mainly motivated by attempts in quantum gravity to understand quantum fluctuations about the classical solutions (for the metric).

Intuitively we can see why nonperturbative treatments are required. The classical solution can be built up by summing all tree graphs.[1,2] In fact in a recent work Duff[3] has explicitly shown this for the Schwarzschild solution using an extended source (since point sources give divergent tree diagrams in his case); so we see that the classical solution is already a nonperturbative one. The natural extension of this program is to sum up all diagrams with one loop. The eventual hope is that a correct treatment of quantum fluctuations will remove the genuine singularities occurring in classical gravity. We have a nontrivial example of this action given by the action functional

$$I=\int\{p(t)\dot{q}(t)-p(t)q^2(t)-V[p(t)]\}dt$$

[with $p(t)>0$]. This example has been studied by Klauder[4] and he calls it an elementary model of quantum gravity. The classical solution of this model exhibits singularities but these disappear in the quantum theory.

The Einstein functional is however incomplete since it gives rise to a nonunitary theory. Fictitious particle contributions are needed. Alternatively this latter contribution can be thought of as a consequence of the presence of the non-Abelian gauge symmetry. One way of summing all one-loop corrections to the tree graphs is to use Feynman path integrals and make a particular approximation for the action. This is a program suggested by Blokhintsev.[5,6] We now give the form of this approximation. Following DeWitt[1] we adopt the notation below.

S: classical action (without source term)

ψ^i: quantum field where i denotes a label representing all the indices both continuous and discrete,

J_i: source function,

$$e^{iW[J]}\equiv\langle 0,\infty\,|\,0,-\infty\rangle_J,$$

$$\left.\frac{\delta W[J]}{\delta J}\right|_{\hbar\to 0}\equiv\hat{\phi}^i,$$

$$\hbar^{1/2}\phi^i\equiv\psi^i-\hat{\phi}^i,$$

$$S_{1\,(\text{loop})}=S[\hat{\phi}]+\tfrac{1}{2}S_{,ij}\,\phi^i\phi^j+J_i\,\phi^i,$$

where

$$_{,ij\cdots k}\equiv\frac{\delta}{\delta\psi^i}\cdots\frac{\delta}{\delta\psi^k}.$$

Owing to the extreme algebraic complications of the Einstein and gauge-breaking parts of the action functional we will not deal with this case. Instead we shall deal with the model

$$L_{\text{int}}=\lambda q^4(t),$$

where t denotes time. This obviously does not have gauge symmetries, and also no problems of renormalization arise, but it is a nonlinear interaction and gives rise to a Hamiltonian with positive definite spectrum. This requirement is quite closely connected with the fact that the signature of the space-time manifold is an invariant under a classical or a quantum action and hence we have a tie-up with Klauder's model [where $p(t)>0$ ensured the invariance of the signature].

We manage to obtain a closed form for the path integral for paths between configurations at time t_a and at time t_b. Some of the finite number of parameters in the solution are given implicitly (i.e., are determined by certain complex equations). However unrealistic the Lagrangian is, it still throws light on possible behaviors of quantum field theories. However, it must be stated that an increase in dimensions may change the nature of the solutions drastically. At the moment unfortunately there is no prospect of solving $L_{\text{int}}=\lambda\phi^4(x)$ with x a 4-vector. Symanzik's[7] Euclidean field theory, although an interesting approach, still has difficulties with renormalizations. It is easy to obtain an expression for the self-energy functional as a determinant of a continuous matrix. The usual Fredholm methods just reproduce the perturbation diagrams of Feynman. Hence the program suggested by Blokhintsev for calculating quantum fluctuations to gravity needs more powerful methods to work out determinants. Our success is that in the case we have considered we have managed a nonperturbative solution for the determinant. Our methods are drastically different from those used by Simon.[8]

Of course the implications for quantum gravity from the study of such a model by necessity have to be somewhat intuitive. The effect of quantum gravity in hadronic and leptonic interactions has recently been stressed by Isham, Salam, and Strathdee.[9] In gravity-modified quantum electrodynamics the effect of gravity is to remove ultraviolet infinities and instead have a factor $\ln\lambda$ (where λ is the gravitational coupling constant). However, no summation in perturbation theory was attempted in our sense. Only the two-point functions were considered, and it was hoped that higher-order calculations, if ever done, would not change the result. We will see in the next section why the behavior with respect to λ may change. Now Weisskopf (as noted by Salam *et al.* in Ref. 9) has interpreted the infinity suppression in electrodynamics as connected with the limiting frequency of a standing photon wave in the curved space-time around an electron. Such a photon has the wavelength given by the Schwarzschild radius and it is not unreasonable to regard this as the definition of the latter. From general relativity this limiting frequency is $1/3m_e$ (where m_e is the electron mass). Using Weisskopf's formula for $\delta m_e/m_e$ we get something proportional to $\ln(1/3m_e\lambda)$. In this we have a sort of connection between the Schwarzschild solution and the analyticity of our S-matrix elements. We can perhaps get some idea about the effect of higher-order quantum corrections on the Schwarzschild solutions from this. If for example the λ behavior becomes $(\ln\lambda)\lambda^{-1/2}$

owing to quantum-loop corrections, the Schwarzschild radius behaves like $\lambda\exp(-\lambda^{-1/2})$.

II. THE PATH INTEGRAL

We consider

$$L=-\tfrac{1}{2}\mu^2[q(\tau)]^2+\tfrac{1}{2}[\dot q(\tau)]^2+\tfrac{1}{4}\lambda[q(\tau)]^4. \quad (2)$$

The equation of motion is

$$\ddot q(\tau)+\mu^2 q(\tau)=\lambda[q(\tau)]^3. \quad (3)$$

A class of solutions[10,11] is given by

$$q_c(\tau)=Ce(\Omega\tau), \quad (4)$$

with

$$\Omega^2=(1-\lambda C^2)\mu^2,$$

$$k^2=-2\lambda C^2(\Omega^2)^{-1},$$

and

$$e(y)\equiv \mathrm{cn}(y,k)+i\,\mathrm{sn}(y,k).$$

As $\lambda\to 0$, $k\to 0$, $e(y)\to e^{iy}$. We define $\eta(t)$ by

$$q(\tau)=q_c(\tau)+\hbar^{1/2}\eta(\tau). \quad (5)$$

Hence

$$S_1[q(\tau)]=S[q_c(\tau)]+\hbar\int_{t_a}^{t_b}d\tau\,\{\tfrac{1}{2}[\dot\eta(\tau)]^2-\tfrac{1}{2}\mu^2[\eta(\tau)]^2+\tfrac{3}{2}\lambda q_c^{\,2}(\tau)\eta^2(\tau)\} \quad (6)$$

since

$$S=\int_{t_a}^{t_b}d\tau\,\{-\tfrac{1}{2}\mu^2[q(\tau)]^2+\tfrac{1}{2}[\dot q(\tau)]^2+\tfrac{1}{4}\lambda[q(\tau)]^4\}$$

and

$$S[q_c]=\frac{1}{\Omega}\int_{\Omega t_a}^{\Omega t_b}d(\Omega\tau)\{\tfrac{1}{2}C^2\Omega^2[\dot e(\Omega\tau)]^2-\tfrac{1}{2}\mu^2C^2[e(\Omega\tau)]^2+\tfrac{1}{4}\lambda C^4[e(\Omega\tau)]^4\}. \quad (7)$$

The details of the calculation of $S[q_c]$ are somewhat long but straightforward and so will be given in an appendix.

$$\Omega S[q_c]=[F(u)]_{u=\Omega t_b}-[F(u)]_{u=\Omega t_a}, \quad (8)$$

where

$$F(u)=E(u)\left[\tfrac{1}{6}C^2\Omega^2\left(1-\frac{2}{k^2}\right)-\frac{\mu^2C^2}{k^2}+2\lambda C^4\left(\frac{k^2-2}{3k^4}\right)\right]$$
$$+u\left[\tfrac{1}{3}C^2\Omega^2\frac{k'^2}{k^2}-\tfrac{1}{2}\mu^2C^2\left(1-\frac{2}{k^2}\right)+\frac{\lambda C^4}{12k^4}(14k'^2+k^2-3k^2k'^2+2)\right]+\mathrm{sn}u\,\mathrm{cn}u\,\mathrm{dn}u\left(-\tfrac{1}{3}C^2\Omega^2+\frac{2\lambda C^4}{3k^2}\right)$$
$$+\mathrm{dn}^3u\left(\tfrac{1}{9}i\,\frac{C^2\Omega^2}{k^4}\right)-i\lambda\frac{C^4}{k^4}\mathrm{dn}^2u+iC^2\,\mathrm{dn}u\left[\frac{(\mu^2+\lambda C^2)k^2+2\lambda C^2k'^2}{k^4}\right]$$

and where

$$E(u)=\int_0^u \mathrm{dn}^2v,\quad k'=(1-k^2)^{1/2}. \quad (9)$$

It is known that[12]

$$\int \mathfrak{D}_F(q)e^{i\epsilon\langle\dot q,\dot q\rangle/2}F\{q\}=\int \mathfrak{D}_W(q)e^{-\epsilon\langle\dot q,\dot q\rangle/2}F\{i^{1/2}q\}, \quad (10)$$

where $\mathfrak{D}_W(q)$ is Wiener measure and $\mathfrak{D}_F(q)$ is Feynman "measure." Now

$$\int \mathfrak{D}_W(\chi)e^{-\epsilon\langle\dot\chi,\dot\chi\rangle/2}\exp\left(\tfrac{1}{2}\int_0^t d\tau\,Q\chi^2\right)=[\pi f(0)]^{-1/2}, \quad (11)$$

for the conditional integral where

$$f''(\tau)+Q(\tau)f(\tau)=0,\quad f(t)=0,\quad f'(t)=-1. \quad (12)$$

(We can take the initial and final times as 0 and t, i.e., $t_a=0$ and $t_b=t$, without loss of generality.) We are interested in

$$K(b;a)=\int \mathfrak{D}_F(\eta)\exp\left[\frac{i}{\hbar}\left(S[q_c]+\hbar\int_{t_a}^{t_b}\{\tfrac{1}{2}[\dot\eta(\tau)]^2+\eta^2(\tau)[\tfrac{3}{2}\lambda q_c{}^2(\tau)-\tfrac{1}{2}\mu^2]\}d\tau\right)\right]$$
$$=\exp\frac{i}{\hbar}S[q_c]\int \mathfrak{D}_F(\eta)\exp\left[i\int_{t_a}^{t_b}\{\tfrac{1}{2}[\dot\eta(\tau)]^2+\eta^2(\tau)[\tfrac{3}{2}\lambda C^2e^2(\Omega\tau)-\tfrac{1}{2}\mu^2]\}\right]d\tau\,. \quad (13)$$

Strictly speaking (10) is a valid equality when $\{q(t)=q(0)=0\}$ or $\{q(0)=0,\ q(t)\text{ unrestricted}\}$. However, the original approach of C. M. DeWitt[12] does not make this restriction and for our considerations this is of no interest. The relevant version[13] of (12) for us is

$$f''(\tau)+[\mu^2-3\lambda C^2e^2(\Omega\tau)]f(\tau)=0\,. \quad (14)$$

The singularities of $e(u)$ within the period parallelogram $(4K, 4K+4K')$ are simple poles at

$$(4K+i3K')\text{ with residue }2i/k$$
$$\text{and} \quad (15)$$
$$(6K+i3K')\text{ with residue }-2i/k\,,$$

where

$$K\equiv\int_0^{\pi/2}(1-k^2\sin^2\theta)^{-1/2}d\theta\,,$$
$$K'\equiv\int_0^{\pi/2}(1-k'^2\sin^2\theta)^{-1/2}d\theta\,. \quad (16)$$

For convenience we let $\omega=4K$ and $\omega'=2K'$. $e^2(u)$ has of course double as well as single poles. From above, within the fundamental parallelogram $(\omega, \omega+2\omega')$

$$e(u)=\frac{2i/k}{u-\omega-i\frac{3}{2}\omega'}-\frac{2i/k}{u-\frac{3}{2}\omega-i\frac{3}{2}\omega'}+A(u) \quad (17)$$

[where $A(u)$ is analytic and free from singularities within the parallelogram]. Consequently within the fundamental parallelogram the double-pole terms are given by

$$-\frac{4}{k^2}[u-(\omega+i\tfrac{3}{2}\omega')]^{-2}-\frac{4}{k^2}[u-(\tfrac{3}{2}\omega+i\tfrac{3}{2}\omega')]^{-2} \quad (18a)$$

and the single-pole terms are given by

$$\frac{2i}{k}A(u)(u-\omega-i\tfrac{3}{2}\omega')^{-1}-\frac{2i}{k}A(u)(u-\tfrac{3}{2}\omega-i\tfrac{3}{2}\omega')^{-1}$$
$$+\frac{8}{k^2}(u-\omega-i\tfrac{3}{2}\omega')^{-1}(u-\omega-i\tfrac{3}{2}\omega')^{-1}\,. \quad (18b)$$

We consider

$$f(\tau)=\frac{B}{(\tau-d)^m}[1+p(\tau-d)+q(\tau-d)^2+\cdots] \quad (19)$$

and then

$$\frac{1}{f}\frac{d^2f}{d\tau^2}=\frac{m+m^2}{(\tau-d)^2}-\frac{2mp}{(\tau-d)}$$
$$+\text{positive powers of }(\tau-d)\,. \quad (20)$$

We require

$$\frac{m+m^2}{(\tau-d)^2}-\frac{2mp}{(\tau-d)}+\cdots=3\lambda C^2e^2(\Omega\tau)-m^2\,. \quad (21)$$

However, we already know the singularity structure of the right-hand side of (21). The solutions of (14) are doubly periodic – at least of the second kind. Hence one possibility is

$$d\equiv\frac{\omega+i\frac{3}{2}\omega'}{\Omega}\equiv d_1\,,$$
$$m+m^2=-\frac{12}{k^2\Omega^2}\lambda C^2\,, \quad (22)$$
$$-2mp=\frac{6i}{k\Omega}A(\omega+i\tfrac{3}{2}\omega')\lambda C^2-\frac{12\lambda C^2}{k^2\Omega^2}\left(\frac{\omega}{\Omega}\right)^{-1}.$$

The other possibility is

$$d\equiv\frac{1}{\Omega}(\tfrac{3}{2}\omega+i\tfrac{3}{2}\omega')\equiv d_2\,,$$
$$m+m^2=-12\lambda C^2/k^2\Omega^2\,, \quad (23)$$
$$-2mp=-\frac{6i\lambda C^2}{k\Omega}A(\tfrac{3}{2}\omega+i\tfrac{3}{2}\omega')+\frac{12\lambda C^2}{k^2\Omega^2}\left(\frac{\omega}{\Omega}\right)^{-1}.$$

It is necessary that m be a positive integer if we are to have doubly periodic solutions. Now

$$m^2+m+\frac{12\lambda C^2}{k^2\Omega^2}=0\,.$$

This implies that

$$m=\frac{1}{2}\left[-1\pm\left(1-\frac{48\lambda C^2}{k^2\Omega^2}\right)^{1/2}\right]. \quad (24)$$

Thus we need

$$\left(1-\frac{48\lambda C^2}{k^2\Omega^2}\right)^{1/2}-1=\text{positive integer}\,. \quad (25)$$

Now k is a variable parameter and so can be taken imaginary (say) to make (25) realizable. We will introduce Weierstrass's σ function:

$$\sigma(z)\equiv z\prod_{-\infty}^{\infty}\prod_{-\infty}^{\infty}\left[\left(1-\frac{z}{\kappa}\right)e^{z/\kappa+z^2/2\kappa^2}\right], \quad (26)$$

where $\kappa=2m\alpha+2m'\alpha'$, the ratio of $\alpha':\alpha$ not being purely real; we have

$$\sigma(z+2m\alpha)=(-1)^me^{2\eta(mz+m^2\alpha)}\sigma(z)\,,$$
$$\sigma(z+2m'\alpha')=(-1)^{m'}e^{2\eta'(m'z'+m'^2\alpha')}\sigma(z)\,, \quad (27)$$

where $\eta'=\mathring\sigma(\alpha')/\sigma(\alpha)$ and $\eta=\mathring\sigma(\alpha)/\sigma(\alpha)$; in our case $\alpha=\omega$, $\alpha'=\omega+2\omega'$. Every doubly periodic function of the first kind must have zeros and infinities within the fundamental parallelogram unless it is a constant. If $f(t)$ is a function with its set of infinities given by $\{c_i \mid 1 \leqslant i \leqslant n\}$ in parallelogram of periods containing t and with a set of zeros given by $\{\gamma_j \mid 1 \leqslant j \leqslant n\}$ (repeated zeros and infinities appearing the relevant number of times in the above sets), we associate a function

$$\hat{f}(t) = \frac{\sigma(t-\gamma_1)\sigma(t-\gamma_2)\cdots\sigma(t-\gamma_n)}{\sigma(t-c_1)\sigma(t-c_2)\cdots\sigma(t-c_n)} e^{\rho t}, \tag{28}$$

where ρ is a constant.

$$\hat{f}(t+2\alpha) = \hat{f}(t) e^{2\eta(\Sigma c_i - \Sigma\gamma_i) + 2\rho\alpha}, \quad \hat{f}(t+2\alpha') = \hat{f}(t) e^{2\eta'(\Sigma c_i - \Sigma\gamma_i) + 2\rho\alpha'}. \tag{29}$$

If we choose

$$\sum_i c_i - \sum_i \gamma_i = a$$

$$\nu = e^{2\eta a + 2\rho\alpha}$$
$$\nu' = e^{2\eta' a + 2\rho\alpha'},$$

then

$$\hat{f}(t+2\alpha) = \nu \hat{f}(t), \quad \hat{f}(t+2\alpha') = \nu' \hat{f}(t) \tag{29'}$$

since

$$f(t+2\alpha) = \nu f(t),$$
$$f(t+2\alpha') = \nu' f(t), \quad \text{by supposition}.$$

$\hat{f}(t)/f(t)$ is a doubly periodic function of the first kind with no zeros and infinities in the finite part of the t plane. Hence

$$\hat{f}(t)/f(t) = \text{constant}. \tag{30}$$

These considerations will form the basis of our solution of (14). Using the quantities defined by (22) and (23) we consider

$$f(t) = \sigma(t-\gamma_1)\sigma(t-\gamma_2)\cdots\sigma(t-\gamma_{2m}) \times [\sigma(t-d_1)]^{-m}[\sigma(t-d_2)]^{-m} e^{\rho t}. \tag{31}$$

Now

$$\ln\sigma(z) = \ln z - \sum_{n=2}^{\infty} \frac{1}{2n} z^{2n} \sum{}' (\kappa^{-2n}) \tag{32}$$

in a neighborhood of $z=0$ [cf. (26)] where $\sum'$ denotes summation for all κ ($\neq 0$). Hence in a neighborhood of $z=0$

$$\begin{aligned} \zeta(z) &= (1/z) - \tfrac{1}{60} g_2 z^3 - \tfrac{1}{140} g_3 z^5 - \cdots - z^{2n-1} \sum{}' \kappa^{-2n} - \cdots, \\ \wp(z) &= (1/z^2) + \tfrac{1}{20} g_2 z^2 + \tfrac{1}{28} g_3 z^4 - \cdots + (2n-1) z^{2n-2} \sum{}' \kappa^{-2n} + \cdots . \end{aligned} \tag{33}$$

Now

$$\frac{1}{f}\frac{df}{dt} = \rho + \sum_{i=1}^{2m} \zeta(t-\gamma_i) - m\zeta(t-d_1) - m\zeta(t-d_2), \tag{34}$$

$$\frac{1}{f}\frac{d^2 f}{dt^2} = \left(\frac{1}{f}\frac{df}{dt}\right)^2 - \left[\sum_{i=1}^{2m} \wp(t-\gamma_i) - m\wp(t-d_1) - m\wp(t-d_2)\right]. \tag{35}$$

The next steps should now be clear. We obtain an expansion of $(1/f)d^2f/dt^2$ in the vicinity of $t=0$ and then compare the coefficients of this expansion with the coefficients of the known expansion of $3\lambda C^2 e^2(\Omega\tau) - m^2$ about the origin. Now

$$\begin{aligned} \wp(u+v) &= -\wp(u) - \wp(v) + \frac{1}{4}\left[\frac{\wp'(u)-\wp'(v)}{\wp(u)-\wp(v)}\right]^2, \\ \zeta(u+v) &= \zeta(u) + \zeta(v) + \frac{1}{2}\left[\frac{\wp'(v)-\wp'(u)}{\wp(v)-\wp(u)}\right]. \end{aligned} \tag{36}$$

We shall consider the case $m=1$. This will illustrate the method of solution adequately.

$$f(\tau) = \frac{\sigma(\tau-\gamma_1)\sigma(\tau-\gamma_2)}{\sigma(\tau-d_1)\sigma(\tau-d_2)} e^{\rho\tau}. \tag{37}$$

It is easy to show that

$$\frac{1}{f}\frac{df}{dt} = V + Xt + Yt^2 + Zt^3 + \cdots, \tag{38}$$

where

$$\begin{aligned} X &= \wp(-d_1) + \wp(-d_2) - \wp(-\gamma_1) - \wp(-\gamma_2), \\ Y &= \tfrac{1}{2}[\wp(-\gamma_1) + \wp(-\gamma_2) - \wp(-d_1) - \wp(-d_2)], \\ Z &= \wp^2(-d_1) + \wp^2(-d_2) - \wp^2(-\gamma_1) - \wp^2(-\gamma_2), \\ V &= \rho + \zeta(-\gamma_1) + \zeta(-\gamma_2) - \zeta(-d_1) - \zeta(-d_2). \end{aligned}$$

Moreover, in a neighborhood of $t=0$

$$3\lambda C^2 e^2(\Omega t) - \mu^2 = (3\lambda C^2 - \mu^2) + 6i\lambda C^2 \Omega t - 6\lambda C^2 \Omega^2 t^2 + \cdots \tag{39}$$

and

$$\frac{1}{f}\frac{d^2 f}{dt^2} = (V^2 + 2XV) + 2t(2VY + X^2 + XV) + 2t^2(3VZ + 3XY + VY + \tfrac{1}{2}X^2) + \cdots. \tag{40}$$

Hence the three conditions which determine ρ, γ_1, γ_2 are

$$\begin{aligned} 3\lambda C^2 - \mu^2 &= V^2 + 2XV, \\ 3i\lambda C^2 \Omega &= 2VY + X^2 + XV, \\ -3\lambda C^2 \Omega^2 &= 3VZ + 3XY + VY + \tfrac{1}{2}X^2. \end{aligned} \tag{41}$$

From general theory we know that (41) gives two different sets of values for $\{\rho, \gamma_1, \gamma_2\}$. The two sets are denoted by

$$\{\rho^{(j)}, \gamma_1^{(j)}, \gamma_2^{(j)}\}, \quad j = 1, 2.$$

The solution of (12) is given by

$$f(\tau)=D_1\sigma(\tau-\gamma_1^{(1)})\sigma(\tau-\gamma_2^{(1)})[\sigma(\tau-d_1)]^{-1}[\sigma(\tau-d_2)]^{-1}\exp(\rho^{(1)}\tau)$$
$$+D_2\sigma(\tau-\gamma_1^{(2)})\sigma(\tau-\gamma_2^{(2)}[\sigma(\tau-d_1)\sigma(\tau-d_2)]^{-1}\exp(\rho^{(2)}\tau)\,, \tag{42}$$

where

$$e^{\rho^{(1)}t}D_1=\frac{\sigma(t-d_1)\sigma(t-d_2)}{\sigma(t-\gamma_1^{(1)})\sigma(t-\gamma_2^{(1)})}[\rho^{(2)}-\rho^{(1)}+\zeta(t-\gamma_1^{(2)})+\zeta(t-\gamma_2^{(2)})-\zeta(t-\gamma_1^{(1)})-\zeta(t-\gamma_2^{(1)})]^{-1}\,,$$

$$e^{\rho^{(2)}t}D_2=-\frac{\sigma(t-d_1)\sigma(t-d_2)}{\sigma(t-\gamma_1^{(2)})\sigma(t-\gamma_2^{(2)})}[\rho^{(2)}-\rho^{(1)}+\zeta(t-\gamma_1^{(2)})-\zeta(t-\gamma_1^{(1)})+\zeta(t-\gamma_2^{(2)})-\zeta(t-\gamma_2^{(1)})]^{-1}\,.$$

Now we have

$$K(b,a)=\pi^{-1/2}\exp\frac{i}{\hbar}S[q_c]$$
$$\times\frac{[\sigma(d_1)\sigma(d_2)]^{1/2}}{[D_1\sigma(\gamma_1^{(1)})\sigma(\gamma_2^{(1)})+D_2\sigma(\gamma_1^{(2)})\sigma(\gamma_2^{(2)})]^{1/2}}\,. \tag{43}$$

(43) is an extremely complex expression. We shall try to understand some of its implications. We shall first elaborate a little more on the classical solution given by (4).

$$q_c(\tau)=\left[\lambda\left(1-\frac{2}{k^2\mu^2}\right)\right]^{-1/2}e\left(\left(\frac{2}{2-k^2\mu^2}\right)^{1/2}\mu\tau\right)\,. \tag{4'}$$

As $\lambda\to 0$ we require $\lambda/k^2\to\alpha$ ($\neq 0$) as $\lambda\to 0$. We then have

$$q_c(\tau)\to\left(-\frac{2\alpha}{\mu^2}\right)^{-1/2}\exp(i\mu\tau)\,. \tag{4''}$$

However, the kinematics in this one-dimensional field theory is not interesting. The interesting thing we can extract (or obtain some idea about) is analyticity of $K(b,a)$ in λ. Inspecting (8), since Ω^2 is $-2/(k^2-2/\mu^2)$ and C^2 is $\lambda^{-1}k^2/(k^2-2/\mu^2)$ we see that $S[q_c]$ is nonanalytic in λ. The other factor contributing to K should represent nonclassical effects. We may well expect similar nonanalyticity in λ to be present.

The Weierstrassian σ function has no singularities in the finite part of the complex plane and this applies also to the exponential function; so singularities occur when the arguments have singularities (i.e., when γ_1, γ_2, ρ have singularities). The factor $[\sigma(d_1)\sigma(d_2)]^{1/2}$ is unimportant in the sense that it is solely a function of k which is a parameter which labels the set of classical solutions that we have used. It is the case, however, that λC^2 is independent of λ, and so the singularities of γ_1, γ_2, and ρ will be in μ and k. It is quite remarkable that the nonclassical terms do not contribute to the nonanalyticity in λ here. Of course if we choose k to be dependent on λ this will no longer hold. We can make our solution more explicit by calculating γ_1, γ_2, and ρ. Taking γ_1,γ_2 small we can replace the second equation in (41) by

$$3i\lambda C^2\Omega\approx\left(\frac{1}{\gamma_1^{\,2}}+\frac{1}{\gamma_2^{\,2}}\right)^2\,. \tag{44}$$

(We should have $1/\gamma_1$, $1/\gamma_1\gamma_2$ terms but we really wish to obtain qualitative insights and these terms merely complicate the algebra.) We can similarly perform approximations for the remaining two equations. We feel that the approximations are meaningful since we have the values of K and K' somewhat at our disposal (in the sense that they are just functions of k which we can vary at will) and basically we can make the d_1 and d_2 such that γ_1 and γ_2 indeed turn out to be small. From (44) and other similar equations we see emerging a cut structure in k and μ. Our framework is particularly appropriate to quantum gravity; in fact DeWitt[1] has shown that functional differentiation with respect to the background field of the functional that we have constructed gives all amplitudes. Since we have been considering a one-dimensional model with no analog of gauges, we must be cautious in extrapolating to quantum gravity and in particular to the infinity suppression in quantum electrodynamics via the cutoff supplied by gravity using an exponential parametrization.[9] However, as studied in Ref. 14, the lower-dimensional analogs are generally less singular in λ than the higher-dimensional cases, and in fact the one-dimensional analog gives two-point functions which are analytic with respect to λ. The above would seem to indicate that such an analytic behavior is destroyed when summation over the major coupling constant is performed. It may well be that the four-dimensional theory which seems usually to be more singular (than lower-dimensional theories) will also give a behavior more singular than $\ln\lambda$ (when we sum an infinite set of diagrams containing chains of superpropagators). However, treatment similar to ours for the three-space-dimensional case seems extremely difficult, and so we can only make intuitive statements.

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APPENDIX

We give here the details of the calculation of $S[q_c]$.

$$S[q_c]=\Omega^{-1}\int_{\Omega t_a}^{\Omega t_b} d(\Omega t)\{\tfrac{1}{2}C^2\Omega^2[\dot{e}(\Omega t)]^2-\tfrac{1}{2}\mu^2C^2[e(\Omega t)]^2+\tfrac{1}{4}\lambda C^4[e(\Omega t)]^4\}. \tag{A1}$$

We shall often have to use relations of the form[11]

$$\int \mathrm{sn}u\,\mathrm{cn}u\,du=-(\mathrm{dn}u)/k^2, \tag{A2}$$

$$\int \mathrm{sn}u\,\mathrm{cn}u\,\mathrm{dn}^m u\,du=\frac{-1}{(m+1)k^2}\mathrm{dn}^{m+1}u, \tag{A3}$$

$$\int \mathrm{cn}^2u\,\mathrm{dn}^2u\,du=\frac{1}{3k^2}[(1+k^2)E(u)-k'^2u+k^2\,\mathrm{sn}u\,\mathrm{cn}u\,\mathrm{dn}u], \tag{A4}$$

and

$$\int \frac{\mathrm{sn}^n u\,\mathrm{cn}^p u}{\mathrm{dn}^{2m}u}du=\frac{1}{k^2}\left[\int \mathrm{sn}^n u\,\mathrm{cn}^{p-2}u\,\mathrm{dn}^{2m-1}u\,du-k'^2\int \mathrm{sn}^n u\,\mathrm{cn}^{p-2}u\,\mathrm{dn}^{2m}u\,du\right], \tag{A5}$$

where $n+p-1-2m\neq 0$. Now

$$\dot{e}(u)=-\mathrm{sn}u\,\mathrm{dn}u+i\,\mathrm{cn}u\,\mathrm{dn}u. \tag{A6}$$

Hence

$$\begin{aligned}\int_{u_a}^{u_b} du[\dot{e}(u)]^2&=\int_{u_a}^{u_b} du\,(\mathrm{sn}^2u\,\mathrm{dn}^2u-\mathrm{cn}^2u\,\mathrm{dn}^2u-2i\,\mathrm{sn}u\,\mathrm{cn}u\,\mathrm{dn}^2u)\\&=\frac{1}{3k^2}A(u)\Big|_{u_a}^{u_b},\end{aligned} \tag{A7}$$

where

$$A(u)=\frac{1}{3k^2}\left[(k^2-2)E(u)+2k'^2u-2k^2\,\mathrm{sn}u\,\mathrm{cn}u\,\mathrm{dn}u+\frac{2i}{3k^2}\mathrm{dn}^3u\right].$$

Now in evaluating the remaining terms in (A1) we shall just expand the integrands as products of sn$u$ and cn$u$ and then use relations such as (A2).

$$\int_{u_a}^{u_b} du[e(u)]^2=\left\{u-\frac{2}{k^2}[u-E(u)]-2i\frac{(\mathrm{dn}u)}{k^2}\right\}_{u=u_a}^{u=u_b}. \tag{A8}$$

Similarly

$$\begin{aligned}\int du[e(u)]^4=&\,8\left(\frac{k^2-2}{3k^4}\right)E(u)+\left(\frac{14k'^2+k^2-3k^2k'^2+2}{3k^4}\right)u\\&+\frac{8}{3k^2}\mathrm{sn}u\,\mathrm{cn}u\,\mathrm{dn}u+4i\left(\frac{k^2+2k'^2}{k^4}\right)\mathrm{dn}u-\left(\frac{4i}{k^4}\right)\mathrm{dn}^2u,\end{aligned} \tag{A9}$$

where we have evaluated $\int \mathrm{sn}u\,\mathrm{cn}^3u\,du$ using (A5) with $m=0$, $n=1$, and $p=3$. Finally collecting together terms, we have

$$S[q_c]=\Omega^{-1}[F(u)]_{u=\Omega t_a}^{u=\Omega t_b}, \tag{A10}$$

where

$$F(u)=E(u)\left[\tfrac{1}{2}C^2\Omega^2\,\frac{k^2-2}{3k^2}-\frac{\mu^2C^2}{k^2}+2\,\frac{(k^2-2)}{3k^4}\,\lambda C^4\right]$$
$$+u\left[\tfrac{1}{2}C^2\Omega^2\left(\frac{2k'^2}{3k^2}\right)-\tfrac{1}{2}\mu^2C^2\left(1-\frac{2}{k^2}\right)+\left(\frac{14k'^2+k^2-3k^2k'^2+2}{3k^4}\right)\frac{\lambda}{4}C^4\right]$$
$$+\operatorname{sn}u\,\operatorname{cn}u\,\operatorname{dn}u\left[\tfrac{1}{2}C^2\Omega^2(-\tfrac{2}{3})+\frac{2\lambda}{3k^2}C^4\right]$$
$$+\operatorname{dn}^3u\left[\tfrac{1}{2}C^2\Omega^2\left(\frac{2i}{(3k^2)^2}\right)\right]+\operatorname{dn}^2u\left(-\frac{\lambda iC^4}{k^4}\right)+\operatorname{dn}u\left[i\,\frac{\mu^2C^2}{k^2}+i\lambda C^4\left(\frac{k^2+2k'^2}{k^4}\right)\right].$$

[1]B. S. DeWitt, *Dynamical Theory of Groups and Fields* (Gordon and Breach, New York, 1965); in *Relativity, Groups, and Topology*, edited by B. S. DeWitt and C. M. DeWitt (Gordon and Breach, New York, 1964).

[2]D. G. Boulware and L. S. Brown, Phys. Rev. 172, 1628 (1968).

[3]M. J. Duff, Phys. Rev. D 7, 2317 (1973).

[4]J. R. Klauder, in *Relativity*, edited by M. Carmeli, S. I. Fickler, and L. Witten (Plenum, New York, 1970).

[5]D. I. Blokhintsev, Nuovo Cimento 2A, 632 (1971).

[6]D. I. Blokhintsev, J.I.N.R. Dubna Report, 1972 (unpublished).

[7]K. Symanzik, in *Proceedings of the International School of Physics, "Enrico Fermi,"* edited by R. Jost (Academic, New York, 1969), Course XLV.

[8]B. Simon, in *Cargèse Lectures in Physics*, edited by D. Bessis (Gordon and Breach, New York, 1972), Vol. 5.

[9]M. J. Duff, J. Huskins, and A. Rothery, Phys. Rev. D 4, 1851 (1971); C. J. Isham, A. Salam, and J. Strathdee, *ibid.* 3, 1805 (1971).

[10]R. Raczka, Lett. Nuovo Cimento 3, 13 (1972).

[11]P. F. Byrd and M. D. Friedman, *Handbook of Elliptic Integrals for Engineers and Physicists* (Springer, New York, 1954).

[12]J. Tarski, in *Lectures in Theoretical Physics*, edited by A. O. Barut and W. E. Brittin (Gordon and Breach, New York, 1968), Vol. XA; C. M. DeWitt, Phys. Rev. 81, 848 (1951).

[13]A. R. Forsyth, *Theory of Differential Equations*, (Cambridge Univ. Press, New York, 1902), part 3.

[14]S. Sarkar, Nucl. Phys. B49, 173 (1972).

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Level-Dependent New Tamm-Dancoff Method

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The coefficients of the Wick expansion relating ordinary operator products to normal products define the transformation taking the single-time Tamm-Dancoff τ-function recursion equation into the new Tamm-Dancoff (NTD) ϕ-function recursion equations. It is shown that the ϕ-function transformation can be constructed from the ground-state τ-function recursion system for the harmonic oscillator, and that an infinite number of other energy-level-dependent transformations can be constructed in the same way. The latter allow for the development of a level-dependent NTD method permitting the accurate determination of transition frequencies through an accumulative level-by-level procedure, assuming knowledge of the ground-state energy. For the sample systems considered, the procedure is self-contained and self-consistent, the ground-state energy can be determined within the context of the NTD method. Numerical results are obtained for several one-dimensional nonlinear oscillators.

I. INTRODUCTION

The recursion formulas of the single-time new-Tamm-Dancoff (NTD) formalism linearly and homogeneously relate the components of an energy eigenstate in a particular representation, the ϕ-function representation.[1-3] In the one-dimensional case the components of the ϕ representative may be considered as ground-state-energy-eigenstate matrix elements of the single-time "generalized