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Explicit field realizations of W algebras

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The fact that certain nonlinear $W_{2,s}$ algebras can be linearized by the inclusion of a spin-1 current can provide a simple way to realize $W_{2,s}$ algebras from linear $W_{1,2,s}$ algebras. In this paper, we first construct the explicit field realizations of linear $W_{1,2,s}$ algebras with double scalar and double spinor, respectively. Then, after a change of basis, the realizations of $W_{2,s}$ algebras are presented. The results show that all these realizations are Romans-type realizations.

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I. INTRODUCTION

After the fundamental work of Zamolodchikov [1] in the middle of the 1980's, W algebras have attracted much attention since they uncover some underlying world sheet symmetries of strings. Many W algebras are known (for review, see [2]) and much work has been carried out on their classification [3–6]. W algebras have many applications and become the subject of great interest in many branches of physics and mathematics, e.g., in W gravity theories [7,8], critical and noncritical W string theories [9,10], Wess-Zumino-Novikov-Witten models [11–13], quantum Hall effect [14], and especially in black holes [15,16], where it was shown that the Hawking radiation can be explained as the fluxes of chiral currents forming a W_∞ algebra.

As we know, W algebras arise from Kac-Moody algebras, which are related to classical Lie algebras. Various free field realizations of W algebras have been extensively studied [17–24]. At quantum level, W algebras are usually nonlinear, which makes it very difficult to give the field realizations of the W algebras. The corresponding W strings were first investigated in Ref. [25] and have been extensively developed since then. Much research on the scalar realizations of $W_{2,s}$ strings has been done [26–34]. Most of the research is based on the grading method, where the Becchi-Rouet-Stora-Tyutin (BRST) charge of $W_{2,s}$ strings is written in the form of $Q_B = Q_0 + Q_1$. This provides an easy way to construct $W_{2,s}$ strings, while it imposes more constrained conditions on the BRST charge. Under the supposition that this grading form still holds true for spinor field realizations, the corresponding works had been done [35–38].

Furthermore, many investigations have been focused on understanding the structure of W algebras [39–42]. It was shown that linear Lie algebras with a finite number of currents may contain some nonlinear W algebras with an arbitrary central charge as subalgebras. Especially for $W_{2,s}$ algebras, they can be linearized by the inclusion of a spin-1 current at $s = 3$ and 4 [39]. After performing a nonlinear change of basis, $W_{2,s}$ algebras can be recast into the form of linear algebras. But for the spin-s current W_0, one has $W_0(z)W_0(\omega) \sim 0$, which indicates that W_0 is a null current. It is exciting that this shines some light on the realizations of the nonlinear $W_{2,s}$ algebras. After constructing the linear bases of $W_{1,2,s}$ algebras and making a change of basis, we can obtain the realizations of the nonlinear $W_{2,s}$ algebras. In fact, the spin-s current W_0 can be set to zero, which will give a Romans-type realization of $W_{2,s}$ algebras. However, in [39], it was shown that the null current W_0 does not need to be set to zero and was first realized with parafermionic vertex operators. It also can be found in [37,40,41] that the null current W_0 was realized with the ghostlike fields. In this paper, we will construct the linear bases of the $W_{1,2,s}$ algebras with double scalar and double spinor, respectively. Through a change of basis, we obtain some new realizations of the nonlinear $W_{2,s}$ algebras. All these results show that there exists no non-Romans-type realization with double scalar or double spinor only. However, we still expect that there exist non-Romans-type realizations of W algebras at some special values of central charge.

The paper is organized as follows. In Sec. II, we give a brief review and analysis of the realizations of the $W_{2,s}$ algebras and the $W_{2,s}$ strings. Then in Sec. III, we introduce the linearization of the $W_{2,s}$ algebras. In Secs. IV and V, we construct the bases of the linear $W_{1,2,s}$ algebras and obtain new realizations of the $W_{2,s}$ algebras with double scalar and double spinor, respectively. Finally, the paper ends with a brief conclusion.

II. NOTE ON THE REALIZATIONS OF THE $W_{2,s}$ ALGEBRAS AND THE $W_{2,s}$ STRINGS

It is known that when extended to the quantum case, the $W_{2,s}$ algebras will become nonlinear. The operator-product expansion (OPE) of two currents with spin s and s' produces terms with spin $(s + s' - 2)$ at leading order. For example, there will be terms with spin-4 and spin-6 cur-

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rents in the OPEs of the $W_{2,3}$ algebra and the $W_{2,4}$ algebra, respectively. However these terms with spin $(s + s' - 2)$ may be interpreted as composite fields built from the products of the fundamental currents with spin s and s'. The $W_{2,3}$ algebra is generated by the spin-2 energy-momentum tensor T and spin-3 current W, which satisfy the OPEs [1]

$$\begin{aligned} T(z)T(\omega) &\sim \frac{C/2}{(z-\omega)^4} + \frac{2T}{(z-\omega)^2} + \frac{\partial T}{z-\omega}, \\ T(z)W(\omega) &\sim \frac{3W}{(z-\omega)^2} + \frac{\partial W}{z-\omega}, \\ W(z)W(\omega) &\sim \frac{C/3}{(z-w)^6} + \frac{2T}{(z-\omega)^4} + \frac{\partial T}{(z-\omega)^3} \\ &\quad + \frac{1}{(z-\omega)^2}\left(2\Theta\Lambda + \frac{3}{10}\partial^2 T\right) \\ &\quad + \frac{1}{(z-\omega)}\left(\Theta\partial\Lambda + \frac{1}{15}\partial^3 T\right), \end{aligned} \tag{1}$$

where the coefficient Θ and composite field Λ (spin 4) are given by

$$\Theta = \frac{16}{22+5C}, \qquad \Lambda = T^2 - \frac{3}{10}\partial^2 T. \tag{2}$$

The constant C is the central charge of the $W_{2,3}$ algebra. It is easy to see that the denominator of Θ at $C = -\frac{22}{5}$ will be zero and the $W_{2,3}$ algebra will become singular. But one can rescale these currents such that the corresponding OPEs are well defined, i.e., there have no divergent coefficients in them (for the detailed discussion see [43,44]).

The $W_{2,4}$ algebra is given by [45]

$$\begin{aligned} T(z)T(\omega) &\sim \frac{C/2}{(z-\omega)^4} + \frac{2T}{(z-\omega)^2} + \frac{\partial T}{z-\omega}, \\ T(z)W(\omega) &\sim \frac{4W}{(z-\omega)^2} + \frac{\partial W}{z-\omega}, \\ W(z)W(\omega) &\sim \left\{\frac{2T}{(z-w)^6} + \frac{\partial T}{(z-\omega)^5} + \frac{\frac{3}{10}\partial^2 T + \sigma_1 U + \sigma_2 W}{(z-\omega)^4} + \frac{1}{15}\frac{\partial^3 T}{(z-\omega)^3} + \frac{1}{84}\frac{\partial^4 T}{(z-\omega)^2} + \frac{1}{560}\frac{\partial^5 T}{(z-\omega)}\right\} \\ &\quad + \sigma_1\left\{\frac{1}{2}\frac{\partial U}{(z-\omega)^3} + \frac{5}{36}\frac{\partial^2 U}{(z-\omega)^2} + \frac{1}{36}\frac{\partial^3 U}{(z-\omega)}\right\} + \sigma_2\left\{\frac{1}{2}\frac{\partial W}{(z-\omega)^3} + \frac{5}{36}\frac{\partial^2 W}{(z-\omega)^2} + \frac{1}{36}\frac{\partial^3 W}{(z-\omega)}\right\} \\ &\quad + \sigma_3\left\{\frac{G}{(z-\omega)^2} + \frac{1}{2}\frac{\partial G}{(z-\omega)}\right\} + \sigma_4\left\{\frac{A}{(z-\omega)^2} + \frac{1}{2}\frac{\partial A}{(z-\omega)}\right\} + \sigma_5\left\{\frac{B}{(z-\omega)^2} + \frac{1}{2}\frac{\partial B}{(z-\omega)}\right\} + \frac{C/4}{(z-\omega)^8}, \end{aligned} \tag{3}$$

where the composite fields U, G, A, and B are defined by

$$\begin{aligned} U &= (TT) - \tfrac{3}{10}\partial^2 T, \\ G &= (\partial^2 TT) - \partial(\partial TT) + \tfrac{2}{9}\partial^2(TT) - \tfrac{1}{42}\partial^4 T, \\ A &= (TU) - \tfrac{1}{6}\partial^2 U, \qquad B = (TW) - \tfrac{1}{6}\partial^2 W, \end{aligned} \tag{4}$$

with normal ordering of products of currents understood. The coefficients $\sigma_i (i = 1\text{–}5)$ are

$$\begin{aligned} \sigma_1 &= \frac{42}{5C+22}, \\ \sigma_2 &= \sqrt{\frac{54(C+24)(C^2-172C+196)}{(5C+22)(7C+68)(2C-1)}}, \\ \sigma_3 &= \frac{3(19C-524)}{10(7C+68)(2C-1)}, \\ \sigma_4 &= \frac{24(72C+13)}{(5C+22)(7C+68)(2C-1)}, \\ \sigma_5 &= \frac{28}{3(C+24)}\sigma_2. \end{aligned} \tag{5}$$

It is worth to point out that, just as the $W_{2,3}$ algebra, the $W_{2,4}$ algebra is singular at $C = -24, \frac{1}{2}, -\frac{22}{5}$, and $-\frac{68}{7}$. After rescaling the spin-4 current W, it can be proved that only the case $C = -24$ satisfies the Jacobi identity [43].

In general, the BRST charge Q_B for a $W_{2,s}$ string is [44,46]

$$Q_B = \oint dz[c(z)T(z) + \gamma(z)W(z)], \tag{6}$$

where the currents T and W generate the corresponding $W_{2,s}$ algebra, and the fermionic ghosts (b, c) and (β, γ) are introduced for the currents T and W, respectively. It is easy to prove that the BRST charge given above does satisfy the nilpotency condition:

$$Q_B^2 = \{Q_B, Q_B\} = 0. \tag{7}$$

A realization for a $W_{2,s}$ algebra means giving an explicit construction of the bases T and W from the basic fields, i.e., scalar fields, spinor fields, or ghost fields. Giving a realization for a nonlinear algebra is difficult and complex. For simplicity, Q_B can generally be expressed as the grading form in many works:

$$Q_B = Q_0 + Q_1, \tag{8}$$

$$Q_0 = \oint dz cT, \tag{9}$$

$$Q_1 = \oint dz \gamma W, \tag{10}$$

where the currents T and W generate the $W_{2,s}$ algebras. The detailed construction of the current T can be found in [47], where T was constructed from scalar, spinor, and ghost fields. Here the ghost fields b, c, β, γ are all fermionic and anticommuting. They satisfy the OPEs

$$b(z)c(\omega) \sim \frac{1}{z-\omega}, \qquad \beta(z)\gamma(\omega) \sim \frac{1}{z-\omega}; \tag{11}$$

in other cases the OPEs vanish. The nilpotency condition of Q_B becomes

$$Q_0^2 = Q_1^2 = \{Q_0, Q_1\} = 0. \tag{12}$$

Although it is easy to construct the $W_{2,s}$ strings in this grading form, one may note that this gives more constrained conditions on Q_B.

One also notes that if we obtain a realization for a $W_{2,s}$ algebra, the BRST charge Q_B of the corresponding $W_{2,s}$ string will be obtained by substituting the explicit forms of currents T and W into (6).

III. LINEARIZATION OF THE $W_{2,s}$ ALGEBRAS FROM THE $W_{1,2,s}$ ALGEBRAS

It was shown that the $W_{2,s}$ algebras can be linearized as the linear $W_{1,2,s}$ algebras generated by currents J, T, and W with spin 1, 2, and s, respectively. The linear $W_{1,2,s}$ algebras for $s = 3, 4$ take the forms [39]

$$\begin{aligned}
T_0(z)T_0(\omega) &\sim \frac{C_0/2}{(z-\omega)^4} + \frac{2T}{(z-\omega)^2} + \frac{\partial T}{z-\omega},\\
T_0(z)W_0(\omega) &\sim \frac{sW}{(z-\omega)^2} + \frac{\partial W}{z-\omega},\\
T_0(z)J_0(\omega) &\sim \frac{C_1}{(z-\omega)^3} + \frac{J_0}{(z-\omega)^2} + \frac{\partial J_0}{z-\omega},\\
J_0(z)J_0(\omega) &\sim -\frac{1}{(z-\omega)^2},\\
J_0(z)W_0(\omega) &\sim \frac{\xi W_0}{z-\omega},\\
W_0(z)W_0(\omega) &\sim 0.
\end{aligned} \tag{13}$$

The coefficients C_0, C_1, and ξ are given by

$$\begin{aligned}
&C_0 = 50 + 24t^2 + \frac{24}{t^2}, \qquad C_1 = -\sqrt{6}\left(t + \frac{1}{t}\right),\\
&\xi = \sqrt{\frac{3}{2}}t \qquad (s = 3), \qquad C_0 = 86 + 30t^2 + \frac{60}{t^2},\\
&C_1 = -3t - \frac{4}{t}, \qquad \xi = t \qquad (s = 4),
\end{aligned} \tag{14}$$

where t is a nonzero constant. From these OPEs (13), it is clear that the current W_0 is a primary field with spin s, while the current J_0 is not unless $C_1 = 0$. Here the spin-s current W_0 is null, and we will construct the most general forms of it in the next section. The results there show that W_0 is zero. It also can be seen that every term on the right-hand side of the OPEs $T_0(z)W_0(\omega)$ and $J_0(z)W_0$ has W_0, so one can consistently set it to zero, though it does not need to be set to zero. In [39], the null current was first realized with parafermionic vertex operators, and later was realized with the ghostlike fields [37,40,41].

The bases T and W of the $W_{2,s}$ algebras were constructed by the linear bases of the $W_{1,2,s}$ algebras in our previous paper [37]. For simplicity, we choose $t = -1$, T and W are given by

$$T = T_0, \tag{15}$$

$$\begin{aligned}
W = {}& W_0 + \frac{7i}{8}\partial^2 J_0 - i\frac{\sqrt{6}}{2}\partial J_0 J_0 + \frac{i}{6}J_0^3 - i\frac{\sqrt{6}}{8}\partial T_0\\
&+ \frac{i}{4}T_0 J_0, \qquad (s = 3)
\end{aligned} \tag{16}$$

$$\begin{aligned}
W = {}& W_0 + \frac{3a}{520}\partial^3 J_0 - \frac{3a}{260}\partial^2 J_0 J_0 - \frac{19a}{1560}(\partial J_0)^2\\
&+ \frac{7a}{780}\partial J_0 (J_0)^2 - \frac{a}{1560}(J_0)^4 - \frac{149}{390a}\partial^2 T_0\\
&- \frac{59}{780a}(T_0)^2 + \frac{a}{390}\partial T_0 J_0 + \frac{a}{260}T_0 \partial J_0\\
&- \frac{a}{780}T_0 (J_0)^2, \qquad (s = 4)
\end{aligned} \tag{17}$$

where $a = \sqrt{\frac{451}{2}}$. In this case, $T = T_0$ implies that the central charges of the linear $W_{1,2,s}$ algebras and the $W_{2,s}$ algebras are equal, i.e., $C = C_0$. One can also shift it with an arbitrary constant $C_{\rm eff}$, the center charge of an effective energy-momentum tensor $T_{\rm eff}$, and rewrite the nonlinear basis as $T = T_0 + T_{\rm eff}$. Here, we have shown that the $W_{1,2,s}$ algebras (13) are linear and contain the $W_{2,s}$ algebras as subalgebras. But one needs to keep in mind that this linearization does not contain the case $C = -\frac{22}{5}$ for $W_{2,3}$, and the cases $C = -24, \frac{1}{2}, -\frac{22}{5}$, and $-\frac{68}{7}$ for $W_{2,4}$, for which these algebras are singular.

IV. DOUBLE-SCALAR REALIZATIONS FOR THE LINEAR $W_{1,2,s}$ ALGEBRAS AND THE $W_{2,s}$ ALGEBRAS

In this section, we would like to construct the bases of the linear $W_{1,2,s}$ algebras with double scalar. Using the fact that the $W_{2,s}$ algebras are contained in the linear $W_{1,2,s}$ algebras as subalgebras, we can obtain new realizations for the $W_{2,s}$ algebras by a change of basis.

A. Realizations for the $W_{1,2,3}$ algebra and the $W_{2,3}$ algebra

First of all, we notice the relation between C_0 and C_1 for $s = 3$ shown in (14):

$$C_0 = 2 + 4C_1^2. \tag{18}$$

A scalar field has spin 0 in conformal field theory, and the OPE of it with itself is given by

$$\phi(z)\phi(\omega) \sim \ln(z - \omega), \tag{19}$$

or expressed as

$$\partial\phi(z)\partial\phi(\omega) \sim -\frac{1}{(z-\omega)^2}. \tag{20}$$

One needs to note that the field ϕ here is real. If φ is a complex scalar field, it is easy to prove that the OPE will be of the form

$$\partial\varphi^\dagger\partial\varphi \sim -\frac{1}{(z-\omega)^2}; \tag{21}$$

in other cases the OPEs vanish.

Now we consider two real scalar fields ϕ_1 and ϕ_2. The OPEs of them with each other are read as

$$\partial\phi_i(z)\partial\phi_j(\omega) \sim -\frac{\delta_{ij}}{(z-\omega)^2}, \qquad (i, j = 1, 2). \tag{22}$$

We would like to construct the explicit forms for the linear bases of the $W_{1,2,3}$ algebra. The most general form of the basis T_0 can be expressed as

$$T_0 = T_{\rm eff} + g_1 T_{\phi_1} + g_2 T_{\phi_2} + g_3 T_{\phi_1\phi_2}, \tag{23}$$

where $T_{\rm eff}$ is an effective energy-momentum tensor with central charge $C_{\rm eff}$. The introduction of $T_{\rm eff}$ will ensure the nontriviality of the solutions. T_{ϕ_1} and T_{ϕ_2} are spin-2 energy-momentum tensors constructed from fields ϕ_1 and ϕ_2, respectively, and $T_{\phi_1\phi_2}$ is constructed from these two scalar fields. The construction is

$$T_{\phi_1} = -\tfrac{1}{2}(\partial\phi_1)^2 - q_1\partial^2\phi_1, \tag{24}$$

$$T_{\phi_2} = -\tfrac{1}{2}(\partial\phi_2)^2 - q_2\partial^2\phi_2, \tag{25}$$

$$T_{\phi_1\phi_2} = \partial\phi_1\partial\phi_2, \tag{26}$$

where q_1 and q_2 are the background charges of T_{ϕ_1} and T_{ϕ_2}, respectively. The other two linear bases are given by

$$J_0 = g_4\partial\phi_1 + g_5\partial\phi_2, \tag{27}$$

$$\begin{aligned} W_0 = {} & g_6\partial T_{\rm eff} + g_7 T_{\rm eff}\partial\phi_1 + g_8 T_{\rm eff}\partial\phi_2 + g_9\partial^3\phi_1 \\ & + g_{10}(\partial\phi_1)^3 + g_{11}\partial^2\phi_1\partial\phi_1 + g_{12}\partial^3\phi_2 \\ & + g_{13}(\partial\phi_2)^3 + g_{14}\partial^2\phi_2\partial\phi_2 + g_{15}(\partial\phi_1)^2\partial\phi_2 \\ & + g_{16}\partial\phi_1(\partial\phi_2)^2 + g_{17}\partial^2\phi_1\partial\phi_2 + g_{18}\partial\phi_1\partial^2\phi_2. \end{aligned} \tag{28}$$

Plugging these linear bases into the OPE's relations (13), we could obtain all the coefficients. One can see that the constant t that appeared in (14) does not take zero, which determines $\xi \neq 0$. This leads to a main result

$$g_i = 0 \quad \text{for } i = 6\text{–}18, \tag{29}$$

which means that the current W_0 is zero. After carefully calculation, we obtain two solutions:

(i) Solution 1

$$\begin{gathered} g_1 = g_2 = 1, \qquad g_3 = 0, \qquad g_4 = g_5 = \frac{\sqrt{2}}{2}h, \\ C_1 = 2\sqrt{2}h, \qquad C_{\rm eff} = 8, \qquad C_0 = 34, \\ q_1 = q_2 = -1, \end{gathered}$$

(ii) Solution 2

$$\begin{gathered} g_1 = g_2 = -g_3 = \frac{1}{2}, \qquad g_4 = g_5 = \frac{\sqrt{2}}{2}h, \\ C_1 = 2\sqrt{2}h, \qquad C_{\rm eff} = 9, \qquad C_0 = 34, \\ q_1 = q_2 = -2, \end{gathered}$$

where h satisfies $h^2 = 1$. The main difference between the above two solutions is whether the energy-momentum tensor $T_{\phi_1\phi_2}$ vanishes. In solution 1, the term $T_{\phi_1\phi_2}$ does not appear. However, in solution 2, the contribution of the term $T_{\phi_1\phi_2}$ to central charge is $\frac{1}{2}$.

Having found two realizations of the linear $W_{1,2,3}$ algebra, we substitute the exact forms of the linear bases T_0 and J_0 into (16) and obtain two new realizations of the $W_{2,3}$ algebra. The first realization is

$$T = T_{\rm eff} + (\partial\phi_1)^2 - \tfrac{1}{2}\partial^2\phi_1 + (\partial\phi_2)^2 - \tfrac{1}{2}\partial^2\phi_2, \tag{30}$$

$$W = \frac{\sqrt{2}i}{48}(6hT_{\rm eff}\partial\phi_1 + 6hT_{\rm eff}\partial\phi_2 - h(\partial\phi_1)^3 + 3h(\partial\phi_1)^2\partial\phi_2 + 9h\partial\phi_1(\partial\phi_2)^2 + (6h - 12\sqrt{3})\partial\phi_1\partial^2\phi_2 - h(\partial\phi_2)^3 + (6h - 6\sqrt{3})\partial^2\phi_1\partial\phi_1 + (6h - 12\sqrt{3})\partial^2\phi_1\partial\phi_2 + (6h - 6\sqrt{3})\partial^2\phi_2\partial\phi_2 - 6\sqrt{3}\partial T_{\rm eff} + (24h - 6\sqrt{3})\partial^3\phi_1 + (6h - 6\sqrt{3})\partial^3\phi_2), \tag{31}$$

and the second one reads

$$T = T_{\rm eff} + (\partial\phi_1)^2 - \tfrac{1}{2}\partial^2\phi_1 + (\partial\phi_2)^2 - \tfrac{1}{2}\partial^2\phi_2 - \tfrac{1}{2}\partial\phi_1\partial\phi_2, \tag{32}$$

$$W = \frac{\sqrt{2}i}{96}(12hT_{\rm eff}\partial\phi_1 + 12hT_{\rm eff}\partial\phi_2 + 3h\partial\phi_1(\partial\phi_2)^2 + (12\sqrt{3} + 12h)\partial^2\phi_1\partial\phi_1 + (12h - 18\sqrt{3})\partial\phi_1\partial^2\phi_2 + h(\partial\phi_2)^3 - 2h(\partial\phi_1)^3 + (12h - 18\sqrt{3})\partial^2\phi_1\partial\phi_2 + (12h - 18\sqrt{3})\partial^2\phi_2\partial\phi_2 - 12\sqrt{3}\partial T_{\rm eff} + (51h - 12\sqrt{3})\partial^3\phi_1 + (48h - 12\sqrt{3})\partial^3\phi_2), \tag{33}$$

where h satisfies $h^2 = 1$. Note that, although $T_{\phi_1\phi_2}$ is absent in the first realization, the energy-momentum tensor T in both realizations has central charge $C = 34$. If plugging these realizations into (6), one will get the BRST charges for the $W_{2,3}$ string.

B. Realizations for the $W_{1,2,4}$ algebra and the $W_{2,4}$ algebra

For the linear $W_{1,2,4}$ algebra, the relation between C_0 and C_1 is

$$C_0 = 1 + \tfrac{1}{24}(85C_1^2 - 5C_1h\sqrt{-48 + C_1^2}). \tag{34}$$

Next, we would like to construct the explicit forms of the linear bases of the $W_{1,2,4}$ algebra. The most general forms of bases T_0 and J_0 are

$$T_0 = f_1T_{\rm eff} + f_2T_{\phi_1} + f_3T_{\phi_2} + f_4T_{\phi_1\phi_2}, \tag{35}$$

$$J_0 = f_5\partial\phi_1 + f_6\partial\phi_2, \tag{36}$$

where the energy-momentum tensors T_{ϕ_1}, T_{ϕ_2}, and $T_{\phi_1\phi_2}$ are given by

$$T_{\phi_1} = -\tfrac{1}{2}(\partial\phi_1)^2 - q_3\partial^2\phi_1, \tag{37}$$

$$T_{\phi_2} = -\tfrac{1}{2}(\partial\phi_2)^2 - q_4\partial^2\phi_2, \tag{38}$$

$$T_{\phi_1\phi_2} = \partial\phi_1\partial\phi_2. \tag{39}$$

For the linear basis W_0 with spin 4, the calculation shows that $W_0 \sim 0$. Under this case, the current W of the $W_{2,4}$ algebra is constructed from the linear bases T_0 and J_0 only. Plugging these linear bases into the OPEs (13), we obtain two solutions, where the energy-momentum tensor $T_{\rm eff}$ vanishes in both cases. These solutions are listed as follows:

(i) Solution 1

$$f_1 = 0, \qquad f_2 = f_3 = 1, \qquad f_4 = 0,$$
$$f_5 = f_6 = \frac{\sqrt{2}}{2}h, \qquad C_1 = i\sqrt{2}h, \qquad C_0 = -4,$$
$$q_3 = q_4 = -i\frac{h}{2},$$

(ii) Solution 2

$$f_1 = 0, \qquad f_2 = f_3 = \frac{1}{2}, \qquad f_4 = -\frac{1}{2},$$
$$f_5 = f_6 = \frac{\sqrt{2}}{2}h, \qquad C_1 = \frac{5\sqrt{3}i}{3}h,$$
$$C_0 = -24, \qquad q_3 = q_4 = -\frac{5i}{6}h,$$

where $h^2 = 1$. It is clear that $f_1 = 0$ in both solutions and this means the vanishing of the energy-momentum tensor $T_{\rm eff}$. Therefore, $C_{\rm eff} = 0$. The coefficient $f_4 = 0$ in solution 1 implies that $T_{\phi_1\phi_2}$ vanishes, however, it does not vanish in solution 2. One may also note that the central charge C_0 in both solutions is negative, which is different from the case of the $W_{1,2,3}$ algebra. The background charges q_3 and q_4 of T_{ϕ_1} and T_{ϕ_2} are all imaginary numbers.

After constructing the explicit forms of the linear bases T_0 and J_0, we would like to substitute them into (17) and obtain new realizations for the $W_{2,4}$ algebra. The realization constructed from solution 1 is

$$T = -\frac{1}{2}(\partial\phi_1)^2 - \frac{1}{2}(\partial\phi_2)^2 - \frac{ih}{2}\partial^2\phi_1 - \frac{ih}{2}\partial^2\phi_2, \tag{40}$$

$$W = b_1(\partial\phi_1)^4 + b_2(\partial\phi_1)^2(\partial\phi_2)^2 + b_3(\partial\phi_1)^2\partial^2\phi_2 + b_4\partial\phi_1\partial^2\phi_2\partial\phi_2 + b_5\partial\phi_1\partial^3\phi_2 + b_6(\partial\phi_2)^4 + b_7\partial^2\phi_1(\partial\phi_1)^2 + b_8\partial^2\phi_1\partial\phi_1\partial\phi_2 + b_9\partial^2\phi_1(\partial\phi_2)^2 + b_{10}(\partial^2\phi_1)^2 + b_{11}\partial^2\phi_1\partial^2\phi_2 + b_{12}\partial^2\phi_2(\partial\phi_2)^2 + b_{13}(\partial^2\phi_2)^2 + b_{14}\partial^3\phi_1\partial\phi_1 + b_{15}\partial^3\phi_1\partial\phi_2 + b_{16}\partial^3\phi_2\partial\phi_2 + b_{17}\partial^4\phi_1 + b_{18}\partial^4\phi_2. \tag{41}$$

These coefficients are

$$b_1 = -\frac{59}{3120a} + \frac{a}{6420}, \quad b_2 = -\frac{59}{1560a} - \frac{a}{3120}, \quad b_3 = \frac{a}{390\sqrt{2}} + \frac{59ih}{1560a} - \frac{iah}{3120}, \quad b_4 = \frac{a}{156\sqrt{2}} - \frac{iah}{1560},$$
$$b_5 = -\frac{3a}{520} + \frac{iah}{780\sqrt{2}}, \quad b_6 = -\frac{59}{3120a} + \frac{a}{6240}, \quad b_7 = \frac{59ih}{1560a} - \frac{iah}{3120}, \quad b_8 = \frac{a}{156\sqrt{2}} - \frac{iah}{1560},$$
$$b_9 = \frac{a}{390\sqrt{2}} + \frac{59ih}{1560a} - \frac{iah}{3120}, \quad b_{10} = \frac{149}{390a} - \frac{19a}{3120} + \frac{iah}{520\sqrt{2}} + \frac{59}{3120a}, \quad b_{11} = -\frac{19a}{1560} + \frac{iah}{260\sqrt{2}} + \frac{59}{1560a},$$
$$b_{12} = \frac{59ih}{1560a} - \frac{iah}{3120}, \quad b_{13} = \frac{149}{390a} - \frac{19a}{3120} + \frac{iah}{520\sqrt{2}} + \frac{59}{3120a}, \quad b_{14} = \frac{149}{390a} - \frac{3a}{520} + \frac{iah}{780\sqrt{2}},$$
$$b_{15} = -\frac{3a}{520} + \frac{iah}{780\sqrt{2}}, \quad b_{16} = \frac{149}{390a} - \frac{3a}{520} + \frac{iah}{780\sqrt{2}}, \quad b_{17} = \frac{3a}{520\sqrt{2}} - \frac{149ih}{780a}, \quad b_{18} = \frac{3a}{520\sqrt{2}} - \frac{149ih}{780a},$$

where $a = \sqrt{\frac{451}{2}}$.

Solution 2 gives a realization of the linear $W_{1,2,4}$ algebra with central charge $C_0 = -24$, which is singular and could not be used to construct the $W_{2,4}$ algebra. But it is indeed a realization of the $W_{1,2,4}$ algebra.

V. DOUBLE-SPINOR REALIZATIONS FOR THE LINEAR $W_{1,2,s}$ ALGEBRAS AND THE $W_{2,s}$ ALGEBRAS

In this section, we would like to construct the bases of the linear $W_{1,2,s}$ algebras with double spinor. The new realizations for the $W_{2,s}$ algebras can be obtained after a change of bases.

A. Realizations for the $W_{1,2,3}$ algebra and the $W_{2,3}$ algebra

A spinor field has spin $\frac{1}{2}$ in conformal field theory, and the OPE $\psi(z)\psi(\omega)$ is given by

$$\psi(z)\psi(\omega) \sim -\frac{1}{z-\omega}. \tag{42}$$

We denote two real spinor fields as ψ_1 and ψ_2, and they satisfy

$$\psi_i(z)\psi_j(\omega) \sim -\frac{\delta_{ij}}{z-\omega}, \qquad (i, j = 1, 2). \tag{43}$$

Next, we would like to construct the explicit forms of the linear bases for the $W_{1,2,3}$ algebra. The most general forms of T_0, J_0, and W_0 can be expressed as

$$\begin{aligned} T_0 &= T_{\rm eff} + h_1 T_{\psi_1} + h_2 T_{\psi_2} + h_3 T_{\psi_1\psi_2}, \\ J_0 &= h_4\psi_1\psi_2, \\ W_0 &= h_5\partial^2\psi_1\psi_1 + h_6\partial^2\psi_2\psi_2 + h_7\partial^2\psi_1\psi_2 \\ &\quad + h_8\psi_1\partial^2\psi_2 + h_9\partial\psi_1\partial\psi_2 + h_{10}\partial T_{\rm eff} \\ &\quad + h_{11}T_{\rm eff}\psi_1\psi_2. \end{aligned} \tag{44}$$

The energy-momentum tensors T_{ψ_1} and T_{ψ_2} with spin 2 are constructed from ψ_1 and ψ_2, respectively, and $T_{\psi_1\psi_2}$ is constructed from these two spinor fields. They are constructed as

$$T_{\psi_1} = \partial\psi_1\psi_1, \tag{45}$$

$$T_{\psi_2} = \partial\psi_2\psi_2, \tag{46}$$

$$T_{\psi_1\psi_2} = \partial\psi_1\psi_2 + \psi_1\partial\psi_2. \tag{47}$$

Plugging these linear bases into the OPE relations (13), we obtain the result:

$$h_1 = h_2 = -\tfrac{1}{2}, \qquad h_3 = 0, \qquad h_4 = 1, \qquad h_i = 0$$
$$(i = 5\text{–}11), \qquad C_1 = 0, \qquad C_0 = 2, \qquad C_{\rm eff} = 1.$$

In this solution, it can be seen that $T_{\psi_1\psi_2}$ and W_0 vanish. The energy-momentum tensor $T_{\rm eff}$ contributes central charge 1.

After constructing the explicit forms of the linear bases T_0, J_0, and W_0, we substitute them into (16) and obtain a realization of the $W_{2,3}$ algebra as follows:

$$T = T_{\mathrm{eff}} - \frac{1}{2}\partial\psi_1\psi_1 - \frac{1}{2}\partial\psi_2\psi_2,$$
$$W = \frac{i}{4}T_{\mathrm{eff}}\psi_1\psi_2 + \frac{7i}{8}\psi_1\partial^2\psi_2 + \frac{\sqrt{6}i}{16}\partial^2\psi_1\psi_1 + \frac{7i}{8}\partial^2\psi_1\psi_2 + \frac{\sqrt{6}i}{16}\partial^2\psi_2\psi_2 - \frac{\sqrt{6}i}{8}\partial T_{\mathrm{eff}}.$$

It is worth remarking that the currents T and W above generate the $W_{2,3}$ algebra with central charge $C = 2$.

B. Realizations for the $W_{1,2,4}$ algebra and the $W_{2,4}$ algebra

For the linear $W_{1,2,4}$ algebra, the bases take the following form:

$$T_0 = T_{\mathrm{eff}} + k_1 T_{\psi_1} + k_2 T_{\psi_2} + k_3 T_{\psi_1\psi_2}, \qquad J_0 = k_4\psi_1\psi_2, \qquad W_0 = 0, \tag{48}$$

where T_{ψ_1}, T_{ψ_2}, and $T_{\psi_1\psi_2}$ are given by (45)–(47). This case gives a precise Romans realization of the $W_{1,2,4}$ algebra, where the basis W_0 is set to zero.

Plugging these linear bases into the OPE relations (13), we obtain two solutions:

(i) Solution 1

$$k_1 = k_2 = -\frac{1}{2}, \qquad k_3 = -\frac{C_1}{2}, \qquad k_4 = -1,$$
$$C_{\mathrm{eff}} = \frac{1}{24}(13C_1^2 + 5C_1 h\sqrt{C_1^2 - 48}),$$
$$C_0 = 1 + \frac{1}{24}(85C_1^2 + 5C_1 h\sqrt{C_1^2 - 48}). \tag{49}$$

(ii) Solution 2

$$k_1 = k_2 = -\frac{1}{2}, \qquad k_3 = \frac{C_1}{2}, \qquad k_4 = 1,$$
$$C_{\mathrm{eff}} = \frac{1}{24}(13C_1^2 + 5C_1 h\sqrt{C_1^2 - 48}),$$
$$C_0 = 1 + \frac{1}{24}(85C_1^2 + 5C_1 h\sqrt{C_1^2 - 48}). \tag{50}$$

The central charges C_0 of both solutions depend on C_1, and this gives realizations of the linear $W_{1,2,4}$ algebra at arbitrary central charge. Then two new realizations of the $W_{2,4}$ algebra from these solutions can be obtained immediately. The first one is given by

$$T = T_{\mathrm{eff}} - \frac{1}{2}\partial\psi_1\psi_1 - \frac{1}{2}\partial\psi_2\psi_2 - \frac{C_1}{2}\partial\psi_1\psi_2 - \frac{C_1}{2}\psi_1\partial\psi_2, \tag{51}$$

$$\begin{aligned} W = -\frac{1}{1560a}(&-2(3a^2 - 59C_1)T_{\mathrm{eff}}\psi_1\partial\psi_2 + 118T_{\mathrm{eff}}\partial\psi_1\psi_1 - 2(3a^2 - 59C_1)T_{\mathrm{eff}}\partial\psi_1\psi_2 + 118C_1 T_{\mathrm{eff}}\partial\psi_2\psi_2 \\ &- 9(a^2 - 32C_1)\psi_1\partial^3\psi_1 - 4a^2\partial T_{\mathrm{eff}}\psi_1\psi_2 + 298\partial^2\psi_2\partial\psi_2 - (59 + 2a^2 - 2a^2C_1^2 + 59C_1^2)\partial\psi_1\psi_1\partial\psi_2\psi_2 \\ &- (27a^2 - 894C_1)\partial\psi_1\partial^2\psi_2 + 298\partial^2\psi_1\partial\psi_1 - (27a^2 - 894C_1)\partial^2\psi_1\partial\psi_2 + 298\partial^3\psi_1\psi_1 + (9a^2 - 298C_1)\partial^3\psi_1\psi_2 \\ &+ 298\partial^3\psi_2\psi_2 - 596\partial^2 T_{\mathrm{eff}} - 118T_{\mathrm{eff}}^2). \end{aligned} \tag{52}$$

The second is

$$T = T_{\mathrm{eff}} - \frac{1}{2}\partial\psi_1\psi_1 - \frac{1}{2}\partial\psi_2\psi_2 - \frac{C_1}{2}\partial\psi_1\psi_2 - \frac{C_1}{2}\psi_1\partial\psi_2, \tag{53}$$

$$\begin{aligned} W = -\frac{1}{1560a}(&+2(3a^2 - 59C_1)T_{\mathrm{eff}}\psi_1\partial\psi_2 + 118T_{\mathrm{eff}}\partial\psi_1\psi_1 + 2(3a^2 - 59C_1)T_{\mathrm{eff}}\partial\psi_1\psi_2 + 118C_1 T_{\mathrm{eff}}\partial\psi_2\psi_2 \\ &+ 9(a^2 - 32C_1)\psi_1\partial^3\psi_1 - 4a^2\partial T_{\mathrm{eff}}\psi_1\psi_2 + 298\partial^2\psi_2\partial\psi_2 - (59 + 2a^2 - 2a^2C_1^2 + 59C_1^2)\partial\psi_1\psi_1\partial\psi_2\psi_2 \\ &+ (27a^2 - 894C_1)\partial\psi_1\partial^2\psi_2 + 298\partial^2\psi_1\partial\psi_1 - (27a^2 - 894C_1)\partial^2\psi_1\partial\psi_2 + 298\partial^3\psi_1\psi_1 - (9a^2 - 298C_1)\partial^3\psi_1\psi_2 \\ &+ 298\partial^3\psi_2\psi_2 - 596\partial^2 T_{\mathrm{eff}} - 118T_{\mathrm{eff}}^2). \end{aligned} \tag{54}$$

Different from the cases of scalar realizations, the results here give two spinor realizations of the $W_{2,4}$ algebra for an arbitrary central charge.

VI. CONCLUSION

In this paper, we obtained the explicit field realizations of the linear $W_{1,2,s}$ algebras and the nonlinear $W_{2,s}$ algebras with double scalar and double spinor, respectively. Owing to the intrinsic nonlinearity of the $W_{2,s}$ algebras, it is hard to construct their field realizations. However, it is proved that the nonlinear $W_{2,s}$ algebras are contained in the linear $W_{1,2,s}$ algebras with three currents J_0, T_0, and W_0 as a subalgebra. With this fact, we first constructed the linear bases of the $W_{1,2,s}$ algebras. Then making a change of basis, we obtained several explicit field realizations of the nonlinear $W_{2,s}$ algebras. All these results imply a symmetry under $\phi_1 \leftrightarrow \phi_2$ or $\psi_1 \leftrightarrow \psi_2$. This method overcomes the difficulty of realizations for a nonlinear algebra.

The spin-s current W_0 of the $W_{1,2,s}$ algebras was considered to be a null current and can be set to zero, then the realizations of the $W_{2,s}$ algebras obtained from the linear $W_{1,2,s}$ algebras are called Romans-type realizations. In fact, it is not necessary to set the current W_0 to zero. In our constructions, we first listed the most general forms of linear bases J_0, T_0, and W_0 with correct spin. Plugging these forms into the OPEs (13), we found that all the coefficients of W_0 vanished for the nonzero constant ξ. These results suggest that there exists no non-Romans-type realization of the $W_{2,s}$ algebra if we use double scalar or double spinor only. However, we expect that there exist non-Romans-type realizations of the $W_{2,s}$ algebras at some value of central charge and more details would be investigated in our future work.

We can also see that all these realizations satisfy $C = C_0$, i.e., the central charge of the $W_{2,s}$ algebras are equal to the central charge of the $W_{1,2,s}$ algebras, which is caused by the assumption $T = T_0$. The central charge C takes some special values for the double-scalar realizations and the double-spinor realizations of the $W_{2,3}$ algebras, while it depends on the value of C_1 for the double-spinor realizations of the $W_{2,4}$ algebra. We also showed that there is no such realization for the $W_{2,3}$ algebra at central charge $C = -\frac{22}{5}$ and for the $W_{2,4}$ algebra at $C = -24, \frac{1}{2}, -\frac{22}{5}$, and $-\frac{68}{7}$, since the $W_{2,s}$ algebras are singular at these values of central charge.

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