Tidal Friction in Slowly Rotating Black Holes

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A stationary distribution of matter outside a Kerr black hole will cause the angular momentum of the black hole to decrease with time provided the distribution is not symmetric about the axis defined by the black hole's angular momentum. The rate of decrease of angular momentum is calculated in the case that (a) the effects of the exterior matter can be treated as a perturbation to the Kerr geometry and (b) the angular momentum of the black hole is small.

I. INTRODUCTION

The detection and observation of black holes will depend not so much on an understanding of the properties of a black hole in isolation but on an understanding of how a black hole interacts with a realistic astrophysical environment. It is appropriate, therefore, to examine in detail the interactions of a black hole with matter which is exterior to it. In this paper we examine one aspect of this interaction – the decrease in angular momentum of a Kerr black hole caused by the presence of a stationary exterior distribution of matter which is not symmetric about the axis defined by the black hole's angular momentum.

That exterior matter could slow down a rotating black hole was already strongly suggested by a theorem of Hawking[1] which shows that any stationary nonstatic black hole must also be axially symmetric. It seemed reasonable to conclude that a nonaxisymmetric black hole, such as a Kerr black hole perturbed by nonaxisymmetric exterior masses, could not be stationary but must tend towards a static black hole, that is, the angular momentum must tend to zero. A simple picture of the origin of the effect may be presented in the following way[2]: One knows that if there is a relative rotation between infinity and a body which is not symmetric about the axis of rotation then the body will emit gravitational radiation to null infinity.[3] The gravitational radiation causes the relative rotation between the body and infinity to decrease with time. There is a close analogy between Penrose's concept of null infinity and the event horizon of a black hole. Both the event horizon and null infinity are null surfaces. Both have the "one-way" property that radiation which reaches or crosses them can never reemerge. Pursuing the analogy, one would expect that if there is a relative rotation between an event horizon and a distribution of matter which is not symmetric about the direction of rotation, then there would be gravitational dissipation which would act to decrease the relative rotation of the matter and the black hole. In particular, even if the matter is stationary with respect to infinity it would slow down a rotating black hole.

The slowing down of a rotating black hole due to the gravitational dissipation produced by exterior matter is analogous to the slowing down of a rotating planet by viscous dissipation due to tides caused by an exterior moon. In effect the gravitational attraction of the exterior matter raises a tide in the event horizon of a black hole. As the black hole rotates under this tide its rotational energy is dissipated gravitationally. Even the form of the expressions for the rate of decrease in the angular momentum in the two cases are similar. It is this close analogy which leads us to call the slowing down of black holes by exterior matter gravitational tidal friction.

In a previous paper[4] a simple general expression was found for the rate of decrease in angular momentum of a Kerr black hole due to a stationary gravitational perturbation,

$$\frac{da}{dt} = -\frac{A}{4am}\int |\sigma^{(1)}|^2 dA . \tag{1.1}$$

Here, a is the Kerr angular momentum parameter, M is the black hole's mass, A its area, and $\sigma^{(1)}$ the shear of the properly normalized null generators of the perturbed event horizon. The area integral extends over the intersection of the perturbed event horizon and a surface of constant t.

This simple and elegant formula is not so simple to evaluate. In this paper we will evaluate Eq. (1.1) for the particularly simple limiting case when the angular momentum of the Kerr black hole is small,

$$a \ll M . \tag{1.2}$$

The equations governing the perturbations can then be expanded simultaneously in powers of the perturbation and in powers of a/M and solved in the lowest relevant orders. The rate of decrease in angular momentum can then be calculated to the lowest nonvanishing order which is second order in the perturbation and first order in a.

In Sec. II we briefly review the derivation of the expression for da/dt. In Sec. III the analogy to tidal friction is pointed out. In Secs. IV and V the general perturbations of the Kerr metric are evaluated to the relevant orders. In Secs. VI and VII the rate of change of angular momentum is evaluated for a general perturbation, and in Sec. VIII the effect for the case of a stationary point particle located outside the black hole is treated. Finally in Sec. IX we discuss possible astrophysical consequences of our results.

II. RATE OF DECREASE OF ANGULAR MOMENTUM

The expression for the rate of decrease of angular momentum of a black hole due to a stationary perturbation [Eq. (1.1)] is so simple to derive that it is appropriate to briefly sketch that derivation here as a guide to making our explicit calculation. The details are in Ref. 4.

A Kerr black hole is characterized by two parameters: the total mass, M, and the specific angular momentum, a. We want to calculate the rates of change of these parameters with respect to any time t such that $\partial/\partial t$ is a timelike Killing vector at infinity caused by a stationary exterior perturbation. A stationary perturbation cannot cause a rate of change of the mass. To see this we can invoke the familiar argument of Edelstein[5]: Find that solution of the everywhere source-free perturbed Einstein equations which agrees with the stationary perturbation under consideration at the horizon. At infinity, this solution must be a superposition of ingoing and outgoing gravitational waves. The energy which is crossing the horizon is the difference between the energy which is being carried inward and that which is being carried outward at infinity. A stationary perturbation, however, will contain no radiation at infinity either ingoing or outgoing. The energy crossing the horizon is, therefore, zero.

The mass, angular momentum, and area of a Kerr black hole are related by

$$A=8\pi M[M+(M^2-a^2)^{1/2}]\ . \tag{2.1}$$

The rate of decrease of angular momentum, J, is thus related to the rate of increase in area by

$$\frac{dJ}{dt}=-\frac{(M^2-a^2)^{1/2}}{8\pi a}\frac{dA}{dt}\quad . \tag{2.2}$$

In turn the rate of increase in area is given by

$$\frac{dA}{dt}=-2\int\rho\, dA\ . \tag{2.3}$$

Here, ρ is the convergence of the null geodesic generators of the event horizon and the integral is over the two surface formed by the intersection of the event horizon with a surface of constant t. If l^μ is the normal to the horizon, normalized so that $l^\mu t_{,\mu}=1$ and m^μ and $\bar{m}^\mu$ are complex conjugate null vectors chosen so that $m^\mu l_\mu=0$ and $m^\mu\bar{m}_\mu=-1$, then

$$\rho=l_{\mu;\nu}\, m^\mu\bar{m}^\nu\ . \tag{2.4}$$

The convergence, ρ, can be calculated from one of the Newman-Penrose[6] equations. In the absence of sources the relevant equation is

$$\begin{aligned}D\rho&=l^\mu\rho_{,\mu}\\&=\rho^2+\sigma\bar{\sigma}+(\epsilon+\bar{\epsilon})\rho\quad .\end{aligned} \tag{2.5}$$

Here,

$$\sigma=l_{\mu;\nu}\, m^\mu m^\nu \tag{2.6}$$

and

$$\epsilon=\tfrac{1}{2}(l_{\mu;\nu}n^\mu l^\nu+m_{\mu;\nu}\bar{m}^\mu l^\nu)\quad , \tag{2.7}$$

where n^μ is a real null vector orthogonal to m^μ and satisfying $l^\mu n_\mu=1$. The vector m^μ is not determined up to a rotation $m^\mu\to e^{i\phi}m^\mu$, and, by proper choice of the function ϕ, ϵ can always be made real. If all quantities in Eq. (2.5) are now expanded in powers of the perturbation the lowest-order (the unperturbed Kerr metric) values on the horizon are $\sigma^{(0)}=\rho^{(0)}=0$ and

$$\epsilon^{(0)}=\frac{(M^2-a^2)^{1/2}}{4[M+(M^2-a^2)^{1/2}]}\quad . \tag{2.8}$$

To first order in the perturbation ρ vanishes. This is reasonable since the flux of angular momentum should be second order in the perturbation [cf. Eqs. (2.2) and (2.3)]. Further, to second order in the perturbation the rate of change of angular momentum and area due to a stationary perturbation should themselves be stationary. Taking Eq. (2.5) to second order in the perturbation, and integrating it over the two-surface formed by intersection of the event horizon and a constant t hypersurface, one finds

$$0=\int|\sigma^{(1)}|^2\,dA+2\epsilon^{(0)}\int\rho^{(2)}\,dA\ . \tag{2.9}$$

Here, $\sigma^{(1)}$ is the shear calculated to first order in the perturbation. The left-hand side of the equation vanished because it is d^2A/dt^2 and the rate of change in area is stationary. Using Eq. (2.3) one has for the rate of increase of area to second order in the perturbation [and thereby through Eq. (2.2) the rate of decrease in angular momentum]

$$\frac{dA}{dt}=\frac{1}{\epsilon^{(0)}}\int|\sigma^{(1)}|^2 dA\ . \tag{2.10}$$

Equation (2.10) is the expression which we wish to evaluate to lowest order in a. The calculation can be done in the following steps: (1) Expand the metric about the Schwarzschild metric simultaneously in powers of a and the perturbation. To keep track of the orders in this expansion we introduce a formal parameter ϵ to keep track of the orders in a and a parameter ζ to keep track of the orders in the perturbation. (Thus, order $\epsilon\zeta$ means first order in a and first order in the perturbation.) (2) Calculate the metric to order ϵ and order ζ (Sec. IV). The metric to order ϵ is simply the Kerr metric. Order ζ contains the stationary perturbations of the *Schwarzschild* geometry caused by the stationary exterior sources. The calculation of the perturbations to order ζ is not difficult because the perturbations separate in the spherically symmetric background Schwarzschild geometry into perturbations transforming under rotations like spherical harmonics with quantum numbers (l,m). To both order ϵ and order ζ the shear will vanish on the horizon; the Kerr geometry is stationary and a stationary perturbation will not cause evolution of a *nonrotating* Schwarzschild black hole. (3) Write out and solve Einstein's equations in order $\epsilon\zeta$ which is the lowest order in which σ is nonvanishing (Sec. V). The equations for the perturbations of this order will contain driving terms which are products of order ϵ and order ζ. The terms of order ϵ transform like $(l=1, m=0)$. An exterior perturbation transforming are (l,m) can therefore produce driving terms in the equations of order $\epsilon\zeta$ which transform only like $(l+1,m)$, (l,m), $(l-1,m)$. It is this simple coupling which makes the calculation of the perturbation in this order feasible. (4) Locate the position of the perturbed horizon (Sec. V). (5) From the perturbations of order $\epsilon\zeta$ calculate the shear $\sigma^{(1)}$ on the perturbed horizon (Sec. VI) and evaluate the rate of decrease of angular momentum (Sec. VII).

III. CONNECTION WITH TIDAL FRICTION

Before proceeding with the calculation of tidal friction in black holes it is appropriate to review briefly the theory of viscous tidal friction in a moon-planet system to bring out the close analogy between the two cases.

Consider a rotating planet covered with a shallow sea of incompressible viscous fluid and possessing a stationary external moon. The gravitational pull of the moon will raise a tide in the sea and the tidal friction between the sea and the rotating planet underneath will cause the rotation of the planet to decrease. We can estimate the rate for this process from the expression for the rate of energy dissipation in a viscous fluid,

$$\frac{d\mathcal{E}}{dt}=-\tfrac{1}{2}\rho\nu\int(\sigma_{ij}\sigma^{ij})\,dV. \tag{3.1}$$

Here, ρ is the fluid density, ν is the viscosity coefficient, and σ_{ij} is the volume shear of the fluid,

$$\sigma_{ij}=\frac{\partial v_i}{\partial x^j}+\frac{\partial v_j}{\partial x^i}\ . \tag{3.2}$$

The integral extends over the whole volume of the sea.

The dissipation of the planet's rotational energy means a decrease in its angular momentum, J. If I is the planet's moment of inertia then $\mathcal{E}=J^2/2I$ and

$$\frac{dJ}{dt}=-\frac{\rho\nu I}{2J}\int(\sigma_{ij}\sigma^{ij})dV. \tag{3.3}$$

Equation (3.3) is similar to Eq. (1.1) except that here the integral is over a volume shear rather than a surface shear.

Suppose that the mass of the planet is M and its angular velocity is Ω. Let the mass of the moon be μ and let it be located a distance R away from the planet's center. If the moon is sufficiently far away only the quadrupole part of the perturbing gravitational potential will significantly influence the tides on the surface of the planet. This perturbing potential is (we are still using $G=1$)

$$\delta\Phi=\mu\frac{r^2}{R^3}P_2(\cos\chi)\ , \tag{3.4}$$

where χ is the angle between the direction of the moon and the direction where $\delta\Phi$ is evaluated and r is the distance from the planet's center. This perturbing gravitational force will change the height of the sea from its original radius R_s to a new radius $R_s+\delta R_s$. The change δR_s must be such that the gravitational potential energy of a fluid element as it rises in the tide remains unchanged,

$$\frac{M}{R_s^2}\delta R_s\sim(\delta\Phi)R_s \sim\mu\frac{R_s^2}{R^3} \tag{3.5}$$

Here we have neglected all angular dependence to obtain the simplest dimensional arguments.

The typical excursion of a given fluid element as it rotates through the tidal distortion will be δR_s and the velocities in the fluid will be of order of magnitude $\Omega\,\delta R_s$. Derivatives of the velocity will then be of order of magnitude $\Omega\,\delta R_s/R_s$, and we have for the square of the shear

$$\sigma_{ij}\sigma^{ij}\sim\Omega^2\left(\frac{\delta R_s}{R_s}\right)^2 \sim\left(\frac{\mu}{M}\right)^2\left(\frac{R_s}{R}\right)^6\Omega^2\ . \tag{3.6}$$

The last line follows from Eq. (3.5). If M_s denotes the mass of the sea, then we have for the rate of decrease of J from Eq. (3.3)

$$\frac{dJ}{dt}\sim-\frac{(\nu M_s I)}{J}\left(\frac{\mu}{M}\right)^2\left(\frac{R_s}{R}\right)^6\Omega^2\ . \tag{3.7}$$

A more detailed analysis bears out these simple dimensional estimates and yields the characteristic $\sin^2\theta$ angular dependence.

The moment of inertia of the planet will be of order of magnitude $MR_s^{\,2}$ and its angular momentum is $I\Omega$. Using this, one has

$$\frac{dJ}{dt}\sim-\left(\frac{\nu M_s R_s^{\,4}}{M}\right)\frac{J\mu^2}{M^2R^6}\ . \tag{3.8}$$

If we were to apply this result to guess the rate of slowing down of a black hole by gravitational tidal friction caused by an external moon, it would seem reasonable to put $M_s\sim R_s\sim M$ and for ν/M, the dimensionless measure of viscosity, a number of order unity. One would then have

$$\frac{dJ}{dt}\sim\frac{J\mu^2M^3}{R^6} \tag{3.9}$$

for an estimate of the tidal friction effect in a black hole. In Sec. VIII we will see that this estimate is borne out by the detailed calculation.

IV. THE METRIC TO FIRST ORDER IN THE ANGULAR MOMENTUM AND FIRST ORDER IN THE PERTURBATION

To begin the program outlined in Sec. II for calculating the slowing-down rate of a slowly rotating black hole we must expand the metric simultaneously in powers of the exterior perturbation and the angular momentum. To lowest order the metric may be taken to be the familiar Schwarzschild metric,

$$(ds^2)_0=\left(1-\frac{2M}{r}\right)dt^2-\left(1-\frac{2M}{r}\right)^{-1}dr^2 -r^2(d\theta^2+\sin^2\theta\, d\phi^2)\ . \tag{4.1}$$

We choose initially to perturb about the metric expressed in these Schwarzschild coordinates even though the coordinates are singular on the horizon. Their advantage is that the expression of time-reversal symmetry has the simple form $t\to -t$ and this is useful in restricting the form of stationary perturbations. Later the coordinate system will be transformed to one which is not singular on the horizon.

The perturbation of order ϵ is simply the Kerr metric. Expressed in terms of Boyer-Lindquist coordinates, the perturbation in this order has the form

$$(ds^2)_\epsilon=\frac{2Ma}{r}\sin^2\theta\, d\phi\, dt\ . \tag{4.2}$$

To calculate the perturbations due to the exterior stationary sources we can draw on the extensive previous studies of the perturbations of the Schwarzschild geometry.[7-9] The perturbations may be decomposed into terms which transform under rotations like the appropriate scalar, vector, and tensor spherical harmonics with quantum numbers (l,m). They may further be decomposed into so-called odd-$[(-1)^{l+1}]$ and even-$[(-1)^l]$ parity parts. Of interest here, however, are not the general perturbations of the Schwarzschild geometry but only the static, time-reversal-invariant perturbations arising from static, time-reversal-invariant sources. These perturbations are independent of t and have vanishing components g_{ti}. In the standard Regge-Wheeler gauge these perturbations have the form[10]

$$(ds^2)_{\zeta,\text{odd parity}}=h_1 dr\left(-\frac{1}{\sin\theta}\frac{\partial Y_l^m}{\partial\phi}d\theta+\sin\theta\frac{\partial Y_l^m}{\partial\theta}d\phi\right), \tag{4.3}$$

$$(ds^2)_{\zeta,\text{even parity}}=\left[\left(1-\frac{2M}{r}\right)H_0dt^2+\left(1-\frac{2M}{r}\right)^{-1}H_2dr^2 +r^2K(d\theta^2+\sin^2\theta\, d\phi^2)\right]Y_l^m\ . \tag{4.4}$$

Here h_1, H_0, H_2, and K are functions of r alone. The index l which should properly label these functions has been suppressed. The vacuum field equations for these functions may easily be found by specializing the appropriate equations of Ref. 9. From Eqs. (8d) and (B3) of Ref. 9 one finds in this static case

$$H_0 = H_2 \equiv H \ , \tag{4.5}$$

$$h_1 = 0 \ . \tag{4.6}$$

Thus, the odd-parity perturbations do not occur in this problem. Combining Eqs. (8a) and (9a) of Ref. 9 to eliminate second derivatives, using Eq. (8c), and making the definition $V(r)=K(r)-H(r)$, one arrives at the following two first-order equations for the unknown metric coefficients:

$$\frac{dH}{dr}+\frac{2(r-M)}{r(r-2M)}H-\frac{V}{M}\left(\frac{l(l+1)}{2}-1\right)=0\ , \tag{4.7}$$

$$\frac{dV}{dr}-\frac{2M}{r(r-2M)}H=0\ . \tag{4.8}$$

The solution of these equations is most readily found by writing a single second-order equation for H in terms of the variable $z=r/M-1$,

$$(1-z^2)\frac{d^2H}{dz^2}-2z\frac{dH}{dz}+\left(l(l+1)-\frac{4}{1-z^2}\right)H=0\ . \tag{4.9}$$

This is a form of Legendre's equation so that the general solution for H is[11]

$$H(r)=AP_l^2(r/M-1)+BQ_l^2(r/M-1), \tag{4.10}$$

where A and B are unknown constants. The associated solution for V can then be found from Eq. (3.7),

$$V(r)=2M[r(r-2M)]^{-1/2}\left[AP_l^1\left(\frac{r}{M}-1\right)+BQ_l^1\left(\frac{r}{M}-1\right)\right]. \tag{4.11}$$

The constants A and B are determined from the distribution of energy and stresses in the sources. Outside the sources H and V must decrease at large r so that space becomes asymptotically flat. At large r, P_l^2 behaves like r^l while Q_l^2 behaves like

$$Q_l^2\left(\frac{r}{M}-1\right)\sim\left(\frac{M}{2r}\right)^{l+1}\frac{\Gamma(l+3)\Gamma(\frac{1}{2})}{\Gamma(l+\frac{3}{2})}\ ,\qquad r\to\infty\ . \tag{4.12}$$

The coefficient of P_l^2 in H must therefore vanish outside the sources.

At $r=2M$, $Q_l^2(r/M-1)$ is infinite while the behavior of P_l^1 and P_l^2 is given by

$$P_l^2(r/M-1)\sim\frac{1}{4M}\frac{(l+2)!}{(l-2)!}(r-2M)\ , \tag{4.13}$$

$$P_l^1(r/M-1)\sim\frac{1}{\sqrt{2M}}\frac{(l+1)!}{(l-1)!}(r-2M)^{1/2}\ . \tag{4.14}$$

In order for the perturbation to be finite at $r=2M$ and in order to avoid a real singularity there the coefficient B in Eqs. (4.10) and (4.11) must vanish inside the sources.

Let the multipole moments M_l^m of the source at infinity be defined by an expansion of g_{tt} in inverse powers of r,

$$g_{tt}=1+2\sum_{lm}\frac{M_l^m Y_l^m(\theta,\phi)}{r^{l+1}}\ . \tag{4.15}$$

In the Newtonian limit the M_l^m become the usual Newtonian multipole moments. The solution outside the sources then has the form

$$H(r)=\frac{2M_l^m}{(2M)^{l+1}}\frac{\Gamma(l+\frac{3}{2})}{\Gamma(\frac{1}{2})\Gamma(l+3)}Q_l^2\left(\frac{r}{M}-1\right)\ , \tag{4.16a}$$

$$V(r)=\frac{2M_l^m}{(2M)^l}\frac{\Gamma(l+\frac{3}{2})}{\Gamma(\frac{1}{2})\Gamma(l+3)}[r(r-2M)]^{1/2}Q_l^1\left(\frac{r}{M}-1\right). \tag{4.16b}$$

It is convenient to define horizon multipole moments S_l^m such that

$$K(r)\to S_l^m,\qquad r\to 2M\ . \tag{4.17}$$

In terms of these the solution inside the sources has the form

$$H(r)=[l(l+1)]^{-1}S_l^mP_l^2(r/M-1), \tag{4.18a}$$

$$V(r)=(4M)^{1/2}[l(l+1)]^{-1}S_l^m[r(r-2M)]^{-1/2}\times P_l^1(r/M-1). \tag{4.18b}$$

As a consequence of Eqs. (4.15) and (4.16) H vanishes on the horizon while V (and hence K) remains finite there.

The constants S_l^m and M_l^m are determined by integrating the perturbed field equations through the source. If the source is confined to a thin shell they are determined by the jump conditions on the metric and its first derivatives there. An example of determining S_l^m and M_l^m will be given in Sec. VIII, but for the present we will confine ourselves to evaluating the rate of increase in area of the horizon in terms of these characteristic parameters of the source.

V. METRIC AND EVENT HORIZON IN ORDER $\epsilon\zeta$

The lowest order in which the shear is nonvanishing is $\epsilon\zeta$ – first order in the perturbation and first order in the angular momentum of the black hole. To calculate $\sigma^{(1)}$ in the approach being used here we must calculate the relevant parts of the metric in this order and locate the event horizon. In this section the form of the metric in order $\epsilon\zeta$ is given and the event horizon located. The relevant parts of the metric and the shear are calculated in the next two sections.

A. Metric in Order $\epsilon\zeta$

An important boundary condition on the perturbation of order $\epsilon\zeta$ is that it be regular on the future component of the event horizon. Regularity on the past component is not necessary since if the black hole arises from a realistic collapse that component will be unphysical.[12] It is, therefore, convenient to discuss the perturbations in a coordinate system which is itself regular on the future horizon. The Schwarzschild, Boyer-Lindquist, and Regge-Wheeler coordinates used to discuss the lower-order perturbations in Sec. IV do not have this property. We will transform the lower-order perturbations to coordinates which are regular on the future event horizon before discussing the perturbations of order $\epsilon\zeta$.

The transformation

$$dt \to du - dr\,(1-2M/r)^{-1}, \tag{5.1a}$$

$$d\phi \to d\phi - (a/r^2)\,dr\,(1-2M/r)^{-1} \tag{5.1b}$$

transforms the metric for a particular multipole (l,m) accurate to orders ϵ and ζ into the form

$$ds^2 = (1-2M/r)(1+HY_l^m)\,du^2 - 2(1+HY_l^m)\,du\,dr + 2HY_l^m(1-2M/r)^{-1}dr^2$$
$$-r^2(1-KY_l^m)(d\theta^2+\sin^2\theta\, d\phi^2) + (4Ma/r)\sin^2\theta\, d\phi\, du - 2a\sin^2\theta\, dr\, d\phi + O(\epsilon\zeta)\,. \tag{5.2}$$

In order ϵ this is just the standard form of the Kerr metric. The coordinates and metric in Eq. (5.2) are clearly regular on the surface $r=2M$ since $H(r)$ vanishes there [Eqs. (4.19) and (4.13)].

The perturbations of order $\epsilon\zeta$ may also be expanded in spherical harmonics and decomposed into even- and odd-parity parts. The perturbations may be written in the Regge-Wheeler gauges *using the coordinates used in* Eq. (5.2). We then write for a particular multipole (l,m)

$$(ds^2)_{\epsilon\zeta} = (ds^2)_{\text{even}} + (ds^2)_{\text{odd}}, \tag{5.3a}$$

$$(ds^2)_{\text{even}} = \left[\left(1-\frac{2M}{r}\right)A\,du^2 + 2B\,du\,dr + C\left(1-\frac{2M}{r}\right)^{-1}dr^2 + Dr^2(d\theta^2+\sin^2\theta\, d\phi^2)\right]Y_l^m, \tag{5.3b}$$

$$(ds^2)_{\text{odd}} = 2(N\,du + Q\,dr)\left(\sin\theta\frac{\partial Y_l^m}{\partial\theta}d\phi - \frac{1}{\sin\theta}\frac{\partial Y_l^m}{\partial\phi}d\theta\right), \tag{5.3c}$$

where A, B, C, D, N, and Q are functions of r alone.

The complete metric to order $\epsilon\zeta$ is given by the sum of Eq. (5.2) and (5.3a) then summed over all (l,m). The functions of r which are involved, A, $B,\ldots$, H, $K,\ldots$, depend on (l,m) but these indices have been suppressed. Which multipoles occur in order ζ depends on the structure of the source. If a given multipole (L,m) occurs in the source and therefore in order ζ, it will combine with the $l=1$, $m=0$ terms of order ϵ to give multipoles $(L+1,m)$, (L,m), and $(L-1,m)$ in order $\epsilon\zeta$ according to the familiar law of addition of angular momenta. The parity of the terms in order ζ is $\pi=(-1)^L$ while order ϵ has $\pi=+1$. Thus, the $(L+1,m)$, $(L-1,m)$ terms will occur only in the odd-parity $[\pi=(-1)^{L+1}]$ part of the expansion of order $\epsilon\zeta$, while the (L,m) terms will occur only in the even-parity part.

B. Location of the Perturbed Event Horizon

The information obtained above and in Sec. III about the perturbation to order ζ is already enough to find the coordinate location of the perturbed event horizon to order $\epsilon\zeta$. The event horizon is, to order $\epsilon\zeta$, the largest closed, stationary null surface. If the equation of the horizon to this order is written

$$F(r,\theta,\phi) = r - 2M + \zeta f_1(\theta,\phi) + \epsilon\zeta f_2(\theta,\phi) = 0\,, \tag{5.4}$$

then the condition that the surface is null is

$$g^{ij}F_{,i}F_{,j} = 0\,. \tag{5.5}$$

To evaluate this relation let us first write

$$g^{\mu\nu} = \gamma^{\mu\nu} + \zeta h_1^{\mu\nu} + \epsilon\zeta h_2^{\mu\nu}\,, \tag{5.6}$$

where $\gamma^{\mu\nu}$ is the Kerr metric to order ϵ, $h_1^{\mu\nu}$ are the perturbations of order ζ found in Sec. IV and $h_2^{\mu\nu}$ are the as yet undetermined perturbations of order $\epsilon\zeta$ whose form is discussed in Sec. V A. Written out to order ζ, Eq. (5.5) becomes

$$(2M)^{-1}f_1(\theta,\phi)+(h_1^{rr})_{r=2M}=0. \tag{5.7}$$

The metric coefficient h_1^{rr} can be found from Eq. (5.3) and is

$$h_1^{rr}=(1-2M/r)H(r)\;Y_l^m(\theta,\phi)\;. \tag{5.8}$$

The function H is regular at $r=2M$ [Eq. (4.19)] so that h_1^{rr} vanishes at $r=2M$. Thus, $f_1(\theta,\phi)=0$ and the coordinate position of the event horizon does not change to order ζ.

The situation is similar in order $\epsilon\zeta$. Taking account of the result that $f_1=0$, Eq. (5.5) in this order is

$$\begin{aligned}(2M)^{-1}f_2(\theta,\phi)&=-(h_2^{rr})_{r=2M}\\&=-[(1-2M/r)A(r)]_{r=2M}\,Y_l^m(\theta,\phi).\end{aligned} \tag{5.9}$$

The function $A(r)$ must be regular at $r=2M$ in order that the perturbation be regular there.[13] Thus $f_2(\theta,\phi)=0$ and the coordinate position of the event horizon does not change to order $\epsilon\zeta$. To the accuracy necessary for this calculation the event horizon is the surface $r=2M$.

C. The Shape of the Perturbed Event Horizon

The fact that the coordinate position of the event horizon remains unchanged by the perturbation is a convenient property of the gauge being used here. The geometry of the event horizon, however *is* changed by the perturbation. A convenient way to describe this change is to embed the two-surface formed by the intersection of the horizon and a constant-time surface in a flat three-dimensional space. The metric on this two-surface is given by [Eqs. (5.2) and (4.18)]

$$d\Sigma^2=4M^2\left(1-\sum_{lm}S_l^m\,Y_l^m(\theta,\phi)\right)(d\theta^2+\sin^2\theta\, d\phi^2)\;, \tag{5.10}$$

where the S_l^m are the horizon multipole moments of the source. In a flat space with polar coordinates $(\bar r,\bar\theta,\bar\phi)$ the line element has the form

$$d\sigma^2=d\bar r^2+\bar r^2\,(d\bar\theta^2+\sin^2\bar\theta\, d\bar\phi^2)\;. \tag{5.11}$$

To order ζ a surface which has the intrinsic geometry of Eq. (5.10) is

$$\bar r=2M\left(1-\tfrac{1}{2}\sum_{lm}S_l^m\,Y_l^m(\bar\theta,\bar\phi)\right). \tag{5.12}$$

The shape of the horizon is clearly distorted by the perturbation. To obtain some insight into *how* a given perturbation distorts the horizon we anticipate the result of Sec. VIII [Eqs. (8.11) and (8.2)] for the horizon multipole moments due to a point particle of mass μ located a coordinate distance R away from the center of the black hole. The polar axis in Eq. (5.10) can be oriented in the direction of the particle. If we further consider the case that $R/M\gg 1$, then only the quadrupole $(L=2)$ moments are significant and the only nonvanishing one of these is

$$S_2^0=-4\mu M^2/R^3\;. \tag{5.13}$$

The equation for the surface of the horizon then becomes

$$\bar r=2M[1+2\mu(M^2/R^3)P_2(\cos\theta)]\;. \tag{5.14}$$

This is the equation for an axially symmetric ellipsoid elongated along the axis connecting the center of the black hole and the point particle. The external point particle thus causes a distortion in the event horizon similar to the tide raised by a moon in an ocean on the surface of a planet.

VI. EXPRESSION FOR THE PERTURBATION OF THE SHEAR

Not every metric coefficient of order $\epsilon\zeta$ given in Eq. (5.3) need be calculated in order to determine the perturbation in the shear $\sigma^{(1)}$ of order $\epsilon\zeta$. To determine what information is needed we use this section to express $\sigma^{(1)}$ in terms of the perturbed metric.

The shear, σ, is defined by Eq. (2.6). For typographical reasons we introduce an alternate notation δf for that perturbation in any quantity f which is first order in the source (first order in ζ) and contains all orders in the angular momentum of the black hole (all orders in ϵ). We will let L^μ and M^μ denote vectors of the unperturbed tetrad associated with the Kerr geometry and a stroke denote covariant differentiation with respect to the Kerr metric. Thus,

$$l_\mu=L_\mu+\delta l_\mu+O(\zeta^2)\;, \tag{6.1a}$$

$$m_\mu=M_\mu+\delta m_\mu+O(\zeta^2)\;. \tag{6.1b}$$

The perturbation in the shear on the horizon is then

$$\sigma^{(1)} \equiv \delta\sigma = -\delta\Gamma^{\alpha}_{\mu\nu} M^{\mu} M^{\nu} L_{\alpha} + \delta l_{\mu|\nu} M^{\mu} M^{\nu} + L_{\mu|\nu}(M^{\mu}\delta m^{\nu} + \delta m^{\mu} M^{\nu}). \quad (6.2)$$

To order $\epsilon\zeta$ the horizon is the surface $r=2M$. The vector l_μ being normal to the horizon has only l_r as a nonvanishing covariant component. Thus, we can put

$$\delta l_{\mu} = A L_{\mu} + O(\epsilon^2), \quad (6.3)$$

for some function A. The second term in Eq. (6.2) then must vanish in the interesting order since the shear of the Kerr metric vanishes on the horizon and $L^{\mu}M_{\mu}=0$.

Perturbation of the relations $l^{\mu}m_{\mu}=m^{\mu}m_{\mu}=0$ and Eq. (6.3) gives

$$\delta m^{\mu} L_{\mu} + O(\epsilon^2) = 0, \quad (6.4a)$$

$$\delta m^{\mu} M_{\mu} + O(\epsilon^2) = 0. \quad (6.4b)$$

The vector δm^{μ} can then be generally written

$$\delta m^{\mu} = B L^{\mu} + C M^{\mu} \quad (6.5)$$

for some functions B and C. Inserting this in Eq. (6.2), recalling that L_μ is tangent to the null *geodesic* generators of the Kerr metric, and that the shear vanishes on the horizon in the Kerr metric, one finds that the last two terms in Eq. (6.2) vanish.

The vector L^{μ} is normalized so that $L^{u}=1$. This implies $L_r=1$ [Eq. (5.2)]. The expression for $\delta\sigma$ may thus be written

$$\delta\sigma = -\delta\Gamma^{r}_{\mu\nu} M^{\mu} M^{\nu} + O(\epsilon^2). \quad (6.6)$$

An explicit form for the vector M^{μ} accurate to order ϵ is

$$M^{u} = M^{r} = 0, \qquad M^{\theta} = \frac{1}{2\sqrt{2}M}, \quad M^{\phi} = \frac{i}{2\sqrt{2}M\sin\theta}. \quad (6.7)$$

Thus, for evaluating σ to order $\epsilon\zeta$, we can write

$$\delta\sigma = -\frac{1}{8M^2}\left(\delta\Gamma^{r}_{\theta\theta} - \frac{1}{\sin^2\theta}\delta\Gamma^{r}_{\phi\phi} + \frac{2i}{\sin\theta}\delta\Gamma^{r}_{\theta\phi}\right) + O(\epsilon^2). \quad (6.8)$$

A short calculation expresses the perturbed Christoffel symbols in terms of the metric of order $\epsilon\zeta$ [Eq. (5.3)]:

$$\delta\Gamma^{r}_{\theta\theta} - \frac{1}{\sin^2\theta}\delta\Gamma^{r}_{\phi\phi} = -2N\frac{\partial^2}{\partial\phi\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial Y^m_l}{\partial\theta}\right) - aK\frac{\partial Y^m_l}{\partial\phi} + O(\epsilon^2), \quad (6.9a)$$

$$\delta\Gamma^{r}_{\theta\phi} = -\frac{1}{2}\frac{1}{\sin\theta}\frac{\partial^2 Y^m_l}{\partial\phi^2} + \tfrac{1}{2}\sin^2\theta\frac{\partial}{\partial\theta}\left(aKY^m_l + N\frac{1}{\sin\theta}\frac{\partial Y^m_l}{\partial\theta}\right) + O(\epsilon^2), \quad (6.9b)$$

all quantities being evaluated on the horizon $r=2M$.

Thus to evaluate σ to order $\epsilon\zeta$ the only metric coefficient of order $\epsilon\zeta$ which need be known is N and that only on the horizon. This greatly simplifies the computation. In the next section we will evaluate N on the horizon from the Einstein equations of order $\epsilon\zeta$.

VII. THE RATE OF INCREASE IN AREA

A. Evaluation of N^m_l

The value of N on the horizon may be obtained from the two Einstein equations $R^A_u=0$ and $A=(\theta,\phi)$. Since R^A_u transforms as a vector under rotations these equations in order $\epsilon\zeta$ must have the general form

$$\sqrt{-g}\,R^A_u = \sum_{lm}\left[\mathcal{E}^m_l \Psi^{mA}_l + \mathcal{O}^m_l \Phi^{mA}_l\right] - \mathfrak{D}^A = 0. \quad (7.1)$$

Here, the terms in the brackets (the homogeneous part) involve the metric perturbations of order $\epsilon\zeta$ while the term $\mathfrak{D}^A$ (the driving term) involves products of the perturbations of order ϵ and order ζ already determined in Sec. IV. The quantities Ψ^{mA}_l and Φ^{mA}_l are, respectively, the even- and odd-parity vector spherical harmonics defined in Appendix A. The coefficients $\mathcal{E}^m_l$ and $\mathcal{O}^m_l$ therefore involve, respectively, only even- and odd-parity radial functions of the expanded metric. In the present case only the odd-parity function N is of interest and it is appropriate therefore to consider only the odd-parity part of Eq. (7.1). This may be separated out by multiplying Eq. (7.1) by $\overline{\Phi}^m_{lA}$ integrating over solid angle $d\Omega$ and using Eq. (A10),

$$l(l+1)\,\mathcal{O}^m_l = \int d\Omega\,\overline{\Phi}^m_{lA}\,\mathfrak{D}^A. \quad (7.2)$$

Using modern algebraic computer languages[14] it is not extremely difficult to evaluate the various pieces of Eq. (7.2). The quantity $\mathcal{O}^m_l$, for example, is most simply found by writing out R^{θ}_u in the case $m=0$. We find

$$\Theta_l^m = -\left(1-\frac{2M}{r}\right)\frac{d^2N}{dr^2} + \left(l(l+1) - \frac{4M}{r}\right)\frac{N}{r^2}. \quad (7.3)$$

The driving terms will be sums over as many multipoles as are contained in the source. For simplicity let us focus on a particular multipole (L,m) and later perform the sum. The driving terms are then

$$\mathfrak{D}^\theta = -\frac{aM}{r^3}\left(4K\cos\theta\frac{\partial^2 Y_L^m}{\partial\theta^2} - 2H\sin\theta\frac{\partial^2 Y_L^m}{\partial\theta\partial\phi}\right), \quad (7.4a)$$

$$\mathfrak{D}^\phi = -\frac{aM}{r^3}\left\{-4K\frac{\partial}{\partial\theta}(\cos\theta\, Y_L^m) + 2r^2\sin\theta\, Y_L^m\left[\left(1-\frac{2M}{r}\right)\frac{d^2H}{dr^2} + \frac{4M}{r^2}\frac{dH}{dr} + \left(l(l+1)+2-\frac{m^2}{\sin^2\theta}\right)\frac{H}{r^2}\right]\right\}. \quad (7.4b)$$

Substituting Eqs. (7.3) and (7.4) into Eq. (7.2) will give us a simple second-order differential equation for N. The boundary conditions for its solution are that N vanish at infinity and be regular at the horizon $r=2M$. On the horizon the driving terms are regular and their form is greatly simplified because H vanishes there. From Eqs. (4.19) and (4.20) we have

$$\mathfrak{D}^\theta = -\frac{aS_L^m}{2M^2}\frac{\partial^2 Y_L^m}{\partial\theta^2} + O(r-2M), \quad (7.5a)$$

$$\mathfrak{D}^\phi = -\frac{aS_L^m}{2M^2}\left(-\cos\theta\frac{\partial Y_L^m}{\partial\theta} + \frac{l(l+1)}{2}Y_L^m\sin\theta\right) + O(r-2M). \quad (7.5b)$$

The above behavior of the driving terms near $r=2M$ shows that the differential equation formed from Eqs. (7.2) and (7.4) will have as a particular solution a power series which is regular at $r=2M$. Let this particular solution be denoted by $\hat{N}_l^m$ The first term in this series is easily found from Eqs. (7.2) and (7.3) and is given by

$$\hat{N}_l^m = \frac{2aS_L^m}{l(l+1)-2}I_l^m(L) + O(r-2M)\ , \quad (7.6)$$

where I_l^m is the angular integral [Eqs. (7.5) and (A5b)]

$$I_l^m(L) = \int d\Omega\left[\frac{\partial \bar{Y}_L^m}{\partial\theta}\left(-\cos\theta\frac{\partial Y_l^m}{\partial\theta} + \frac{l(l+1)}{2}\sin\theta\, Y_L^m\right) - \frac{m^2}{\sin^2\theta}\bar{Y}_l^m Y_L^m\right]. \quad (7.7)$$

The most general solution of the differential equation is a sum of this particular solution and a linear combination of the two independent solutions to the homogeneous equation $\Theta_l^m=0$. Near the horizon the two independent solutions of the homogeneous equation behave like $(r-2M)$ and like [constant + $(r-2M)\ln(r-2M)$]. The latter behavior leads to a divergence in the physical component of the Riemann tensor on the horizon. [For example,

$$R_{(r)(\phi)(r)(t)} = (1-2M/r)^{1/2}(r^2\sin\theta)^{-1/2}R_{r\phi rt} \sim (1-2M/r)^{1/2}\partial^2 g_{\phi t}/\partial r^2 \sim (1-2M/r)^{-1/2}$$

near the horizon.] The general solution therefore must consist of a homogeneous solution which vanishes on the horizon and the particular solution whose behavior near $r=2M$ is given in Eq. (7.6). Thus, one concludes that on the horizon

$$N_l^m = \hat{N}_l^m + O(r-2M)\ . \quad (7.8)$$

The angular integral [Eq. (7.7)] can be evaluated to yield the following result for N_l^m on the horizon:

$$N_{L+1}^m = \frac{aS_L^m}{2L+1}\left(\frac{L-M+1}{L+1}\right), \quad (7.9a)$$

$$N_{L-1}^m = -\frac{aS_L^m}{2L+1}\left(\frac{L+M}{L}\right), \quad (7.9b)$$

$$N_l^m = 0,\ l\neq L\pm 1\ . \quad (7.9c)$$

As expected a multipole (L,m) of the source produces an odd-parity perturbation in order $\epsilon\zeta$ with only multipoles $(L\pm1,m)$. This result is now to be used with Eqs. (6.8), (6.9), and (2.10) to find the rate of increase in area of the horizon.

B. Evaluation of the Perturbation in the Shear and the Rate of Increase in Area

The important quantity in Eqs. (6.8) and (6.9) is the sum $\sum_{lm}N_l^m Y_l^m$. Equations (7.9) and elementary identities for the derivatives of spherical harmonics can be used to evaluate this as

$$\sum_{lm}N_l^m Y_l^m = \frac{aS_L^m}{L(L+1)}\sin\theta\frac{\partial Y_L^m}{\partial\theta}\ . \quad (7.10)$$

The expression for $\delta\sigma\equiv\sigma^{(1)}$ may then be evaluated. Reinstating the sum over the source multipoles (L,m), one has[15]

$$\delta\sigma = -\sum_{Lm} \frac{imaS_L^m}{4L(L+1)M^2} \left[\Upsilon^m_{L(\theta)(\theta)} + i\Upsilon^m_{L(\theta)(\phi)}\right] \tag{7.11}$$

where $\Upsilon^m_{L(A)(B)}$ are the physical components of the tensor spherical harmonics defined in Appendix A [Eqs. (A14)]. This result is now to be inserted in the expression for dA/dt, Eq. (2.10). The resulting angular integral can be done using Eqs. (A14a) and (A13),

$$\frac{dA}{dt} = \frac{a^2}{8M}\sum_{Lm} m^2\left(\frac{L(L+1)-2}{L(L+1)}\right)|S_L^m|^2 . \tag{7.12}$$

Equation (7.12) is the central result of this paper. It expresses the rate of increase of area of the horizon in terms of the horizon multipoles S_L^m of the source. Several features of this expression should be noted: (1) When the source is axially symmetric S_L^m vanishes for $m\neq 0$ and dA/dt therefore vanishes as expected. (2) The source multipoles $L=0$ and $L=1$ do not contribute to dA/dt. The $L=0$ multipole is of course axially symmetric and therefore could not contribute. The $L=1$ multipole does not contribute because it represents a uniform gravitational field perturbation of the horizon; only the "tidal" $L\geq 2$ multipoles contribute to the rate of area increase.

VIII. THE RATE OF SLOWING DOWN DUE TO AN EXTERNAL MOON

A. Boundary Conditions

The general result of the previous section for the rate of slowing down of a slowly rotating black hole due to an external perturbation will be applied in this section to the simple case when the perturbation is due to a stationary point particle of mass μ, i.e., an external moon.

A point particle will not remain stationary in the gravitational field of the black hole without some stresses to support it. If the particle is located at Schwarzschild coordinates $r=R$, $\theta=\Theta$, and $\phi=\Phi$, we will take these stresses to be localized in the spherical shell of radius R. The radial gravitational pull of the black hole is then to be balanced by the tangential stresses in the shell. The radial components of the stress vanish. The stress-energy tensor for the source is then, to order ζ,

$$T^t_t \equiv \rho = \mu\left(1-\frac{2M}{R}\right)^{1/2}\frac{1}{R^2\sin\Theta}\,\delta(r-R)\,\delta(\theta-\Theta)\,\delta(\phi-\Phi), \tag{8.1a}$$

$$T^B_A = -t^B_A\,\delta(r-R), \tag{8.1b}$$

where A and B run over the tangential directions θ and ϕ and all other components of T^ν_μ vanish. The multipole moments of this source may be found by expanding T^ν_μ in the appropriate spherical harmonics. In particular,

$$\rho^m_L = \mu\left(1-\frac{2M}{R}\right)^{1/2}\frac{1}{R^2}\,\bar{Y}^m_L(\Theta,\Phi)\,\delta(r-R) \equiv \sigma^m_L\,\delta(r-R) . \tag{8.2}$$

The expansion of the t^B_A we will simply denote by $(t^B_A)^m_L$.

The horizon multipole moments, S_L^m, which enter into the expression [Eq. (7.12)] for the rate of increase in the area of the black hole must be found by matching at the shell the perturbation which is regular on the horizon [Eq. (4.18)] to one which is regular at infinity [Eq. (4.16)]. The matching conditions at the shell could, of course, be found from an examination of the Einstein equations using the stress energy given in Eq. (8.1). We prefer to use the equivalent general results of Israel[16] on the boundary conditions across thin shells because of their ease of application.

The Regge-Wheeler gauge is not suitable for applying Israel's boundary conditions because the coordinates are singular on the shell. Perhaps the easiest way to see this is to evaluate

$$R^\theta_\theta - R^\phi_\phi = -8\pi(t^\theta_\theta - t^\phi_\phi)\,\delta(r-R)$$

and notice that this requires $H_0 - H_2 \propto \delta(r-R)$ so that some of the metric coefficients must be singular on the shell. We therefore first make a transformation to what Israel calls "natural coordinates" in which the perturbation in g_{rr} vanishes. If we write the transformation

$$x^\mu \to x^\mu + \eta^\mu(x) , \tag{8.3}$$

where η^μ is of order ζ, then the vector η^μ which makes g_{rr} vanish has in a particular multipole the components

$$\eta_t = 0, \tag{8.4a}$$

$$\eta_r = \left(1-\frac{2M}{r}\right)^{1/2} F Y^m_L , \tag{8.4b}$$

$$\eta_A = r^2 G \Psi^m_{LA} . \tag{8.4c}$$

Here, F and G are functions of r alone defined by

$$\frac{dF}{dr} = -\tfrac{1}{2}H_2\left(1-\frac{2M}{r}\right)^{1/2} , \tag{8.5}$$

$$\frac{dG}{dr} = -r^{-2}\left(1-\frac{2M}{r}\right)^{1/2} F \ . \tag{8.6}$$

The perturbation in the metric of order $\zeta(h_{1\mu\nu})$ then becomes

$$h_{1tt} = \left(1-\frac{2M}{r}\right)\left[H - \frac{2M}{r^2}\left(1-\frac{2M}{r}\right)^{-1/2} F\right] Y_L^m \ , \tag{8.7a}$$

$$h_{1AB} = r^2\left[K + \frac{2}{r}\left(1-\frac{2M}{r}\right)^{-1/2} F\right]\Phi_{LAB}^m + 2r^2 G \Psi_{LAB}^m \ , \tag{8.7b}$$

where all other components vanish and Φ_{LAB}^m and Ψ_{LAB}^m are the standard Regge-Wheeler tensor spherical harmonics discussed in Appendix A. In these coordinates Israel shows that $h_{1\mu\nu}$ is continuous across the shell. The discontinuities in the derivatives are given in terms of the stress energy in the shell S_μ^ν,

$$\mathrm{Disc}\left[\frac{\partial h_{1\mu\nu}}{\partial r}\right] = 16\pi(S_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}S) \ , \tag{8.8}$$

where $T_{\mu\nu} = S_{\mu\nu}\,\delta(r-R)$, $S = S_\mu^\mu$, and Disc [ ] denotes the discontinuity of a function. That part of the Eq. (8.8) which involves ρ alone serves to determine the discontinuity in the derivatives of the metric in terms of the mass μ. The rest of the relations may be thought of as fixing the stresses t_A^B in the shell necessary to support the moon. The relation which involves ρ alone is from Eq. (8.1),

$$\mathrm{Disc}\left[\frac{\partial h_{1A}^A}{\partial r}\right] = -16\pi\sigma_L^m, \tag{8.9}$$

where the indices on $h_{1\mu\nu}$ are raised with the unperturbed metric and the summation is over $A = \theta, \phi$. Using Eqs. (8.7) these conditions may be translated into discontinuity relations for our functions H and K,

$$\left(1-\frac{2M}{R}\right)\mathrm{Disc}\,[H] + \frac{M}{R}\mathrm{Disc}\,[K] = 0, \tag{8.10a}$$

$$\mathrm{Disc}\left[\frac{dK}{dr}\right] + \frac{1}{M}\left(\frac{l(l+1)}{2} + 1 - \frac{2M}{R}\right)\mathrm{Disc}\,[H] = 8\pi\sigma_L^m \, . \tag{8.10b}$$

Thus in the Regge-Wheeler gauge the metric coefficients themselves are not continuous across the shell. In the Newtonian limit when $2M/R \ll 1$ and H and K approach twice the perturbation in the Newtonian potential $\delta\Phi$ these relations imply correctly Disc $[\delta\Phi] = 0$, Disc $[d(\delta\Phi)/dr] = 4\pi\sigma_L^m$.

B. The Rate of Slowing Down

The relations (8.10) are sufficient to determine the two unknown constants S_L^m and M_L^m [Eqs. (3.16) and (3.18)] in the solution for the perturbation of order ζ. A tedious calculation yields

$$S_L^m = 4\pi M \sigma_L^m \, U_L\left(\frac{R}{M} - 1\right) \ , \tag{8.11}$$

where $U_L(z)$ is the function

$$U_L(z) = (z^2-1)\left[z^2 Q_L^2(z) + 2(z^2-1)^{-1/2} Q_L^1(z)\right]\left(\tfrac{1}{2}[L(L+1)-2]\,(z^2-1) + 2\frac{z-1}{z+1}\right)^{-1} . \tag{8.12}$$

Inserting (8.11) into the expression for dA/dt [Eq. (7.12)] and using Eq. (8.2) for σ_L^m, we find the sum over m may be done. The relevant sum is

$$\begin{aligned}\sum_m m^2 \,|Y_L^m(\Theta,\Phi)|^2 &= -\left[\frac{\partial^2}{\partial\lambda^2}\sum_m \bar{Y}_L^m(\Theta,\Phi+\lambda)\,Y_L^m(\Theta,\Phi)\right]_{\lambda=0} \\ &= -\frac{2L+1}{4\pi}\left[\frac{\partial^2}{\partial\lambda^2}P_L(\cos^2\Theta + \sin^2\Theta\,\cos\lambda)\right]_{\lambda=0} \\ &= \frac{2L+1}{4\pi}\,\frac{L(L+1)}{2}\sin^2\Theta \ . \end{aligned} \tag{8.13}$$

The final expression for dA/dt due to a point external mass μ at (R,Θ) is

$$\frac{dA}{dt} = \frac{\pi a^2}{4M}\left(\frac{\mu}{M}\right)\left(\frac{M}{R}\right)^4 \sin^2\Theta\, \mathfrak{F}\left(\frac{R}{M} - 1\right) \ , \tag{8.14}$$

where $\mathfrak{F}(z)$ is the sum

$$\mathfrak{F}(z) = \left(\frac{z-1}{z+1}\right)\sum_{L=2}^{\infty}(2L+1)[L(L+1)-2]\,|U_L(z)|^2 \tag{8.15}$$

and $U_L(z)$ is given by Eq. (8.12). We have evaluated the function $\mathfrak{F}(z)$ numerically and the results are given in Fig. 1 and Table I.

The factor $\sin^2\Theta$ appearing in the result means

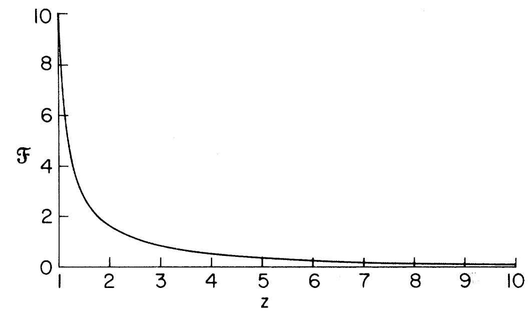

FIG. 1. The function $\mathcal{F}(z)$ which governs the magnitude of the tidal-friction effect due to a single external moon at a distance of a Schwarzschild coordinate radius $R = M(z+1)$ from the center of a slowly rotating black hole. $\mathcal{F}$ is defined in Eq. (8.15). For large values of z it falls off as $1/z^2$ leading to the $1/R^6$ falloff in the rate of slowdown of the black hole characteristic of tidal-friction processes. The horizon is located at $z=1$. If the moon is close to the horizon the tidal-friction effect becomes large.

TABLE I. The function $\mathcal{F}(z)$.

z	$\mathcal{F}(z)$	z	$\mathcal{F}(z)$
1.0	∞	5.0	0.356
1.1	7.40	5.5	0.303
1.2	4.88	6.0	0.261
1.3	3.79	6.5	0.227
1.4	3.14	7.0	0.200
1.5	2.70	7.5	0.179
2.0	1.59	8.0	0.156
2.5	1.10	8.5	0.142
3.0	0.822	9.0	0.128
3.5	0.641	9.5	0.116
4.0	0.516	10.0	0.106
4.5	0.425		

that the slowing down rate vanishes when the mass μ is on the axis of rotation (axial symmetry) and is a maximum when the source is at the equator.

The limiting cases which are of interest are when the source is far from the horizon and when it is close to it. For large values of r/M, z is large and Q_L^m behaves like z^{-L-1}. The $L=2$ (quadrupole) term becomes the dominant one in the sum for $\mathcal{F}$ and using Eq. (4.12) one finds

$$\frac{dA}{dt}=\left(\frac{16}{5}\right)\frac{\pi a^2}{M}\left(\frac{\mu}{M}\right)^2\sin^2\Theta\left(\frac{M}{R}\right)^6,\quad \frac{M}{R}\ll 1. \tag{8.16}$$

This is precisely the combination of a^2, μ^2, M and R predicted in Ref. 4 on simple dimensional grounds. Translating the result in Eq. (8.16) into a result for dJ/dt through Eq. (2.2) one has

$$\frac{dJ}{dt}=-\frac{2}{5}\frac{J\mu^2M^3}{R^6}\sin^2\Theta,\quad \frac{M}{R}\ll 1. \tag{8.17}$$

This result has the same form as the expression [Eq. (3.8)] for the slowing down of a planet due to the tidal friction caused by a stationary external moon in the case that the planet's radius and mass are comparable and when the dimensionless measure of viscosity is a number of order unity. Both expressions have the same dependence on J, μ, M, R, and Θ characteristic of tidal-friction processes. This similarity justifies the use of the tidal-friction analogy for the slowing down of a rotating black hole due to exterior matter.

When the source is close to the horizon z is near unity. Near $z=1$, $Q_L^2(z)=(z-1)^{-1}$ and $Q_L^1(z)=[2(z-1)]^{1/2}$ so that the series for $\mathcal{F}$ diverges. The dominant behavior near $z=1$ of the relevant sums of products of $Q_L^m(z)$'s is calculated in Appendix B. Using the results of Appendix B one finds

$$\frac{dA}{dt}=\frac{\pi a^2}{16M}\left(\frac{\mu}{M}\right)^2\sin^2\Theta\,\ln\left(\frac{R}{M}-2\right)\left(1-\frac{2M}{R}\right)^{-1},\quad R\sim 2M. \tag{8.18}$$

The slow-down rate thus becomes large when the perturbation approaches the horizon.

IX. CONCLUSIONS

The interactions of a black hole and its exterior environment is an important class of problems for astrophysics and for relativity as well. Only through a study of such interactions will the role, if any, which black holes play in puzzles such as quasars, x-ray sources, and Weber's pulses of gravitational radiation be understood. Further, it is only through such interactions that the black hole will be detected. Recently great progress has been made in the study of the motion of test particles in the field of black holes and the possible astrophysical consequences of the resulting electromagnetic and gravitational radiation.[17]

In this paper the slowing down of a slowly rotating black hole due to an exterior moon has been calculated. This effect represents a dynamical coupling between the black hole and its exterior environment. As has been emphasized by Press[18] (see also Ref. 4) the slowing-down effect is unlikely to be detectable in binary systems in which the components are widely separated $R\gg M$. In such systems the gravitational radiation provides a much more efficient mechanism for the dissipation of the system's angular momentum. When R becomes close to $2M$ however the slowing-down effect becomes large (cf. Fig. 1). The evolution of binary systems so closely spaced is likely to be rapid and therefore the slowing-down effect difficult to disentangle from the observations.

The calculations presented in this paper have been limited to the slowly rotating-black-hole case, $a \ll M$. This limitation did not arise out of any physical circumstance but only because of the mathematical tractability of solution by expansion about the Schwarzschild geometry. Simple dimensional estimates of Hawking[4] indicate that the slowing-down effect could be very large if there were a resonance between the orbital angular velocity of a perturbing mass and the effective angular velocity $(1/2M)$ of a maximal, $a=M$, black hole. It would be of great interest to investigate this case in detail and the recent work of Teukolsky[19] separating the wave equation for the radiative perturbations to the Kerr geometry may provide us with a way to do this.

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APPENDIX A: ORTHOGONALITY AND NORMALIZATION OF REGGE-WHEELER SPHERICAL HARMONICS

The overlap and normalization integrals for the Regge-Wheeler spherical harmonics occur frequently in problems in which the Schwarzschild geometry is perturbed. Here, we show how a covariant approach allows these integrals to be evaluated simply.

Let upper case Latin indices $A, B, \ldots$ run over the two coordinates which locate a point on the surface of a sphere. The metric on the sphere γ_{AB} is given in a standard spherical coordinate system $x^1=\theta$, $x^2=\phi$ by

$$\gamma_{11}=1, \quad \gamma_{12}=\gamma_{21}=0, \quad \gamma_{33}=\sin^2\theta. \tag{A1}$$

Covariant differentiation with respect to this metric is denoted by a stroke, viz., $v_{A|B}$. The two-dimensional alternating symbol ϵ_{AB} is defined by

$$\epsilon_{BA}=-\epsilon_{AB}, \quad \epsilon_{12}=-\sin\theta. \tag{A2}$$

Scalar spherical harmonics are denoted by Y_l^m and are assumed normalized so that

$$\int d\Omega\, \bar{Y}_l^m Y_{l'}^{m'} = \delta_{ll'}\,\delta_{mm'} \;, \tag{A3}$$

where the integral ranges over all solid angle. The spherical harmonics satisfy

$$\gamma^{AB} Y^m_{l\,|AB} = -l(l+1)Y_l^m \;. \tag{A4}$$

The Regge-Wheeler vector and tensor harmonics are constructed from the differentiated scalar harmonics, the γ_{AB} and the ϵ_{AB}. In the notation of Thorne and Campolattaro,[9]

$$\Psi^m_{lA} = Y^m_{l\,|A} \;, \tag{A5a}$$

$$\Phi^m_{lA} = \epsilon_A{}^B\, Y^m_{l\,|B} \;, \tag{A5b}$$

$$\Phi^m_{lAB} = \gamma_{AB} Y_l^m \;, \tag{A5c}$$

$$\Psi^m_{lAB} = Y^m_{l\,|AB} \;, \tag{A5d}$$

$$\chi^m_{lAB} = \tfrac{1}{2}(\epsilon_A{}^C \Psi^m_{lCB} + \epsilon_B{}^C \Psi^m_{lCA}). \tag{A5e}$$

We denote the scalar product of two general harmonics $H^m_{lAB\cdots}$ and $K^m_{lAB\cdots}$ by

$$(H^m_l, K^{m'}_{l'}) = \int d\Omega\, H^m_{lAB\cdots} K_l^{mAB\cdots} \;, \tag{A6}$$

all indices being contracted. We now evaluate all the integrals of the form (A6) for the harmonics defined in (A5). The technique in every case is the same. $\int d\Omega$ is written $\int d^2x\sqrt{\gamma}$. Covariant integration by parts is performed in the sense that when

$$(t^{ABC\cdots} s_{BC\cdots})_{|A} = t^{ABC\cdots}{}_{|A} s_{BC\cdots} + t^{ABC\cdots} s_{BC\cdots\,|A} \tag{A7}$$

is integrated over the sphere, the left-hand side vanishes for any two tensors $t^{ABC\cdots}$ and $s_{BC\cdots}$. Where appropriate, use is made of the identities

$$\epsilon_{AB\,|C}=0, \quad \gamma_{AB\,|C}=0, \tag{A8a}$$

$$\epsilon_{AB}\epsilon^{CB} = \delta_A^C \;, \tag{A8b}$$

$$v_{A\,|BC} = v_{A\,|CB} + R^D{}_{ABC} v_D \;, \tag{A8c}$$

$$R_{ABCD} = \epsilon_{AB}\epsilon_{CD} \tag{A8d}$$

for arbitrary vector v_A. We will not reproduce the details of the calculations in every case but confine ourselves to an illustrative case useful here. For the tensor harmonics Ψ^m_{lAB},

$$(\Psi_l^m, \Psi_{l'}^{m'})$$
$$= \int d^2x\sqrt{\gamma}\,\gamma^{AC}\gamma^{BD}\bar{Y}_l^m{}_{|AB} Y_{l'}^{m'}{}_{|CD}$$
$$= -\int d^2x\sqrt{\gamma}\,\gamma^{AC}\gamma^{BD}\bar{Y}_l^m{}_{|ABD} Y_{l'}^{m'}{}_{|C}$$
$$= -\int d^2x\sqrt{\gamma}\,\gamma^{AC}\gamma^{BD}(\bar{Y}_l^m{}_{|BDA} + R^E{}_{BAD}\bar{Y}_l^m{}_{|E}) Y_{l'}^{m'}{}_{|C}$$
$$= \int d^2x\sqrt{\gamma}(\gamma^{AC}\gamma^{BD}\bar{Y}_l^m{}_{|BD} Y_{l'}^{m'}{}_{|AC} - \gamma^{AB}\bar{Y}_l^m{}_{|A} Y_{l'}^{m'}{}_{|B})$$
$$= l(l+1)[l(l+1)-1]\,\delta_{ll'}\,\delta_{mm'} \;. \tag{A9}$$

For the vector harmonics the results are

$$(\Psi_l^m, \Psi_{l'}^{m'}) = l(l+1)\delta_{ll'}\,\delta_{mm'} \;, \tag{A10a}$$

$$(\Psi_l^m, \Phi_{l'}^{m'}) = 0 \;, \tag{A10b}$$

$$(\Phi_l^m, \Phi_{l'}^{m'}) = l(l+1)\delta_{ll'}\,\delta_{mm'} \;. \tag{A10c}$$

For the tensor spherical harmonics one finds

$$(\Phi_l^m, \Phi_{l'}^{m'}) = 2\delta_{ll'}\,\delta_{mm'} \;, \tag{A11a}$$

$$(\Phi_l^m, \Psi_{l'}^{m'}) = -l(l+1)\delta_{ll'}\,\delta_{mm'} \;, \tag{A11b}$$

$$(\Phi_l^m, \chi_{l'}^{m'}) = 0 \;, \tag{A11c}$$

$$(\Psi_l^m, \Psi_{l'}^{m'}) = l(l+1)[l(l+1)-1]\,\delta_{ll'}\,\delta_{mm'} \;, \tag{A11d}$$

$$(\Psi_l^m, \chi_{l'}^{m'}) = 0 \;, \tag{A11e}$$

$$(\chi_l^m, \chi_{l'}^{m'}) = \tfrac{1}{2}l(l+1)[l(l+1)-2]\,\delta_{ll'}\,\delta_{mm'} \;. \tag{A11f}$$

Thus for the particular combination which occurs in the work of Price and Thorne and here

$$\Upsilon_{lAB}^m = \tfrac{1}{2}l(l+1)\,\Phi_{lAB}^m + \Psi_{lAB}^m \;, \tag{A12}$$

we have

$$(\Upsilon_l^m, \Upsilon_{l'}^{m'}) = \tfrac{1}{2}l(l+1)[l(l+1)-2]\,\delta_{ll'}\,\delta_{mm'} \;. \tag{A13}$$

The tensor harmonics Φ_{lAB}^m, Υ_{lAB}^m, and χ_{lAB}^m form an orthogonal set in contrast to the Regge-Wheeler ones. Furthermore, both Υ_{lAB}^m and χ_{lAB}^m are traceless.

The physical components of the tensor spherical harmonics are defined for example by

$$\Upsilon_{l(A)(B)}^m = |\gamma^{AA}|^{1/2}|\gamma^{BB}|^{1/2}\Upsilon_{lAB}^m \quad \text{(no sum).} \tag{A14}$$

The explicit form of these particular harmonics which are useful in the body of the paper are

$$\Upsilon_{l(\theta)(\theta)}^m = -\Upsilon_{l(\phi)(\phi)}^m$$
$$= \tfrac{1}{2}l(l+1)Y_l^m + \frac{\partial^2 Y_l^m}{\partial\theta^2} \;, \tag{A14a}$$

$$\Upsilon_{l(\theta)(\phi)}^m = \frac{1}{\sin\theta}\left(\frac{\partial Y_l^m}{\partial\theta\alpha\phi} - \cot\theta\,\frac{\partial Y_l^m}{\partial\phi}\right). \tag{A14b}$$

APPENDIX B: SUMS RELEVANT TO SLOWDOWN RATE FOR MASS NEAR THE HORIZON

The sum for $\mathfrak{F}$ near the horizon can be decomposed into sums of products of Q_L^1 and Q_L^2 with themselves and with each other. At $z=1$ the sums diverge like $\sum_L(1/L)$. The divergent behavior of the sums for z near unity is thus governed by the large-L behavior of the series and is the same as

$$\mathcal{S}_{mn}(z) = \sum_{L=0}^{\infty}(L+1)^{-1}Q_L^m(z)\,Q_L^n(z) \;. \tag{B1}$$

Using the Rodrigues's formula for the Q_L^m [Ref. 11, Eq. (3.6.5)] and the dispersion integral representation for $Q_L(z)$ [Ref. 11, Eq. (3.6.29)], we can write

$$\mathcal{S}_{mn}(z) = \tfrac{1}{4}(-1)^{m+n}(z^2-1)^{(m+n)/2}\int_{-1}^{1}dx\int_{-1}^{1}dx'\,(z-x)^{-m-1}(z-x')^{-n-1}\,\mathcal{P}(x,x') \;, \tag{B2}$$

where

$$\mathcal{P}(x,x') = \sum_{L=0}^{\infty}(L+1)^{-1}P_L(x)P_L(x') \;. \tag{B3}$$

Using the addition formula [Ref. 11, Eq. (3.11.1)] and the generating function for Legendre functions [Ref. 11, Eq. (3.6.33)], this can be expressed as

$$\mathcal{P}(x,x') = \frac{1}{2\pi}\int_0^{2\pi}d\psi\int_0^1 dy(1-2yx+y^2)^{-1/2} = \frac{1}{2\pi}\int_0^{2\pi}d\psi\ln\{1-[\tfrac{1}{2}(1-X)]^{-1/2}\} \;, \tag{B4}$$

where

$$X = xx' + (1-x^2)^{1/2}(1-x'^2)^{1/2}\cos\phi\ . \tag{B5}$$

Near $z=1$ the divergent part of the integral in Eq. (B2) comes from x and x' near unity. Writing $z=1+\epsilon$, $x=1+\epsilon\xi$, and $x'=1+\epsilon\xi'$, we find that for small ϵ

$$\mathcal{S}_{mn} = \frac{1}{4}\left(\frac{2}{\epsilon}\right)^{(m+n)/2}\int_0^{2/\epsilon} d\xi \int_0^{2/\epsilon} d\xi'\,(1+\xi)^{-m-1}(1+\xi')^{-n-1}\frac{1}{2\pi}\int_0^{2\pi} d\phi\left[-\tfrac{1}{2}\ln\epsilon + \ln\left\{\epsilon^{1/2} + \left(\frac{1}{2\epsilon}(1-X)\right)^{-1/2}\right\}\right] \tag{B6}$$

The last term remains finite and integrable as $\epsilon \to 0$ so that doing the remaining simple integral we have for $\mathcal{S}_{mn}$ as $\epsilon\to 0$

$$\mathcal{S}_{mn} = \frac{(-1)^{m+n}}{mn}\left(\frac{2}{\epsilon}\right)^{m+n}(-\ln\epsilon). \tag{B7}$$

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[1]S. W. Hawking, Commun. Math. Phys. 25, 152 (1972). See also S. W. Hawking and G. F. R. Ellis, Singularities, Causality and Cosmology (Cambridge Univ. Press, Cambridge, England, to be published). Appreciation is expressed to Brandon Carter for discussions of this consequence of Hawking's result.

[2]See also J. Ipser, Phys. Rev. Letters 27, 529 (1971).

[3]R. Penrose, Phys. Rev. Letters 10, 66 (1963).

[4]S. W. Hawking and J. B. Hartle, Commun. Math. Phys. 27, 283 (1972).

[5]L. Edelstein (unpublished).

[6]E. Newman and R. Penrose, J. Math. Phys. 3, 566 (1962).

[7]T. Regge and J. A. Wheeler, Phys. Rev. 108, 1063 (1957).

[8]L. Edelstein and C. V. Vishveshwara, Phys. Rev. D 1, 3514 (1970).

[9]K. Thorne and A. Campolattaro, Astrophys. J. 149, 591 (1967).

[10]We are following here the notation of Ref. 9.

[11]In conventions regarding Legendre functions we follow A. Erdélyi *et al.*, *Higher Transcendental Functions* (McGraw-Hill, New York, 1953), Vol. I.

[12]See Ref. 4 for more discussion on this point.

[13]A more mechanical argument which leads to the same result is to compute the equation $R^{\theta}_{\theta} - R^{\phi}_{\phi} = 0$ to order $\epsilon\xi$. It implies $A(r) = C(r)$. Since C must vanish at $r=2M$ in order for the perturbation to be regular on the horizon [cf. Eq. (5.3)], A must vanish there also. The combination which occurs in Eq. (5.9) must therefore certainly vanish.

[14]See, e.g., J. Fletcher, R. Clemens, R. Matzner, K. Thorne, and B. Zimmerman, Astrophys. J. Letters 148, L91 (1967).

[15]In manipulating these expressions it is useful to remember that since S_L^m represents a real source, then $\bar{S}_L^m = S_L^{-m}$, and hence sums like $\Sigma_m S_L^m \Upsilon_{LAB}^m$ are always real.

[16]W. Israel, Nuovo Cimento 44B, 1 (1966).

[17]See, e.g., M. Davis, R. Ruffini, W. Press, and R. Price, Phys. Rev. Letters 27, 1466 (1971); C. W. Misner, *ibid.* 28, 994 (1972); W. Press, Astrophys. J. Letters 170, L105 (1971); C. Goebel, *ibid.* 172, L95 (1972); C. Misner, R. Breuer, D. Brill, P. Chrzanowski, H. Hughes, III, and C. Periera, Phys. Rev. Letters 28, 998 (1972); J. M. Bardeen, W. Press, and S. Teukolsky (unpublished).

[18]W. Press, Astrophys. J. 175, 243 (1972).

[19]S. Teukolsky, Phys. Rev. Letters 29, 1114 (1972).