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Conserved quantities in non-Abelian monopole fields

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Van Holten's covariant Hamiltonian framework is used to find conserved quantities for an isospin-carrying particle in a non-Abelian monopolelike field. For a Wu-Yang monopole we find the most general scalar potential such that the combined system admits a conserved Runge-Lenz vector. In the effective non-Abelian field for nuclear motion in a diatomic molecule due to Moody, Shapere, and Wilczek, a conserved angular momentum is constructed, despite the nonconservation of the electric charge. No Runge-Lenz vector has been found.

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I. INTRODUCTION

In a recent paper van Holten [1] outlined an algorithm for deriving conserved quantities for a particle in a given external field, based on the use of Killing tensors. An example is provided by an isospin-carrying particle in the field of a Wu-Yang (WY) monopole [2–8] for which the conclusion is that no Runge-Lenz type vector exists. This reminds one to an earlier result by Fehér [9], who proved that a charged particle in a Dirac monopole field can not have a globally defined Runge-Lenz vector. In the latter case there is a way out, though [10–14]: adding a fine-tuned inverse square potential removes the obstruction, providing us with a conserved Runge-Lenz vector. Below we find, using van Holten's recipe, the most general additional potential, which allows for a conserved Runge-Lenz vector in non-Abelian monopolelike fields.

Similar results hold in the effective field of a diatomic molecule, considered before by Moody, Shapere, and Wilczek [15–17]. Despite the nonconservation of the electric charge, we can construct a conserved angular momentum. No Runge-Lenz vector exists in general, though.

II. CONSERVED QUANTITIES

We start with the equations of motion for an isospin-carrying particle in a static non-Abelian gauge field [18]

$$\dot{\pi}_i = I^a F^a_{ij}\dot{x}^j - D_i V, \qquad \dot{I}^a = -\epsilon_{abc} I^b \left(A^c_j \dot{x}^j - \frac{\partial V}{\partial I^c}\right), \tag{1}$$

where $\pi_i = \dot{x}_i$, and a scalar potential, $V = V(\vec{r}, \vec{I})$ has also been included for later convenience. Defining the covariant Poisson bracket and Hamiltonian as [1]

$$\{f, g\} = D_j f \frac{\partial g}{\partial \pi_j} - \frac{\partial f}{\partial \pi_j} D_j g + I^a F^a_{jk} \frac{\partial f}{\partial \pi_j}\frac{\partial g}{\partial \pi_k} - \epsilon_{abc}\frac{\partial f}{\partial I^a}\frac{\partial g}{\partial I^b} I^c, \tag{2}$$

$$H = \tfrac{1}{2}\vec{\pi}^2 + V(\vec{r}, \vec{I}), \tag{3}$$

where D_j is the covariant derivative

$$D_j f = \partial_j f - \epsilon_{abc} I^a A^b_j \frac{\partial f}{\partial I^c}. \tag{4}$$

Let us record the commutation relation of the covariant derivatives

$$[D_i, D_j] = -\epsilon_{abc} I^a F^b_{ij} \frac{\partial}{\partial I^c}. \tag{5}$$

Equations (1) can be obtained in a Hamiltonian framework $\dot{x}_i = \{x_i, H\}$, $\dot{\pi}_i = \{\pi_i, H\}$, $\dot{I}^a = \{I^a, H\}$.

Following van Holten [1], constants of the motion can conveniently be sought for in the form of an expansion into powers of the covariant momentum

$$Q = C(\vec{r}, \vec{I}) + C_i(\vec{r}, \vec{I})\pi_i + \frac{1}{2!} C_{ij}(\vec{r}, \vec{I})\pi_i \pi_j + \ldots \tag{6}$$

Requiring Q to Poisson commute with the Hamiltonian yields a series of constraints,

$$\begin{aligned}
&C_i D_i V + \epsilon_{abc} I^a \tfrac{\partial C}{\partial I^b}\tfrac{\partial V}{\partial I^c} = 0, && \text{order } 0\\
&D_i C = I^a F^a_{ij} C_j + C_{ij} D_j V + \epsilon_{abc} I^a \tfrac{\partial C_i}{\partial I^b}\tfrac{\partial V}{\partial I^c}, && \text{order } 1\\
&D_i C_j + D_j C_i = I^a (F^a_{ik} C_{kj} + F^a_{jk} C_{ki}) + C_{ijk} D_k V \\
&\quad + \epsilon_{abc} I^a \tfrac{\partial C_{ij}}{\partial I^b}\tfrac{\partial V}{\partial I^c}, && \text{order } 2\\
&\quad\vdots \qquad\qquad \vdots \qquad\qquad \vdots
\end{aligned} \tag{7}$$

The expansion can be truncated at a finite order, provided the covariant Killing equation is satisfied, $D_{(i_1} C_{i_2 \ldots i_n)} = 0$ when we can set $C_{i_1 \ldots i_n \ldots} = 0$. For $n = 1$, we have a Killing vector. For example, we have, for any unit vector $\hat{n}$, the generator of a rotation around $\hat{n}$,

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$$\vec{C} = \hat{n} \times \vec{r}. \tag{8}$$

Then van Holten's recipe yields a conserved angular momentum. Similarly, for any unit vector $\hat{n}$,

$$C_{ij} = 2\delta_{ij}\hat{n} \cdot \vec{r} - (n_i x_j + n_j x_i) \tag{9}$$

is a Killing tensor of order 2, associated with the Runge-Lenz vector of planetary motion.

III. WU-YANG MONOPOLE

The WY monopole is given by the non-Abelian gauge potential with a "hedgehog" magnetic field [7],

$$A_i^{aWY} = \epsilon_{iak}\frac{x_k}{r^2}, \qquad F_{ij}^a = \epsilon_{ijk}\frac{x_k x_a}{r^4}. \tag{10}$$

Let us now consider an isospin-carrying particle moving in a Wu-Yang monopole field augmented by a scalar potential, and inquire about conserved quantities. For an arbitrary potential, van Holten's procedure yields that

$$Q = \vec{I} \cdot \hat{r} \qquad \left(\hat{r} = \frac{\vec{r}}{r}\right) \tag{11}$$

is covariantly constant, $\vec{D}Q = 0$, and Q is, therefore, a constant of the motion. It is identified with the electric charge. A similar calculation yields, for the Killing vector (8), the angular momentum [1],

$$\vec{J} = \vec{r} \times \vec{\pi} - Q\hat{r}. \tag{12}$$

Let us now turn to quadratic quantities. Inserting (9) into (7), from the 2nd-order equation we find, therefore,

$$\vec{C} = \vec{n} \times (Q\hat{r}). \tag{13}$$

Now we identify the "good" potentials. For this, we observe that the first-order constraint in (7) can be written as

$$D_i C = \frac{Q^2}{r^2}\left((\hat{n} \cdot \hat{r})\frac{x_i}{r} - n_i\right) + C_{ij}D_j V. \tag{14}$$

Putting $V = V_0 + V_1$, the first term on the right-hand side can now be removed by choosing a fine-tuned potential, V_0

$$\vec{D}V_0 = -\frac{Q^2}{r^3}\hat{r} \Rightarrow V_0 = \frac{Q^2}{2r^2}. \tag{15}$$

Assuming that V_1 only depends on r, $V = V_1(r)$, $\vec{D}_i V_1 = \vec{\nabla}V_1$ is radial. This leaves us with

$$D_i C = C_{ij}\partial_j V_1 = -r^2 V_1' \vec{\nabla}(\hat{n} \cdot \hat{r}), \tag{16}$$

where $V_1' = dV_1/dr$, which can be solved by

$$V_1 = \frac{\alpha}{r} + \beta, \qquad C = \alpha(\hat{n} \cdot \hat{r}). \tag{17}$$

Hence,

$$V = \frac{Q^2}{2r^2} + \frac{\alpha}{r} + \beta \quad \text{and} \quad C = \alpha\vec{n} \cdot \hat{r}, \tag{18}$$

where α and β are arbitrary constants. Collecting our results,

$$\vec{K} = \vec{\pi} \times \vec{J} + \alpha\hat{r} \tag{19}$$

is a conserved Runge-Lenz vector for an isospin-carrying particle in the Wu-Yang monopole field combined with the potential (18) [19].

The physical interpretation of the previous result is for large r, the field of a *self-dual* non-Abelian monopole of charge m is that of Wu-Yang, Eq. (10), augmented with a hedgehog Higgs field

$$\Phi^a = (1 - \frac{m}{r})\frac{x^a}{r}.$$

The equations of motion of an isospin-carrying particle in such a field are precisely those considered above [12].

IV. DIATOMIC MOLECULE

In Ref. [15] Moody, Shapere, and Wilczek have shown that in the Born-Oppenheimer approximation nuclear motion in a diatomic molecule can be described by the effective non-Abelian gauge field and Hamiltonian

$$A_i^a = (1 - \kappa)\epsilon_{iaj}\frac{x_j}{r^2}, \qquad H = \frac{1}{2}\vec{\pi}^2 + V, \tag{20}$$

where κ is a real parameter. This can be achieved by applying a suitable gauge transformation with respect to the original form [16].

The field strength

$$F_{ij}^a = (1 - \kappa^2)\epsilon_{ijk}\frac{x_k x_a}{r^4}$$

resembles that of the monopole aligned into the third internal direction $F_{\theta\phi} = (1 - \kappa^2)\sin\theta T_3$, except for the parameter κ being unquantized. The potential (20) is that of a Wu-Yang [i.e., an imbedded Dirac] monopole of unit charge when $\kappa = 0$; for other values of κ, it is a truly non-Abelian configuration—except for $\kappa = \pm 1$, when the field strength vanishes and (20) is a gauge transform of the vacuum.

Turning to the conserved quantities, we note that when $\kappa \neq 0$ the used-to-be electric charge Q in (11) is *not more covariantly conserved* in general,

$$\{H, Q\} = -\vec{\pi} \cdot \vec{D}Q, \qquad D_j Q = \frac{\kappa}{r}\left(I^j - Q\frac{x_j}{r}\right). \tag{21}$$

Nor is Q^2 conserved, $\{H, Q^2\} = -2\kappa Q(\vec{\pi} \cdot \vec{D}Q)$. Note for further reference that unlike Q^2, the length of isospin I^2 *is* conserved, $\{H, I^2\} = 0$.

The gauge field (20) is rotationally symmetric and an isospin-carrying particle moving in it admits a conserved angular momentum [15,16]. Its form is, however, some-

what unconventional, and we rederive it, therefore, in detail.

Our starting point is the first-order condition in (7). We take first $V = 0$; then this is the only condition. Evaluating the right-hand side with F^a_{jk} as given in (20), the equation to be solved is

$$D_i C = (1-\kappa^2)\frac{Q}{r}\left((\hat{n}\cdot\hat{r})\frac{x_i}{r} - n_i\right). \tag{22}$$

In the Wu-Yang case, $\kappa = 0$, this equation was solved by $C = -\hat{n}\cdot Q\hat{r}$. But for $\kappa \neq 0$ the electric charge Q is not conserved, and using (21), as well as the relations

$$D_i I^j = \frac{(1-\kappa)}{r}\left(Q\delta_{ij} - I^i\frac{x_j}{r}\right),$$

$$D_i(Q\hat{n}\cdot\hat{r}) = \frac{Q}{r}\left(n_i + (\vec{n}\cdot\hat{r})\left(\kappa I_i - (1+\kappa)\frac{r_i}{r}\right)\right),$$

$$I^a F^a_{ij} = (1-\kappa^2)Q\frac{\epsilon_{ijk}x_k}{r^3}, \tag{23}$$

we find

$$(\kappa-1)D_i(Q\hat{n}\cdot\hat{r}) = \kappa D_i I^j n_j + (1-\kappa^2)\left((\hat{n}\cdot\hat{r})\frac{x_i}{r} - n_i\right).$$

Comparing with (22) allows us to infer that

$$C = -\hat{n}\cdot((1-\kappa)Q\hat{r} + \kappa\vec{I}). \tag{24}$$

The conserved angular momentum is, therefore,

$$\vec{J} = \vec{r}\times\vec{\pi} - \vec{\Psi}, \tag{25}$$

$$\vec{\Psi} = (1-\kappa)Q\hat{r} + \kappa\vec{I} = Q\hat{r} + \kappa(\hat{r}\times\vec{I})\times\hat{r}, \tag{26}$$

consistently with the results in [15,16,20]. Note the "replacement rule" $Q\hat{r}\to\vec{\Psi}$.

For $\kappa = 0$ we recover the Wu-Yang expression, (12). Eliminating $\vec{\pi}$ in favor of $\vec{p} = \vec{\pi} + \vec{A}$ allows us to rewrite the total angular momentum as $\vec{J} = \vec{r}\times\vec{p} - \vec{I}$ making manifest the celebrated "spin from isospin term" [21].

Restoring the potential, we see that, again due to the nonconservation of Q, $D_j V \neq 0$ in general. The zeroth-order condition $\vec{C}\cdot\vec{D}V = 0$ in (7) is, nevertheless, satisfied if V is a radial function independent of $\vec{I}$, $V = V(r)$, since then $\vec{D}V = \vec{\nabla}V$, which is perpendicular to infinitesimal rotations $\vec{C}$. Alternatively, a direct calculation, using the same formulae (21)–(23), allows us to confirm that $\vec{J}$ commutes with the Hamiltonian, $\{J_i, H\} = 0$.

Multiplying (26) by $\hat{r}$ yields, once again, $\vec{J}\cdot\hat{r} = -Q$, as in the Wu-Yang case. This is, however, less useful as before, since Q is not a constant of the motion so that the angle between $\vec{J}$ and the radius vector $\vec{r}(t)$ is not more constant.

Let us now turn to searching for a Runge-Lenz vector. Inserting the Killing tensor (9) into the 2nd-order equations in (7) yields

$$\vec{C} = \hat{n}\times\vec{\Psi}. \tag{27}$$

Our next step would be to identify a good potential. The first-order constraint in (7) can be written as

$$D_i C = (1-\kappa^2)\frac{Q^2}{r^2}\left((1-\kappa)(\hat{n}\cdot\hat{r})\frac{x_i}{r} - n_i\right) + \kappa(1-\kappa^2)\frac{Q}{r^2}(\hat{n}\cdot\hat{r})I^i + C_{ij}D_j V. \tag{28}$$

In the WY case, $\kappa = 0$, our clue has been to remove the first term by a fine-tuned term in the potential. This could again be attempted, namely, by putting $V = V_0 + V_1$,

$$\vec{D}V_0 = -(1-\kappa^2)\frac{Q}{r^3}\vec{\Psi}. \tag{29}$$

Assuming that such a potential does exist, $C_{ij}D_j V_0$ would cancel the upper term in (28) [but contributes to others], leaving us with

$$D_i C = \kappa(1-\kappa^2)\frac{Q}{r^2}(\hat{n}\times(\hat{r}\times\vec{I}))_i + C_{ij}D_j V_1. \tag{30}$$

Our remaining task would now be to integrate Eqs. (29) and (30)—which we have not been able to do yet in general, except for $\kappa = 0$ when the integrability of Eq. (30) can also be studied as follows: By (5), we must have

$$-\epsilon_{abc}I^a F^b_{ij}\frac{\partial C}{\partial I^c} = (n_i x_j - n_j x_i)\Delta V_1. \tag{31}$$

Assuming that C only depends on $\vec{r}$, the left-hand side vanishes, and this condition merely requires

$$\triangle V_1 = 0 \Rightarrow V_1 = \frac{\alpha}{r} + \beta, \tag{32}$$

as we found before.

We should remark, however, that even if we succeeded to integrate (29), the resulting potential would *break the rotational invariance*. The zeroth-order condition in (7) requires in fact that the $D_j V$ be *perpendicular* to $\vec{C}$. But the $\vec{C}$ of angular momentum and the one appropriate for the Runge-Lenz vector, namely, the infinitesimal rotation in (8) and (27), respectively, have different orientations, so that the two conditions can not be simultaneously satisfied.

V. DISCUSSION

It is worth mentioning that van Holten's algorithm is equivalent to the approach of Forgacs-Manton-Jackiw [5,22,23], as has been proved in an earlier version of this paper.

Our results have a nice interpretation in terms of fiber bundles [24]. For $\kappa \neq 0, \pm 1$, it is truly non-Abelian, i.e., *not reducible* to one on an $U(1)$ bundle. No covariantly constant direction field, and, therefore, *no conserved electric charge* exists in this case.

The field is nevertheless radially symmetric, but the conserved angular momentum (26) has a nonconventional form [16].

In the truly non-Abelian case, the consistency condition involves the covariant, rather than ordinary derivative and covariantly constant sections only exist in exceptional cases, namely, when the bundle is reducible. Thus, only some (noncentral extension) act on the bundle.

We have not been able to derive a Runge-Lenz vector for diatomic molecules, except for $\kappa = 0$.

Let us emphasize that the derivation of the non-Abelian field configuration (20) from molecular physics [15] indicates that our analysis may not be of purely academic interest. The situation could well be analogous to what happened before with the non-Abelian Aharonov-Bohm experiment [25,26], which became recently accessible experimentally [27,28].

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