Landscape of superconducting membranes

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The AdS/CFT correspondence may connect the landscape of string vacua and the "atomic landscape" of condensed matter physics. We study the stability of a landscape of IR fixed points of $\mathcal{N} = 2$ large N gauge theories in $2 + 1$ dimensions, dual to Sasaki-Einstein compactifications of M theory, toward a superconducting state. By exhibiting instabilities of charged black holes in these compactifications, we show that many of these theories have charged operators that condense when the theory is placed at a finite chemical potential. We compute a statistical distribution of critical superconducting temperatures for a subset of these theories. With a chemical potential of 1 mV, we find critical temperatures ranging between 0.24 and 165 K.

I. A TALE OF TWO LANDSCAPES

This paper will explore the relation between quantum critical phenomena in condensed matter systems and the landscape of string vacua. The connection between these will be the AdS/CFT correspondence [1].

String theory has infinitely many compactifications to four dimensions. Of those, googols may lead to low energy physics compatible with observations [2–7]. The existence of this landscape of string theory vacua has lead to a revival of anthropic reasoning in cosmology and particle physics, together with associated philosophical conundrums and worries about the scientific status and predictability of string theory. Against this background, it would be appealing if the string landscape could be related to a different set of physical systems than particle physics and cosmology.

Whereas particle physics and cosmology give us direct experimental access to only one vacuum and its associated low energy effective field theory, in condensed matter physics there is a virtually unlimited supply of "vacua" and corresponding field theories. Typical examples are crystal lattices. These are metastable ground states of a single underlying microscopic theory, the standard model, translation invariant at large distance scales and with low energy excitations described by effective field theories. Material science is in essence the exploration of this vast landscape. In addition, an increasing range of lattice Hamiltonians can be engineered and controlled in tabletop experiments, for instance using optical lattices [8].

While the systems arising in the "atomic landscape" are generally sensitive to their underlying discreteness, as a function of couplings they can undergo second order phase transitions at zero temperature, called quantum phase transitions. At the quantum critical point the long distance physics is sometimes described by a continuum "relativistic" conformal field theory (CFT), e.g. [9,10]. We will focus on such relativistic quantum critical theories as they are the cases in which AdS/CFT is best understood. Note however that the AdS/CFT correspondence can be adapted to nonconformal relativistic theories (see e.g. [11] for a review) and also to theories with a nonrelativistic scale invariance [12–14]. We will furthermore focus in this paper on $2 + 1$ dimensional systems.

The AdS/CFT correspondence [1,15,16] implies the existence of a $2 + 1$ dimensional conformal field theory for every $3 + 1$ dimensional theory of quantum gravity in an asymptotically anti–de Sitter (AdS) spacetime. The string landscape provides an immense number of such theories. Therefore, the string landscape also provides a wealth of new quantum critical, that is, scale invariant, theories. Whether any of these theories can be used to model the physics associated to quantum phase transitions in experimentally realizable discrete systems is an important question for future work. In this paper we initiate a study of their properties.

Given a vacuum of a theory, for instance with a finite chemical potential, two immediate questions are first to characterize low energy excitations about the vacuum and second to enquire about the stability of the vacuum configuration. These two issues can be directly related. For instance, in conventional superconductivity an instability of the vacuum with unbroken gauge symmetry arises due to interactions between low energy phonons and (dressed) electrons.

For generic lattice structures there is by now a very well-developed set of techniques for identifying the low lying degrees of freedom and their dynamics. Some examples are shown in Table I. However, at quantum critical points the system is not describable in terms of conventional quasi-particle degrees of freedom. The critical point describes the dynamics of highly nonlocal entangled states of matter, in which different competing orders are finely balanced [10]. There is no preferred energy scale and generically no weak

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TABLE I.   Comparison of two landscapes.

	Atomic landscape	String landscape
Microscopic theory	Standard model	M theory
Fundamental excitations	Leptons, quarks, photons, etc.	?
Typical vacuum	Atomic lattice	Compactification
Low energy excitations	Dressed electrons, phonons, spinons, triplons, etc.	Gravitons, gauge bosons, moduli, intersectons, etc.
Low energy theory	Various quantum field theories	Various supergravities

coupling. The lesson of the AdS/CFT correspondence is that, at least in a "large N" limit, there can be a dual semiclassical description of quantum critical physics.[1] Examples of dual low energy excitations are also shown in Table I.

Table I suggests a complementary relationship between the string and the atomic landscapes. The string landscape may supply tractable models of quantum critical points in the atomic landscape. Furthermore, studying the string landscape in its totality may lead to the identification of universal or typical properties and also novel exotic behaviors. One is also lead to wonder whether the atomic landscape might have implications for string theory. We will speculate on this latter connection at the end of the paper.

The dynamical property of CFTs with string vacuum duals that we shall investigate in this paper is the potential instability toward a superconducting phase. We show that a large class of string compactifications do indeed have such instabilities. *Inter alia* these backgrounds provide explicit string theory realizations of holographic s wave superconductors [17–19], including cases in which the dual field theories are known.[2] In particular, the theories are those arising on M2 branes placed at the tip of a Calabi-Yau cone. These are the IR fixed points of $\mathcal{N} = 2$ supersymmetric gauge theories in $2 + 1$ dimensions. Among these, we find a superconducting instability in the maximally supersymmetry $\mathcal{N} = 8$ CFT in $2 + 1$ dimensions at a finite chemical potential.

We will begin by reviewing the framework of holographic superconductivity. We then go on to discuss a subset of the landscape given by $\mathcal{N} = 2$ Freund-Rubin Sasaki-Einstein compactifications of M theory. These theories can be consistently truncated to Einstein-Maxwell theory on a four-dimensional space with negative cosmological constant. We show that there exist minimally coupled charged pseudoscalar modes that decouple from all other fluctuation modes at the linearized level, in arbitrary backgrounds solving the Einstein-Maxwell equations. They correspond to modes of the M theory 3-form obtained by reducing certain harmonic 4-forms on the Calabi-Yau cone over the Sasaki-Einstein manifold. We show that these modes lead to instabilities toward a superconducting phase of the dual CFT at low temperatures for a large number of Sasaki-Einstein compactifications, and we obtain a distribution of critical temperatures on this landscape.

II. HOLOGRAPHIC SUPERCONDUCTORS

A. General framework

Holographic superconductors are a class of quantum critical theories which have an instability to a superconducting phase at low temperatures when held at a finite chemical potential μ [17–19]. One can equivalently work with a fixed charge density ρ. Scale invariance and dimensional analysis imply that the critical temperature $T_c \propto \mu$. Our objective is to show that a large number of simple string vacua are holographic superconductors and to determine T_c/μ for these theories.

The minimal bulk action for a holographic superconductor must describe the dynamics of the metric, a Maxwell field, and at least one charged field that can condense and spontaneously break the $U(1)$ symmetry.[3] We focus in this work on the case in which the charged field is a scalar in AdS$_4$. In general the full nonlinear action is complicated, as a consistent embedding into string theory will typically involve many coupled fields. Physically this implies that there will be many condensates at low temperature. In this work we avoid this problem by only considering the scalar equations of motion to linearized order, at which many fields decouple. This is sufficient to determine the critical temperature.

The bulk action for a minimally coupled scalar field to quadratic order in the scalar is

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[1]It is important to emphasize that unlike in the large N limit of, for instance, the $O(N)$ model, the AdS/CFT theories are always strongly coupled in the gravity regime. This may make them better models for strongly coupled finite N real world systems.

[2]Making approximations to the non-Abelian Dirac-Born-Infeld action, holographic p wave superconductors [20–22] can be obtained in string theory using coincident D branes [22–24].

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[3]The $U(1)$ symmetry to be broken in the boundary field theory is a global symmetry. From the point of view of superconductivity, this corresponds to neglecting virtual photons. This approximation is correctly made in all standard microscopic theories of superconductivity (such as the Bardeen-Cooper-Schrieffer theory).

$$\mathcal{L} = \frac{M^2}{2} R + \frac{3M^2}{L^2} - \frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu}$$
$$- |\nabla\phi - iqA\phi|^2 - m^2|\phi|^2. \qquad (1)$$

There are four dimensionless quantities in this action: the AdS radius in Planck units $(ML)^2$, the mass squared of the scalar field $(mL)^2$, the Maxwell coupling g, and the charge of the scalar field q. We will show in the following section that this action can be consistently obtained from M theory Freund-Rubin compactifications. The internal geometry of the compactification will fix the values of these coefficients. The dimensionless quantities have the following field theory interpretations:

   (i) The central charge of the CFT is

$$c = 192(ML)^2, \quad \text{where} \quad s = \frac{c\pi^3}{54} T^2. \qquad (2)$$

   Here s is the entropy density. Recall that for a $2 + 1$ CFT, the central charge can be defined in two ways [25]. Either as a parametrization of the energy momentum tensor two point function, or as a parametrization of the entropy density, as we have used in (2). It was noted in [26] that these two notions agree for theories with classical gravity duals.[4]

   (ii) The electrical conductivity at zero momentum is frequency independent [27]

$$\sigma \equiv \sigma_{xx} = \frac{1}{g^2}. \qquad (3)$$

   This is the conductivity appearing in Ohm's law $j = \sigma E$. Recall that conductivity is dimensionless in $2 + 1$ dimensions, and so σ may also be thought of as a central charge.

   (iii) The scaling dimension of the charged operator $\mathcal{O}$ dual to the bulk field ϕ is [15,16]

$$\Delta(\Delta - 3) = (mL)^2. \qquad (4)$$

   Both roots to this equation are admissible [28] so long as they satisfy the unitarity bound $\Delta \geq \frac{1}{2}$.

   (iv) The charge q is the charge of the dual operator $\mathcal{O}$. We will consider cases in which the gauge group is $U(1)$ (rather than $\mathbb{R}$) and work in units in which the charges take integer values.

The quantum critical theory at finite temperature and chemical potential is dual to the bulk theory in an AdS-Reissner-Nordstrom black hole background. This has metric

$$ds^2 = -f dt^2 + \frac{dr^2}{f} + \frac{r^2}{L^2}(dx^2 + dy^2), \qquad (5)$$

and scalar potential

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[4]In Eq. (2) we are using the normalization of [26] for the central charge. In this normalization, the central charge of a massless free boson is $c = 81\zeta(3)/\pi^4 \approx 0.9996$.

$$A_0 = \mu\left(1 - \frac{r_+}{r}\right). \qquad (6)$$

The function f is given by

$$f = \frac{r^2}{L^2} - \left(\frac{r_+^2}{L^2} + \frac{\mu^2}{2g^2 M^2}\right)\frac{r_+}{r} + \frac{\mu^2}{2g^2 M^2}\frac{r_+^2}{r^2}, \qquad (7)$$

where the horizon radius r_+ is related to the temperature through

$$T = \frac{1}{8\pi r_+}\left(\frac{6r_+^2}{L^2} - \frac{\mu^2}{g^2 M^2}\right). \qquad (8)$$

Here T and μ are the temperature and chemical potential of the field theory, respectively. The charge density of the field theory is

$$\rho = \frac{\mu r_+}{g^2 L^2} = \mu\sigma T\left(\frac{2\pi}{3} + \sqrt{\left(\frac{2\pi}{3}\right)^2 + \frac{32\sigma}{c}\frac{\mu^2}{T^2}}\right). \qquad (9)$$

To see whether the theory develops superconductivity we need to check the stability of this background against fluctuations of the scalar field.

B. Criterion for instability of minimally coupled scalars

The equations of motion for the charged scalar field following from (1) are

$$- (\nabla_\mu - iqA_\mu)(\nabla^\mu - iqA^\mu)\phi + m^2\phi = 0. \qquad (10)$$

Looking for an unstable mode of the form $\phi = \phi(r)e^{-i\omega t}$ one obtains

$$- \phi'' - \left(\frac{2}{r} + \frac{f'}{f}\right)\phi' - \frac{[r\omega + q\mu(r - r_+)]^2}{r^2 f^2}\phi + \frac{m^2}{f}\phi = 0. \qquad (11)$$

The AdS-Reissner-Nordstrom black hole will be unstable if there is a normalizable solution to this equation, with ingoing boundary conditions at the horizon, such that ω has a nonzero positive imaginary part.

We will shortly solve (11) numerically. A few prior comments are in order. It is useful to introduce the ratio[5]

$$\gamma^2 \equiv \frac{c}{96\sigma} = 2g^2(ML)^2. \qquad (12)$$

A ratio of central charges, γ might be thought of as quantifying the efficiency of charge transport in the theory. The Bogomol'nyi-Prasad-Sommerfeld (BPS) bound[6] for

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[5]Essentially this ratio was also considered in [29].
[6]This bound can be derived from the superconformal algebra when the $U(1)$ under consideration is the R symmetry in this algebra, as will in fact be the case for the Sasaki-Einstein compactifications we will consider. Unlike in asymptotically flat space, the BPS bound lies strictly below the black hole extremality bound [30], except in the limit $\gamma q \to 0$ in which the black holes become small and the flat space result is recovered. Extremal black holes with any finite charge do not preserve any supersymmetry.

charged scalars can then be written as

$$\Delta \geq \gamma q. \tag{13}$$

The normalization can be obtained, for instance, from the extremality condition of black holes with spherical horizons that are much smaller than the AdS radius in the theory (1). Recall that q is quantized to be integer. We further observe, allowing ourselves to rescale the radial coordinate, that Eq. (11) depends only on the following three dimensionless quantities: Δ, γq, and $\gamma T/\mu$. Fixing the first two of these, the mass and charge, we solve (11) to obtain the critical temperature T_c below which there is an instability. In more detail, the numerical algorithm proceeds as follows. We fix $\gamma T/\mu$, γq, and Δ, and start by constructing the solution in the very near horizon region obtained by Taylor series expansion to third order in the coordinate distance from the horizon. We then numerically solve the linear differential equation (11) out to a sufficiently large value of r. The equation is solved with $\omega = 0$ as we are looking for the onset of an instability. Finally, we match this to the general large r asymptotic solution, obtained by power series expansion to seventh order. (Working to such high order is necessary to get accurate results across the full parameter range.) This procedure thus yields two coefficients as a function of T, multiplying the solutions with $r^{-\Delta}$ and $r^{\Delta-3}$ leading asymptotics. Solving for the largest value of T for which the coefficient multiplying the $r^{\Delta-3}$ branch vanishes gives us T_c at the given values of Δ and γq. This is then repeated for a fine grid of values of Δ and γq. The result is shown in Fig. 1.

The zero temperature result of this plot can be understood analytically. If we look for a threshold unstable mode, with $\omega = 0$, at zero temperature, then near the horizon we find the behavior

$$\phi \sim (r - r_+)^{(-3 \pm \sqrt{3}\sqrt{3 - q^2\gamma^2 + 2\Delta(\Delta-3)})/6}. \tag{14}$$

On general grounds one expects an instability to arise when the field oscillates infinitely many times before reaching the horizon [31]. From (14) we see that this requires

$$q^2\gamma^2 \geq 3 + 2\Delta(\Delta - 3). \tag{15}$$

Therefore we expect an instability when the charge of the scalar field is sufficiently large as given by (15). If the charge is lower than the critical value there will never be an instability, as raising the temperature acts to stabilize the theory. The black line in Fig. 1, obtained numerically, is precisely the curve (15) separating stable backgrounds from backgrounds that become unstable below some temperature. This is a line of quantum critical points. It would be interesting to study in detail the dynamics close to these points.

The instability criterion (15) reduces to the inequality noted in [19] for the case of neutral scalar fields ($q = 0$). There the result was obtained by comparing the mass squared of the field to the Breitenlohner-Freedman bound
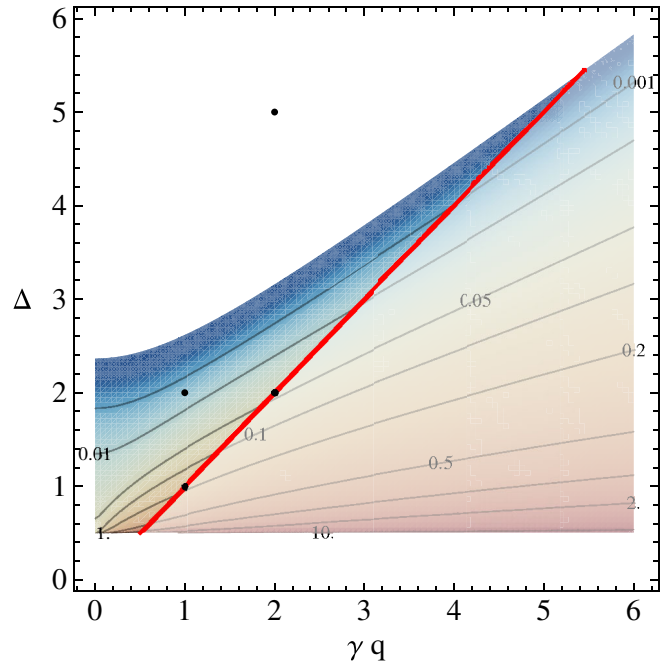

FIG. 1 (color online). The critical temperature T_c for a minimally coupled scalar as a function of the charge γq and dimension Δ of the dual operator. Contours are labeled by values of $\gamma T_c/\mu$. This plot is obtained numerically from Eq. (11). The BPS line $\Delta = \gamma q$ is shown as a heavy black (red) line; the shaded triangle to the left of it is the window of unstable values compatible with the BPS bound. The top boundary $q^2\gamma^2 = 3 + 2\Delta(\Delta - 3)$ is a line of quantum critical points separating superconducting and normal phases at $T = 0$. The bottom boundary is the unitarity bound $\Delta = 1/2$, where T_c diverges. The black dots indicate special cases which we will see arise in the context of $\mathcal{N} = 2$ M2 brane theories.

in the AdS$_2$ near horizon region. The full result (15) may be obtained by requiring the near horizon effective mass squared, including the coupling to the Maxwell field [32], to be below the AdS$_2$ Breitenlohner-Freedman bound.

The remaining noteworthy feature of Fig. 1 is that the critical temperature diverges as $\Delta \to \frac{1}{2}$. This divergence is exhibited clearly in Fig. 2, which shows the critical tem-
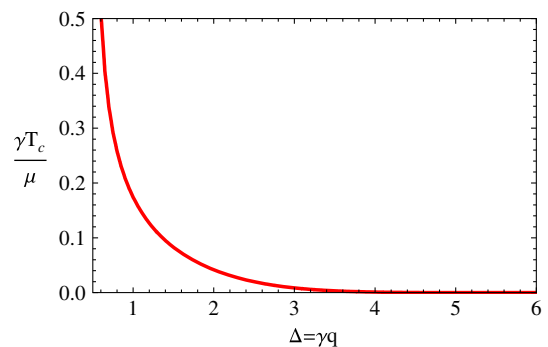

FIG. 2 (color online). Critical temperature $\gamma T_c/\mu$ as a function of Δ for operators on the BPS line $\Delta = \gamma q$.

perature as a function of operator dimension along the BPS line $\Delta = \gamma q$. It is presumably related to the fact that $\Delta = \frac{1}{2}$ modes form singleton representations of the AdS_4 isometry group. These modes can be gauged to the boundary of AdS [33], which one thinks of as the UV of the field theory, and hence are not sensitive to the temperature, which only affects the IR physics. Thus the superconducting instability can never be stabilized by the temperature in this case. The field theory statement of this fact is that these modes are decoupled from all others and therefore do not acquire a thermal mass.

If we wish to find string theory realizations of holographic superconductivity, we need to find compactifications of string theory that have charged scalars with masses and charges that fall inside the shaded region to the left of the BPS line in Fig. 1.

C. The weak gravity bound

A priori it is not obvious that there exist compactifications with charged scalars that lie in the left-hand region of Fig. 1. An argument in favor of the generic presence of an instability comes from the conjectured "weak gravity" bound [34]. Perhaps the sharpest of the statements in that paper was the requirement that extremal black holes should be able to decay in consistent theories of quantum gravity. In asymptotically Minkowski spacetime, a simple kinematic argument shows that this requirement implies that there must exist a charged particle in the theory that has mass and charge related by $m \leq \sqrt{2}gqM$, where m, g, q, and M have the same meanings as they did in the previous sections. The interest of this statement is that if $gq \ll 1$, then the charged particle is much lighter than would be predicted from standard effective field theory logic.

In asymptotically anti–de Sitter spacetimes it is less straightforward to make kinematic arguments for a weak gravity bound, as particles may not scatter out to infinity. However, the criterion (15) for a classical instability was obtained from only the near horizon geometry of the extremal black hole. If the preferred decay mode of the black hole is through a minimally coupled scalar, as we have been assuming, then (15) is a natural candidate for the correct weak gravity bound. Namely, in any consistent asymptotically AdS theory of quantum gravity there should exist a charged particle with charge q and energy Δ such that (15) is satisfied and extremal black holes can decay. We note that (15) does not reduce to the Minkowski space bound when $\Delta \gg 1$. We are only considering large AdS black holes; smaller black holes can require a more stringent condition in order to decay.

A caveat to the above statement is the possibility of decaying through charged modes that are not minimally coupled scalars. Given a field with a specified spin and coupling to the Maxwell field, it is easy to rerun the above argument involving the near horizon Breitenlohner-Freedman bound and obtain an instability criterion analogous to (15). The weak gravity bound would only require the existence of one unstable mode, of any spin and coupling.

The instability we are describing is essentially Schwinger pair production. Although this is initially a quantum mechanical effect, once there is sufficient condensate accumulated it is described as a classical instability in terms of macroscopic fields. Whatever the microscopic mechanism for emission of charge from the black hole, it seems likely that the classical field instability considered here is the correct description once the number of quanta involved becomes large. Furthermore, numerical investigations in [19] suggested (but not conclusively) that at the endpoint of the extremal black hole instability, if the charge of the scalar field is nonzero, all of the charge is carried by the scalar field condensate. Therefore this instability leads to the complete decay of the extremal black hole, as required by the weak gravity conjecture.

An interesting exception to the statements in the previous paragraph might arise if the preferred decay mode of the black hole were to charged fermionic particles. In the absence of a pairing mechanism these will not develop macroscopic occupation numbers, but rather build up a Fermi surface. This could lead to novel black holes with charged fermionic hair.

Whether or not one believes in the weak gravity bound, we shall now show that there indeed exist a large set of vacua in which extremal AdS-Reissner-Nordstrom black holes are unstable. Note that extremal AdS-Reissner-Nordstrom black holes are not supersymmetric and do not saturate the BPS bound.

III. CHARGED SCALARS FROM SASAKI-EINSTEIN VACUA

A. $\mathcal{N} = 2$ Freund-Rubin compactifications of M theory

The M theory bosonic action is (in the conventions of [35])

$$S = \frac{1}{2\kappa^2} \int d^{11}x\sqrt{-g}R$$
$$- \frac{1}{\kappa^2} \int \left[G \wedge \star G + \frac{2}{3}C \wedge G \wedge G \right], \quad (16)$$

with $G = dC$. We are interested in Freund-Rubin vacua with a background electromagnetic field in four dimensions. See for instance [27,36]. The metric ansatz is

$$ds_{11}^2 = L^2 ds_{M_4}^2 + 4L^2[(a[d\psi + A] + \sigma)^2 + ds_{M_6}^2], \quad (17)$$

supported by the flux

$$G = \frac{3L^3}{2}\mathrm{vol}_{M_4} - 4L^3 a\omega \wedge \star_4 F, \quad (18)$$

where $\frac{1}{2}d\sigma = \omega$ is the Kähler form on M_6, which is taken to be a six real dimensional Kähler-Einstein manifold

satisfying $\text{Ric}_{M_6} = 8g_{M_6}$, and $F = dA$. In (17) the coefficient a is such that ψ has range 2π, and we have chosen the four-dimensional gauge connection A to be normalized so that excitations have integer charges.

One can check that (17) and (18) solve the 11-dimensional equations of motion if and only if the four-dimensional metric $g^{(4)}$ and gauge field A solve the four-dimensional Einstein-Maxwell-AdS equations of motion. These come from the effective four-dimensional Lagrangian density

$$\mathcal{L}^{(4)} = \frac{1}{2\kappa_4^2}\left[R^{(4)} + \frac{6}{L^2} - 4L^2 a^2 F_{\mu\nu} F^{\mu\nu} \right], \qquad (19)$$

where

$$\frac{1}{2\kappa_4^2} = \frac{(2L)^7 \text{Vol}(M_7)}{2\kappa^2}. \qquad (20)$$

In this expression M_7 refers to the Sasaki-Einstein manifold[7]

$$ds_{M_7}^2 = (ad\psi + \sigma)^2 + ds_{M_6}^2, \qquad (21)$$

with unit radius, that is, $ds_{M_7}^2$ is such that the cone

$$ds_{M_8}^2 = dr^2 + r^2 ds_{M_7}^2, \qquad (22)$$

is Ricci flat, i.e. a Calabi-Yau fourfold. The construction of Sasaki-Einstein manifolds as $U(1)$ fibrations over Kähler-Einstein manifolds is reviewed with differing emphases in [37–40]. The simplest example is $M_7 = S^7$, the round 7-sphere, for which $M_8 = \mathbb{C}^4$, $M_6 = \mathbb{CP}^3$, $ds_{M_6}^2$ the Fubini-Study metric, and $a = 1$. The $U(1)$ symmetry is the R symmetry of the dual $\mathcal{N} = 2$ field theory.

In checking that this ansatz indeed provides a consistent truncation to four-dimensional Einstein-Maxwell with a negative cosmological constant, it is important to be precise about orientations. We are taking the Sasaki-Einstein metric to be oriented such that its volume form is

$$\text{vol}_{M_7} = +\frac{a}{6} d\psi \wedge \omega \wedge \omega \wedge \omega. \qquad (23)$$

This implies, for instance, that

$$\star_7 (\omega \wedge \omega) = +2(ad\psi + \sigma) \wedge \omega, \qquad (24)$$

which is an equation one uses in confirming consistency.

Comparing the effective action (19) to our general expression in Sec. II above we find that for these theories the "ratio of central charges"

$$\gamma = \sqrt{\frac{c}{96\sigma}} = \frac{1}{2a}. \qquad (25)$$

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[7]We will only consider quasiregular Sasaki-Einstein manifolds, i.e. those for which the orbits of the Killing vector close. Hence the fibration is $U(1)$ rather than $\mathbb{R}$, and charges are quantized.

The coefficient γ is therefore determined by a single component of the Sasaki-Einstein metric, giving the (constant) radius of the canonical $U(1)$ fibration. This radius is determined topologically. Concretely:

$$\gamma = \frac{2k}{\text{gcd}c_1(M_6)}, \qquad (26)$$

where k is a positive integer and $\text{gcd}c_1(M_6)$ is the greatest integer by which the first Chern class $c_1(M_6)$ can be divided such that it remains an integral (orbifold) cohomology class [40]. The freedom to choose k corresponds to the freedom to quotient the circle by $\mathbb{Z}_k$. For example for $M_7 = S^7/\mathbb{Z}_k$, since $\text{gcd}c_1(\mathbb{CP}^3) = 4$, we get $\gamma = k/2$. There is a constraint on the values of k that are compatible with supersymmetry. The Killing spinor has a ψ dependence of the form $e^{i2a\psi}$ [37]. In order for the spinor to be well defined we must therefore have $4a \in \mathbb{Z}$. This constrains k not to be too large, given $\text{gcd}c_1(M_6)$.

By comparison with Sec. II we can also obtain the central charge

$$c = \frac{192L^2}{\kappa_4^2} = \frac{32\pi}{\sqrt{6}\text{Vol}(M_7)^{1/2}} N^{3/2}. \qquad (27)$$

In this expression we introduced the M2 brane charge $N \propto \int[*G + C \wedge G]$, which is a positive integer.[8] The dual 2 + 1-dimensional CFT, to be discussed below, will have an ultraviolet description as a gauge theory with an $SU(N)$ gauge group. Like the fiber radius a, the normalized volume $\text{Vol}(M_7)$ can be computed topologically [40]. Bishop's theorem implies that $\text{Vol}(M_7) \leq \text{Vol}(S^7) = \pi^4/3$. Therefore the central charge (27) is always larger than the central charge of the maximally supersymmetric theory, $c_{\mathcal{N}=8} \approx 7.2N^{3/2}$.

B. Examples of Sasaki-Einstein manifolds: Brieskorn-Pham links

A rich landscape of examples of Sasaki-Einstein manifolds is provided by links of Calabi-Yau hypersurface singularities. These are constructed as follows. Consider a weighted homogeneous polynomial $F(z)$ in $\mathbb{C}^5$. That is, satisfying

$$F(\lambda^{w_1} z_1, \ldots, \lambda^{w_5} z_5) = \lambda^d F(z_1, \ldots, z_5), \qquad (28)$$

where w_i and d are positive integers. An example is

$$F(z) = z_1^2 + z_2^5 + z_3^6 + z_4^7 + z_5^8 = 0, \qquad (29)$$

which has $\mathbf{w} = (420, 168, 140, 120, 105)$ and $d = 840$. The scaling action implies that the zero set $F(z) = 0$ is a four complex dimensional cone in $\mathbb{C}^5$. By definition, if the hypersurface supports a conical Ricci flat Kähler metric

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[8]Specifically, $N = 3(2L)^6 \text{Vol}(M_7)/(2\pi^3\kappa^4)^{1/3}$. This normalization can be obtained from the Dirac quantization condition for M2 and M5 branes in M theory.

as in (22), the base (link) of the cone is Sasaki-Einstein. The $U(1)$ acting as $\psi \to \psi + \Delta \psi$ on (21) acts as $z_i \to e^{iw_i \Delta \psi} z_i$ on the coordinates z_i. Thus the integrally quantized charge of the coordinate z_i is precisely w_i. This will shortly enable us to obtain the integrally quantized charge q of various 3-form modes from the weights $\{w_i\}$.

For these Sasaki-Einstein spaces the quantities a and $\mathrm{Vol}(M_7)$ introduced above, and therefore γ and c, are known explicitly [40,41]:

$$a = \frac{\sum\limits_i w_i - d}{4}, \qquad (30)$$

$$\mathrm{Vol}\,(M_7) = \frac{\pi^4 a^4 d}{3 \prod\limits_i w_i}. \qquad (31)$$

Not every cone constructed in this manner supports a Ricci flat Kähler metric, and correspondingly not every link supports a Sasaki-Einstein metric. A necessary condition for existence is [42] $\min_i w_i \geq a > 0$, with a given by (30). The CFT interpretation of this bound is quite pretty [42]: it is the unitarity bound $\Delta \geq \frac{1}{2}$ for chiral primaries corresponding to holomorphic functions on the cone. An example that violates this condition is the A_k singularity $F(z) = z_1^{k+1} + z_2^2 + z_3^2 + z_4^2 + z_5^2 = 0$ for $k > 2$. A second necessary condition is the bound following from Bishop's theorem [42] $\mathrm{Vol}(M_7) \leq \mathrm{Vol}(S^7) = \pi^4/3$, with $\mathrm{Vol}(M_7)$ given by (31).

A sufficient condition can be formulated [43] for the special case of Brieskorn-Pham cones, defined by Fermat type polynomials

$$F(z) = z_1^{m_1} + \cdots + z_5^{m_5} = 0. \qquad (32)$$

These are weighted homogeneous polynomials as in (28) above with

$$d = \mathrm{lcm}(m_i \mid i = 1 \cdots 5), \qquad w_i = \frac{d}{m_i}. \qquad (33)$$

According to [43], if the coefficients satisfy the following two conditions, then the link is Sasaki-Einstein:

$$1 < \sum_i \frac{1}{m_i} < 1 + \frac{4}{3} \min_{i,j} \left\{ \frac{1}{m_i}, \frac{1}{b_i b_j} \right\}. \qquad (34)$$

In this expression

$$b_j = \gcd(m_j, c_j), \qquad c_j = \mathrm{lcm}(m_i | i \neq j). \qquad (35)$$

Furthermore, two such Sasaki-Einstein manifolds, corresponding to different exponents $\{m_i\}$ and $\{m_i'\}$, are isomorphic if and only if the two sets of exponents are permutations of each other. These conditions are sufficient but not necessary for existence. A general necessary and sufficient condition is not known.

The example given in (29) satisfies the conditions (34). It yields a Sasaki-Einstein manifold with $a = 28.25$ and

$\mathrm{Vol}(M_7) \approx 0.1396$, so $\gamma \approx 0.0177$ and $c \approx 110 N^{3/2}$. The results reviewed in [44] imply that this manifold is homotopy equivalent (and therefore, by the generalized Poincaré conjecture, homeomorphic) but not diffeomorphic to the standard sphere S^7. That is, it can be continuously deformed into the round S^7 and there is a continuous but not smooth one to one map to the round S^7. Some further remarkable results about Sasaki-Einstein spaces constructed in this way may be found in [43–45].

C. Minimally coupled pseudoscalars from 3-form modes

Given the 11-dimensional background (17) and (18) of the previous section, we wish to know whether the AdS-Reissner-Nordstrom black hole (5) is unstable against charged excitations of the background. To answer this question systematically one should consider the general linearized perturbation of the 11-dimensional metric and 3-form about the background. While the spectrum of perturbations about neutral Freund-Rubin compactifications is a well-developed subject [35], the analysis is substantially complicated by the presence of a background four-dimensional Maxwell field. Generically the various modes that appear diagonally in the spectrum about the neutral vacuum are not minimally coupled to the background Maxwell field and furthermore get mixed among each other. For example, one may get couplings such as $|\phi|^2 F^2$, $\phi F^{\mu\nu} \partial_\mu v_\nu$, or $\phi \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} b_{\rho\sigma}$, where v_μ is some charged vector mode and $b_{\mu\nu}$ a charged 2-form mode. Moreover, such nonminimal couplings tend to qualitatively alter the stability analysis of Sec. II B.

Rather then perform the full stability analysis we shall focus on particular 3-form modes which, remarkably, turn out to be only minimally coupled to the Maxwell field and decouple from all other perturbations at the linearized level, in any background satisfying the Einstein-Maxwell equations. We will then show that these modes are sufficient to establish instabilities in a large number of Sasaki-Einstein vacua at finite chemical potential. It should be borne in mind however that there may be more unstable modes than the ones we find. Therefore our results for critical temperatures should be taken as lower bounds only.

In order to describe these modes, it is useful to start with the eight-dimensional Calabi-Yau cone (22). Consider a closed self-dual or anti-self-dual 4-form $\hat{Y}_4$ on the cone. That is

$$d\hat{Y}_4 = 0, \qquad \star_8 \hat{Y}_4 = s\hat{Y}_4, \qquad s = \pm 1. \qquad (36)$$

These conditions imply that $\hat{Y}_4$ is harmonic. Assume furthermore that the 4-form is homogeneous with degree n on the cone. That is

$$\mathcal{L}_{r\partial_r} \hat{Y}_4 = n\hat{Y}_4. \qquad (37)$$

Here $\mathcal{L}$ denotes the Lie derivative. Then we can decom-

pose the form as

$$\hat{Y}_4 = r^n\left(\frac{dr}{r} \wedge Y_3 + Y_4\right), \qquad (38)$$

with Y_3 and Y_4 being forms on the Sasaki-Einstein manifold M_7, and (36) implies

$$\star_7 dY_3 = snY_3, \qquad d\star_7 Y_3 = 0. \qquad (39)$$

Now consider the 3-form fluctuation

$$\delta C = \phi Y_3 + \text{c.c.}, \qquad (40)$$

where ϕ only depends on the four-dimensional spacetime coordinates. The field ϕ will be a pseudoscalar because the 3-form field changes sign under space or time reflections [35]. From (39) one shows that in a neutral background with $A_\mu = 0$ [35]:

$$\nabla^\mu\nabla_\mu\phi = m^2\phi, \qquad m^2 = \frac{n(n+6s)}{4L^2}. \qquad (41)$$

Moreover this mode does not source any other Kaluza-Klein modes at linear order [35].

The question now is what happens when A_μ is nonzero. We claim the following:

If $\hat{Y}_4$ is a primitive and closed (4,0)- or (3,1)-form on the Calabi-Yau fourfold, and $\{g_{\mu\nu}, A_\mu\}$ solve the 4D Einstein-Maxwell equations, then the covariantization of the mode (40) linearly decouples from all other Kaluza-Klein modes and satisfies the covariantized equation of motion (41), with $s = +1$ for (4, 0)-forms and $s = -1$ for (3, 1)-forms.

Before sketching the proof, let us clarify the claim. Recall that a primitive middle dimensional form on a Kähler manifold is one that satisfies

$$\hat{\omega} \wedge \hat{Y}_4 = 0 \quad \text{or equivalently} \quad \hat{\omega} \cdot \hat{Y}_4 = 0, \qquad (42)$$

where $\hat{\omega}$ is the Kähler form on the Calabi-Yau cone. Covariantization means replacing, in the coordinates of (21), $d\psi \to d\psi + A$ in (40), and $\nabla_\mu \to \nabla_\mu - iqA_\mu$ in (41), where we assumed the mode to have a definite charge q under the canonical $U(1)$ symmetry of the cone:

$$\mathcal{L}_{\partial_\psi}\hat{Y}_4 = iq\hat{Y}_4. \qquad (43)$$

This charge will be directly inherited by Y_3 and Y_4. Thus, explicitly, we take

$$\delta C = \phi Y_3^A + \text{c.c.}, \qquad (44)$$

where Y_3^A is obtained from Y_3 by replacing $d\psi$ by $d\psi + A$. In components

$$\delta C_{mnp} = \phi Y_{3mnp}, \qquad \delta C_{\mu mn} = \phi A_\mu Y_{3mn\psi}, \qquad (45)$$

where m, n, and p are indices on M_7 and μ on M_4.

We will discuss the existence of modes $\hat{Y}_4$ satisfying all of the above conditions in the next section. For the moment we assume existence. To prove our claim, first note that the Kähler form on the cone may be decomposed as (see e.g. [40])

$$\hat{\omega} = r^2\left(\frac{dr}{r} \wedge \eta + \omega\right), \qquad (46)$$

where $d\eta = 2\omega$, and ω is as before the Kähler form of M_6. In terms of the metric we wrote in (21) above, $\eta = ad\psi + \sigma$. The primitivity condition (42) is easily seen to imply

$$\omega \cdot Y_3 = 0, \qquad \wedge Y_3 + s\eta \wedge \star_7 Y_3 = 0. \qquad (47)$$

Here we also used (36). By plugging the mode (44) into the 11-dimensional equations of motion and using (47), we obtain[9] the following three results:

Decoupling from metric fluctuations.—The 3-form mode (44) does not source any linearized metric fluctuations provided that

$$F \wedge F(\omega_n{}^q Y_{3mq\psi} + \omega_m{}^q Y_{3nq\psi}) = 0. \qquad (48)$$

Decoupling from other 3-form modes.—The 3-form mode (44) does not source any other linearized 3-form fluctuations provided that

$$(s+1)\omega \wedge Y_3 = 0. \qquad (49)$$

Equation of motion for the pseudoscalar.—If decoupling occurs, then the four-dimensional pseudoscalar field satisfies

$$(\nabla^\mu - iqA^\mu)(\nabla_\mu - iqA_\mu)\phi = m^2\phi,$$
$$m^2 = \frac{n(n+6s)}{4L^2}. \qquad (50)$$

Solving this equation is sufficient to solve the full 11D linearized supergravity equations.

We now proceed to characterize forms for which (48) and (49) hold. In our electrically charged AdS-Reissner-Nordstrom background, $F \wedge F$ vanishes. Therefore the decoupling of metric fluctuations will be automatic. However, one might certainly wish to consider dyonic black holes also (for instance to study phenomena such as the Hall or Nernst effects [47–49]) for which this term does not vanish. Therefore in order to solve (48) we will require that $\omega_n{}^q Y_{3mq\psi} + \omega_m{}^q Y_{3nq\psi} = 0$. There are (at least) four interesting cases in which this is true. These are if $\hat{Y}_4$ is a (4, 0)-, (0, 4)-, (3, 1)-, or (1, 3)-form on the eight-dimensional Calabi-Yau cone. Let us consider these cases one at a time.

If $\hat{Y}_4$ is a (4, 0)-form, then $Y_{3mq\psi}$ is zero. This follows from the fact that $dr \wedge \eta$ is a (1, 1)-form on the Calabi-Yau cone, see for instance (46). If Y_3 had a $d\psi$ component (i.e. an η component), then $\hat{Y}_4$ in (38) would necessarily have an antiholomorphic component and could not be (4, 0). Hence $Y_{3mq\psi}$ is zero.

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[9]We will not reproduce the straightforward but tedious computations here. We verified our results using the abstract tensor calculus package XACT[46].

If $\hat{Y}_4$ is a (3, 1)-form, then $Y_{3mq\psi}dx^m \wedge dx^q$ is a (2, 0)-form. This again follows from the decomposition of $\hat{Y}_4$ in (38) and the fact that $dr \wedge \eta$ is a (1, 1)-form. Given that both m and q are holomorphic indices it follows that $\omega_n{}^q Y_{3mq\psi} + \omega_m{}^q Y_{3nq\psi} = i(Y_{3mn\psi} + Y_{3nm\psi}) = 0$. The first of these equalities follows from the fact that $\omega_m{}^q$ is proportional to the complex structure while the second equality follows from antisymmetry of Y_3.

These arguments clearly go through identically when $\hat{Y}_4$ is (0, 4) or (1, 3). They do not work however when $\hat{Y}_4$ is a (2, 2)-form. We can recall at this point that (4, 0)-forms are always primitive [from (42)] and self-dual whereas primitive (3, 1)-forms are anti-self-dual, in the canonical orientation with which we are working.

In order for (49) to vanish and other 3-form modes to decouple, we need that either $s = -1$ or that $\omega \wedge Y_3 = 0$. The first of these will hold if and only if $\hat{Y}_4$ is anti-self-dual whereas the second holds if $\hat{Y}_4$ is a (4, 0)-form. This last statement follows from noting that the structure of the eight-dimensional Kähler form (46) implies that $\frac{dr}{r} + i\eta$ is a holomorphic 1-form on the Calabi-Yau cone. Therefore in order for $\hat{Y}_4$ to be (4, 0) the decomposition (38) must take the form $\hat{Y}_4 = r^n(\frac{dr}{r} \pm i\eta) \wedge Y_3$, with Y_3 a (3, 0) on the six-dimensional Kähler-Einstein base of the Sasaki-Einstein manifold. However, if Y_3 is a (3, 0)-form, then $\omega \wedge Y_3$ is zero.

This proves our claim. Summarizing: The mode (44) decouples from all other perturbations if the closed 4-form $\hat{Y}_4$ is a (4, 0)- or primitive (3, 1)-form on the Calabi-Yau cone. It is described by a minimally coupled pseudoscalar in four dimensions with charge q and mass squared

$$L^2 m_{(4,0)}^2 = \left(\frac{n}{2} + 3\right)\frac{n}{2}, \qquad (51)$$

$$L^2 m_{(3,1)}^2 = \frac{n}{2}\left(\frac{n}{2} - 3\right). \qquad (52)$$

The same expressions hold for (0,4)- and (1,3)-forms, respectively. Using the relation $(Lm)^2 = \Delta(\Delta - 3)$, we can read off the possible conformal dimensions of the dual operators.

D. Existence

We will now establish the existence of modes in the classes described above and confirm that in many examples they lead to instabilities and superconductivity at low temperatures.

All Calabi-Yau cones admit a canonical holomorphic (4, 0)-form. This form is thus closed and self-dual. If we introduce holomorphic vielbeins θ^a, a runs from 1 to 4, such that the metric is written $ds_{M_8}^2 = \theta^a \bar{\theta}^a$, then the form is given by

$$\hat{Y}_4 = \hat{\Omega}_4 \equiv \theta^1 \wedge \theta^2 \wedge \theta^3 \wedge \theta^4. \qquad (53)$$

It is immediate that this form has scaling dimension $n = 4$ under the homothetic vector $r\partial_r$, as the metric has scaling dimension 2 and hence the θ^a have scaling dimension 1. Furthermore, we can easily obtain the charge $q = 4a$ by noting that for this mode

$$\mathcal{L}_{\partial_\psi}\hat{Y}_4 = \partial_\psi \cdot d\hat{Y}_4 + d(\partial_\psi \cdot \hat{Y}_4) = d(\partial_\psi \cdot \hat{Y}_4)$$
$$= aid(r^4 Y_3) = 4ai\hat{Y}_4. \qquad (54)$$

In the third and fourth equalities we used the fact noted previously that holomorphic 4-forms must take the form $\hat{Y}_4 = r^n(\frac{dr}{r} + i\eta) \wedge Y_3$. In the last equality we also used the first expression in (47). It is clear that this argument will apply to any closed (4, 0)-form with scaling dimension n, giving charge $q = na$. Such forms are readily obtained by multiplying $\hat{\Omega}_4$ by a homogeneous holomorphic function of degree $n - 4$.

It follows from (25) that all of the Sasaki-Einstein vacua have a decoupled pseudoscalar mode with charge $\gamma q = 2$ and, from (51), mass squared $m^2 L^2 = 10$. This corresponds to an operator of dimension $\Delta = 5$. Comparing with Fig. 1 or Eq. (15) we see that this mode never leads to an instability.

The recent results of [50] imply that this mode is part of a long vector[10] OSp(2|4) supermultiplet (the $E_0 = 4$, $y = 0$ case in Table I of [51]) which consistently decouples from all other Kaluza-Klein modes even at the nonlinear level. There are no other charged scalars in this multiplet.

Before moving on to consider a general class of (3, 1)-forms, we can consider the special case of $M_7 = S^7$ for which the Calabi-Yau cone is simply $M_8 = \mathbb{C}^4$. A (3, 1)-form on $\mathbb{C}^4$ is given by, for instance,

$$\hat{Y}_4 = d\bar{z}_1 \wedge dz_2 \wedge dz_3 \wedge dz_4. \qquad (55)$$

This is a closed, primitive, anti-self-dual (3, 1)-form with $n = 4$ and $\gamma q = 1$, recalling that $a = 1$ for the seven sphere. From (51) the four-dimensional mass will be $m^2 L^2 = -2$, corresponding to $\Delta = 2$ or $\Delta = 1$. This is precisely the value of the mass studied in detail in [18,19]. The two different dimensions of the dual operator correspond to theories that are related via a renormalization group flow generated by a double trace deformation [52]. From Fig. 1 or Eq. (15) we see that this mode does condense at low temperatures. Therefore, the IR conformal fixed point of $\mathcal{N} = 8$ $SU(N)$ Yang-Mills theory at large N spontaneously breaks $U(1)_R$ and becomes a superconductor at low temperatures and nonzero chemical potential. Taking $\Delta = 2$, we numerically find that the critical tem-

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[10]And hence not part of a short hypermultiplet as was claimed in [51].

perature is $T_c \approx 0.007\mu$. For $\Delta = 1$, we get $T_c \approx 0.35\mu$. We recall this is a lower bound.[11]

We now turn to our main source of examples, namely, (3, 1)-forms associated to complex structure moduli of the Calabi-Yau fourfold cone. Consider a metric deformation δg_{ab}, with a and b both holomorphic indices, preserving Ricci flatness. Then this is a Lichnerowicz zero mode and

$$\hat{Y}_4 \equiv \delta g_{\bar{a}\bar{e}} \hat{\Omega}^{\bar{e}}_{4bcd} d\bar{z}^{\bar{a}} \wedge dz^b \wedge dz^c \wedge dz^d, \quad (56)$$

is a harmonic (3, 1)-form [55]. In this equation a bar denotes an antiholomorphic index. It is easy to see that this form is furthermore primitive. We thus get an example of an anti-self-dual closed (3, 1)-form, as considered above. Calabi-Yau metric deformations which preserve the cone structure (21) and (22), and are therefore moduli of the Sasaki-Einstein manifold, have the same scaling dimension as the metric and are neutral under the $U(1)$ isometry (otherwise they would not preserve the isometry and the metric would no longer be Sasaki-Einstein). Thus the associated $\hat{Y}_4$ has the same scaling dimension $n = 4$ as $\hat{\Omega}_4$, and the same charge $\gamma q = 2$. The mass formula (51) now implies $\Delta_+ = 2$, saturating the BPS bound.[12] Such modes always condense at low temperature, with (see Fig. 2)

$$T_c \approx 0.0416 \frac{\mu}{\gamma}. \quad (57)$$

Therefore: The IR fixed point of $\mathcal{N} = 2$ $SU(N)$ Yang-Mills theories at large N with Sasaki-Einstein duals with at least one metric modulus becomes superconducting at temperatures below (57). As previously, this is a lower bound on T_c, and there may be other unstable modes with higher critical temperatures.

Not all Sasaki-Einstein metrics have deformation moduli. For example the round sphere has none. However, many of the Brieskorn-Pham links introduced in Sec. III B have plenty of moduli, obtained as polynomial deformations of the same weight d as the original polynomial (28). The number of such moduli equals the number of monomials of weight d minus the number of coordinate transformations respecting the weights [56], that is

$$N_{\text{mod}} = N_{\text{mon}}(d) - \sum_i N_{\text{mon}}(w_i), \quad (58)$$

where $N_{\text{mon}}(w)$ stands for the number of monomials of weight w. For the example (29), $N_{\text{mod}} = 1$: There is precisely one deformation which cannot be reabsorbed in a weight preserving coordinate transformation, namely $\delta F(z) = \epsilon z_3^3 z_5^4$. We shall look more systematically at the existence of moduli in the following section.

One could also consider deformations $\delta F = \epsilon F'$ of the defining equation (28) with weight $d' \neq d$. Such deformations do not preserve the cone structure, and so they are not moduli of the Sasaki-Einstein space. However if the fluctuation preserves the Ricci flatness of the cone metric to linear order, then (56) still gives a harmonic (3, 1)-form, and the corresponding pseudoscalar mode still satisfies all the required properties to be minimally coupled. To determine the charge of such a metric fluctuation it is useful to formally associate a charge $q_\epsilon = w_\epsilon = d - d'$ to ϵ. This way the polynomials F and δF would have the same charge d. The charge of the metric mode $\delta g_{ab} = \partial_\epsilon g_{ab}|_{\epsilon=0}$ is thus seen to be $-q_\epsilon = d' - d$. The associated form mode (56) thus has charge q and radial scaling dimension n given by

$$\frac{n}{2} = \gamma q = 2 + \gamma(d' - d). \quad (59)$$

As an example, consider the deformation $\delta F = \epsilon z_1 z_2^2$ of (29). This has $d' = 756$, and so, using $\gamma = \frac{2}{113}$, we get $\frac{n}{2} = \gamma q = \frac{58}{113} \approx 0.5132$. If this truly corresponded to a Calabi-Yau preserving deformation, it would give rise to a minimally coupled BPS pseudoscalar with this value of $\Delta = \gamma q$. This leads to $T_c = 1.473\,18\frac{\mu}{\gamma}$, substantially higher than the cone-preserving modes (57). Determining in general when such modes are indeed Calabi-Yau preserving appears to be an interesting open mathematical problem [57]. We shall not address this problem here, but note that it could lead to higher values of T_c than the ones we will discuss.

E. Comment on the dual field theories and operators

The gravity backgrounds that we have been describing are dual to $\mathcal{N} = 2$ superconformal field theories. The supersymmetry and conformality follow directly from the global (super)symmetries of the gravitational solutions. In special cases there may be an enhancement of supersymmetry. For instance, when $M_7 = S^7$ the theory has $\mathcal{N} = 8$ supersymmetry and if M_7 is tri-Sasakian then the theory will have $\mathcal{N} = 3$ supersymmetry.

More specifically, the dual field theory is that describing the world volume dynamics of N M2 branes placed at the tip of a Calabi-Yau fourfold cone over the Sasaki-Einstein manifold M_7 [58]. Until recently, this relationship was not useful for obtaining an explicit description of the field theory degrees of freedom. On general grounds one might expect the M2 brane theories to arise as IR fixed points of multiple D2 brane gauge theories in a background obtained by dimensionally reducing the M theory geometry along

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[11]There is another known instability for the case of $M_7 = S^7$, the Gubser-Mitra instability [53,54]. That instability corresponds to the charge becoming redistributed among the more than one $U(1)$ symmetry in the theory, and does not induce superconductivity, as all the operators involved are neutral. In our units $T_{\text{G-M}} = \mu/\pi \approx 0.32\mu$. Thus in the $\Delta = 1$ case the superconducting instability kicks in before the Gubser-Mitra instability.

[12]This mode is thus the lowest component of an $OSp(2, 4)$ hypermultiplet. Its scalar superpartner is the metric modulus fluctuation, which has $\gamma q = 0$ and $\Delta_+ = 3$, as expected for a marginal deformation.

the $U(1)$ isometry of the Sasaki-Einstein metric [59]. This reduction will break all the manifest supersymmetry of the background for generic ($\mathcal{N} = 2$) Sasaki-Einstein manifolds. This occurs because the Killing spinor is charged under the $U(1)$ isometry, as we recalled below (26).

A different brane construction for the case $M_7 = S^7/\mathbb{Z}_k$ was presented in [60] (ABJM), following the renewed interest in multiple M2 brane theories initiated by [61–63]. The construction involves two NS5 branes, N D3 branes, and k D5 branes. Upon T dualizing and lifting to M theory one obtains N multiple M2 branes probing a geometry that has local $\mathbb{C}^4/\mathbb{Z}_k$ singularities. The brane construction allowed Aharony *et al.* [60] to identify the field theory as a specific superconformal $U(N) \times U(N)$ Chern-Simons theory at levels k and $-k$.

The ABJM brane construction was generalized to a family of $\mathcal{N} = 3$ field theories in [64]. These are dual to backgrounds in which M_7 is a tri-Sasakian manifold. The field theory dual for general $\mathcal{N} = 2$ theories is not yet available; it appears that the most tractable subset of $\mathcal{N} = 2$ theories are those in which the Calabi-Yau cone M_8 is toric (the Brieskorn-Pham cones we considered above are generally not toric, and the dual field theories are not known explicitly). Combining the extensive intuition gained from toric $\mathcal{N} = 1$ superconformal field theories in $3 + 1$ dimensions and the ABJM construction, it has been proposed that the world volume theory of M2 branes probing toric Calabi-Yau cones is given by a quiver Chern-Simons theory [65–70]. These have large gauge symmetries with associated gauge fields A_i and complex scalar fields ϕ_a that are charged under the gauge symmetries. The supermultiplets are then completed with additional scalar and spinor fields. The action takes the form

$$S = \sum_i \frac{k_i}{4\pi} \int d^3x \, \mathrm{Tr}\Big(A_i \wedge dA_i + \frac{2}{3} A_i \wedge A_i \wedge A_i$$
$$+ \text{superpartners}\Big) + \sum_a \int d^3x \Big(|D\phi_a|^2 - \Big|\frac{\partial W}{\partial \phi_a}\Big|^2$$
$$+ \text{superpartners}\Big). \tag{60}$$

We are being somewhat schematic. The superpotential W is a holomorphic function of the ϕ_a. A thorough discussion of $\mathcal{N} = 2$ Chern-Simons theories may be found in [71]. The point we would like to emphasize is that concrete field theory duals have been proposed for certain Sasaki-Einstein manifolds. One can therefore hope to identify the precise operator $\mathcal{O}$ which condenses at the superconducting instability.

The first mode we discussed in Sec. III D was obtained from the canonical holomorphic (4, 0)-form on the Calabi-Yau cone. Although this mode did not lead to an instability, it is instructive to consider its dual field theory operator. The mode must be dual to a canonical operator which is present in all $\mathcal{N} = 2$ theories. The most natural candidate

is the superpotential itself: $\mathcal{O} = W$.[13] As well as being holomorphic and canonical, this mode had charge $\gamma q = 2$ which is also the R charge of the superpotential. However, the mode has dimension $\Delta = 5$, whereas the superpotential has classical dimension $\Delta = 2$, as a consequence of being chiral. This identification would therefore imply that the dimension of the superpotential is renormalized in these $2 + 1$ theories. This is consistent with the fact, mentioned in Sec. III D, that this mode is part of a long multiplet [50], so its dimension is not protected.

The second set of modes we discussed were (3, 1)-forms corresponding to complex moduli deformations of the Calabi-Yau cone. These must be canonically dual to deformations of the field theory that preserve supersymmetry and conformality. The most natural candidate dual operators are deformations of the superpotential, $\mathcal{O} = \delta W$. In this case our bulk mode was BPS, with charge and dimension $\Delta = \gamma q = 2$, equal to those of bare superpotentials. These are relevant charged operators. This identification would indicate that whereas the overall superpotential is renormalized, deformations of the superpotential (if they exist) are not.

We also noted in Sec. III D that the (3, 1)-form modes lie in a hypermultiplet which contained a scalar superpartner with $\gamma q = 0$ and $\Delta = 3$. This mode will be dual to a marginal deformation of the Lagrangian. If our previous identification with deformations of the superpotential is correct, these operators will be of the form $\mathcal{O} = \int d^2\theta \, \delta W + \text{c.c.} = \partial_{\phi^a} \partial_{\phi^b} \delta W \psi^a \psi^b + \cdots$, with ψ^a fermionic superpartners of the ϕ^a.

It is certainly of interest to flesh out these identifications further for cases in which the superpotential and its deformations are known explicitly. We will leave this for future work.

F. Comment on skew whiffing

Given a (neutral) Freund-Rubin compactification from 11 to four dimensions, a different solution may be constructed by skew whiffing [35]. One way to describe the skew-whiffed solution is to change the sign of the 3-form background with everything else held fixed. In terms of the ansatz (17) and (18), with $A = 0$, this corresponds to letting $L \rightarrow -L$. In terms of brane constructions, this means that instead of N M2 branes at the tip of a Calabi-Yau cone, one takes N anti-M2 branes. This operation is not as innocuous as it might seem. With the exception of the case $M_7 = S^7$, only one of the two solutions can be supersymmetric [35]. At the strict classical level, skew-whiffed solutions obtained from supersymmetric Freund-Rubin compactifications give examples of stable nonsupersymmetric vacua [35]. Stability beyond the classical level is not known.

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[13]An analogous identification is implicitly made in the AdS$_5$/CFT$_4$ case with $\mathcal{N} = 1$ supersymmetry in $3 + 1$ dimensions in, for instance, [40].

In the skew-whiffed backgrounds ($L \rightarrow -L$) it turns out that the construction of Sec. III A above does not give a consistent reduction to Einstein-Maxwell theory in general. This is because a relative sign changes between the kinetic and Chern-Simons term in the 3-form equations of motion. However, for a purely electric (or purely magnetic) background, such as the AdS-Reissner-Nordstrom black holes of interest to us, the Chern-Simons term vanishes and one does obtain a solution.

Perturbing the skew-whiffed charged background by our mode (44) one finds that both the decoupling conditions (48) and (49) and the equation of motion for the pseudo-scalar (50) are changed by $s \rightarrow -s$. It follows from our previous arguments that only the modes obtained from closed (4, 0)- and (0, 4)-forms on the Calabi-Yau cone decouple in this case. Their mass squared is now given by

$$m^2_{(4,0)} = m^2_{(0,4)} = \frac{n(n-6)}{4L^2}. \tag{61}$$

We recalled above that all Calabi-Yau cones admit a closed (4, 0)-form with $n = 4$ and charge $\gamma q = 2$. Therefore, all of the skew-whiffed backgrounds have a minimally coupled pseudoscalar with $m^2 = -2$, corresponding to $\Delta = 2$ or $\Delta = 1$. We noted above (see Fig. 1) that these values of the charge and Δ lead to a superconducting instability at low temperatures. Therefore all theories dual to skew-whiffed Sasaki-Einstein compactifications of M theory are superconducting at low temperatures when placed at a finite chemical potential.

IV. A DISTRIBUTION OF CRITICAL TEMPERATURES

In this section we consider Sasaki-Einstein manifolds obtained as Brieskorn-Pham links, as discussed in Sec. III B, and which have unstable 3-form modes of the type considered in Sec. III D. For these theories, all the quantities in the four-dimensional action (1) can be explicitly computed. This allows us to obtain a distribution of critical temperatures.

More specifically, we will focus on the 3-form modes associated to metric moduli. Their critical temperature T_c is given by (57).[14] Notice that T_c/μ is proportional to γ^{-1}, with constant of proportionality independent of the theory. Therefore, in order to obtain a distribution of critical temperatures T_c at fixed μ, it is sufficient to obtain a distribution of values of $\gamma^{-1} = 2a \in \frac{1}{2}\mathbb{Z}_k$. We shall now note various features of this distribution for Brieskorn-Pham cones, putting aside momentarily the question of whether or not the manifolds have metric moduli.

The lowest value of a is clearly $a = 1/4$. From (57), this corresponds to $T_c \approx 0.0208\mu$. To gain some intuition for this result, it is useful to express this relation in physical units. The only quantity that we need to reintroduce is the Boltzmann constant $k_B = 8.617 \times 10^{-5}$ eV K^{-1}, which we have thus far set to unity. Furthermore we recall that 1 V is V = eV e^{-1} and that we have set the fundamental charge $e = 1$.[15] The lowest critical temperature we find is therefore

$$\frac{T_c}{[\text{K}]}\bigg|_{\text{min}} \approx 0.241\frac{\mu}{[\text{mV}]}. \tag{62}$$

Thus, for instance, if we put the membrane CFT at a chemical potential of 1 mV, the critical temperature would be 0.24 K. If (62) is taken literally, then by increasing the chemical potential we can make T_c arbitrarily high. Of course, in actual theories arising at quantum critical points in a real-life crystal, other factors such as impurities and interactions with background ions would influence the onset of superconductivity.

Less obvious, there is also an upper bound on a and hence an upper bound on the critical temperatures within this class of Sasaki-Einstein duals:

Lemma 1.—For the Brieskorn-Pham links constructed in [43] and reviewed in Sec. III B, the metric coefficient a has an upper bound. Thus the critical temperature at fixed chemical potential (57) is bounded above in these models.

This proof of this result is in the Appendix. The largest value of a that we found by scanning numerically (over $m_i < 100$) is $a = 2039/4$. However, this manifold does not have moduli. The largest value of a we found for a manifold with moduli is $a = 683/4$. The defining polynomial (32) for this case is $F = z_1^2 + z_2^3 + z_3^7 + z_4^{37} + z_5^{99}$. There is a single modulus $\delta F = z_2^2 z_5^{33}$. This value of a leads to $T_c \approx 14.2\mu$. Introducing physical units as above leads to

$$\frac{T_c}{[\text{K}]}\bigg|_{\text{max}} \approx 165\frac{\mu}{[\text{mV}]}. \tag{63}$$

Thus T_c is 165 K if the system is at a chemical potential of 1 mV. This is likely not the maximum T_c attainable; rather it is the largest value we found by scanning numerically.

A second interesting result is that while there are infinitely many Brieskorn-Pham links that lead to Sasaki-Einstein manifolds, only a handful of values of a occur infinitely many times.

Lemma 2.—There are precisely 19 values of a which occur infinitely many times in the Brieskorn-Pham links. These are

$$a = \frac{n}{4}, \tag{64}$$

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[14]Although we will only consider the distribution of the critical temperatures for this particular mode, we should keep in mind that there may be other modes that become unstable at higher temperatures.

[15]If these theories were to be realized in a lab, the identification $e = 1$ would only be correct if the unit of charge in the (emergent) CFT coincided with the (standard model) electron charge.

where $n = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 15, 18, 20, 21, 24, 30, 42\}$.

The proof of this result is again in the Appendix. This series suggests the resolution of a puzzle raised in [72].

It is straightforward to scan numerically through different values of the exponents $\{m_i\}$ in the defining polynomials for the Brieskorn-Pham cones, and to check whether they satisfy the condition (34) for being Sasaki-Einstein. We then need to check whether the Sasaki-Einstein space has metric moduli. Each time we find a solution with metric moduli, we can compute a and hence T_c, via (57). In Fig. 3 we show the solutions obtained for a scan over all $m_i < 100$. This scan led to 7278 distinct Sasaki-Einstein manifolds, 6190 of which had metric moduli. As noted below (26), we can also consider quotients of these manifolds by $\mathbb{Z}_k$, with k a divisor of $4a$. After considering quotients of the manifolds with moduli, we obtain 11 821 solutions. The critical temperatures of these manifolds are shown in Fig. 3. Of the 7278 manifolds found, only around 350 belong to the infinite families of Theorem 2. Removing them does not change the distribution noticeably. It seems therefore that Fig. 3 accurately captures the distribution of critical temperatures in the finitely many theories which do not belong to infinite families.

In Fig. 3 we see that the critical temperatures cluster around the lowest value $T_c/\mu \approx 0.241$ K/mV. The clustering appears to roughly follow a power law.

We close this discussion by noting that the instability we found for the maximally supersymmetric ($\mathcal{N} = 8$) theory in Sec. III D, which is not due to a modulus mode and not included in Fig. 3, gives the following critical temperatures in physics units:

$$\frac{T_c}{[\mathrm{K}]}\bigg|_{\mathcal{N}=8} \approx 0.081\,\frac{\mu}{[\mathrm{mV}]} \quad \text{or} \quad 4.1\,\frac{\mu}{[\mathrm{mV}]}, \qquad (65)$$
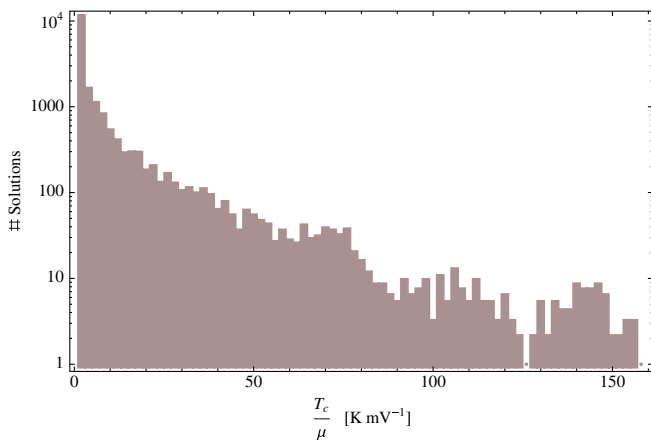

FIG. 3 (color online). A logarithmic distribution of critical temperatures over the chemical potential, in units of degrees Kelvin per milliVolt. The distribution is obtained from a scan over Brieskorn-Pham cones admitting Sasaki-Einstein metrics with moduli, along with allowed $\mathbb{Z}_k$ quotients. The solutions have been binned into ranges of width 2 K/mV.

corresponding to $\Delta = 2$ and $\Delta = 1$ for the operator that condenses, respectively. We noted in footnote 11 that in the $\Delta = 1$ case, this instability occurs at a higher temperature than the Gubser-Mitra instability of the $\mathcal{N} = 8$ theory at a finite chemical potential.

V. DISCUSSION

In this paper we have given the first explicit string theory realizations of the onset of an s wave superconducting phase in strongly coupled field theories at finite chemical potential as considered in [17–19]. The main technical result that made this possible was the identification of charged modes in Sasaki-Einstein compactifications which decoupled from other modes at a linearized level, even in the presence of a background Maxwell field. Our results sit at the intersection of three directions of current string theory research: the string landscape, AdS/CFT duality for M2 brane theories, and applications of AdS/CFT to condensed matter physics. This leads to future research questions with differing flavors.

In order to obtain a more complete picture of the superconducting physics of these $\mathcal{N} \geq 2$ theories there are two important questions we have not addressed. Still at the linearized level, one should perform a complete stability analysis with all of the coupled scalar, vector, and tensor modes. This way one can identify the most unstable mode, obtain the precise critical temperature, and determine whether or not *all* Sasaki-Einstein compactifications become superconducting. If the most unstable mode is a charged vector or tensor, one might obtain p wave (cf. [20–22]) or d wave superconductors, respectively. Beyond the linearized level, one would ultimately like to find the endpoint of the instability well below T_c. These will be hairy black hole solutions of M theory. Given the full solution there will be many properties to investigate, starting with the possible existence of a mass gap.

The recent progress in constructing field theory duals to AdS$_4$ backgrounds opens various interesting future directions. One would like to identify precisely the operators which condense and ultimately gain some dynamical understanding of what is driving the instability. Also, if the field theory admits a weak coupling limit, one can ask whether the superconducting phase continues to weak coupling. In fact, it is rather natural that a weakly coupled theory with massless charged bosonic degrees of freedom become superconducting when placed at a finite chemical potential. This is because the chemical potential acts as a negative mass squared. It would also be interesting, therefore, if there are theories that are superconducting at weak coupling but not strong coupling. In this vein, because the $\mathcal{N} = 2$ theories we have considered have exact moduli spaces of vacua, it is important to understand to what extent the theories have a true zero temperature ground state at finite chemical potential.

In terms of field theory duals, one is not restricted to AdS_4/CFT_3. It seems likely that Sasaki-Einstein compactifications to AdS_5 will have similar instabilities. If so, this will lead to superconducting phases in very well-studied field theories with AdS_5 duals. It was checked in [73] that the basic mechanism of holographic superconductivity generalizes to AdS_5.

Regarding the string landscape, we have considered here only the simplest (Freund-Rubin) flux compactifications of string/M theory. As we noted in Sec. II C, the logic behind the weak gravity bound, if correct, suggests that theories dual to generic AdS_4 flux compactifications should have a superconducting phase when considered at a finite chemical potential. A natural question is to scan the wider string theory landscape in search of superconductors. As in this work, the main technical difficulty will be to identify a sector for which the stability analysis becomes tractable.

In the Introduction we highlighted a parallel between the string landscape and the atomic landscape of condensed matter physics. It would be fascinating if this connection could be made literal by actually engineering a (large N) supersymmetric gauge theory in a lab. Emergent gauge fields are known to occur in certain lattice systems, see e.g. [74]. One conceptually interesting consequence of such a connection would be that a standard model lattice vacuum would provide a nonperturbative definition of string theory (with specific AdS asymptotic boundary conditions), thus inverting the traditional roles of string theory and the standard model.

APPENDIX: PROOFS FOR THE DISTRIBUTION OF CONDUCTIVITIES

Lemma 1.—For the Brieskorn-Pham links,

$$a = \frac{\text{lcm}(m_i \mid i = 1 \cdots 5)}{4} \left(\sum \frac{1}{m_i} - 1 \right), \quad (A1)$$

has an upper bound.

Proof.—For a to be unbounded, clearly at least one of the m_i, call it m_5, must become arbitrarily large.

Suppose that $\sum_{i=1}^{4} 1/m_i < 1$. It can be shown [44] that given that the m_i are positive integers, this requires $\sum_{i=1}^{4} 1/m_i \leq 1805/1806$. The first inequality in (34) now requires that $1/1806 < 1/m_5$, and hence $m_5 < 1806$ is bounded.

Suppose instead that $\sum_{i=1}^{4} 1/m_i = 1 + X$, with $X \geq 0$. The second inequality in (34) implies that $X < 1/(3m_5)$. In a couple of paragraphs later, we show that for $i \neq 5$ we must have $m_i \leq 42$. It follows that if $X > 0$, then X cannot be made arbitrarily small, and hence $m_5 < 1/(3X)$ gives a bound for m_5.

The remaining case to consider is $X = 0$, that is, $\sum_{i=1}^{4} 1/m_i = 1$. Here m_5 is not bounded. However, the formula (A1) for a in this case implies that $4a \leq m_1 m_2 m_3 m_4$. Because $m_i \leq 42$, for $i \neq 5$, then this is bounded.

To complete the proof we need to show that $m_i \leq 42$, for $i \neq 5$, when $\sum_{i=1}^{4} 1/m_i \geq 1$. First, note that $\sum_{i=1}^{3} 1/m_i < 1$ because otherwise the second inequality in (34) implies $1/m_4 + 1/m_5 < 4/(3m_5) < 2/m_5$ which contradicts the fact that $m_5 \geq m_4$. From this inequality it can be shown [44] that $\sum_{i=1}^{3} 1/m_i \leq 41/42$. Combining this fact with $\sum_{i=1}^{4} 1/m_i \geq 1$ implies that $m_4 \leq 42$. Swapping the labels around, this argument gives m_1, m_2, m_3, and $m_4 \leq 42$, as required.

Lemma 2.—There are precisely 19 values of a which occur infinitely many times in the Brieskorn-Pham links. These are

$$a = \frac{n}{4}, \quad (A2)$$

where $n = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 15, 18, 20, 21, 24, 30, 42\}$.

Proof.— We noted in the proof of our previous lemma that the largest exponent m_5 can only become unbounded if $\sum_{i=1}^{4} 1/m_i = 1$. However, there are only 14 different sets of (m_1, m_2, m_3, m_4) for which this is possible. Namely: (2, 3, 7, 42), (2, 3, 8, 24), (2, 3, 9, 18), (2, 3, 10, 15), (2, 3, 12, 12), (2, 4, 5, 20), (2, 4, 6, 12), (2, 4, 8, 8), (2, 5, 5, 10), (2, 6, 6, 6), (3, 3, 4, 12), (3, 3, 6, 6), (3, 4, 4, 6), (4, 4, 4, 4). For sufficiently large integer k, any of these sets together with $m_5 = k$ solves the conditions (34). It is then simple to use the formula (A1) to obtain the 19 values of n that appear in the statement of this theorem.

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