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Invariant Mappings and Classically Conserved Quantities

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We determine the properties of the most general (infinitesimal) mapping in phase space which preserves a congruence of classical trajectories. Although such mappings need not be canonical, we find that they can nevertheless be associated in a unique fashion with constants of the motion. The application of these results to the problem of determining the observables of the general theory of relativity is indicated.

I. INTRODUCTION

It is well known that the infinitesimal invariant canonical mappings of classical trajectories in phase space, that is, those infinitesimal canonical mappings which preserve the form of the Hamiltonian, are in one-to-one correspondence with constants of motion of the system.[1] In fact, it is the constants of the motion which generate the corresponding mappings. In this paper we shall consider more general infinitesimal mappings in phase space which preserve the congruence of classical trajectories. Our two principal results are: (1) There exists a special class canonical transformation not generated by constants of the motion which preserve the congruence of trajectories; and (2) the noncanonical invariant mappings of trajectories are also associated with constants of the motion. It is this second result which is of particular interest to us as it affords a new method of constructing constants of the motion. In the concluding section we shall indicate how this result can be employed to construct observables for the general theory of relativity.

II. INVARIANT MAPPINGS

Let us denote the standard coordinates of the classical phase by Z^α, where, for a system of n degrees of freedom, α ranges from 1 to $2n$. We shall employ the symplectic form, $\epsilon^{\alpha\beta}$, to raise and lower indices of vector fields over the phase space. With this notation the Poisson bracket of two fields A and B may be written

$$[A,B] \equiv A_{,\mu}\epsilon^{\mu\nu}B_{,\nu}\ , \tag{2.1}$$

where the comma denotes differentiation with respect to the dynamical variables which coordinatize the phase space, Z^α, and a summation convention is understood on repeated indices. Trajectories of a classical theory are determined by the functional form of a given scalar field over the phase space, $H(Z^\alpha)$, the Hamiltonian, via the Hamilton equations of motion

$$\begin{aligned}\dot{Z}^\alpha &= \epsilon^{\alpha\mu}H_{,\mu}\\ &= [Z^\alpha, H]\,.\end{aligned} \tag{2.2}$$

We now wish to consider infinitesimal mappings

$$Z'^\alpha = Z^\alpha + \xi^\alpha \tag{2.3}$$

which preserve the congruence of trajectories determined by Eq. (2.2). With an eye toward the eventual application of our considerations to the general theory of relativity it would be sufficient to consider ξ^α as a vector field defined on the $2n$-dimensional phase space; however, for generality we shall consider ξ^α to be defined on the product of phase space with time, so that ξ^α may be explicitly time-dependent. Thus, we have in general

$$\xi^\alpha = \xi^\alpha(Z^\beta, t)\,. \tag{2.4}$$

Starting at some initial point in the product space, $Z^\alpha(t)$, we may consider the two mappings given by Eqs. (2.2) and (2.3) applied sequentially. The condition that the trajectories of Eq. (2.2) be preserved under the mappings of Eq. (2.3) is that the order in which the two mappings are applied should be irrelevent. First applying the Hamilton equations for an infinitesimal time interval δt, immediately followed by the second mapping, we arrive at the point

$$\begin{aligned}Z'^\alpha &= Z^\alpha + \delta t\epsilon^{\alpha\mu}H_{,\mu} + \xi^\alpha(Z^\beta + \delta t\epsilon^{\beta\mu}H_{,\mu}, t+\delta t)\\ &= Z^\alpha + \xi^\alpha(Z^\beta, t)\\ &\quad + \delta t\left(\epsilon^{\alpha\mu}H_{,\mu} + \xi^\alpha{}_{,\nu}\epsilon^{\nu\mu}H_{,\mu} + \frac{\partial \xi^\alpha}{\partial t}\right),\end{aligned} \tag{2.5}$$

where we have kept terms which are at most first order in δt. Reversing the order of the two mappings, to the same order in δt we arrive at the point

$$\begin{aligned}Z'^\alpha &= Z^\alpha + \xi^\alpha(Z^\beta, t) + \delta t\epsilon^{\alpha\mu}H_{,\mu}(Z^\beta + \xi^\beta)\\ &= Z^\alpha + \xi^\alpha(Z^\beta, t) + \delta t(\epsilon^{\alpha\mu}H_{,\mu} + \epsilon^{\alpha\mu}H_{,\mu\nu}\xi^\mu)\,.\end{aligned} \tag{2.6}$$

(Note that in this case it is not equivalent to propagate in time from $Z^\alpha + \xi^\alpha$ via $\delta t[Z^\alpha + \xi^\alpha, H]$.)

Comparing Eqs. (2.5) and (2.6), we find that the condition that the infinitesimal mapping described by ξ^α preserves the congruence of trajectories determined by the Hamiltonian H is

$$\epsilon^{\alpha\mu} H_{,\mu\nu} \xi^\nu = \xi^\alpha{}_{,\nu} \epsilon^{\nu\mu} H_{,\mu} + \frac{\partial \xi^\alpha}{\partial t} . \tag{2.7}$$

This expression may appear more illuminating if we rearrange it by lowering the free index, α, and performing a "parts integration" on the first term. Thus, recalling that $\epsilon^{\alpha\beta}$ is antisymmetric, we find

$$(H_{,\mu} \xi^\mu)_{,\alpha} = (\xi_{\alpha,\nu} - \xi_{\nu,\alpha}) \epsilon^{\nu\mu} H_{,\mu} + \frac{\partial \xi_\alpha}{\partial t} . \tag{2.8}$$

The canonical mappings are defined as those mappings, (2.3), which leave the symplectic form, $\epsilon^{\alpha\beta}$, invariant when regarding that form as a tensor field over the phase space. It is easy to check that this condition for canonicity may be written

$$\xi_{\alpha,\beta} - \xi_{\beta,\alpha} = 0 , \tag{2.9}$$

or equivalently

$$\xi_\alpha = \xi_{,\alpha} . \tag{2.10}$$

Substituting Eq. (2.10) into Eq. (2.8), we obtain

$$(H_{,\mu} \epsilon^{\mu\nu} \xi_{,\nu})_{,\alpha} = \left(\frac{\partial \xi}{\partial t}\right)_{,\alpha} , \tag{2.11}$$

which upon integration yields

$$[\xi, H] + \frac{\partial \xi}{\partial t} = c , \tag{2.12}$$

where c is a constant over the entire phase space, not just a constant along a classical trajectory. Should c vanish, ξ becomes a constant in time along each trajectory and generates the canonical transformation which leaves the functional form of the Hamiltonian invariant. We recover in this fashion the well-known theorem referred to in the introductory section of this paper.

However, if the constant c does not vanish, ξ will generate a canonical transformation [via Eqs. (2.10) and (2.3)] which preserves the classical trajectories, but not the form of the Hamiltonian. In that event ξ will not be a constant of the motion, as is evident from Eq. (2.12). The simple example of the one-dimensional free particle will clarify the significance of such a generator. Thus for

$$H = p^2/2m , \tag{2.13}$$

the generator

$$\xi = x/p \tag{2.14}$$

satisfies Eq. (2.12), where

$$c = 1/m . \tag{2.15}$$

It is clear from this trivial example that such ξ provides a measure of intrinsic time for the dynamical system in question. They generate canonical mappings which merely add a constant to the functional form of the Hamiltonian, thereby preserving trajectories.

If we now return to the general case where the mapping described by ξ^α is not necessarily canonical, the more interesting result is obtained by taking the divergence of Eq. (2.8). Again recalling that $\epsilon^{\alpha\beta}$ is antisymmetric, we find

$$[\xi^\alpha{}_{,\alpha}, H] + \frac{\partial}{\partial t} (\xi^\alpha{}_{,\alpha}) = 0 \tag{2.16}$$

or

$$\xi^\alpha{}_{,\alpha} = \text{const.} \tag{2.17}$$

When the transformation described by ξ^α is canonical, we see that Eq. (2.17) is trivially satisfied by $\xi^\alpha{}_{,\alpha} = 0$. However, for noncanonical invariant mappings, $\xi^\alpha{}_{,\alpha}$ will in general be a nontrivial constant of the motion. The canonical mapping which this constant generates,

$$\delta Z^\alpha = \epsilon^{\alpha\mu} \xi^\nu{}_{,\nu\mu} , \tag{2.18}$$

seems to bear very little relationship to the original mapping described by ξ^α.

III. APPLICATION TO GENERAL RELATIVITY

The observables of the general theory of relativity, which are to be realized as Hermitian operators on a Hilbert space in the transition to quantum theory, must commute with the constraints of that theory, and hence be constants of the motion.[2] Thus far no one has succeeded in exhibiting such constants of the motion as local functionals on the phase space of that theory. In order to apply the principal result of this present paper to obtain such observables, it is necessary to obtain, in advance, a vector field on the phase space which is not canonical, but which preserves trajectories.

In a recent paper[3] it was shown that the group of curvilinear point mappings of a four-dimensional Riemannian manifold into itself, that is, the group of infinitesimal coordinate transformation

$$x'^\mu = x^\mu + \lambda^\mu \tag{3.1}$$

(where μ now ranges from 1 to 4) such that the descriptor λ^μ is purely a function of position and *not* a function of the dynamical varibles of space-time, is not in general a canonical mapping. However, since it is merely a coordinate transformation, it certainly provides a vector field on the phase space which preserves trajectories. Such mappings should therefore provide nontrivial constants of the motion. In fact, since these resulting constants are intimately associated with the preferred group of space-time coordinate transformations they appear to be particularly appropriate for constructing a quantum theory.[4]

In order to exhibit the required constants for the general theory of relativity, the results of this paper will have to be extended from finite-dimensional phase spaces to the infinite-dimensional function space required by a field theory. In addition an added difficulty arises due to the presence of constraints. In effect this results in the symplectic form becoming singular in the neighborhood of the constraint hypersurface. Much, though not all, of what is required for the extension of our results to this situation, is straightforward. It will be developed in a subsequent paper.

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Gravitational-Wave Antenna Design to Detect Random Gravitational Waves*

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A design is proposed for an antenna system capable of detecting random gravitational waves and separating their effects from random fluctuations in the antenna. The strongest known signal should be the random signal from all of the binary stars in the galaxy. This design may allow one to penetrate the background noise of the Earth itself and recover this signal.

I. INTRODUCTION AND RESULTS

According to general relativity, in a system excited by gravitational waves (GW) the proper acceleration and the second time derivative of the proper strain are not always equal. An earlier paper[1] suggested that this property should be used in the design of GW detectors. Here we will study one possible design exploiting this property, and will pay special attention to the effects of thermal noise. We will see that this property can be used to design a detector capable of distinguishing between random gravitational waves and thermal noise, and for separating a GW signal from a background of thermal noise. A look at typical parameters shows the design to be both reasonable and interesting.

This design is shown in Fig. 1. The intention is to use the outer mass-spring oscillators as acceleration sensors for the central mass-spring oscillator. Although we will spend most of our time on the case where the outer masses are much smaller than the central masses, the results apply also to systems which are not so cleanly separable into oscillator plus accelerometer. Formulas for the general case will be given. The particular configuration shown in Fig. 1 was chosen with an eye to providing both an acceleration signal and a strain signal, while measuring only strains or displacements.

This antenna system incorporates several stochastic (random) elements. The circles in Fig. 1 represent generators of random forces, and are to model thermal fluctuations. We will study the response of this antenna to a plane, linearly polarized, random-amplitude gravitational wave incident normal to the antenna. The response of the antenna to pulsed signals requires a separate analysis not attempted here.

The reader may be surprised that I chose to concentrate on random gravitational waves. One reason is the simplicity of the presentation, which must include in any case random thermal noise. Another reason is my feeling that the neglect of random GW reflects the opinion that it will always be hopelessly confused with thermal noise. Here