PHYSICAL REVIEW D **79,** 126009 (2009)

String spectra near some null cosmological singularities

Kallingalthodi Madhu[1] and K. Narayan[2]

[1]*Tata Institute of Fundamental Research, Homi Bhabha Road, Colaba, Mumbai 400005, India*
[2]*Chennai Mathematical Institute, SIPCOT IT Park, Padur PO, Siruseri 603103, India*
(Received 12 May 2009; published 15 June 2009)

We construct cosmological spacetimes with null Kasner-like singularities as purely gravitational solutions with no other background fields turned on. These can be recast as anisotropic plane-wave spacetimes by coordinate transformations. We analyze string quantization to find the spectrum of string modes in these backgrounds. The classical string modes can be solved for exactly in these time-dependent backgrounds, which enables a detailed study of the near-singularity string spectrum, (time-dependent) oscillator masses, and wave functions. We find that for low-lying string modes (finite oscillation number), the classical near-singularity string mode functions are nondivergent for various families of singularities. Furthermore, for any infinitesimal regularization of the vicinity of the singularity, we find a tower of string modes of ultrahigh oscillation number which propagate essentially freely in the background. The resulting picture suggests that string interactions are non-negligible near the singularity.

DOI: 10.1103/PhysRevD.79.126009 PACS numbers: 11.25.−w, 98.80.−k

I. INTRODUCTION

Understanding cosmological singularities in string theory is an important goal, and has been the subject of several investigations, e.g. [1–29].

Our work in this paper has been in part motivated by investigations [23,24] involving generalizations of AdS/CFT where the bulk contains null or spacelike cosmological singularities, with a nontrivial dilaton e^{Φ} that vanishes at the location of the cosmological singularity, the curvatures behaving as $R_{MN} \sim \partial_M \Phi \partial_N \Phi$. The gauge theory duals are $\mathcal{N} = 4$ super Yang-Mills theories with a time-dependent gauge coupling $g_{\rm YM}^2 = e^{\Phi}$, and [23,24] describe aspects of the dual descriptions of the bulk cosmological singularities. From the bulk point of view, supergravity breaks down and possible resolutions of the cosmological singularity stem from stringy effects. Indeed, noting $\alpha' \sim \frac{1}{g_{\rm YM}^2 N}$ from the usual AdS/CFT dictionary and extrapolating naively to these time-dependent cases with a nontrivial dilaton, we have $\alpha' \sim \frac{1}{e^{\Phi} N}$, indicating vanishing effective tension for stringy excitations, when $e^{\Phi} \to 0$ near the singularity. While this is perhaps wrong in detail, we expect that stringy effects are becoming important near the bulk singularity, corresponding to possible gauge coupling effects in the dual gauge theory. It is therefore interesting to understand world-sheet string effects in the vicinity of the singularity. Owing to the technical difficulties with string quantization in an AdS background with Ramond-Ramond (RR) flux, we would like to look for simpler, purely gravitational backgrounds as toy models whose singularity structure shares some essential features with the backgrounds in the AdS/CFT investigations. We first find such spacetime backgrounds as "near-singularity" solutions to type II supergravity (preserving a fraction of light-cone supersymmetry). In general, these are null Kasner-like solutions with null cosmological singularities (we also find approximate solutions that extrapolate from these near-singularity solutions to flat space asymptotically). These can be recast as anisotropic plane-wave-like spacetimes by a coordinate transformation, and we outline arguments in these coordinates, suggesting the absence of higher derivative curvature corrections to these spacetimes (which are essentially plane-wave backgrounds).

We then perform an analysis of string quantization to find the spectrum of string modes in these backgrounds. We find it convenient to use (Rosen-like) coordinates where the null cosmology interpretation is manifest. With these lightlike backgrounds, it is natural to use light-cone gauge. The classical string modes can be solved for exactly in these time-dependent backgrounds, which enables a detailed study of the near-singularity string spectrum. For various families of singularities, the classical string oscillation amplitudes for low-lying oscillation number n are nondivergent near the singularity, with asymptotic time dependence similar to the center-of-mass modes. From the Hamiltonian, we find time-dependent masses for these string oscillator modes. However, for any infinitesimal regularization of the vicinity of the singularity, say $\tau \lesssim \tau_\epsilon$, we find string modes of ultrahigh oscillation number $n \gg \frac{1}{\tau_\epsilon^{a+1}}$ which propagate essentially freely in the background. The near-singularity region thus appears to be filled with such highly stringy modes. There have been several investigations of string quantization in plane-wave backgrounds with singularities [8–10,28,29], and our string analysis has some overlap with [8] in particular.

In Sec. II, we describe the spacetime backgrounds. Section III describes the string quantization. Section IV contains a discussion and open questions. In Appendix A, we describe some properties of the spacetime backgrounds, while Appendix B outlines string quantization in coordinates corresponding to a different time parameter.

II. THE SPACETIME BACKGROUNDS

We are interested in spacetime backgrounds that have a big-bang or big-crunch type of singularity at some value of the lightlike time coordinate x^+. We also want to restrict attention to purely gravitational solutions for simplicity, i.e. with unexcited dilaton and RR/Neveu-Schwarz–Neveu-Schwarz (RR/NSNS) fields. This means we want to solve the equations $R_{MN} = 0$. Null-time dependence reduces these equations to $R_{++} = 0$.

Let us begin by considering a spacetime background with two scale factors, of the form

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = e^{f(x^+)}(-2dx^+dx^- + dx^i dx^i) + e^{h(x^+)}dx^m dx^m, \quad (1)$$

where $i = 1, 2$, $m = 3, \ldots, D-2$. One may also think of the x^m directions as compactified, representing, say, a T^{D-4}. For the critical superstring with $D = 10$, we could alternatively replace this six-dimensional transverse space by some Ricci-flat space such as a Calabi-Yau 3-fold. The intuition here is that the time dependence of the ‘‘internal’’ space induces time dependence in the four-dimensional spacetime as well, as in e.g. [30,31]. Another perspective is that the internal space scale factor is the analog of the dilaton in the AdS/CFT cosmological solutions of [23,24], as we will elaborate on below.

Simple classes of singularities in this system are obtained for spacetimes whose limiting form in the vicinity of $x^+ = 0$ is null Kasner-like,[1]

$$ds^2 = (x^+)^a(-2dx^+dx^- + dx^i dx^i) + (x^+)^b dx^m dx^m, \quad a > 0. \quad (2)$$

More generally, consider spacetimes of a general Kasner-like form,

$$ds^2 = (x^+)^a(-2dx^+dx^- + dx^i dx^i) + (x^+)^{b_m} dx^m dx^m, \quad a > 0; \quad (3)$$

i.e. the individual internal dimensions x^m evolve independently according to their Kasner exponents b_m appearing in the individual scale factors $e^{h_m(x^+)} \to (x^+)^{b_m}$ as $x^+ \to 0$.

The coordinate transformation $x^I = (x^+)^{-a_I/2}y^I$, where $a_I \equiv a, b_m$, gives

$$(x^+)^{a_I}(dx^I)^2 = (dy^I)^2 - \frac{a_I dx^+ y^I dy^I}{x^+} + \frac{a_I^2 (y^I)^2 (dx^+)^2}{4(x^+)^2}. \quad (4)$$

Then the metric (3) becomes of manifest plane-wave form,

$$ds^2 = -2(x^+)^a dx^+ dy^- + \left[\sum_I \left(\frac{a_I^2}{4} - \frac{a_I(a+1)}{2}\right)(y^I)^2\right] \times \frac{(dx^+)^2}{(x^+)^2} + (dy^I)^2, \quad (5)$$

where we have redefined $y^- = x^- + (\frac{\sum_I a_I (y^I)^2}{4(x^+)^{a+1}})$. For $a_I = a, b_m$ distinct, these are, in general, anisotropic plane waves with singularities [after further redefining $(x^+)^a dx^+ = d\lambda$]. In what follows, we will find it convenient to work in the (Rosen) coordinates (2) and (3), where the null cosmology interpretation is manifest, but as we will see below, there are close parallels with various previous studies on plane-wave spacetimes with singularities, most notably [8] (see also [9,28,29]).

The spacetimes (2) have nonvanishing Riemann curvature components (with e.g. $f' \equiv \frac{df}{dx^+}$)

$$\begin{aligned} R_{+i+i} &= \frac{1}{4}((f')^2 - 2f'')e^{f(x^+)} = \frac{a(a+2)}{4}(x^+)^{a-2}, \\ R_{+m+m} &= \frac{1}{4}(2f'h'_m - (h'_m)^2 - 2h''_m)e^{h_m(x^+)} \\ &= \frac{b(2a+2-b)}{4}(x^+)^{b-2}. \end{aligned} \quad (6)$$

For these spacetimes to be Ricci-flat solutions of the Einstein equations, the equation of motion $R_{++} = 0$ must hold, giving

$$\begin{aligned} R_{++} &= \frac{1}{2}(f')^2 - f'' + \frac{1}{2}\sum_m(-2h''_m - (h'_m)^2 + 2f'h'_m) = 0 \\ &\Rightarrow a^2 + 2a + \frac{1}{2}\sum_m(-b_m^2 + 2b_m + 2ab_m) = 0. \end{aligned} \quad (7)$$

This relates the various (null) Kasner-like exponents a, b_m. The equation in terms of the general scale factors shows that the curvature for the 4D scale factor e^f is sourced by those for the internal scale factors e^{h_m}: indeed, the h_m are the analogs of the dilaton scalar in the AdS/CFT cosmological context [23,24] where the corresponding equation was $R_{++}^{(4)} = \frac{1}{2}(\partial_+\Phi)^2$. That is, the kinetic terms $(\partial_+ h_m)^2$ (and related cross terms) play the role of the dilaton in driving the singular behavior of the 4D part of the spacetime.

In what follows, we will specialize to the symmetric case here, i.e. all $b_m \equiv b$ equal (and $e^{h_m} \equiv e^h$). Then $R_{++} = 0$ simplifies to

[1] A spacetime of the form (2) and (3) with $a < 0$ can be transformed by a change of coordinates to one with $a > 0$ by redefining $y^+ = \frac{1}{x^+}$. This recasts $g_{+-} = (x^+)^{-|a|} = (y^+)^{|a|}$ and moves the singularity at $x^+ \to \infty$ in the spacetimes with $a < 0$ to $y^+ = 0$. Thus it is sufficient to study spacetimes (2) and (3) with $a > 0$.

$$\frac{1}{2}(f')^2 - f'' + \frac{D-4}{2}(-2h'' - (h')^2 + 2f'h') = 0$$
$$\Rightarrow a^2 + 2a + \frac{D-4}{2}(-b^2 + 2b + 2ab) = 0. \quad (8)$$

If $b = a$, this equation (assuming $D > 2$) simplifies to give the solutions $b = a = 0, -2$, in which case the Riemann curvature components are seen to identically vanish [the solution $(-2, -2)$ can be shown to be flat space by the coordinate transformation to plane-wave form]. Thus an interesting solution requires that the internal x^m space either grows or shrinks faster than the spatial part of the four-dimensional cosmology. For any $b \neq a$, the equation of motion above is a quadratic in a, which admits various solutions with

$$2a = -2 - (D-4)b \pm \sqrt{4 + (D-4)(D-2)b^2}. \quad (9)$$

Taking the positive radical, it can be seen that restricting $a > 0$ for our solutions implies $b > 2$ or $b < 0$. Furthermore $a + 1 - b > 0$ if $b < 0$ or $|b| < \frac{\sqrt{2}}{D-2}$.

Suppose we focus on finding solutions with a, b, being even integers, so that the metric allows unambiguous analytic continuation from $x^+ < 0$ to $x^+ > 0$ across the singularity. One may imagine that this is a coordinate-dependent choice of the time parameter x^+ and therefore not sacrosanct: however, if we do take this choice, a, b being even integers seems natural. This is more restrictive: we must consider the above as Diophantine quadratic equations with solutions over integers, which are, in general, rarer. We then need to look for those b for which the radical above is integral. For the cases of obvious interest, i.e. the bosonic string ($D = 26$) and the superstring ($D = 10$), the radicals simplify to $2\sqrt{1 + 132b^2}$ and $2\sqrt{1 + 12b^2}$, respectively. It is then straightforward to check that

$$(a, b) = (0, 2), (44, -2), (44, 92), (2068, -92)\ldots \quad [D = 26],$$
$$(a, b) = (0, 2), (12, -2), (12, 28), (180, -28), (180, 390), \ldots \quad [D = 10] \quad (10)$$

are solutions. Our analysis of these solutions in what follows will not depend on these detailed values though.

These solutions can be thought of as the "near-singularity" limiting regions of more general spacetimes where the scale factors e^f, e^{h_m} are not necessarily of power-law type.[2] Since the various scale factors e^f, e^{h_m} are related by the single equation of motion (7), a generic choice of e^f admits a solution to (7) for the remaining scale factors e^{h_m}. For instance, with a single scale factor $e^{h_m} = e^h$, taking $e^f = \tanh^a(x^+)$, we can, in principle, solve for e^h. In the limiting near-singularity region, we have already seen null-Kasner-like solutions with (8) relating the Kasner exponents. In the asymptotic region of large x^+, it can be checked that

$$e^f = \tanh^a(x^+) \to 1 - 2ae^{-2x^+}, \qquad e^h \sim \text{const} + \frac{2a}{D-4}e^{-2x^+} \quad (11)$$

is an approximate solution to (8) (dropping the subleading nonlinear terms).

We now make a few comments on the cosmological singularities in these spacetimes. No curvature invariants diverge due to the lightlike nature of this system, since no nontrivial contraction is nonzero. However, there are diverging tidal forces for null geodesic congruences. Consider, for instance, a simple class of null geodesic congruences propagating solely along x^+ (at constant x^-, x^i, x^m), with a cross section along the x^i or x^m directions. These are described by ($\Gamma^+_{++} = f' = \frac{a}{u}$ is the only nonzero Γ^+_{ij})

$$\frac{d^2x^+}{d\lambda^2} + \Gamma^+_{ij}\left(\frac{dx^i}{d\lambda}\right)\left(\frac{dx^j}{d\lambda}\right) = \frac{d^2x^+}{d\lambda^2} + \Gamma^+_{++}\left(\frac{dx^+}{d\lambda}\right)^2 = 0. \quad (12)$$

This gives the affine parameter along these null geodesics,

$$\lambda = \text{const}\int dx^+ e^{f(x^+)} = \text{const}\int dx^+ (x^+)^a = \text{const}\frac{(x^+)^{a+1}}{a+1}, \quad (13)$$

and the tangent vector

$$\xi = \partial_\lambda = \left(\frac{dx^+}{d\lambda}\right)\partial_+ \equiv \xi^+\partial_+. \quad (14)$$

The relative acceleration of neighboring geodesics in a null congruence can be calculated using the geodesic deviation equation giving

$$a^M = g^{MN}R_{NCBD}\xi^C\xi^D n^B \quad (15)$$

where $n = n^B\partial_B$ is the separation vector along a cross section of the congruence. For our system, this gives

$$a^i = g^{ii}R_{+i+i}(\xi^+)^2n^i = \frac{a(a+2)n^i}{4(x^+)^{2a+2}}, \qquad a^m = g^{mm}R_{+m+m}(\xi^+)^2n^m = \frac{b(2a+2-b)n^m}{4(x^+)^{2a+2}}. \quad (16)$$

The corresponding invariant acceleration norms are

$$|a^i|^2 = g_{ii}a^ia^i \sim \frac{1}{(x^+)^{3a+4}}, \qquad |a^m|^2 = g_{mm}a^ma^m \sim \frac{1}{(x^+)^{-b+4a+4}}. \quad (17)$$

[2] These solutions also arise as certain Penrose limits starting with some cosmological spacetimes and adding a spectator dimension [8] (see also [32]).

So we see diverging tidal forces as $x^+ \to 0$ for spacetimes satisfying the conditions (restricting to $a > 0$)

$$b < 4a + 4, \qquad a > 0, \tag{18}$$

indicating a singularity.[3] Since tidal forces diverge (for a, b, satisfying both these conditions) along both the x^i and the x^m directions, the locus of the singularity is the eight-dimensional space spanned by the x^i, x^m. From the point of view of a Penrose-like diagram, we see that the singularity locus extends all the way to $x^- \to \infty$. We will see reflections of this later in the string world-sheet analysis.

In Appendix A, we show that these spacetime backgrounds preserve 16 real (light-cone) supercharges. This is not a feature we use however, and our world-sheet analysis below does not appear to depend crucially on spacetime supersymmetry of these backgrounds.

We also mention that these spacetimes appear to not admit α' corrections due to higher order curvature terms, as is often the case with lightlike backgrounds. This is perhaps not surprising in light of the coordinate transformation that casts these null cosmologies in the form of anisotropic plane waves, which are known to be devoid of higher derivative corrections.

In general, these spacetimes are slightly different from those studied by e.g. [2–6] which were time-orbifold-like spacetimes. Although there are conceptual similarities, the detailed structure of the spacetimes are different and, in particular, there is no issue of backreaction due to several orbifold "images" [5,6].

In what follows, we analyze the string spectrum in the vicinity of the cosmological singularities of these spacetimes.

III. A STRING WORLD-SHEET ANALYSIS

We will now describe a world-sheet analysis of string propagation in these backgrounds. Consider the world-sheet action for the closed string propagating in such backgrounds,

$$S = -\frac{1}{4\pi\alpha'}\int d\tau d\sigma \sqrt{-h} h^{ab}\partial_a X^\mu \partial_b X^\nu g_{\mu\nu}(X). \tag{19}$$

The world-sheet metric h_{ab} has the signature $(-1, 1)$. It is convenient in the world-sheet analysis to use light-cone gauge $x^+ = \tau$, in keeping with the null structure of the spacetimes in question here. Unlike flat space however, it is not possible, in general, to use both light-cone gauge $x^+ = \tau$ and conformal gauge $h_{ab} \propto \eta_{ab}$ since that is one gauge condition too many, as we will see below. Let us therefore begin by setting $h_{\tau\sigma} = 0$, to simplify the world-sheet action, as in [33] (see also [34]).[4] Then the world-sheet Lagrangian becomes

$$\mathcal{L} = -\frac{1}{4\pi\alpha'}\int d\sigma\Big(-E g_{IJ}\partial_\tau X^I \partial_\tau X^J + \frac{1}{E} g_{IJ}\partial_\sigma X^I \partial_\sigma X^J - 2E g_{+-}\partial_\tau X^-\Big), \tag{20}$$

where we have defined $E(\tau, \sigma) = \sqrt{-\frac{h_{\sigma\sigma}}{h_{\tau\tau}}}$. Since X^- is not dynamical, we can eliminate this and reduce the system to the physical transverse degrees of freedom. Now if $E = 1$ is allowed, then we have $h_{\tau\tau} = -h_{\sigma\sigma}$, which is equivalent to conformal gauge being compatible with light-cone gauge. However, since the momentum conjugate to X^- is $p_- = \frac{E g_{+-}}{2\pi\alpha'}$, which is a τ-independent constant, we have $E = -\frac{1}{g_{+-}}$ (setting $p_- = -\frac{1}{2\pi\alpha'}$ by a τ-independent reparametrization invariance). Thus we see that conformal gauge is disallowed[5] since $g_{+-} \neq -1$. The action for our background simplifies to

$$S = \frac{1}{4\pi\alpha'}\int d^2\sigma((\partial_\tau X^i)^2 - e^{2f(\tau)}(\partial_\sigma X^i)^2 + e^{h(\tau)-f(\tau)}(\partial_\tau X^m)^2 - e^{h(\tau)+f(\tau)}(\partial_\sigma X^m)^2). \tag{21}$$

This action contains only the physical transverse oscillation modes $X^I \equiv X^i$, X^m of the string. In effect, all the gauge freedom and corresponding constraints have been used up, with $X^- = x_0^- + p_-\tau$.

The corresponding Hamiltonian $-p_+$, reexpressing the momenta Π^I in terms of $\partial_\tau X^I$, is

$$H = \frac{1}{4\pi\alpha'}\int d\sigma[(\partial_\tau X^i)^2 + e^{2f(\tau)}(\partial_\sigma X^i)^2 + e^{h(\tau)-f(\tau)}(\partial_\tau X^m)^2 + e^{h(\tau)+f(\tau)}(\partial_\sigma X^m)^2]. \tag{22}$$

In general, one might imagine that a time-dependent background pumps in energy and excites string modes, and the classical Hamiltonian above does reflect this. For spacetimes satisfying $e^{2f} \to 0$ near the singularity $\tau \to 0$, the potential energy of the X^I modes due to the $e^{2f(\tau)}$ factor becomes vanishingly small near $x^+ = 0$ (for $a > 0$). This could be taken to mean that it costs vanishingly little energy to create long strings as we approach $x^+ = \tau = 0$, the effective tension of string modes becoming vanishingly small near the singularity. However, this appears to be misleading: what is relevant is e.g. the ratio $\frac{e^{2f}(\partial_\sigma X^i)^2}{(\partial_\tau X^i)^2}$. This has a more detailed form involving nontrivial τ dependence stemming from both g_{IJ} and from the asymptotic behavior of string modes X^I, which we can solve for exactly in this background. Furthermore, since this Hamiltonian corresponds to x^+ translations and $g_{+-} \neq -1$, the string oscillator masses, which are coordinate

[3] This is true except when the coefficients of all a^I vanish: this happens for the spacetimes $(a, b) = (0, 0)$, (0,2).

[4] Reference [35] studies some aspects of string quantization in Brinkman coordinates.

[5] Appendix B contains a discussion with the affine parameter λ being the time parameter: in this case, $g_{+-} = -1$, and conformal gauge is compatible with light-cone gauge.

invariant, are $m^2 \sim g^{+-}p_+p_-$, whose τ dependence is different from that of the Hamiltonian. In the case of affine parameter quantization (Appendix B), the time dependence of the Hamiltonian translates directly to that of the oscillator masses.

Heuristically one might imagine that the string gets highly excited and breaks up into bits propagating independently near the singularity: in a sense, this is akin to a world-sheet analog of the observations of BKL [36] on ultralocality near a cosmological singularity. It would be interesting to understand this better. We will find some parallels with this in our analysis later, which will reveal distinctly stringy behavior.

In the next section, we will study quantum string propagation in this background in detail. We will focus on the symmetric case, i.e. all $b_m = b$ equal, giving two exponents a, b, but it is straightforward to generalize our analysis to the general case.

A. String modes and quantization

We are interested in studying the behavior of string modes as we approach the singularity from the past, i.e. $\tau < 0$. For notational convenience, we will simply use τ to denote $|\tau| = -\tau$ in the expressions below. The equations of motion from the world-sheet action above are

$$\partial_\tau^2 X^i - e^{2f(\tau)}\partial_\sigma^2 X^i = 0,$$
$$\partial_\tau^2 X^m + (\partial_\tau h - \partial_\tau f)\partial_\tau X^m - e^{2f(\tau)}\partial_\sigma^2 X^m = 0, \qquad (23)$$

which simplify in the near-singularity region of spacetime to

$$\partial_\tau^2 X^i - \tau^{2a}\partial_\sigma^2 X^i = 0,$$
$$\partial_\tau^2 X^m + \frac{b-a}{\tau}\partial_\tau X^m - \tau^{2a}\partial_\sigma^2 X^m = 0. \qquad (24)$$

Decomposing the X^I as $f_n^I(\tau)e^{in\sigma}$, we can show that the time-dependent mode solutions of these equations are given in terms of arbitrary linear combinations of two Bessel functions,[6]

$$f_n^i(\tau) = c_{n1}^i \sqrt{n\tau} J_{1/(2a+2)}\left(\frac{n\tau^{a+1}}{a+1}\right) + c_{n2}^i \sqrt{n\tau} Y_{1/(2a+2)} \times \left(\frac{n\tau^{a+1}}{a+1}\right),$$
$$f_n^m(\tau) = c_{n1}^m \sqrt{n}\tau^\nu J_{\nu/(a+1)}\left(\frac{n\tau^{a+1}}{a+1}\right) + c_{n2}^m \sqrt{n}\tau^\nu Y_{\nu/(a+1)} \times \left(\frac{n\tau^{a+1}}{a+1}\right),$$
$$\nu = \frac{a+1-b}{2}. \qquad (25)$$

These expressions are valid for $\nu > 0$, while similar Bessel functional forms with the index $\frac{|\nu|}{a+1}$ hold for $\nu < 0$. The Bessel index in f_n^i is thus always less than $\frac{1}{2}$ since $a > 0$, while for $b < 0$, the Bessel index in f_n^m is always greater than $\frac{1}{2}$. The complex coefficients c_{n1}^I, c_{n2}^I can be taken to indicate the choice of a vacuum by defining positive/negative frequency modes. For now, we keep them as two independent unfixed constants: we will comment on specific choices at appropriate points in what follows.

Note the similarity between these string world-sheet mode solutions and the well-known Hankel function description of spacetime scalar modes propagating in four-dimensional de Sitter backgrounds. Spacetime scalar modes in the present null Kasner-like backgrounds are somewhat different from these however.[7]

We can also examine the behavior of the zero modes or center-of-mass modes. For $n = 0$, the equations of motion (24) for $X_0^I(\tau)$ can be solved to give

$$X_0^i(\tau) = \frac{x_0^i}{\sqrt{2\pi}} + \sqrt{2\pi}\alpha' p_{i0}\tau,$$
$$X_0^m(\tau) = \frac{x_0^m}{\sqrt{2\pi}} + \sqrt{2\pi}\alpha' p_{m0}\tau^{2\nu}, \qquad (26)$$

where p_{I0} are the center-of-mass momenta defined later (29). These show that for singularities with $2\nu \geq 0$, the center of mass of the string is not driven to infinity by the singularity. We will find parallels of this with the asymp-

[6] Setting $f_n^i \to \sqrt{\tau} f_n^i$, $f_n^m \to \tau^\nu f_n^m$, transforms the equations of motion (24) to the standard Bessel forms

$$t^2 f_n^{i\prime\prime} + t f_n^{i\prime} + \left(t^2 - \frac{1}{4(a+1)^2}\right) f_n^i = 0,$$
$$t^2 f_n^{m\prime\prime} + t f_n^{m\prime} + \left(t^2 - \frac{\nu^2}{(a+1)^2}\right) f_n^m = 0, \qquad t = \frac{n\tau^{a+1}}{a+1}.$$

[7] Consider a massive scalar ϕ in the background (2), with action $S = \int d^D x \sqrt{-g}(-g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - m^2\phi^2)$, and equation of motion $\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi) - m^2\phi = 0$. Taking modes $\phi = f(x^+)e^{ik_- x^- + ik_i x^i + ik_m x^m}$, this simplifies to $\frac{1}{f}\frac{df}{dx^+} = \frac{i}{2k_-}(k_i^2 + k_m^2 (x^+)^{a-b} + m^2 (x^+)^a + \frac{2a+(D-4)b}{2x^+})$, which can be solved to give

$$\phi(x^\mu) = \exp\left[\frac{i}{2k_-}\left(k_i^2 x^+ + k_m^2 \frac{(x^+)^{a+1-b}}{a+1-b} + m^2 \frac{(x^+)^{a+1}}{a+1} + \frac{2a+(D-4)b}{2}\log x^+\right)\right].$$

Thus generically these modes have a phase that oscillates "wildly" near the singularity $x^+ \to 0$.

totics of low-lying string oscillation modes. This is to be contrasted with the divergences for spacetimes with $2\nu < 0$. Note that the zero mode behavior is essentially point-particle-like. Thus the centers of mass of, say, a collection of infalling strings would appear to exhibit diverging tidal forces through geodesic deviation. However, the crucial point is that the oscillations of the string are now non-negligible (even if finite). Thus neighboring strings would appear to have large spatial overlap, and string interactions become important near the singularity.

The mode expansion for the spacetime coordinates of the string is

$$X^I(\tau,\sigma) = X_0^I(\tau) + \sum_{n=1}^{\infty}(k_n^I f_n^I(\tau)(a_n^I e^{in\sigma} + \tilde{a}_n^I e^{-in\sigma}) + k_n^{I*} f_n^{I*}(\tau)(a_{-n}^I e^{-in\sigma} + \tilde{a}_{-n}^I e^{in\sigma})). \tag{27}$$

The constant k_n^I will be fixed by demanding canonical commutation relations for the creation-annihilation operators. The momentum conjugates $\Pi^I = \frac{\partial\mathcal{L}}{\partial(\partial_\tau X^I)}$ are

$$\Pi^i(\tau,\sigma) = \frac{1}{2\pi\alpha'}\partial_\tau X^i, \qquad \Pi^m(\tau,\sigma) = \frac{\tau^{b-a}}{2\pi\alpha'}\partial_\tau X^m. \tag{28}$$

We define the center-of-mass momenta p_{I0} as

$$p_{i0} = \int_0^{2\pi}\frac{d\sigma}{\sqrt{2\pi}}\Pi^i = \frac{1}{\sqrt{2\pi}\alpha'}\dot{X}_0^i(\tau), \qquad p_{m0} = \int_0^{2\pi}\frac{d\sigma}{\sqrt{2\pi}}\Pi^m = \frac{\tau^{b-a}}{\sqrt{2\pi}\alpha'}\dot{X}_0^m(\tau). \tag{29}$$

Then we see that imposing the nonzero commutation relations

$$[x_0^I, p_{J0}] = i\delta_J^I, \qquad [a_n^I, a_{-m}^J] = n\delta^{IJ}\delta_{nm}, \qquad [\tilde{a}_n^I, \tilde{a}_{-m}^J] = n\delta^{IJ}\delta_{nm} \tag{30}$$

implies the equal time commutation relations, e.g.

$$[X^I(\tau,\sigma), \Pi^J(\tau,\sigma')] = \frac{i}{2\pi}\delta^{IJ}\left(1 + \sum_{n=1}^{\infty}(e^{in(\sigma-\sigma')} + e^{-in(\sigma-\sigma')})\right) = i\delta^{IJ}\delta(\sigma-\sigma'), \tag{31}$$

using the Fourier series representation for the Dirac δ function, with the constant k_n^I being (this agrees with the conventions of [37] for flat space, except for a reversal of left/right movers)

$$k_n^I = \frac{i}{n}\sqrt{\frac{\pi\alpha'}{2|c_{n0}^I|(a+1)}}, \qquad c_{n0}^I = c_{n1}^I c_{n2}^{I*} - c_{n1}^{I*} c_{n2}^I, \tag{32}$$

where c_{n0}^I is the Wronskian. We have used above the expressions for the derivatives of the mode functions f_n^I and some recursion relations for the Bessel functions[8] to calculate the Wronskian of f_n^I, $\dot{f}_n^I$.

Let us now discuss level matching. The operator that generates σ translations is the world-sheet momentum P given by the stress tensor

$$T_{ab} \sim -\frac{1}{\sqrt{-h}}\frac{\delta\mathcal{L}}{\delta h^{ab}} \sim -\left(g_{IJ}\partial_a X^I\partial_b X^J - \frac{1}{2}h_{ab}h^{cd}g_{IJ}\partial_c X^I\partial_d X^J\right). \tag{33}$$

Then the σ-translation gauge invariance is fixed by demanding that the momentum operator vanishes on the physical states, i.e. $P = \int d\sigma T_{\tau\sigma} = 0$. From our action above and our light-cone gauge condition $h_{\tau\sigma} = 0$, we have

$$P = \int d\sigma(\tau^a\partial_\tau X^i\partial_\sigma X^i + \tau^b\partial_\tau X^m\partial_\sigma X^m). \tag{34}$$

Using the mode expansion (27), this can be evaluated as

$$P \sim \tau^a\sum_n n((a_{-n}^i a_n^i - \tilde{a}_{-n}^i\tilde{a}_n^i) + (a_{-n}^m a_n^m - \tilde{a}_{-n}^m\tilde{a}_n^m)), \tag{35}$$

where we have used the Bessel recursion relations and the expressions for $\dot{f}_n^I$ (suppressing some overall unimportant numerical factors). This recovers the level matching conditions $N = \tilde{N}$.

Now we calculate the string Hamiltonian. Using the mode expansion (27), we first evaluate

[8] We have used the following, the Bessel function argument being $(\frac{n\tau^{a+1}}{a+1})$,

$$\frac{df_n^i(\tau)}{d\tau} = n\sqrt{n}\tau^{a+(1/2)}(c_{n1}^i J_{(1/(2a+2))-1} + c_{n2}^i Y_{(1/(2a+2))-1}),$$

$$\frac{df_n^m(\tau)}{d\tau} = n\sqrt{n}\tau^{a+\nu}(c_{n1}^m J_{(\nu/(a+1))-1} + c_{n2}^m Y_{(\nu/(a+1))-1}),$$

$$J_{\nu-1}(z) + J_{\nu+1}(z) = \frac{2\nu}{z}J_\nu(z),$$

$$J_{\nu-1}(z) - J_{\nu+1}(z) = 2\frac{dJ_\nu(z)}{dz},$$

$$Y_{\nu-1}(z) + Y_{\nu+1}(z) = \frac{2\nu}{z}Y_\nu(z),$$

$$Y_{\nu-1}(z) - Y_{\nu+1}(z) = 2\frac{dY_\nu(z)}{dz},$$

$$J_\nu(z)Y_{\nu-1}(z) - J_{\nu-1}(z)Y_\nu(z) = \frac{2}{\pi z}.$$

$$\frac{1}{2\pi}\int_0^{2\pi} d\sigma(\partial_\tau X^I)^2 = (\dot{X}_0^I)^2 + \sum_n |k_n|^2(|\dot{f}_n^I|^2(\{a_n^I, a_{-n}^I\} + \{\tilde{a}_n^I, \tilde{a}_{-n}^I\}) - (\dot{f}_n^I)^2\{a_n^I, \tilde{a}_n^I\} - (\dot{f}_n^{I*})^2\{a_{-n}^I, \tilde{a}_{-n}^I\}),$$
$$\frac{1}{2\pi}\int_0^{2\pi} d\sigma(\partial_\sigma X^I)^2 = \sum_n n^2|k_n|^2(|f_n^I|^2(\{a_n^I, a_{-n}^I\} + \{\tilde{a}_n^I, \tilde{a}_{-n}^I\}) - (f_n^I)^2\{a_n^I, \tilde{a}_n^I\} - (f_n^{I*})^2\{a_{-n}^I, \tilde{a}_{-n}^I\}). \tag{36}$$

The Hamiltonian (22) then simplifies to

$$H = \frac{1}{2\alpha'}((\dot{X}_0^i)^2 + \tau^{b-a}(\dot{X}_0^m)^2) + \frac{1}{2\alpha'}\sum_n |k_n|^2((\{a_n^i, a_{-n}^i\} + \{\tilde{a}_n^i, \tilde{a}_{-n}^i\})(|\dot{f}_n^i|^2 + n^2\tau^{2a}|f_n^i|^2) - \{a_n^i, \tilde{a}_n^i\}((\dot{f}_n^i)^2 + n^2\tau^{2a}(f_n^i)^2)$$
$$- \{a_{-n}^i, \tilde{a}_{-n}^i\}((\dot{f}_n^{i*})^2 + n^2\tau^{2a}(f_n^{i*})^2)) + \frac{1}{2\alpha'}\sum_n |k_n|^2((\{a_n^m, a_{-n}^m\} + \{\tilde{a}_n^m, \tilde{a}_{-n}^m\})(\tau^{b-a}|\dot{f}_n^m|^2 + n^2\tau^{b+a}|f_n^m|^2)$$
$$- \{a_n^m, \tilde{a}_n^m\}(\tau^{b-a}(\dot{f}_n^m)^2 + n^2\tau^{b+a}(f_n^m)^2) - \{a_{-n}^m, \tilde{a}_{-n}^m\}(\tau^{b-a}(\dot{f}_n^{m*})^2 + n^2\tau^{b+a}(f_n^{m*})^2)). \tag{37}$$

In the next section, we will examine free string behavior in the vicinity of the singularity.

B. Strings in the near-singularity region

Let us now understand the behavior of the string mode functions near the singularity. It turns out that the near-singularity limit $\tau \to 0$ must be taken with care. We define a cutoff $\tau = \tau_\epsilon \sim 0$ as a short time regulator in the vicinity of the singularity $\tau = 0$. Then we define $n_\epsilon \equiv \frac{1}{\tau_\epsilon^{a+1}}$ as a cutoff on the world-sheet oscillation number. We then see sharp differences between the behavior near $\tau = \tau_\epsilon$ of string modes with "low-lying" oscillation numbers $n \lesssim n_\epsilon$ (i.e. $n\tau_\epsilon^{a+1} \ll 1$), and highly oscillating string modes with $n \gg n_\epsilon$ (i.e. $n\tau_\epsilon^{a+1} \gg 1$).

Noting the asymptotics $J_{\pm\nu}(x) \sim x^{\pm\nu}$ for $x \sim 0$, and $Y_\nu = \cot(\pi\nu)J_\nu - \mathrm{cosec}(\pi\nu)J_{-\nu}$, we see that, near $\tau = 0$, the $f_n^I(\tau)$ approach

$$f_n^i \to \lambda_{n0}^i + \lambda_{n\tau}^i\tau, \qquad f_n^m \to \lambda_{n0}^m + \lambda_{n\tau}^m\tau^{2\nu} \qquad (\tau \to 0), \tag{38}$$

for modes with low-lying oscillation numbers $n \lesssim n_\epsilon$. The constant coefficients are (from the asymptotic Bessel expressions)

$$\lambda_{n\tau}^i = \sqrt{n}\left(\frac{n}{2a+2}\right)^{1/(2a+2)}\frac{c_{n1}^i + c_{n2}^i\cot\frac{\pi}{2a+2}}{\Gamma(\frac{2a+3}{2a+2})},$$
$$\lambda_{n0}^i = -c_{n2}^i\sqrt{n}\left(\frac{n}{2a+2}\right)^{-1/(2a+2)}\frac{\mathrm{cosec}\frac{\pi}{2a+2}}{\Gamma(\frac{2a+1}{2a+2})},$$
$$\lambda_{n\tau}^m = \sqrt{n}\left(\frac{n}{2a+2}\right)^{\nu/(a+1)}\frac{c_{n1}^m + c_{n2}^m\cot\frac{\nu\pi}{a+1}}{\Gamma(\frac{a+\nu+1}{a+1})},$$
$$\lambda_{n0}^m = -c_{n2}^m\sqrt{n}\left(\frac{n}{2a+2}\right)^{-\nu/(a+1)}\frac{\mathrm{cosec}\frac{\nu\pi}{a+1}}{\Gamma(\frac{a+1-\nu}{a+1})}. \tag{39}$$

Thus we see that the asymptotic τ dependence of such finite n string oscillation modes near $\tau \to 0$ is essentially the same as for the center-of-mass modes of the string (26). Thus the (classical) string mode amplitudes are nondivergent near the singularity for cosmological solutions with $2\nu = a + 1 - b \geq 0$. The string oscillation amplitude in such a curved spacetime is perhaps better defined as $g_{mm}(f_n^m)^2$: this gives the asymptotics to be nondivergent for $2a + 2 \geq b$. In what follows, we will find the Wronskian combinations useful for the λ_{n0}^I, $\lambda_{n\tau}^I$,

$$\Lambda_{n,0\tau}^i \equiv \lambda_{n0}^i\lambda_{n\tau}^{i*} - \lambda_{n\tau}^i\lambda_{n0}^{i*} = nc_{n0}^i\frac{\mathrm{cosec}\frac{\pi}{2a+2}}{\Gamma(\frac{2a+3}{2a+2})\Gamma(\frac{2a+1}{2a+2})},$$
$$\Lambda_{n,0\tau}^m \equiv \lambda_{n0}^m\lambda_{n\tau}^{m*} - \lambda_{n\tau}^m\lambda_{n0}^{m*} = nc_{n0}^m\frac{\mathrm{cosec}\frac{\nu\pi}{a+1}}{\Gamma(\frac{a+1+\nu}{a+1})\Gamma(\frac{a+1-\nu}{a+1})},$$
$$c_{n0}^I = c_{n1}^I c_{n2}^{I*} - c_{n1}^{I*}c_{n2}^I. \tag{40}$$

On the other hand, consider now modes with $n \gg n_\epsilon$. Then we can see from the Bessel mode functions (25) [or directly from the equations of motion (24)] that these are oscillatory near the singularity: the argument $\frac{n\tau^{a+1}}{a+1}$ cannot be taken to be small and the asymptotics (38) above are not valid. For instance, choosing linear combinations c_{n1}^I, $c_{n2}^I = 1, \pm i$, gives modes that are the analogs of ingoing or outgoing plane waves, i.e. the f_n^I are Hankel functions dressed with powers of τ, with asymptotics[9] for $\tau \to 0$,

$$f_n^i \sim \frac{1}{\tau^{a/2}}e^{\pm in\tau^{a+1}/(a+1)},$$
$$f_n^m \sim \frac{1}{\tau^{b/2}}e^{\pm in\tau^{a+1}/(a+1)}, \qquad n \gg n_\epsilon. \tag{41}$$

Note that for any regulator τ_ϵ, however small, in the vicinity of the singularity, there exist modes of sufficiently high oscillation n such that the corresponding modes f_n^I are of this form (41). Since the string oscillation number n can be arbitrarily large, such modes exist uniformly for all singularities, with $2\nu \gtrless 0$, and are in a sense trans-Planckian: they are reminiscent of high frequency scalar modes propagating in an inflationary background. This behavior, somewhat different from the finite n mode behavior, is distinctly stringy.

[9]This is also the asymptotic behavior near $\tau \to \infty$ of the modes (25) for *any* n.

We first analyze the case $2\nu = a + 1 - b \geq 0$. Using the asymptotic forms of the mode functions f_n^I near the singularity $\tau \to 0$, for finite $n \lesssim n_\epsilon$ modes,

$$f_n^i \to \lambda_{n0}^i, \qquad \dot{f}_n^i \to \lambda_{n\tau}^i, \qquad f_n^m \to \lambda_{n0}^m, \qquad \dot{f}_n^m \to \lambda_{n\tau}^m (2\nu)\tau^{2\nu-1}, \tag{42}$$

the Hamiltonian (37) simplifies to

$$\begin{aligned} H = {}& \frac{1}{2\alpha'}((\dot{X}_0^i)^2 + \tau^{b-a}(\dot{X}_0^m)^2) + \frac{1}{2\alpha'}\sum_n |k_n|^2((((\{a_n^i, a_{-n}^i\} + \{\tilde{a}_n^i, \tilde{a}_{-n}^i\})|\lambda_{n\tau}^i|^2 - \{a_n^i, \tilde{a}_n^i\}(\lambda_{n\tau}^i)^2 - \{a_{-n}^i, \tilde{a}_{-n}^i\}(\lambda_{n\tau}^{i*})^2) \\ & + n^2\tau^{2a}((\{a_n^i, a_{-n}^i\} + \{\tilde{a}_n^i, \tilde{a}_{-n}^i\})|\lambda_{n0}^i|^2 - \{a_n^i, \tilde{a}_n^i\}(\lambda_{n0}^i)^2 - \{a_{-n}^i, \tilde{a}_{-n}^i\}(\lambda_{n0}^{i*})^2)) + \sum_n \frac{|k_n|^2}{2\alpha'}(\tau^{a-b}(2\nu)^2((\{a_n^m, a_{-n}^m\} \\ & + \{\tilde{a}_n^m, \tilde{a}_{-n}^m\})|\lambda_{n\tau}^m|^2 - \{a_n^m, \tilde{a}_n^m\}(\lambda_{n\tau}^m)^2 - \{a_{-n}^m, \tilde{a}_{-n}^m\}(\lambda_{n\tau}^{m*})^2) + n^2\tau^{b+a}((\{a_n^m, a_{-n}^m\} + \{\tilde{a}_n^m, \tilde{a}_{-n}^m\})|\lambda_{n0}^m|^2 \\ & - \{a_n^m, \tilde{a}_n^m\}(\lambda_{n0}^m)^2 - \{a_{-n}^m, \tilde{a}_{-n}^m\}(\lambda_{n0}^{m*})^2)). \end{aligned} \tag{43}$$

Note that there are "interaction terms" of the form $a_n^I \tilde{a}_n^I$ and $a_n^{I\dagger}\tilde{a}_n^{I\dagger}$ besides the diagonal number-operator terms. The interaction terms have the same τ-dependent coefficients as the diagonal terms so that they are not unimportant and cannot be ignored.

The corresponding calculation for flat space (a, $b = 0$) involves sine and cosine modes (the analogs of the Bessel-J, Y), the Hamiltonian having no time dependence. Analyzing this near $\tau \to 0$, we see that the coefficients of the $a_n^I\tilde{a}_n^I$ and $a_{-n}^I\tilde{a}_{-n}^I$ terms are of the form $(c_1^2 + c_2^2)$ and $(c_1^{*2} + c_2^{*2})$, while that of the diagonal terms is $(|c_1|^2 + |c_2|^2)$, where c_1, c_2 are the coefficients of the sine, cosine: then we see that choosing the usual positive frequency modes, with c_1, c_2 being 1, $-i$, results in just the diagonal term in the Hamiltonian. In the present case, due to the extra τ dependences in the Hamiltonian, the resulting expressions do not simplify and the "interaction" terms remain. A similar calculation with different choices of the basis modes (e.g. Hankel functions) yields equivalent results.

This Hamiltonian (43), corresponding to the choice of x^+ as a time coordinate,[10] can now be recast as

$$\begin{aligned} H = {}& \pi\alpha'((p_{i0})^2 + \tau^{a-b}(p_{m0})^2) + \sum_n \frac{\pi}{2(a+1)n^2} \\ & \times \bigg(\frac{1}{|c_{n0}^i|}(b_{n\tau}^{i\dagger}b_{n\tau}^i + n^2\tau^{2a}b_{n0}^{i\dagger}b_{n0}^i) \\ & + \frac{1}{|c_{n0}^m|}((2\nu)^2\tau^{a-b}b_{n\tau}^{m\dagger}b_{n\tau}^m + n^2\tau^{b+a}b_{n0}^{m\dagger}b_{n0}^m)\bigg), \end{aligned} \tag{44}$$

where we have defined new oscillator modes (and their Hermitian conjugates)

$$\begin{aligned} b_{n0}^I &= \lambda_{n0}^I a_n^I - \lambda_{n0}^{I*}\tilde{a}_{-n}^I, \\ b_{n\tau}^I &= \lambda_{n\tau}^I a_n^I - \lambda_{n\tau}^{I*}\tilde{a}_{-n}^I, \qquad I = i, m. \end{aligned} \tag{45}$$

The string oscillator masses are Lorentz invariant expressions

$$m^2 = -2g^{+-}p_+p_- - g^{II}(p_{I0})^2. \tag{46}$$

From the above expressions, and recalling that $p_- = -\frac{1}{2\pi\alpha'}$, $-p_+ = H$, we see that the center-of-mass terms cancel, resulting in the time-dependent masses for these low-lying $n \lesssim n_\epsilon$ oscillation string modes,

$$\begin{aligned} m^2(\tau) = {}& \frac{1}{2\alpha'(a+1)}\sum_{i,m;n\lesssim n_\epsilon}\bigg(\frac{1}{\tau^a}\frac{N_{n\tau}^i}{n^2|c_{n0}^i|} + \tau^a\frac{N_{n0}^i}{|c_{n0}^i|} \\ & + \frac{(2\nu)^2}{\tau^b}\frac{N_{n\tau}^m}{n^2|c_{n0}^m|} + \tau^b\frac{N_{n0}^m}{|c_{n0}^m|}\bigg), \qquad [2\nu \geq 0], \end{aligned} \tag{47}$$

defining

$$\begin{aligned} N_{n\tau}^i &= b_{n\tau}^{i\dagger}b_{n\tau}^i, \qquad N_{n0}^i = b_{n0}^{i\dagger}b_{n0}^i, \\ N_{n\tau}^m &= b_{n\tau}^{m\dagger}b_{n\tau}^m, \qquad N_{n0}^m = b_{n0}^{m\dagger}b_{n0}^m. \end{aligned} \tag{48}$$

These expressions should be understood as valid in the vicinity of the singularity, but only up to the regulator $\tau \lesssim \tau_\epsilon$.

The original left- and right-moving oscillator operators can be reexpressed in terms of b_n^I as

$$\begin{aligned} a_n^I &= \frac{1}{\Lambda_{n,0\tau}^I}(\lambda_{n\tau}^{I*}b_{n0}^I - \lambda_{n0}^{I*}b_{n\tau}^I), \\ \tilde{a}_n^I &= \frac{1}{\Lambda_{n,0\tau}^{I*}}(\lambda_{n\tau}^{I*}b_{n0}^{I\dagger} - \lambda_{n0}^{I*}b_{n\tau}^{I\dagger}), \qquad I = i, m, \end{aligned} \tag{49}$$

and the level matching condition (35) is recast as

$$0 = \sum_n n(a_{-n}^I a_n^I - \tilde{a}_{-n}^I\tilde{a}_n^I) = \sum_n \frac{n}{\Lambda_{n,0\tau}^I}(b_{n\tau}^{I\dagger}b_{n0}^I - b_{n\tau}^I b_{n0}^{I\dagger}). \tag{50}$$

The commutation relations satisfied by the b_{n0}^I, $b_{n\tau}^I$ are

[10] The affine parameter quantization, Appendix B, yields similar results as we describe here.

$$[b^I_{m0}, b^{J\dagger}_{n0}] = 0 = [b^I_{m\tau}, b^{J\dagger}_{n\tau}] = [b^I_{m0}, b^J_{n\tau}],$$
$$[b^I_{m0}, b^{J\dagger}_{n\tau}] = n\Lambda^I_{n,0\tau}\delta^{IJ}\delta_{mn} = -[b^I_{n\tau}, b^{J\dagger}_{n0}],$$
$$[N^I_{m0}, b^J_{n\tau}] = n\Lambda^I_{n,0\tau}b^I_{n0}\delta^{IJ}\delta_{mn},$$
$$[N^I_{m0}, b^{J\dagger}_{n\tau}] = n\Lambda^I_{n,0\tau}b^{I\dagger}_{n0}\delta^{IJ}\delta_{mn}, \tag{51}$$
$$[N^I_{m\tau}, b^J_{n0}] = -n\Lambda^I_{n,0\tau}b^I_{n\tau}\delta^{IJ}\delta_{mn},$$
$$[N^I_{n\tau}, b^{I\dagger}_{n0}] = -n\Lambda^I_{n,0\tau}b^{I\dagger}_{n\tau},$$
$$[N^I_{m0}, N^J_{n\tau}] = n\delta^{IJ}\delta_{mn}\Lambda^I_{n,0\tau}(b^{I\dagger}_{n0}b^I_{n\tau} + b^{I\dagger}_{n\tau}b^I_{n0}),$$

using the left- and right-moving a, $\tilde{a}$-oscillator commutators (30), and the Wronskian combinations $\Lambda^I_{n,0\tau}$ from (40). Since the b^I_{n0}, $b^I_{n\tau}$ operators commute with their conjugates, the N^I_{n0}, $N^I_{n\tau}$ operators do not have a number-operator-like interpretation on states annihilated by b^I_{n0}, $b^I_{n\tau}$. From the expression for the time-dependent masses, it is tempting to speculate that states that have e.g. vanishing $\langle N^i_{n\tau}\rangle$ but nonzero $\langle N^i_{n0}\rangle$ will become massless near the singularity $\tau \to 0$. However, since $N^I_{n\tau}$, N^I_{n0} do not commute,[11] these are generically not simultaneous eigenstates of N^I_{n0} and $N^I_{n\tau}$, or energy eigenstates. If such a possibility can be validated for these b^I states, then the b^i_{n0}, $b^m_{n\tau}$-oscillator states are light near the singularity while the $b^i_{n\tau}$, b^m_{n0}-oscillator states are massive near $\tau = x^+ \to 0$, for the singularities with $2\nu \geq 0$, $b < 0$ (while for $b > 0$ singularities, the b^I_{n0}-oscillator states are light and the $b^I_{n\tau}$ states are massive). All these are light relative to the typical curvature scale however, as we will outline later. Some description of the b^I states is given in the next subsection: it would be interesting to develop this further.

Let us now consider the case $2\nu = a + 1 - b < 0$. Then the modes f^m_n behave near $\tau \to 0$ as $f^m_n \to \lambda^m_{n\tau}\tau^{2\nu}$, while $\dot{f}^m_n \to \lambda^m_{n\tau}(2\nu)\tau^{2\nu}$. Thus λ^m_{n0} does not appear in the Hamiltonian (37) evaluated near $\tau \to 0$, which thus shows all a^m, $\tilde{a}^m$ terms having identical asymptotics with time dependence as $\tau \to 0$, e.g.

$$\tau^{b-a}|\dot{f}^m_n|^2 + n^2\tau^{b+a}|f^m_n|^2 \sim \tau^{a-b}((2\nu)^2 + n^2\tau^{2a+2}) \to \tau^{a-b} \tag{52}$$

in the coefficients. It is therefore not particularly insightful to recast a^m, $\tilde{a}^m$ in terms of the b^m operators. The invariant oscillator masses thus grow as $\frac{1}{\tau^a}$ and $\frac{1}{\tau^b}$ for the $b^i_{n\tau}$- and a^m-oscillator states. The b^I_{n0} states are light as before.

Now let us consider the high oscillation modes with $n \gg n_\epsilon = \frac{1}{\tau_\epsilon^{a+1}}$: these have a uniform behavior for both $2\nu \gtrless 0$. Then, using the asymptotics (41) for such modes (with $c^I_{n1} = 1$, $c^I_{n2} = -i$, which are positive frequency), we see that

$$\dot{f}^{\,i}_n \sim \left(-in\tau^a - \frac{a}{2\tau}\right)\frac{e^{-in\tau^{a+1}/(a+1)}}{\tau^{a/2}},$$
$$\dot{f}^m_n \sim \left(-in\tau^a - \frac{b}{2\tau}\right)\frac{e^{-in\tau^{a+1}/(a+1)}}{\tau^{b/2}}. \tag{53}$$

This is very similar to the asymptotics of the modes (25) at early times $|\tau| \to \infty$: however, we are considering a different limit here, with large n, small τ, and $n\tau^{a+1} \gg 1$, so it is worth elaborating a little. In this limit, we calculate the expressions in (37) and express them as

$$\frac{1}{n^2}((\dot{f}^i_n)^2 + n^2\tau^{2a}(f^i_n)^2)$$
$$\sim \tau^a\left(\frac{a^2}{4(n\tau^{a+1})^2} + \frac{ia}{(n\tau^{a+1})}\right)e^{-2in\tau^{a+1}/(a+1)},$$
$$\frac{1}{n^2}(\tau^{b-a}(\dot{f}^m_n)^2 + n^2\tau^{b+a}(f^m_n)^2)$$
$$\sim \tau^a\left(\frac{b^2}{4(n\tau^{a+1})^2} + \frac{ib}{(n\tau^{a+1})}\right)e^{-2in\tau^{a+1}/(a+1)}, \tag{54}$$
$$\frac{1}{n^2}(|\dot{f}^i_n|^2 + n^2\tau^{2a}|f^i_n|^2) \sim 2\tau^a,$$
$$\frac{1}{n^2}(\tau^{b-a}|\dot{f}^m_n|^2 + n^2\tau^{b+a}|f^m_n|^2) \sim 2\tau^a.$$

The expressions in the first two equations are vanishingly small relative to the ones in the rest, and the Hamiltonian (37) simplifies to

$$H_{n\gg n_\epsilon} \sim \tau^a \sum_{I;n\gg n_\epsilon}\frac{1}{a+1}(a^I_{-n}a^I_n + \tilde{a}^I_{-n}\tilde{a}^I_n + n), \tag{55}$$

as for free string propagation. The overall factor τ^a arises as before from the fact that we are using x^+ as a time coordinate [with $g_{+-} = -(x^+)^a$]. The oscillator masses for these highly stringy modes, using (46), become

$$m^2(\tau) \sim -g^{+-}H\frac{1}{\alpha'} = \sum_{I;n\gg n_\epsilon}\frac{1}{a+1}(N^I_n + \tilde{N}^I_n + n), \tag{56}$$

as for free strings in flat space. The zero point energy has an ultraviolet completion as in that case. Thus these highly stringy modes exhibit essentially free propagation in these backgrounds. Comparing the mass $\sqrt{\frac{n}{\alpha'}}$ of a typical single excitation state with the typical curvature scale set by the tidal forces $|a^i|$, $|a^m|$, in this region, we have $\frac{n}{\alpha'|a^i|^2} \sim \frac{n\tau^{3a+4}}{\alpha'}$, $\frac{n}{\alpha'|a^m|^2} \sim \frac{n\tau^{4a+4-b}}{\alpha'}$. Thus states satisfying $\frac{1}{\tau^{a+1}} \ll n \ll \frac{1}{\tau^{3a+4}}$ and $\frac{1}{\tau^{a+1}} \ll n \ll \frac{1}{\tau^{4a+4-b}}$ are light relative to the local curvature scale. Similar comparisons for the $n \lesssim n_\epsilon$ states with the τ dependences $\tau^{\pm a}$ and $\tau^{\pm b}$ relative to the typical curvature scale hold if $b < 2a + 2$.

[11]In terms of the original a, $\tilde{a}$ operators, this expression is

$$[N^I_{n0}, N^J_{n\tau}] = n\delta^{IJ}[(\lambda^I_{n0}\lambda^{I*}_{n\tau} + \lambda^I_{n\tau}\lambda^{I*}_{n0})(a^I_{-n}a^I_n + \tilde{a}^I_n\tilde{a}^I_{-n}) - 2\lambda^I_{n0}\lambda^I_{n\tau}a^I_n\tilde{a}^I_n - 2\lambda^{I*}_{n0}\lambda^{I*}_{n\tau}a^I_{-n}\tilde{a}^I_{-n}].$$

For any finite, if infinitesimal, value of the near-singularity cutoff τ_ϵ, such highly stringy modes exist, for oscillation number $n \gg n_\epsilon = \frac{1}{\tau_\epsilon^{a+1}}$, although naively removing the cutoff would suggest the absence of any such modes.

C. Near-singularity string states and wave functions

We describe here some aspects of string states near the singularity using our discussion in the previous section, beginning with the low-lying oscillation mode b^I states.

Noting that the b^I operators are complex linear combinations of the a^I, $\tilde{a}^{I\dagger}$, and recalling Bogolubov transformations, a b^I vacuum $|\phi\rangle$ (annihilated by the b^I) would appear to be a multiparticle state in terms of the original a^I, $\tilde{a}^I$ operators, and vice versa. Indeed we have

$$\langle 0|\sum_n N^I_{n\rho}|0\rangle = \sum_n n|\lambda^I_{n\rho}|^2, \qquad \rho = 0, \tau, \tag{57}$$

where $a^I_n|0\rangle = 0 = \tilde{a}^I_n|0\rangle$. Similarly, defining $|b_0\rangle = b^i_{n0}|0\rangle$, $|b_0^\dagger\rangle = b^{i\dagger}_{n0}|0\rangle$, $|b_\tau\rangle = b^i_{n\tau}|0\rangle$, $|b_\tau^\dagger\rangle = b^{i\dagger}_{n\tau}|0\rangle$, it is straightforward to show that the lowest excited states have

$$\langle b_p|b^{i\dagger}_{mq}b^i_{mq}|b_r^\dagger\rangle = 0,$$
$$\langle b_p|b^{i\dagger}_{mq}b^i_{mq}|b_r\rangle = n\lambda^{i*}_{nr}\lambda^i_{np}\left(\sum_m m|\lambda^i_{mq}|^2 + n|\lambda^i_{nq}|^2\right),$$
$$p, q, r = 0, \tau, \tag{58}$$

using the expressions for the b^I in terms of the a^I, $\tilde{a}^I$, and their commutation relations.

Now it can be shown that $[N^I_{m0}, (b^{J\dagger}_{n\tau})^l] = l(n\Lambda^{I*}_{n,0\tau})\delta^{IJ}\delta_{nm}(b^{I\dagger}_{n\tau})^{l-1}b^{I\dagger}_{n0}$, using the b^I-oscillator algebra (51). Assuming the existence of a b^I_0 vacuum, defining an excited state $|\Psi_l\rangle = (b^{I\dagger}_{n\tau})^l|\phi\rangle$ gives $N^I_{n0}|\Psi_l\rangle = l(n\Lambda^{I*}_{n,0\tau})b^{I\dagger}_{n0}|\Psi_{l-1}\rangle$. Thus we see heuristically that, starting with the b^I_0 vacuum and constructing a Fock space using $b^{I\dagger}_{n\tau}$, we obtain states with nonzero $\langle N^I_0\rangle$. Similarly, possible coherent states of the form $|s\rangle = e^{sb^{I\dagger}_{n\tau}}|\phi\rangle$ have $b_0|s\rangle \sim s|s\rangle$, up to numerical factors. Since $[b^I_0, b^{I\dagger}_0] = 0$, we see that $b^{I\dagger}_{n0}|s\rangle$ is also a coherent state with the same eigenvalue.

Note that these are not eigenstates of the Hamiltonian $H_{n\le n_\epsilon}$ since N^I_0, N^I_τ do not commute, so generically such states mix under time evolution. Consider the Schrodinger equation $i\frac{d}{d\tau}|\Psi\rangle = H|\Psi\rangle$, with $|\Psi\rangle = \sum_l c_l|\Psi^i_l\rangle$ constructed using only $b^{i\dagger}_{n\tau}$ oscillators. This gives $i\frac{d}{d\tau}|\Psi\rangle \sim \sum_{i,m,n}(N^i_{n\tau} + \tau^{2a}N^i_{n0})|\Psi\rangle$. This suggests that the time dependence of these states is regular near the singularity $\tau \to 0$.

Along similar lines, we can, more simply, construct states of the form $(b^{i\dagger}_0)^{l_i}(b^{m\dagger}_0)^{l_m}|\phi\rangle$, starting with the b^I_0 vacuum. These states have vanishing $\langle N^I_0\rangle$ but nonzero $\langle N^I_\tau\rangle$. The Schrodinger equation for such states is of the form $i\frac{d}{d\tau}|\Psi\rangle \sim \sum_{i,m,n}(N^i_{n\tau} + \tau^{a-b}N^m_{n\tau})|\Psi\rangle$. In accord with level matching (50), we can construct states of the form $(b^I_\tau)^l(b^{J\dagger}_0)^m|\phi\rangle$: then, since b^I_0, b^J_τ commute, these states again have vanishing $\langle N^I_0\rangle$ and nonzero $\langle N^I_\tau\rangle$.

We have described states constructed in terms of the b^I_0 vacuum so far: similarly, assuming formally the existence of a vacuum annihilated by $b^I_{n\tau}$, we can construct excited states along the lines of arguments similar to the ones above.

To obtain some rudimentary intuition for the spacetime description of these states, let us now describe position-space wave functions near the singularity. We will analyze the wave functions for the reduced quantum mechanics of string modes with σ momentum n,

$$\begin{aligned} x^I_n &= i|k^I_n|(f^I_n(\tau)a^I_n - f^{I*}_n(\tau)a^I_{-n}),\\ \tilde{x}^I_n &= i|k^I_n|(f^I_n(\tau)\tilde{a}^I_n - f^{I*}_n(\tau)\tilde{a}^I_{-n}),\\ \Pi^i_n &= \frac{i|k^i_n|}{2\pi\alpha'}(\dot{f}^i_n(\tau)a^i_n - \dot{f}^{i*}_n(\tau)a^i_{-n}),\\ \Pi^m_n &= \frac{i|k^m_n|\tau^{b-a}}{2\pi\alpha'}(\dot{f}^m_n(\tau)a^m_n - \dot{f}^{m*}_n(\tau)a^m_{-n}), \end{aligned} \tag{59}$$

from the string coordinate mode expansion (27) and the momentum conjugates (28) (we have suppressed explicitly writing the left-moving momenta $\tilde{\Pi}^I_n$).

Transforming to a position-space Schrodinger representation, we set $\Pi^I_n = -i\partial_{x^I_n}$, $\tilde{\Pi}^I_n = -i\partial_{\tilde{x}^I_n}$. It is then straightforward to obtain the expressions

$$\begin{aligned} a^i_n &= \frac{\dot{f}^{i*}_n x^i_n - 2\pi\alpha' f^{i*}_n(-i\partial_{x^i_n})}{i|k^i_n|(f^i_n\dot{f}^{i*}_n - \dot{f}^i_n f^{i*}_n)},\\ a^m_n &= \frac{\dot{f}^{m*}_n x^m_n - 2\pi\alpha'\tau^{a-b}f^{m*}_n(-i\partial_{x^m_n})}{i|k^m_n|(f^m_n\dot{f}^{m*}_n - \dot{f}^m_n f^{m*}_n)}, \end{aligned} \tag{60}$$

and their conjugates, with similar expressions for the $\tilde{a}^I_n$. We can obtain expressions for the b^I oscillators from the definitions (45), mixing the left- and right-moving terms.

$$\begin{aligned} b^i_{n0} &= \frac{\lambda^i_{n0}\lambda^{i*}_{n\tau}x^i_n - \lambda^{i*}_{n0}\lambda^i_{n\tau}\tilde{x}^i_n + i|\lambda^i_{n0}|^2 2\pi\alpha'(\partial_{x^i_n} - \partial_{\tilde{x}^i_n})}{i|k^i_n|\Lambda^I_{n,0\tau}},\\ b^i_{n\tau} &= \frac{|\lambda^i_{n\tau}|^2(x^i_n - \tilde{x}^i_n) + i2\pi\alpha'(\lambda^{i*}_{n0}\lambda^i_{n\tau}\partial_{x^i_n} - \lambda^i_{n0}\lambda^{i*}_{n\tau}\partial_{\tilde{x}^i_n})}{i|k^i_n|\Lambda^I_{n,0\tau}}. \end{aligned} \tag{61}$$

Then the ground state wave function $|0\rangle$ defined as $a^I_n|0\rangle = 0$, $\tilde{a}^I_n|0\rangle = 0$, for low-lying oscillation modes, satisfies, near the singularity,

$$\begin{aligned} &(\lambda^{i*}_{n\tau}x^i_n - 2\pi\alpha'\lambda^{i*}_{n0}(-i\partial_{x^i_n})\psi^i_0(x^I_n) = 0\\ &\quad = (\lambda^{i*}_{n\tau}\tilde{x}^i_n - 2\pi\alpha'\lambda^{i*}_{n0}(-i\partial_{\tilde{x}^i_n})\psi^i_0(x^I_n), \end{aligned} \tag{62}$$

giving $\psi^i_0(x^I_n) \sim \exp[i\frac{\lambda^{i*}_{n\tau}}{2\pi\alpha'\lambda^{i*}_{n0}}((x^i_n)^2 + (\tilde{x}^i_n)^2)]$. For positive frequency modes with $c^i_{n1} = 1$, $c^i_{n2} = -i$, we see that this simplifies to a real Gaussian part (as expected for a set of harmonic oscillators) and a phase containing $\cos(\frac{\pi}{2a+2}) \times \frac{\Gamma(\frac{2a+1}{2a+2})}{\Gamma(\frac{2a+3}{2a+2})}$ (this phase vanishes for flat space $a = 0$). Note that there is no explicit τ dependence here: the wave function is regular near the singularity $\tau \to 0$. Similar statements hold for the x^m part of the wave function (if $2\nu > 0$). Excited states can be constructed using either the a, $\tilde{a}$ or the b^I oscillators: these generically mix, as can be seen either from the interaction terms in the Hamiltonian, or alternatively by noting that the b^I do not commute.

The highly stringy states are simpler to describe: they are simply states of the form

$$|k_n^I, \tilde{k}_n^J\rangle \equiv \prod_{I,J;n\gg n_\epsilon} (a_{-n}^I)^{k_n^I} (\tilde{a}_{-n}^I)^{\tilde{k}_n^I} |0\rangle. \tag{63}$$

These are, in fact, eigenstates of the Hamiltonian $H_{n\gg n_\epsilon}$ so their time evolution is relatively simple, with the Schrodinger equation giving

$$i\frac{d}{d\tau}|k_n^I, \tilde{k}_n^J\rangle \sim \tau^a(nk_n^I + \tilde{k}_n^J + n)|k_n^I, \tilde{k}_n^J\rangle. \tag{64}$$

This can be recast as $i\frac{d}{d\lambda}|\Psi\rangle = H_\lambda|\Psi\rangle$ in terms of the affine parameter (13), with the corresponding quantization discussed in Appendix B. This equation is essentially of the same form as in flat space, with the time parameter being the affine parameter: the time evolution is essentially given by phases of the form $e^{-iE_{(k_n^I,\tilde{k}_n^J)}(\tau^{a+1})/(a+1)} = e^{-iE_{(k_n^I,\tilde{k}_n^J)}\lambda}$.

For the position-space description of the highly stringy states, we need to evaluate the expressions taking the limit in question carefully: using (53) and (60), the ground state $\psi_0^{n\gg n_\epsilon}(x_n^I)$ annihilated by a_n^I, $\tilde{a}_n^I$ satisfies

$$\begin{aligned}\bigg(\bigg(in\tau^a - \frac{a}{2\tau}\bigg)x_n^i - 2\pi\alpha'(-i\partial_{x_n^i})\bigg)\psi_0^{n\gg n_\epsilon} &= 0\\ = \bigg(\bigg(in\tau^a - \frac{a}{2\tau}\bigg)\tilde{x}_n^i - 2\pi\alpha'(-i\partial_{\tilde{x}_n^i})\bigg)\psi_0^{n\gg n_\epsilon},\end{aligned} \tag{65}$$

giving

$$\begin{aligned}\psi^{n\gg n_\epsilon} &\sim \exp\bigg[-\bigg(n\tau^a + \frac{ia}{2\tau}\bigg)((x_n^i)^2 + (\tilde{x}_n^i)^2)\bigg]\\ &= \exp\bigg[-n\tau^a\bigg(1 + \frac{ia}{2n\tau^{a+1}}\bigg)((x_n^i)^2 + (\tilde{x}_n^i)^2)\bigg].\end{aligned} \tag{66}$$

Note that the phase, containing $\frac{1}{\tau}$, oscillates wildly as $\tau \to 0$. However, from the second expression, we see that in the limit we are considering, $n\tau^{a+1} \gg 1$, the phase oscillation is slower than the damping of the real Gaussian part of the wave function.

Similarly, $a_n^m\psi_0(x) = 0 = \tilde{a}_n^m\psi_0(x)$ gives $\psi_0(x) \sim \exp[-n\tau^a\tau^{b-a}(1 + \frac{ib}{2n\tau^{a+1}})\frac{((x_n^m)^2+(\tilde{x}_n^m)^2)}{2}]$. Thus the overall factor $n\tau^a\tau^{b-a} = n\tau^b$ is heavily damped for $b < 0$, while the phase of the wave function is $\tau^{-2\nu}$, but damped relative to its real Gaussian part.

Excited states can be constructed by acting with the creation operators: e.g. the first excited states are e.g. $a_{-n}^i\tilde{a}_{-n}^j\psi_0(x) \sim x_n^i\tilde{x}_n^i e^{-[(2in\tau^{a+1})/(a+1)]}\psi_0(x)$.

As we have mentioned in the previous subsection, the near-singularity limit we are considering appears subtle. In particular, as time evolves towards the singularity and τ_ϵ shrinks, the world-sheet oscillation number cutoff n_ϵ increases and these highly stringy states are no longer eigenstates, except for n larger than the increased value of the cutoff $n_\epsilon(\tau_\epsilon - \delta\tau_\epsilon)$. A state with some $n_0 \gg n_\epsilon(\tau_\epsilon)$ at some later time crosses the cutoff threshold and ceases to be highly stringy: it then becomes part of the set of b^I states and interacts nontrivially with them. Since there is an infinity of highly stringy modes, it would appear that this process will continue indefinitely: making the description of changing the cutoff more precise might draw parallels with the renormalization group. It would be interesting to understand this better.

IV. DISCUSSION

We have constructed cosmological spacetimes with null Kasner-like singularities: the Kasner exponents satisfy algebraic conditions following from the Einstein equations satisfied by the backgrounds. These near-singularity spacetimes can be extrapolated to approximate solutions that are asymptotically flat at early times. It is possible to recast these as anisotropic plane-wave spacetimes, with the corresponding α'-exactness properties of higher derivative corrections.

We have found that the classical string modes admit exact solutions in terms of Bessel functions. Using the near-singularity behavior of the string mode functions, we can analyze the light-cone string world-sheet spectrum through the Hamiltonian and calculate the oscillator masses. The near-singularity region, regulated by, say, $\tau < \tau_\epsilon$, always contains highly stringy modes with oscillation number $n \gg \frac{1}{\tau_\epsilon^{a+1}}$ that propagate essentially freely in the background. On the other hand, low-lying string modes (finite $n \lesssim \frac{1}{\tau_\epsilon^{a+1}}$) have asymptotic near-singularity τ dependence similar to the center-of-mass mode. The oscillator masses are time dependent and can be recast in terms of two new sets of oscillators, one of which becomes light. It would be interesting to explore this further. This suggests that the vicinity of the singularity is filled with "stringy fuzz," comprising highly stringy modes. We expect string interactions are non-negligible near the singularity.

Our analysis is essentially from the bosonic parts of the string world-sheet theory. Since the world-sheet fermion terms are quadratic (with covariant derivatives) for these purely gravitational backgrounds, we expect that including them will not qualitatively change our results here. It would be interesting to carry out the superstring analysis in detail. Relatedly, several aspects of the matrix string analysis in these backgrounds have been studied in [18].

We have largely been studying the near-singularity Kasner-like spacetimes. Consider the case where the spacetime scale factor $e^f \to 1$ asymptotically (so that the spacetime is flat at early times). To elaborate, note that from the equation of motion $R_{++} = 0$, there is a function-worth of solutions, i.e. for a generic e^f, although perhaps not always (for the general Kasner case, there is one equation relating several scale factors e^f, e^{h_m}). For such a scale factor e^f that is asymptotically $e^f \to 1$, e.g. $e^f = \tanh^a(x^+)$, one can, in principle, find a solution for e^h. Indeed, an approximate solution of this kind (in the asymptotic region) is given[12] by Eq. (11). Choosing e^f that is asymptotically $e^f \to 1$, the spectrum of masses of string states is asymptotically as in flat space, while the near-singularity spectrum is as discussed above.

Some oscillator states becoming increasingly massive is reminiscent of [29] who argue for finite energy of free string propagation across plane-wave singularities. We note, however, that we have essentially analyzed the free string spectrum in the vicinity of the singularity in these backgrounds. Although formally it is possible to continue the string mode expansion across the singularity, it would seem that the physically relevant question would be to try and understand the role of string interactions in the vicinity of the singularity, to obtain a better understanding of string propagation across the singularity.

If string interactions generate a nontrivial (semiclassical) dilaton profile, say $\Phi(x)$ (that is regular), then presumably this is one way the background is desingularized, e.g. if the backreacted background satisfies an equation of the form $R_{MN} \sim \partial_M \Phi \partial_N \Phi$. The solutions of these equations coincide with the singular background for $\Phi = 0$ and are regular when a nonzero Φ is generated (although possibly of string scale curvature).

Now we make a few comments on drawing insights into the AdS/CFT cosmological investigations [23,24] from our analysis here. We have essentially used the scale factors $h_m(x^+)$ in our solutions here to simulate the role of the dilaton there; i.e. the internal $h_m(x)$ scale factors shrinking effectively drive the singularity in the x^i directions, just as the time-varying dilaton drives the singularity in the AdS/CFT cosmological context. It would then seem that interaction effects between the various string modes could become non-negligible near the null singularity in the bulk, although the original classical bulk background might possess α'-exactness properties. This would be dual to possible nontrivial corrections to the gauge theory effective potential stemming from loop effects, the time-dependent gauge coupling being $g_{\rm YM}^2 = g_s = e^\Phi$. It would be interesting to explore this.

Finally, it is interesting to ask if there are universal features in the behavior of string oscillator modes near generic time-dependent singularities. For example, internal six-dimensional spaces with intrinsic time dependence, e.g. due to closed string tachyon instabilities, will give rise to 4D cosmological dynamics. Consider the case of unstable noncompact conifoldlike singularities [38] embedded in some compact space (say, a nonsupersymmetric orbifold of a Calabi-Yau space). Phase diagrams obtained in the noncompact limit from appropriate gauged linear sigma models show evolution from one of the two classical phases corresponding to small resolutions to the other more stable one through a flip transition [38,39], involving the blowdown of a 2-cycle and a blowup of the topologically distinct 2-cycle. From the point of view of the four-dimensional effective field theory, the sizes of these cycles are time-dependent scalars whose spontaneous time evolution governs the four-dimensional cosmology. In particular, one might imagine that as we approach a flip singularity in the internal space, a time-dependent four-dimensional singularity develops. While a direct stringy analysis of such a transition and resulting four-dimensional cosmology seems *a priori* difficult, it would be interesting to ask if simple models using internal scale factors of the form studied here can be used to mimic the internal time dependence of collapsing/growing cycles and to study the resulting string dynamics, possibly along the lines of [40].

[12] The exponent in the near-singularity form of e^h may not be integral of course.

ACKNOWLEDGMENTS

It is a pleasure to thank S. Das, R. Gopakumar, S. Govindarajan, P. Mukhopadhyay, S. Minwalla, and N. Suryanarayana for useful discussions and conversations. K. N. thanks S. Das and S. Trivedi for an enjoyable collaboration, some of the discussions from [23] in particular leading to this investigation. K. M. thanks the hospitality of CMI and IMSc over Nov.-Dec. '07 and Feb.-May '08, respectively. We thank the organizers of the Monsoon Workshop, TIFR, June '08, and the ISM08 Workshop, Pondicherry, Dec '08, for hospitality and a stimulating environment. The work of K. N. is partially supported by a Ramanujan grant, DST, Government of India.

APPENDIX A: SOME PROPERTIES OF THE SPACETIME BACKGROUNDS

1. Light-cone supersymmetry of the backgrounds

Here we analyze the supersymmetry of the Kasner-like backgrounds described here, although we have not really used this in our analysis in the paper. Choose the obvious diagonal orthonormal frame $e^+ = e^{f/2}dx^+$, $e^- = e^{f/2}dx^-$, $e^i = e^{f/2}dx^i$, $e^m = e^{h_m/2}dx^m$. The spin connection one-forms are defined by $de^a + \omega^a{}_b \wedge e^b = 0$, where raising/lowering is performed by the flat space frame metric. This gives the spin connection one-forms as

$$\omega_{-+} = -\tfrac{1}{2}f' dx^+, \qquad \omega_{+i} = -\tfrac{1}{2}f' dx^+, \qquad \omega_{+m} = -\tfrac{1}{2}h'_m dx^m. \tag{A1}$$

(It can be checked that these give the coordinate basis curvature components given previously.) Taking the supersymmetry parameter ϵ to be a function only of the light-cone time, $\epsilon(x^+)$, the supersymmetry variation of the dilatino is trivially zero in this purely gravitational background with unexcited dilaton and RR/NSNS fluxes. The supersymmetry variation of the gravitino $\delta\psi_M$ [using Eqs. (2.1, 2.2) of [41]; see also [42]] reduces to

$$D_M\epsilon = (\partial_M + \tfrac{1}{8}\omega_M^{ab}[\Gamma_a, \Gamma_b])\epsilon = 0, \qquad \text{(A2)}$$

where Γ_a are flat space Γ matrices satisfying $\{\Gamma_a, \Gamma_b\} = 2\eta_{ab}$, the curved space ones being $\gamma_\mu = e^a_\mu\Gamma_a$. Taking ϵ to be x^i, x^m independent is consistent with $D_M\epsilon = 0$, $M \neq +$. Along with $D_+\epsilon = 0$, this gives

$$\Gamma^+\epsilon = 0, \qquad (\partial_+ - \tfrac{1}{4}f')\epsilon = 0 \qquad \text{(A3)}$$

which can be solved as $\epsilon = e^{f/4}\eta$, where η is a constant spinor satisfying $\Gamma^+\eta = 0$ (it can be taken to be $\eta \sim \Gamma^+\chi$ where χ is some arbitrary constant spinor). Closure of the algebra gives the equations of motion $R_{MN} = 0$. Thus these spacetime backgrounds preserve 16 real (light-cone) supercharges.

2. Higher derivative curvature corrections

As is often the case with lightlike backgrounds, these spacetimes do not appear to admit α' corrections due to higher order curvature terms. This is expected since these are, after a coordinate transformation, anisotropic plane-wave-like backgrounds (5) which are known to have such α'-exactness properties [1]. We outline below some rudimentary analysis of the vanishing of higher derivative terms in the cosmological coordinates (2) and (3), mainly for completeness.

At the level of the action, this is straightforward to see: with R_{++} alone being nonzero, there are no nonzero contractions since there are no tensors with two or more upper $+$ components. At the level of the equations of motion, one could ask if there are corrections to $R_{++} = 0$ from higher order curvature terms. In this regard, various straightforward checks do in fact suggest the absence of corrections, although we do not prove this in a theorematic way.

To elaborate a little, it is straightforward to see that no corrections of the form $f(R)R_{++}$ can arise where $f(R)$ is a complete contraction since the latter vanishes. Let us therefore consider possible higher order terms of the form $A_{++} = R_{+M+N}T^{MN} = R_{+M+N}g^{MP}g^{NQ}T_{PQ}$, where T_{PQ} is some tensor built out of R_{MN}, R_{MNPQ}, etc. Analyzing the possible values for the indices forced by the contractions, we see that if T_{PQ} has only T_{++} nonzero, then A_{++} vanishes since the background has $R_{+-+-} = 0$. Thus, e.g., a possible correction at $\mathcal{O}(R^2)$ of the form $R_{+M+N}R^{MN}$ vanishes. It is straightforward to further show that any correction A_{++} with e.g. $T_{PQ} \equiv R^{(k)} = R_{PP_1}R^{P_1}{}_{P_2}R^{P_2}{}_{P_3}\ldots R^{P_k}{}_Q$ vanishes: this can be seen by expanding $T_{PQ} \equiv R^{(k)}$ to obtain the form $g^{P_1Q_1}g^{P_2Q_2}\ldots R_{PP_1}R_{Q_1P_2}R_{Q_2P_3}\ldots$, and noting that $g^{++} = 0$ and R_{++} alone is nonzero. Thus all higher order corrections with T_{PQ} built from the Ricci tensor vanish. Similarly, it is possible to show that a correction of the form e.g. $R_{+M+N}R^{MPLQ}R^N{}_{PLQ}$ vanishes. It would seem that this would be possible to generalize to all orders as well.

The backgrounds in question have nonvanishing Weyl components C_{+i+i}, C_{+m+m}, with an index structure as for R_{ijkl}. Thus higher derivative corrections involving the Weyl tensor are similar in structure and also vanish.

APPENDIX B: AN ALTERNATIVE TIME PARAMETER AND QUANTIZATION

We have been working with x^+ as the time parameter so far. We will now outline the analysis of this system with a canonical time parameter with $g_{+-} = -1$, and indicate results similar to the ones we have discussed so far. Consider a coordinate transformation to the affine parameter λ as the time parameter transforming the metric (2) to

$$ds^2 = -2d\lambda dx^- + \lambda^{a'}dx^i dx^i + \lambda^{b'}dx^m dx^m, \qquad \text{(B1)}$$

where $a' = \frac{a}{a+1}$, $b' = \frac{b}{a+1}$. Null congruences now have a natural time parameter here with $\xi = \frac{d}{d\lambda}$. Thus the geodesic deviation equation gives the acceleration norms as

$$|a^i|^2 \sim \frac{1}{\lambda^{4-a'}}, \qquad |a^m|^2 \sim \frac{1}{\lambda^{4-b'}}, \qquad \text{(B2)}$$

giving a singularity for a', $b' < 4$, which are the same as the conditions (18).

Performing a light-cone gauge string quantization with $\tau \equiv \lambda$, here we obtain $E = -\frac{1}{g_{+-}} = 1$, so that in this case we effectively have conformal gauge also, as discussed earlier (see Sec. III). Now the world-sheet action becomes

$$S = \frac{1}{4\pi\alpha'}\int d^2\sigma(g_{II}(\partial_\tau X^I)^2 - g_{II}(\partial_\sigma X^I)^2). \qquad \text{(B3)}$$

The equations of motion for the time-dependent modes f_n^I now are $\partial_\tau(\tau^{A_I}\partial_\tau f_n^I) + n^2\tau^{A_I}f_n^I = 0$, where $A_I \equiv a'$, b'. These give the mode functions

$$f_n^I(\tau) = c_{n1}^I\sqrt{n}\tau^{(1-A_I)/2}J_{(A_I-1)/2}(n\tau) + c_{n2}^I\sqrt{n}\tau^{(1-A_I)/2}Y_{(A_I-1)/2}(n\tau). \qquad \text{(B4)}$$

For low-lying oscillation modes with finite $n \lesssim \frac{1}{\lambda}$, these have the asymptotics $f_n^I \to c_{n0}^I + c_{n\tau}^I\tau^{1-A_I}$. These string mode amplitudes thus do not diverge for X_n^i since $a' = \frac{a}{a+1} < 1$ always, while the X_n^m mode amplitudes normalized with the metric behave as $\tau^{b'}(X_n^m)^2$ which is finite if $b' < 2$, i.e. the same conditions ($b < 2a + 2$) as before.

The conjugate momenta are $\Pi^I = \frac{1}{2\pi\alpha'}g_{II}(\partial_\tau X^I)$. The Hamiltonian for this system (rewriting the Π^I in terms of $\partial_\tau X^I$),

$$H = \frac{1}{4\pi\alpha'}\int d\sigma(g_{II}(\partial_\tau X^I)^2 + g_{II}(\partial_\sigma X^I)^2), \quad \text{(B5)}$$

is the generator of λ translations, rather than x^+ translations. For the zero modes, the center-of-mass momenta are $p_{I0}(\tau) = \int d\sigma \Pi^I = \frac{1}{\alpha'} g_{II}\dot{X}_0^I(\tau)$. Then the zero mode terms in the expression for the masses cancel between $2g^{+-}(-H_0)(p_-) - g^{II}(p_{I0})^2$.

The oscillator contributions for the low-lying modes can be calculated near the singularity as before, using their limiting expressions $f_n^I \to c_{n0}^I$, $\dot{f}_n^I \to c_{n\tau}^I\tau^{-A_I}$. Thus these low-lying oscillator terms in the Hamiltonian above can again be simplified using (36) and rewritten in terms of one set of operators with coefficient $g_{II}(\dot{f}_n^I)^2 \to \tau^{A_I}\tau^{-2A_I} = \tau^{-A_I}$, and another set of operators with coefficient $g_{II} = \tau^{A_I}$. This is identical in form to the expression for the masses (47) earlier, after resubstituting $A_I \equiv a'$, $b' = \frac{a}{a+1}, \frac{b}{a+1}$.

Now we consider the highly stringy modes: for any cutoff τ_ϵ, there are modes with $n \gg n_\epsilon = \frac{1}{\tau_\epsilon} = \frac{1}{\lambda}$ whose asymptotics is essentially like a plane wave, with $f_n^I \to \tau^{-(A_I/2)}e^{in\tau}$. Using these, the Hamiltonian above simplifies using (36) to

$$H_\lambda \sim \frac{1}{\alpha'}\sum_n \frac{1}{n^2}((a_{-n}^I a_n^I + \tilde{a}_{-n}^I\tilde{a}_n^I + n)(g_{II}|\dot{f}_n^I|^2 + n^2 g_{II}|f_n^i|^2) - a_n^I\tilde{a}_n^I(g_{II}(\dot{f}_n^I)^2 + n^2 g_{II}(f_n^i)^2) - a_{-n}^I\tilde{a}_{-n}^I((\dot{f}_n^{I*})^2 + n^2 g_{II}(f_n^{I*})^2)). \quad \text{(B6)}$$

Using the f_n^I asymptotics, the terms in the second line are vanishingly small while the τ-dependent terms in the first line are $\tau^{A_I}(2n^2)\tau^{-A_I}$. This Hamiltonian has the same form as $g^{+-}H_{x^+}$, using the expression (55) for H_{x^+}, for the highly stringy modes near the singularity.

We now describe some aspects of string quantization in the Brinkman coordinates (5), after redefining to the affine parameter λ. The metric is $ds^2 = -2d\lambda dy^- + \sum_I \chi_I(y^I)^2\frac{d\lambda^2}{\lambda^2} + (dy^I)^2$, with $\chi_I = \frac{a_I}{2(a+1)}(\frac{a_I}{2(a+1)} - 1)$. The string action is $S = \frac{1}{4\pi\alpha'}\int d^2\sigma((\partial_\tau y^I)^2 - (\partial_\sigma y^I)^2 + \sum_I \frac{\chi_I}{\tau^2}(y^I)^2)$. The equations of motion give the mode functions

$$f_n^I(\tau) = \sqrt{n\tau}(c_{n1}^I J_{(\sqrt{1+4\chi_I})/2}(n\tau) + c_{n2}^I Y_{(\sqrt{1+4\chi_I})/2}(n\tau)), \quad \text{(B7)}$$

resulting in a mode expansion similar to (27), with $k_n^I = \frac{i}{n}\sqrt{\frac{\pi\alpha'}{4}}$. The highly stringy modes are defined by the limit of small τ, large n, and $n\tau \gg 1$. Then $f_n^I \sim e^{-in\tau}$ for $c_{n1}^I = 1$, $c_{n2}^I = -i$, and the Hamiltonian from the action above reduces to

$$H \sim \sum_{n\gg 1/\tau}\left(1 - \frac{\chi_I}{2n^2\tau^2}\right)(a_{-n}^I a_n^I + \tilde{a}_{-n}^I\tilde{a}_n^I + n) - \frac{\chi_I\pi}{4n^2\tau^2}(a_n^I\tilde{a}_n^I(f_n^I)^2 + a_{-n}^I\tilde{a}_{-n}^I(f_n^{I*})^2). \quad \text{(B8)}$$

Thus the Hamiltonian for the highly stringy modes exhibits similar behavior here as earlier.[13]. Similarly, defining the b^I oscillators as before, the Hamiltonian for the low-lying oscillator modes is $H \sim \sum_{n\lesssim 1/\tau}\frac{\pi}{4n^2}(b_{n\tau}^{I\dagger}b_{n\tau}^I + (n^2 - \frac{\chi_I}{\tau^2})b_{n0}^{I\dagger}b_{n0}^I)$.

[13] Similar expressions arise from the corresponding limit in [8] (Sec. 6).

[1] G. T. Horowitz and A. R. Steif, Phys. Rev. Lett. **64**, 260 (1990); Phys. Rev. D **42**, 1950 (1990).
[2] V. Balasubramanian, S. F. Hassan, E. Keski-Vakkuri, and A. Naqvi, Phys. Rev. D **67**, 026003 (2003).
[3] L. Cornalba and M. S. Costa, Phys. Rev. D **66**, 066001 (2002).
[4] H. Liu, G. Moore, and N. Seiberg, J. High Energy Phys. 06 (2002) 045; 10 (2002) 031.
[5] A. Lawrence, J. High Energy Phys. 11 (2002) 019.
[6] G. Horowitz and J. Polchinski, Phys. Rev. D **66**, 103512 (2002).
[7] B. Craps, D. Kutasov, and G. Rajesh, J. High Energy Phys. 06 (2002) 053.
[8] G. Papadopoulos, J. G. Russo, and A. A. Tseytlin, Classical Quantum Gravity **20**, 969 (2003).
[9] J. David, J. High Energy Phys. 11 (2003) 064.
[10] M. Blau, M. Borunda, M. O'Loughlin, and G. Papadopoulos, Classical Quantum Gravity **21**, L43 (2004).
[11] A. Giveon, E. Rabinovici, and A. Sever, Fortschr. Phys. **51**, 805 (2003).
[12] M. Berkooz and B. Pioline, J. Cosmol. Astropart. Phys. 11

(**2003**) 007.

[13] M. Berkooz, B. Durin, B. Pioline, and D. Reichmann, J. Cosmol. Astropart. Phys. 10 (2004) 002; B. Durin and B. Pioline, in *Cargese 2004, String Theory: From Gauge Interactions to Cosmology* (Springer, New York, 2006).

[14] J. Karczmarek and A. Strominger, J. High Energy Phys. 04 (2004) 055; S. R. Das, J. L. Davis, F. Larsen, and P. Mukhopadhyay, Phys. Rev. D **70**, 044017 (2004); S. R. Das and J. L. Karczmarek, Phys. Rev. D **71**, 086006 (2005); S. R. Das, arXiv:hep-th/0503002.

[15] J. McGreevy and E. Silverstein, J. High Energy Phys. 08 (2005) 090.

[16] B. Craps, S. Sethi, and E. Verlinde, J. High Energy Phys. 10 (2005) 005; D. Robbins and S. Sethi, J. High Energy Phys. 02 (2006) 052; B. Craps, A. Rajaraman, and S. Sethi, Phys. Rev. D **73**, 106005 (2006); D. Robbins, E. Martinec, and S. Sethi, J. High Energy Phys. 08 (2006) 025.

[17] M. Li, Phys. Lett. B **626**, 202 (2005); M. Li and W. Song, J. High Energy Phys. 10 (2005) 073; 08 (2006) 089.

[18] M. Blau and M. O Loughlin, J. High Energy Phys. 09 (2008) 097.

[19] M. Berkooz, Z. Komargodski, D. Reichmann, and V. Shpitalnik, J. High Energy Phys. 12 (2005) 018; Y. Hikida, R. R. Nayak, and K. L. Panigrahi, J. High Energy Phys. 09 (2005) 023; B. Chen, Phys. Lett. B **632**, 393 (2006); S. R. Das and J. Michelson, Phys. Rev. D **72**, 086005 (2005); J. H. She, J. High Energy Phys. 01 (2006) 002; B. Chen, Y.-l. He, and P. Zhang, Nucl. Phys. **B741**, 269 (2006); T. Ishino, H. Kodama, and N. Ohta, Phys. Lett. B **631**, 68 (2005); S. R. Das and J. Michelson, Phys. Rev. D **73**, 126006 (2006); H. Chen and B. Chen, Phys. Lett. B **638**, 74 (2006); T. Ishino and N. Ohta, Phys. Lett. B **638**, 105 (2006); R. Nayak and K. Panigrahi, Phys. Lett. B **638**, 362 (2006); H. Kodama and N. Ohta, Prog. Theor. Phys. **116**, 295 (2006); see also the review, B. Craps, Classical Quantum Gravity **23**, S849 (2006).

[20] E. Silverstein, Phys. Rev. D **73**, 086004 (2006).

[21] K. Dasgupta, G. Rajesh, and S. Sethi, J. High Energy Phys. 03 (2003) 041.

[22] T. Hertog and G. T. Horowitz, J. High Energy Phys. 07 (2004) 073.T. Hertog and G. T. Horowitz, J. High Energy Phys. 04 (2005) 005.

[23] S. Das, J. Michelson, K. Narayan, and S. Trivedi, Phys. Rev. D **74**, 026002 (2006); **75**, 026002 (2007).

[24] A. Awad, S. Das, K. Narayan, and S. Trivedi, Phys. Rev. D **77**, 046008 (2008); A. Awad, S. Das, S. Nampuri, K. Narayan, and S. Trivedi, Phys. Rev. D **79**, 046004 (2009).

[25] C. S. Chu and P. M. Ho, J. High Energy Phys. 04 (2006) 013.02 (2008) 058.

[26] F.-L. Lin and W.-Y. Wen, J. High Energy Phys. 05 (2006) 013; F. L. Lin and D. Tomino, J. High Energy Phys. 03 (2007) 118.

[27] N. Turok, B. Craps, and T. Hertog, arXiv:0711.1824; B. Craps, T. Hertog, and N. Turok, arXiv:0712.4180.

[28] B. Craps and O. Evnin, J. High Energy Phys. 04 (2008) 021; B. Craps, F. de Roo, and O. Evnin, J. High Energy Phys. 04 (2008) 036;

[29] B. Craps, F. de Roo, and O. Evnin, J. High Energy Phys. 03 (2009) 105.

[30] P. Townsend and M. Wohlfarth, Phys. Rev. Lett. **91**, 061302 (2003); I. Neupane, Classical Quantum Gravity **21**, 4383 (2004); I. Neupane and D. Wiltshire, Phys. Lett. B **619**, 201 (2005); N. Ohta, Phys. Rev. Lett. **91**, 061303 (2003); Prog. Theor. Phys. **110**, 269 (2003); Int. J. Mod. Phys. A **20**, 1 (2005); M. Wohlfarth, Phys. Lett. B **563**, 1 (2003); Phys. Rev. D **69**, 066002 (2004); S. Roy, Phys. Lett. B **567**, 322 (2003); L. Cornalba, M. Costa, and C. Kounnas, Nucl. Phys. **B637**, 378 (2002); R. Emparan and J. Garriga, J. High Energy Phys. 05 (2003) 028.

[31] P. Chen, K. Dasgupta, K. Narayan, M. Shmakova, and M. Zagermann, J. High Energy Phys. 09 (2005) 009; T. Ishino, H. Kodama, and N. Ohta, Phys. Lett. B **631**, 68 (2005); R. Nayak, K. Panigrahi, and S. Siwach, Phys. Lett. B **640**, 214 (2006); N. Ohta and K. Panigrahi, Phys. Rev. D **74**, 126003 (2006); K. Bamba, Z. Guo, and N Ohta, Prog. Theor. Phys. **118**, 879 (2007); K. Maeda, N. Ohta, M. Tanabe, and R. Wakebe, arXiv:0903.3298.

[32] M. Blau, J. Figueroa-O'Farrill, and G. Papadopoulos, Classical Quantum Gravity **19**, 4753 (2002).

[33] J. Polchinski and L. Susskind, arXiv:hep-th/0112204.

[34] R. R. Metsaev and A. A. Tseytlin, Phys. Rev. D **63**, 046002 (2001); R. R. Metsaev, C. B. Thorn, and A. A. Tseytlin, Nucl. Phys. **B596**, 151 (2001).

[35] M. Blau and S. Weiss, Classical Quantum Gravity **25**, 125014 (2008).

[36] E. Lifshitz, V. Belinskii, and I. Khalatnikov, Adv. Phys. **19**, 525 (1970) [see L. D. Landau and E. M. Lifshitz, *Course of Theoretical Physics*, Classical Theory of Fields Vol. 2 (Pergamon, New York, 1987) for a lucid discussion]; C. W. Misner, Phys. Rev. **186**, 1319 (1969); T. Damour, M. Henneaux, and H. Nicolai, Classical Quantum Gravity **20**, R145 (2003).

[37] J. Polchinski, *String Theory* (Cambridge University Press, Cambridge, England, 1998), Vols. 1–2.

[38] K. Narayan, J. High Energy Phys. 03 (2006) 036; Phys. Rev. D **75**, 066001 (2007).

[39] D. Morrison, K. Narayan, and M. R. Plesser, J. High Energy Phys. 08 (2004) 047; D. Morrison and K. Narayan, J. High Energy Phys. 02 (2005) 062.

[40] S. Hellerman and I. Swanson, Phys. Rev. D **77**, 126011 (2008); J. High Energy Phys. 07 (2008) 022; I. Swanson, Phys. Rev. D **78**, 066020 (2008).

[41] M. Graña and J. Polchinski, Phys. Rev. D **63**, 026001 (2000); **65**, 126005 (2002).

[42] J. Schwarz and P. West, Phys. Lett. **126B**, 301 (1983); J. Schwarz, Nucl. Phys. **B226**, 269 (1983); P. Howe and P. West, Nucl. Phys. **B238**, 181 (1984).