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Interaction between two nonthreshold bound states

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A general nonthreshold Bogomol'nyi-Prasad-Sommerfield (F, D_p) [or (D_{p-2}, D_p)] bound state can be described by a boundary state with a quantized world-volume electric (or magnetic) flux and is characterized by a pair of integers (m, n). With this, we calculate explicitly the interaction amplitude between two such nonthreshold bound states with a separation Y when each of the states is characterized by a pair of integers (m_i, n_i) with $i = 1, 2$. With this result, one can show that the nondegenerate (i.e., $m_i n_i \neq 0$) interaction is, in general, attractive for the case of (D_{p-2}, D_p), but this is true and for certain only at large separation for the case of (F, D_p). In either case, this interaction vanishes only if $m_1/n_1 = m_2/n_2$ and $n_1 n_2 > 0$. We also study the analytic structure of the corresponding amplitude and calculate, in particular, the rate of pair production of open strings in the case of (F, D_p).

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I. INTRODUCTION

It is well known by now that two parallel Dp-branes separated by a distance feel no force between them, independent of their separation, when they are both at rest. This is due to the Bogomol'nyi-Prasad-Sommerfield (BPS) nature or the preservation of a certain number of space-time supersymmetries of this system and goes by the name "no-force" condition. This was shown initially for brane supergravity configurations through a probe [1–3] and later through the string-level computations as an open string one-loop annulus diagram with one end of the open string located at one D-brane and the other end at the other D-brane making use of the "usual abstruse identity" [4]. With this feature, one can easily infer that, when one of the branes in the above is replaced by the corresponding antibrane, there must be a separation-dependent nonvanishing force to arise since such a system is not a BPS one and breaks all space-time supersymmetry. The corresponding forces can easily be computed given our knowledge of computing forces between two identical branes. In general, no separation-dependent force arising is a good indication that the underlying system preserves a certain number of space-time supersymmetries.

In addition to the simple strings and simple D-branes, i.e., extended objects charged under only one Neveu-Schwarz–Neveu-Schwarz (NS-NS) potential or one Ramond-Ramond (R-R) potential, there also exist their supersymmetry-preserving bound states such as (F, D_p) [5–12] and (D_{p-2}, D_p) [13–15], i.e., extended objects charged under more than one potential. It would be interesting to know how to compute the forces between two such bound states separated by a distance. Since each of the bound states involves at least two kinds of branes, the force structure is richer and more interesting to explore. In this paper, we will focus on the above-mentioned two types of the so-called nonthreshold BPS bound states, namely, (F, D_p) and (D_{p-2}, D_p), with even p in IIA and odd p in IIB, respectively.

The nonthreshold BPS bound state (F, D_p), charged under both the NS-NS 2-form potential and the R-R $(p + 1)$-form potential, is formed from the fundamental strings and D_p-branes by lowering the system energy through dissolving the strings in the D_p-branes, turning the strings into an electric flux. A similar picture applies to the nonthreshold BPS (D_{p-2}, D_p) bound state charged under both the R-R $(p - 1)$-form potential and the R-R $(p + 1)$-form potential, where the initial D_{p-2}-branes dissolve in D_p-branes, turning into a magnetic flux. Dirac charge quantization implies that the two potentials for either bound state are characterized by their corresponding quantized charges, and therefore each bound state is characterized by a pair of integers (m, n). When the pair of integers is coprime, the system is stable (otherwise, it is marginally unstable) [16]. In this paper, we will use the description of a boundary state with a quantized world-volume flux given in [12,15,17] for the bound state to calculate explicitly the interaction between two nonthreshold (F, D_p) [or (D_{p-2}, D_p)] bound states separated by a distance.[1] Here each state is characterized by an arbitrary pair of integers (m_i, n_i) with $i = 1, 2$. We find that the

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[1]In this paper, we limit our consideration of the two fluxes, with each in a bound state, being along the same plane. Two of the present authors also considered after this work the rest of the cases including the two fluxes being along different planes and even being different in nature, i.e., one electric and the other magnetic, in [18]. The basic structure of amplitude is more general, and there also appear new instabilities. In particular, the pair-production rate of open strings can be greatly enhanced in certain cases.

nondegenerate (i.e., $m_i n_i \neq 0$) force is, in general, attractive for the case of (D_{p-2}, D_p), but this is only certain at large separation for the case of (F, D_p). This interaction in either case vanishes only if $m_1/n_1 = m_2/n_2$ and $n_1 n_2 > 0$. The expected vanishing interaction for the special case of two identical (F, D_p) bound states was previously shown in [12].

This paper is organized as follows. In Sec. II, using the known couplings obtained from bulk and the world-volume effective field theories for the respective massless modes [12], we calculate the long-range interaction between two (F, D_p) [or (D_{p-2}, D_p)] bound states separated by a distance Y with each state characterized by an arbitrary pair of integers (m_i, n_i) $(i = 1, 2)$ and study the underlying properties. In Sec.n III, we calculate the interaction at the string level between two arbitrary (F, D_p) [or (D_{p-2}, D_p)] bound states placed parallel to each other with a separation Y using the closed string boundary state approach. We summarize the results in Sec. IV.

II. THE LONG-RANGE INTERACTIONS

In this section, we will calculate the lowest-order contribution to the interaction between two arbitrary (F, D_p) [or (D_{p-2}, D_p)] bound states placed parallel to each other at a given separation Y due to the exchanges of massless modes, therefore representing the force at large separation. We will employ the couplings of the bound state to the bulk massless modes in type II theories as given in [12] to fulfill this purpose.[2] As mentioned in the introduction, the lower dimensional brane in the bound state can be represented by the corresponding flux on the D_p-brane world volume. For the present case, the F strings in (F, D_p) can be represented by an electric flux along a given direction on the p-brane world volune, while the D_{p-2}-branes in (D_{p-2}, D_p) can be represented by a magnetic flux similarly.

Let us begin with the nonthreshold (F, D_p) states. We choose the constant electric flux $\hat{F}$ the following way:

$$\hat{F} = \begin{pmatrix} 0 & -f & & & & \\ f & 0 & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & 0 \end{pmatrix}_{(p+1)\times(p+1)}. \tag{1}$$

The couplings given in [12] are for a single D_p-brane in the bound state, and for multiple D_p-branes, we should replace the c_p by nc_p in the couplings with n an integer. The constant flux is also quantized and is given for an electric flux as [12]

$$-\frac{nf}{\sqrt{1-f^2}} = mg_s \tag{2}$$

with m an integer. This gives $f = -m/\Delta^{1/2}_{(m,n)}$, where we have defined

$$\Delta_{(m,n)} = m^2 + \frac{n^2}{g_s^2}. \tag{3}$$

Then we have

$$-\det(\eta + \hat{F}) = 1 - f^2 = \frac{n^2/g_s^2}{\Delta_{(m,n)}} \tag{4}$$

and

$$V \equiv (\eta + \hat{F})^{-1} = \begin{pmatrix} -\frac{1}{1-f^2} & -\frac{f}{1-f^2} & & & & \\ \frac{f}{1-f^2} & \frac{1}{1-f^2} & & & & \\ & & 1 & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & \cdot \\ & & & & & & 1 \end{pmatrix}_{(p+1)\times(p+1)} = \begin{pmatrix} -\frac{g_s^2\Delta_{(m,n)}}{n^2} & \frac{mg_s^2\Delta^{1/2}_{(m,n)}}{n^2} & & & & \\ -\frac{mg_s^2\Delta^{1/2}_{(m,n)}}{n^2} & \frac{g_s^2\Delta_{(m,n)}}{n^2} & & & & \\ & & 1 & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & \cdot \\ & & & & & & 1 \end{pmatrix}_{(p+1)\times(p+1)}. \tag{5}$$

With the above, we have now the explicit couplings as

$$J^i_h = -c_p V_{p+1} \frac{n_i^2}{g_s \Delta^{1/2}_{(m_i,n_i)}} V_i^{\alpha\beta} h_{\beta\alpha},$$
$$J^i_\phi = \frac{c_p}{2\sqrt{2}} V_{p+1} \frac{(3-p)n_i^2 - 2m_i^2 g_s^2}{g_s \Delta^{1/2}_{(m_i,n_i)}} \phi, \tag{6}$$
$$J^i_B = -\frac{c_p}{\sqrt{2}} V_{p+1} \frac{n_i^2}{g_s \Delta^{1/2}_{(m_i,n_i)}} V_i^{\alpha\beta} B_{\beta\alpha}$$

for the NS-NS fields and

$$J^i_{C_{p+1}} = \sqrt{2} c_p V_{p+1} n_i C_{01\ldots p},$$
$$J^i_{C_{p-1}} = c_p V_{p+1} \frac{\sqrt{2} n_i m_i}{\Delta^{1/2}_{(m_i,n_i)}} C_{23\ldots p} \tag{7}$$

for the R-R fields. Here i denotes the respective bound state with $i = 1, 2$.

We now calculate the long-range interaction (in momentum space) between two parallel (F, D_p) bound states

[2] We here replace the T_p in [12] by c_p to avoid its possible confusion with the usual brane tension.

separated by a transverse distance Y with each state characterized by a pair of integers (m_i, n_i), respectively. The gravitational contribution due to the exchange of graviton is

$$U_h = \frac{1}{V_{p+1}} \underbrace{J_h^{(1)} J_h^{(2)}} = c_p^2 V_{p+1} \frac{n_1^2 n_2^2}{g_s^2 \Delta^{1/2}_{(m_1,n_1)} \Delta^{1/2}_{(m_2,n_2)}} V_1^{\alpha\beta} V_2^{\gamma\delta} \underbrace{h_{\beta\alpha} h_{\delta\gamma}}, \tag{8}$$

where the propagator reads

$$\underbrace{h_{\beta\alpha} h_{\delta\gamma}} = \left[\frac{1}{2}(\eta_{\beta\delta}\eta_{\alpha\gamma} + \eta_{\alpha\delta}\eta_{\beta\gamma}) - \frac{1}{8}\eta_{\alpha\beta}\eta_{\gamma\delta}\right]\frac{1}{k_\perp^2} \tag{9}$$

for the canonically normalized graviton propagating in the transverse directions in the de Donder (harmonic) gauge. The explicit expression for the interaction can be obtained using the matrix V in the third line of (5) as

$$U_h = \frac{c_p^2}{8g_s^2} \frac{V_{p+1}}{k_\perp^2} \frac{12 g_s^4 m_1^2 m_2^2 + 2(7-p) g_s^2 (m_1^2 n_2^2 + m_2^2 n_1^2) + (7-p)(p+1) n_1^2 n_2^2}{\Omega} \tag{10}$$

with

$$\Omega \equiv \Delta^{1/2}_{(m_1,n_1)} \Delta^{1/2}_{(m_2,n_2)} = \sqrt{\left(m_1^2 + \frac{n_1^2}{g_s^2}\right)\left(m_2^2 + \frac{n_2^2}{g_s^2}\right)}. \tag{11}$$

The contribution to the interaction due to the exchange of dilaton can be calculated as

$$U_\phi = \frac{1}{V_{p+1}} \underbrace{J_\phi^1 J_\phi^2} = \frac{c_p^2}{8g_s^2} V_{p+1} \frac{4 g_s^4 m_1^2 m_2^2 - 2(3-p) g_s^2 (m_1^2 n_2^2 + n_1^2 m_2^2) + (3-p)^2 n_1^2 n_2^2}{\Omega} \underbrace{\phi\phi}, \tag{12}$$

where Ω is given in (11) and the dilaton propagator is

$$\underbrace{\phi\phi} = \frac{1}{k_\perp^2}. \tag{13}$$

So we have

$$U_\phi = \frac{c_p^2}{8g_s^2} \frac{V_{p+1}}{k_\perp^2} \frac{4 m_1^2 m_2^2 - 2(3-p) g_s^2 (m_1^2 n_2^2 + n_1^2 m_2^2) + (3-p)^2 n_1^2 n_2^2}{\Omega}. \tag{14}$$

The contribution due to the exchange of Kalb-Ramond field can be calculated similarly as

$$U_B = \frac{1}{V_{p+1}} \underbrace{J_B^1 J_B^2} = \frac{c_p^2}{2g_s^2} V_{p+1} \frac{n_1^2 n_2^2}{\Omega} V_1^{\alpha\beta} V_2^{\gamma\delta} \underbrace{B_{\beta\alpha} B_{\delta\gamma}}. \tag{15}$$

Using the propagator for the Kalb-Ramond field

$$\underbrace{B_{\beta\alpha} B_{\delta\gamma}} = (\eta_{\beta\delta}\eta_{\alpha\gamma} - \eta_{\alpha\delta}\eta_{\beta\gamma})\frac{1}{k_\perp^2} \tag{16}$$

and the explicit expression for the matrices V_i, we have

$$U_B = \frac{c_p^2}{8g_s^2} \frac{V_{p+1}}{k_\perp^2}(-16 m_1 m_2 g_s^4). \tag{17}$$

We now turn to the calculations of the contributions from R-R fields. The contribution from the exchange of R-R potential $C_{01\ldots p}$ is

$$U_{C_{p+1}} \equiv \frac{1}{V_{p+1}} \underbrace{J^1_{C_{p+1}} J^2_{C_{p+1}}} = 2 c_p^2 V_{p+1} n_1 n_2 \underbrace{C_{01\ldots p} C_{01\ldots p}}. \tag{18}$$

Using the propagator for the rank-$(p+1)$ R-R potential

$$\underbrace{C_{01\ldots p} C_{01\ldots p}} = -\frac{1}{k_\perp^2}, \tag{19}$$

we have

$$U_{C_{p+1}} = \frac{c_p^2}{8g_s^2} \frac{V_{p+1}}{k_\perp^2}(-16 n_1 n_2 g_s^2). \tag{20}$$

Similarly, we have

$$U_{C_{p-1}} \equiv \frac{1}{V_{p+1}} \underbrace{J^1_{C_{p-1}} J^2_{C_{p-1}}} = 2c_p^2 V_{p+1} \frac{m_1 m_2 n_1 n_2}{\Omega} \underbrace{C_{23\ldots p} C_{23\ldots p}} = \frac{c_p^2}{8g_s^2} \frac{V_{p+1}}{k_\perp^2} \frac{16 m_1 m_2 n_1 n_2 g_s^2}{\Omega}, \quad (21)$$

where we have used the propagator for the rank-$(p-1)$ R-R potential

$$\underbrace{C_{23\ldots p} C_{23\ldots p}} = \frac{1}{k_\perp^2}. \quad (22)$$

Note that, apart from the overall factor $c_p^2 \frac{V_{p+1}}{k_\perp^2}$, the form field contributions are independent of the dimensionality of the bound state, while this is not case for either the graviton or the dilaton contribution.

We would like to point out that each of the components calculated above agrees completely with what has been given in [12] when we set $(m_1, n_1) = (m_2, n_2)$ and $g_s = 1$, i.e., when the two bound states are identical with string coupling set to one. We here generalize the calculations there for two arbitrary bound states which are characterized by their respective pair of integers (m_i, n_i) with $i = 1, 2$. The total contribution to the interaction from the NS-NS sector is

$$U_{\text{NS-NS}} = U_h + U_\phi + U_B = c_p^2 \frac{V_{p+1}}{k_\perp^2} \frac{2g_s^4 m_1^2 m_2^2 + g_s^2(m_1^2 n_2^2 + m_2^2 n_1^2) + 2n_1^2 n_2^2 - 2m_1 m_2 g_s^4 \Omega}{g_s^2 \Omega} = c_p^2 \frac{V_{p+1}}{k_\perp^2} U_{\text{NS}}(m_1, n_1; m_2, n_2), \quad (23)$$

where in the last line we have made use of the explicit expression for Ω given in (11) and

$$U_{\text{NS}}(m_1, n_1; m_2, n_2) = \frac{g_s^4 m_1^2 m_2^2 + n_1^2 n_2^2 + g_s^4 \Omega^2 - 2m_1 m_2 g_s^4 \Omega}{g_s^2 \Omega}, \quad (24)$$

while the total R-R contribution is

$$U_{\text{R-R}} = U_{C_{p+1}} + U_{C_{p-1}} = -c_p^2 \frac{V_{p+1}}{k_\perp^2} U_{\text{R}}(m_1, n_1; m_2, n_2), \quad (25)$$

where

$$U_{\text{R}}(m_1, n_1; m_2, n_2) = \frac{2n_1 n_2 (\Omega - m_1 m_2) g_s^2}{g_s^2 \Omega}. \quad (26)$$

Note that, although either the graviton or the dilaton contribution apart from the factor $c_p^2 \frac{V_{p+1}}{k_\perp^2}$ depends on the dimensionality of the brane, their addition is not. This has to be so since the form field contributions are independent of the dimensionality and the no-force condition holds once we set the two bound states identical. The total contribution from both sectors is

$$U = U_{\text{NS-NS}} + U_{\text{R-R}} = c_p^2 \frac{V_{p+1}}{k_\perp^2} \frac{[(g_s^2 m_1 m_2 + n_1 n_2) - g_s^2 \Omega]^2}{g_s^2 \Omega} \geq 0. \quad (27)$$

This clearly shows that the interaction is, in general, attractive[3] and vanishes only if

$$g_s^2 m_1 m_2 + n_1 n_2 = g_s^2 \Omega > 0. \quad (28)$$

For the nondegenerate case, i.e., $m_i n_i \neq 0$ with $i = 1, 2$, the above implies $m_1/n_1 = m_2/n_2$ and $n_1 n_2 > 0$. In showing this, we also have made use of the explicit expression for Ω given in (11). The vanishing result for the special case of $(m_1, n_1) = (m_2, n_2)$ was previously shown in [12], and we here generalize it.

We now turn to the case for the nonthreshold (D_{p-2}, D_p) bound state. The calculations are similar, and we list below only the necessary steps and the main results. The constant magnetic flux $\hat{F}$ on the world volume is chosen as

$$\hat{F} = \begin{pmatrix} 0 & & & & \\ & \cdot & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & 0 & -f \\ & & & f & 0 \end{pmatrix}_{(p+1)\times(p+1)}. \quad (29)$$

Here again we need to replace the c_p for a single D_p-brane in the bound state by nc_p for multiple branes with n an integer (also due to charge quantization) in the couplings. The constant magnetic flux is also quantized and in the present case is given by $-nf = m$, which gives $f = -m/n$. So we have now

$$-\det(\eta + \hat{F}) = 1 + f^2 = \frac{n^2 + m^2}{n^2} \quad (30)$$

and

[3]We choose conventions here that $U > 0$ means attractive which differs from the standard one by a sign.

$$V \equiv (\eta + \hat{F})^{-1} = \begin{pmatrix} -1 & & & & & \\ & 1 & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & \frac{1}{1+f^2} & \frac{f}{1+f^2} \\ & & & & & -\frac{f}{1+f^2} & \frac{1}{1+f^2} \end{pmatrix}_{(p+1)\times(p+1)} = \begin{pmatrix} -1 & & & & & \\ & 1 & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & \frac{n^2}{m^2+n^2} & -\frac{nm}{m^2+n^2} \\ & & & & & \frac{nm}{n^2+m^2} & \frac{n^2}{n^2+m^2} \end{pmatrix}_{(p+1)\times(p+1)}. \tag{31}$$

We then have the explicit couplings for the respective bound state denoted by index i with $i = 1, 2$ for the present case as

$$\begin{aligned} J_h^i &= -c_p V_{p+1}\sqrt{m_i^2 + n_i^2} V_i^{\alpha\beta} h_{\beta\alpha}, \\ J_\phi^i &= \frac{c_p}{2\sqrt{2}} V_{p+1} \frac{(3-p)(n_i^2 + m_i^2) + 2m_i^2}{\sqrt{m_i^2 + n_i^2}} \phi, \\ J_B^i &= -\frac{c_p}{\sqrt{2}} V_{p+1}\sqrt{m_i^2 + n_i^2} V_i^{\alpha\beta} B_{\beta\alpha} \end{aligned} \tag{32}$$

for the NS-NS couplings and

$$\begin{aligned} J_{C_{p+1}}^i &= \sqrt{2} c_p V_{p+1} n_i C_{01\ldots p}, \\ J_{C_{p-1}}^i &= \sqrt{2} c_p V_{p+1} m_i C_{01\ldots p-2} \end{aligned} \tag{33}$$

for the R-R couplings. The long-range interaction due to the exchange of each respective massless field is

$$\begin{aligned} U_\phi &= \frac{c_p^2}{8} \frac{V_{p+1}}{k_\perp^2} \frac{(5-p)^2 m_1^2 m_2^2 + (5-p)(3-p)(m_1^2 n_2^2 + n_1^2 m_2^2) + (3-p)^2 n_1^2 n_2^2}{\tilde{\Omega}}, \\ U_h &= \frac{c_p^2}{8} \frac{V_{p+1}}{k_\perp^2} \frac{(9-p)(p-1) m_1^2 m_2^2 + (7-p)(p-1)(m_1^2 n_2^2 + n_1^2 m_2^2) + (7-p)(p+1) n_1^2 n_2^2}{\tilde{\Omega}}, \\ U_B &= \frac{c_p^2}{8} \frac{V_{p+1}}{k_\perp^2} \frac{16 m_1 m_2 n_1 n_2}{\tilde{\Omega}} \end{aligned} \tag{34}$$

for the NS-NS fields and

$$\begin{aligned} U_{C_{p+1}} &= c_p^2 \frac{V_{p+1}}{k_\perp^2}(-2n_1 n_2), \\ U_{C_{p-1}} &= c_p^2 \frac{V_{p+1}}{k_\perp^2}(-2m_1 m_2) \end{aligned} \tag{35}$$

for the R-R fields. In the above, we have defined

$$\tilde{\Omega} = \sqrt{(m_1^2 + n_1^2)(m_2^2 + n_2^2)}. \tag{36}$$

We again have that the interaction contribution due to the exchange of the dilaton or the graviton in the NS-NS sector apart from the factor $c_p^2 \frac{V_{p+1}}{k_\perp^2}$ still depends on the dimensionality of the world volume, while this is not the case for any form field in either the NS-NS sector or the R-R sector. The total contribution to the interaction from the NS-NS sector is

$$\begin{aligned} U_{\text{NS-NS}} &= U_\phi + U_h + U_B \\ &= c_p^2 \frac{V_{p+1}}{k_\perp^2} U_{\text{NS}}(m_1, n_1; m_2, n_2), \end{aligned} \tag{37}$$

where

$$U_{\text{NS}}(m_1, n_1; m_2, n_2) = \frac{2m_1^2 m_2^2 + (m_1^2 n_2^2 + n_1^2 m_2^2) + 2n_1^2 n_2^2 + 2m_1 m_2 n_1 n_2}{\tilde{\Omega}}, \tag{38}$$

independent of the dimensionality of the world volume. The total interaction from the R-R sector is

$$U_{\text{R-R}} = U_{C_{p+1}} + U_{C_{p-1}} = -c_p^2 \frac{V_{p+1}}{k_\perp^2} U_{\text{R}}(m_1, n_1; m_2, n_2), \tag{39}$$

where

$$U_{\text{R}}(m_1, n_1; m_2, n_2) = 2(n_1 n_2 + m_1 m_2). \tag{40}$$

The total interaction from both sectors is now

$$U = U_{\text{NS-NS}} + U_{\text{R-R}} = c_p^2 \frac{V_{p+1}}{k_\perp^2} \frac{(m_1 m_2 + n_1 n_2 - \tilde{\Omega})^2}{\tilde{\Omega}} \geq 0, \tag{41}$$

where in the second equality we have used the explicit expression for $\tilde{\Omega}$ given in (36). This also clearly shows that the interaction is, in general, attractive and vanishes only if

$$m_1 m_2 + n_1 n_2 = \tilde{\Omega} \tag{42}$$

which again implies $m_1/n_1 = m_2/n_2$ and $n_1 n_2 > 0$ for the nondegenerate case, i.e., $m_i n_i \neq 0$ with $i = 1, 2$, the expected supersymmetry-preserving result.

We can use Fourier transformation to obtain the corresponding interaction in coordinate space when $p < 7$ as

$$U(Y) = \int \frac{d^\perp k_\perp}{(2\pi)^\perp} e^{-ik_\perp \cdot Y} U(k_\perp) = \frac{C(m_1, n_1; m_2, n_2)}{Y^{7-p}}, \tag{43}$$

where

$$C(m_1, n_1; m_2, n_2) = \frac{c_p^2 V_{p+1} U(m_1, n_1; m_2, n_2)}{(7-p)\Omega_{8-p}} \tag{44}$$

with

$$U(m_1, n_1; m_2, n_2) = \begin{cases} \frac{[(g_s^2 m_1 m_2 + n_1 n_2) - g_s^2 \Omega]^2}{g_s^2 \Omega} & \text{for the case of } (F, D_p), \\ \frac{[(m_1 m_2 + n_1 n_2) - \tilde{\Omega}]^2}{\tilde{\Omega}} & \text{for the case of } (D_{p-2}, D_p), \end{cases} \tag{45}$$

and $Y^2 = Y_i Y^i$ with the summation index i along the transverse directions. In the above, we have used the following relation:

$$\int \frac{d^\perp k_\perp}{(2\pi)^\perp} \frac{e^{-ik_\perp \cdot Y}}{k_\perp^2} = \frac{1}{(7-p) Y^{7-p} \Omega_{8-p}}, \tag{46}$$

where $\Omega_q = 2\pi^{(q+1)/2}/\Gamma((q+1)/2)$ is the volume of the unit q sphere.

III. THE STRING-LEVEL FORCE CALCULATIONS

We want to go one step further to calculate the forces between two (F, D_p) or (D_{p-2}, D_p) bound states at a separation Y at the string level as the corresponding interaction vacuum amplitude.[4] In addition, we will use the results to discuss certain properties of the underlying systems such as the analytic structure of the amplitude and to calculate the rate of pair production of open strings in the open string channel.

The interaction vacuum amplitude can be calculated via

$$\Gamma = \langle B(m_1, n_1) | D | B(m_2, n_2) \rangle, \tag{47}$$

where the bound state with a constant world-volume field in each sector has been given explicitly in [12] and is characterized by a pair of integers (m_i, n_i) with $i = 1, 2$ and D is the closed string propagator defined as

$$D = \frac{\alpha'}{4\pi} \int_{|z|^2 \le 1} \frac{d^2 z}{|z|^2} z^{L_0} \bar{z}^{\tilde{L}_0}. \tag{48}$$

Here L_0 and $\tilde{L}_0$ are the respective left and right mover total zero-mode Virasoro generators of matter fields, ghosts, and superghosts. For example, $L_0 = L_0^X + L_0^\psi + L_0^{gh} + L_0^{sgh}$, where L_0^X, L_0^ψ, L_0^{gh}, and L_0^{sgh} represent contributions from matter fields X^μ, matter fields ψ^μ, ghosts b and c, and superghosts β and γ, respectively, and their explicit expressions can be found in any standard discussion of superstring theories, for example, in [19] and therefore will not be presented here even though we will need them in our following calculations. The above total vacuum amplitude has contributions from both NS-NS and R-R sectors, respectively, and can be written as $\Gamma = \Gamma_{\rm NS} + \Gamma_{\rm R}$. In calculating either $\Gamma_{\rm NS}$ or $\Gamma_{\rm R}$, we need to keep in mind that the boundary state used should be the Gliozzi-Scherk-Olive (GSO) projected one and is related to the usual two boundary states $|B, \eta\rangle$ with $\eta = \pm$ in each sector, respectively, as

$$|B\rangle_{\rm NS} = \tfrac{1}{2}[|B, +\rangle_{\rm NS} - |B, -\rangle_{\rm NS}] \tag{49}$$

in the NS-NS sector and

$$|B\rangle_{\rm R} = \tfrac{1}{2}[|B, +\rangle_{\rm R} + |B, -\rangle_{\rm R}] \tag{50}$$

in the R-R sector. For this purpose, we need to calculate first the following amplitude:

$$\Gamma(\eta', \eta) = \langle B^1, \eta' | D | B^2, \eta \rangle \tag{51}$$

in each sector with $\eta' \eta = \pm$ and $B^i = B(m_i, n_i)$. In doing the calculations, we can set $\tilde{L}_0 = L_0$ in the above propagator due to the fact that $\tilde{L}_0 |B\rangle = L_0 |B\rangle$, which can be used to simplify the calculations. Given the structure of the boundary state in [12], the amplitude $\Gamma(\eta', \eta)$ can be factorized as

$$\Gamma(\eta', \eta) = \frac{n_1 n_2 c_p^2}{4} \frac{\alpha'}{4\pi} \times \int_{|z| \le 1} \frac{d^2 z}{|z|^2} A^X A^{bc} A^\psi(\eta', \eta) A^{\beta\gamma}(\eta', \eta), \tag{52}$$

where we have also replaced the c_p in the boundary state by $n c_p$ with n an integer to count the multiplicity of the D_p-branes in the bound state. In the above,

[4] Actually, it is the vacuum free energy.

$$\begin{aligned} A^X &= \langle B_X^1 | |z|^{2L_0^X} | B_X^2 \rangle, \\ A^\psi(\eta', \eta) &= \langle B_\psi^1, \eta' | |z|^{2L_0^\psi} | B_\psi^2, \eta \rangle, \\ A^{bc} &= \langle B_{gh}^1 | |z|^{2L_0^{gh}} | B_{gh}^2 \rangle, \\ A^{\beta\gamma}(\eta', \eta) &= \langle B_{sgh}^1, \eta' | |z|^{2L_0^{sgh}} | B_{sgh}^2, \eta \rangle. \end{aligned} \tag{53}$$

Note that the various components of the boundary state in the ghost part are independent of flux and the basic structure of various components of the boundary state in the matter part remains the same whether there is a flux present or not. All of the information about the flux in the matter part is encoded in the zero modes and the following S matrix:

$$S = ([(\eta - \hat{F})(\eta + \hat{F})^{-1}]_{\alpha\beta}, -\delta_{ij}), \tag{54}$$

where the Greek indices $\alpha, \beta, \ldots$ label the world-volume directions $0, 1, \ldots, p$ along which the D_p-brane extends, while the Latin indices $i, j, \ldots$ label the directions transverse to the brane, i.e., $p+1, \ldots, 9$. We also define $\hat{F} = 2\pi\alpha' F$ with F the flux. In order to perform the calculations, using the explicit expressions of various components of the boundary state in the matter part (in addition to the ghost part) in [12], we need to specify the world-volume gauge field and the above S matrix for both (F, D_p) and (D_{p-2}, D_p) bound states, respectively. For the case of (F, D_p), we need to use (1) for $\hat{F}$ with f determined by (2), i.e., $f = -m/\triangle_{(m,n)}^{1/2}$ through the charge quantization. The corresponding longitudinal part of the S matrix as given in (54) is now

$$S_{\alpha\beta} = \begin{pmatrix} -\frac{1+f^2}{1-f^2} & \frac{2f}{1-f^2} & & & & \\ -\frac{2f}{1-f^2} & \frac{1+f^2}{1-f^2} & & & & \\ & & 1 & & & \\ & & & \ddots & & \\ & & & & & 1 \end{pmatrix}_{(p+1)\times(p+1)} = \begin{pmatrix} -\frac{g_s^2(\Delta_{(m,n)}+m^2)}{n^2} & -\frac{2mg_s^2\Delta_{(m,n)}^{1/2}}{n^2} & & & & \\ \frac{2mg_s^2\Delta_{(m,n)}^{1/2}}{n^2} & \frac{g_s^2(\Delta_{(m,n)}+m^2)}{n^2} & & & & \\ & & 1 & & & \\ & & & \ddots & & \\ & & & & & 1 \end{pmatrix}_{(p+1)\times(p+1)}, \tag{55}$$

while for (D_{p-2}, D_p), we need to use (29) for $\hat{F}$ with the quantized $f = -m/n$. Now we have the longitudinal part of S matrix as

$$S_{\alpha\beta} = \begin{pmatrix} -1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & \frac{1-f^2}{1+f^2} & \frac{2f}{1+f^2} \\ & & & -\frac{2f}{1+f^2} & \frac{1-f^2}{1+f^2} \end{pmatrix}_{(p+1)\times(p+1)} = \begin{pmatrix} -1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & \frac{n^2-m^2}{m^2+n^2} & -\frac{2nm}{m^2+n^2} \\ & & & \frac{2nm}{n^2+m^2} & \frac{n^2-m^2}{n^2+m^2} \end{pmatrix}_{(p+1)\times(p+1)}. \tag{56}$$

With the above preparations, we are now ready to perform rather straightforward calculations for the various matrix elements specified in (53) in either the NS-NS or R-R sector for either of the bound states under consideration, using the explicit expressions given in [12] for the boundary states with $\hat{F}$ and the matrix S given in (54) as just described for either of the bound states. We have now

$$\begin{aligned} A^X &= C_F V_{p+1} e^{-Y^2/2\pi\alpha' t}(2\pi^2\alpha' t)^{-(9-p)/2} \\ &\quad \times \prod_{n=1}^{\infty} \frac{1}{(1-\lambda|z|^{2n})(1-\lambda^{-1}|z|^{2n})(1-|z|^{2n})^8}, \\ A^{bc} &= |z|^{-2} \prod_{n=1}^{\infty}(1-|z|^{2n})^2 \end{aligned} \tag{57}$$

for both the NS-NS and R-R sectors,

$$\begin{aligned} A_{\rm NS}^{\beta\gamma}(\eta', \eta) &= |z| \prod_{n=1}^{\infty} \frac{1}{(1+\eta'\eta|z|^{2n-1})^2}, \\ A_{\rm NS}^{\psi} &= \prod_{n=1}^{\infty}(1+\eta'\eta\lambda|z|^{2n-1})(1+\eta'\eta\lambda^{-1}|z|^{2n-1}) \\ &\quad \times (1+\eta'\eta|z|^{2n-1})^8 \end{aligned} \tag{58}$$

for the NS-NS sector, and

$$\begin{aligned} A_{\rm R}^{\beta\gamma}(\eta', \eta) A_{\rm R}^{\psi}(\eta', \eta) &= -2^4|z|^2 D_F \delta_{\eta'\eta,+} \prod_{n=1}^{\infty}(1+\lambda|z|^{2n}) \\ &\quad \times (1+\lambda^{-1}|z|^{2n})(1+|z|^{2n})^6 \end{aligned} \tag{59}$$

for the R-R sector. Note that we have $|z| = e^{-\pi t}$ above, and in (59) we have followed the prescription given in [19,20] not to separate the contributions from matter fields ψ^μ and superghosts in the R-R sector in order to avoid the complication due to the respective zero modes. Also in the above, we have

$$C_F = \begin{cases} \sqrt{(1-f_1^2)(1-f_2^2)} & \text{for } (F, D_p), \\ \sqrt{(1+f_1^2)(1+f_2^2)} & \text{for } (D_{p-2}, D_p), \end{cases} \tag{60}$$

$$D_F = \begin{cases} \frac{1-f_1 f_2}{\sqrt{(1-f_1^2)(1-f_2^2)}} & \text{for } (F, D_p), \\ \frac{1+f_1 f_2}{\sqrt{(1+f_1^2)(1+f_2^2)}} & \text{for } (D_{p-2}, D_p), \end{cases} \tag{61}$$

and

$$\lambda + \lambda^{-1} = 2(2D_F^2 - 1) = \begin{cases} 2\frac{(1+f_1^2)(1+f_2^2)-4f_1 f_2}{(1-f_1^2)(1-f_2^2)} & \text{for } (F, D_p), \\ 2\frac{(1-f_1^2)(1-f_2^2)+4f_1 f_2}{(1+f_1^2)(1+f_2^2)} & \text{for } (D_{p-2}, D_p), \end{cases} \tag{62}$$

with the previously given

$$f_i = \begin{cases} -\frac{m_i}{\triangle^{1/2}_{(m_i,n_i)}} & \text{for } (F, D_p), \\ -\frac{m_i}{n_i} & \text{for } (D_{p-2}, D_p), \end{cases} \tag{63}$$

where $i = 1, 2$ and the explicit expression for $\Delta_{(m_i,n_i)}$ is given in (3).

In calculating A^X and $A^\psi(\eta', \eta)$ as given explicitly above, we have made use of an important property for the S matrix:

$$S^T{}_\mu{}^\rho S_\rho{}^\nu = \delta_\mu{}^\nu, \tag{64}$$

with T denoting the transpose. We can check this using, for example, the explicit expression (54) for $S_{\mu\nu}$ with the indices raised or lowered using the corresponding metric. This property enables us to perform unitary transformations of the respective operators in the boundary states such that the S matrix appearing in one of the boundary states, for example, in the boundary state originally denoted as "1" above, completely disappears, while leaving the other one (originally denoted as "2") with a new S matrix as $S = S_2 S_1^T$, in the course of evaluating the respective A^X or A^ψ. This new S matrix shares the same property (64) as the original S_1 and S_2 do, but its determinant is always equal to one. Therefore this S matrix under consideration can always be diagonalized to give two eigenvalues λ and λ^{-1} with their sum as given in (62) above and the other eight eigenvalues all equal to one. This is the basis for the structure appearing in the contributions due to the respective oscillators to the A^X and $A^\psi(\eta', \eta)$ as given in (57)–(59) above.

We can now have the vacuum amplitude in the NS-NS sector as

$$\begin{aligned}\Gamma_{\rm NS} &= {}_{\rm NS}\langle B^1|D|B^2\rangle_{\rm NS} \\ &= \frac{n_1 n_2 c_p^2 V_{p+1} C_F}{32\pi(2\pi^2\alpha')^{(7-p)/2}} \int_0^\infty \frac{dt}{t} e^{-Y^2/2\pi\alpha' t} t^{-(7-p)/2} |z|^{-1} \Bigg[\prod_{n=1}^\infty \frac{(1+\lambda|z|^{2n-1})(1+\lambda^{-1}|z|^{2n-1})(1+|z|^{2n-1})^6}{(1-\lambda|z|^{2n})(1-\lambda^{-1}|z|^{2n})(1-|z|^{2n})^6} \\ &\quad - \prod_{n=1}^\infty \frac{(1-\lambda|z|^{2n-1})(1-\lambda^{-1}|z|^{2n-1})(1-|z|^{2n-1})^6}{(1-\lambda|z|^{2n})(1-\lambda^{-1}|z|^{2n})(1-|z|^{2n})^6}\Bigg],\end{aligned} \tag{65}$$

where we have used the GSO projected boundary state in (49) for $|B^i\rangle_{\rm NS}$ ($i = 1, 2$) with B^i as defined previously and have made use of the matrix elements in (57) and (58). Also we have used in the above

$$\int_{|z|\le 1} \frac{d^2 z}{|z|^2} = 2\pi^2 \int_0^\infty dt, \tag{66}$$

with $|z| = e^{-\pi t}$. The corresponding vacuum amplitude in the R-R sector is now

$$\Gamma_R = {}_{\rm R}\langle B^1|D|B^2\rangle_{\rm R} = -\frac{n_1 n_2 c_p^2 V_{p+1} C_F D_F}{2\pi(2\pi^2\alpha')^{(7-p)/2}} \int_0^\infty \frac{dt}{t} e^{-Y^2/2\pi\alpha' t} t^{-(7-p)/2} \prod_{n=1}^\infty \frac{(1+\lambda|z|^{2n})(1+\lambda^{-1}|z|^{2n})(1+|z|^{2n})^6}{(1-\lambda|z|^{2n})(1-\lambda^{-1}|z|^{2n})(1-|z|^{2n})^6}, \tag{67}$$

where we have used the GSO projected boundary state in (50) for $|B^i\rangle_{\rm R}$ ($i = 1, 2$) again with B^i as defined previously and made use of the matrix elements in (57) and (59) as well as Eq. (66). In the above, we always assume both n_1 and n_2 are positive integers and the p-branes in the nonthreshold bound states are both D_p-branes (or both anti-D_p-branes). In the case when the p-branes in either of the nonthreshold bound states (but not both) are anti-D_p-branes, the corresponding $\Gamma_{\rm R}$ will switch sign from the one above but the $\Gamma_{\rm NS}$ will remain the same. In what follows, we will focus on the fact that the p-branes in both nonthreshold bound states are D_p-branes; i.e., (65) and (67) are valid. The case when the p-branes in either of the bound states are anti-D_p-branes can be similarly analyzed.

We would like to pause here to make a few checks of the above results (65) and (67) against known ones. When we set $n_1 = n_2 = 1$ and switch off the world-volume gauge fields, i.e., setting $f_1 = f_2 = 0$ (therefore $C_F = D_F = 1$ and $\lambda = \lambda^{-1} = 1$), our above $\Gamma_{\rm NS}$ and $\Gamma_{\rm R}$ agree with the well-known results between two identical Dp-branes placed parallel to each other and separated by a distance Y. For example, our results completely agree with the calculations given in Eqs. (9.285) and (9.289) in [19] when we set $p = p'$, i.e., $\nu = 0$, in their case if we notice that

$$\frac{c_p^2}{32\pi(2\pi^2\alpha')^{(7-p)/2}} = \frac{1}{(8\pi^2\alpha')^{(p+1)/2}} \times \frac{1}{2}, \tag{68}$$

where we have used $c_p = \sqrt{\pi}(2\pi\sqrt{\alpha'})^{3-p}$. For the case of the (F, D_p) bound state, when two such bound states are identical, i.e., $f_1 = f_2 = -m/\Delta^{1/2}_{(m,n)}$, the results for $\Gamma_{\rm NS}$ and $\Gamma_{\rm R}$ with the string coupling set to unity were given in [12] as mentioned earlier. Applying the same conditions to our calculations for the (F, D_p) case, we again find perfect agreements if we make use of (68) and notice the following: (i) $S_1 = S_2$, therefore the matrix $S = S_1 S_2^T$ is now a unit matrix, and so $\lambda = \lambda^{-1} = 1$; (ii)

$$D_F = 1, \qquad C_F = 1 - f^2 = \frac{n^2}{g_s^2 \triangle_{(m,n)}} \tag{69}$$

with $\Delta_{(m,n)}$ given in (3) and g_s set equal to unity; (iii) Their integration variable t is π times ours.

The total vacuum amplitude is now

$$\begin{aligned}\Gamma &= \Gamma_{\rm NS} + \Gamma_{\rm R} \\ &= \frac{n_1 n_2 V_{p+1} C_F}{2(8\pi^2\alpha')^{(1+p)/2}} \int_0^\infty \frac{dt}{t} e^{-Y^2/2\pi\alpha' t} t^{-(7-p)/2} \Big\{ |z|^{-1} \Big[\prod_{n=1}^{\infty} \frac{(1+\lambda|z|^{2n-1})(1+\lambda^{-1}|z|^{2n-1})(1+|z|^{2n-1})^6}{(1-\lambda|z|^{2n})(1-\lambda^{-1}|z|^{2n})(1-|z|^{2n})^6} \\ &\quad - \prod_{n=1}^{\infty} \frac{(1-\lambda|z|^{2n-1})(1-\lambda^{-1}|z|^{2n-1})(1-|z|^{2n-1})^6}{(1-\lambda|z|^{2n})(1-\lambda^{-1}|z|^{2n})(1-|z|^{2n})^6} \Big] - 2^4 D_F \prod_{n=1}^{\infty} \frac{(1+\lambda|z|^{2n})(1+\lambda^{-1}|z|^{2n})(1+|z|^{2n})^6}{(1-\lambda|z|^{2n})(1-\lambda^{-1}|z|^{2n})(1-|z|^{2n})^6} \Big\},\end{aligned} \tag{70}$$

where we have used (68). This is our basic result of this paper in addition to the long-distance one given in the previous section. At first look, this is completely different from the calculation given in [21] for $p = 1$, i.e., the D-string case in the Wick rotated version using the light-cone boundary state. In what follows, we will show that our result above is indeed the same as theirs for $p = 1$ using various θ-function relations. For this purpose, let us express our amplitude (70) in terms of θ functions and the Dedekind η function with their standard definitions as given, for example, in [22]. We then have

$$\begin{aligned}\Gamma = \frac{n_1 n_2 V_{p+1} C_F \sin\pi\nu}{(8\pi^2\alpha')^{(1+p)/2}} \int_0^\infty \frac{dt}{t} e^{-Y^2/2\pi\alpha' t} t^{-(7-p)/2} \frac{1}{\eta^9(it)} \\ \times \Big[\frac{\theta_3(\nu|it)\theta_3^3(0|it)}{\theta_1(\nu|it)} - \frac{\theta_4(\nu|it)\theta_4^3(0|it)}{\theta_1(\nu|it)} \\ - \frac{\theta_2(\nu|it)\theta_2^3(0|it)}{\theta_1(\nu|it)} \Big],\end{aligned} \tag{71}$$

where we have defined $\lambda = e^{2\pi i\nu}$ and used the fact that $\cos\pi\nu = D_F$ which can be obtained from $\lambda + \lambda^{-1} = 2(2D_F^2 - 1)$ as given in (62). Note that $\nu = i\nu_0$ with $0 \leq \nu_0 < \infty$ for the case of (F, D_p), while $\nu = \nu_0$ with $0 \leq \nu_0 < 1$ for (D_{p-2}, D_p). Further $\nu_0 \to \infty$ when $f_1 \neq f_2$ and either of $|f_i| \to 1$ (or both $|f_i| \to 1$ when $f_1 = -f_2$) in the former case while $\nu_0 \to 1$ when $f_1 = -f_2$ with $|f_i| \to \infty$ in the latter case but $\nu_0 = 0$ when $f_1 = f_2$ in both cases. Now we use the following identify for θ functions:

$$\begin{aligned}2\theta_1^4(\nu|\tau) = \theta_3(2\nu|\tau)\theta_3^3(0|\tau) - \theta_4(2\nu|\tau)\theta_4^3(0|\tau) \\ - \theta_2(2\nu|\tau)\theta_2^3(0|\tau),\end{aligned} \tag{72}$$

which is obtained from (iv) on page 468 in [23].[5] With the identity (72), the amplitude (70) is greatly simplified to

[5]In obtaining the above identity from the more general one (iv) there, we have made choices of variables $x' = y' = z' = 0$ and $w' = 2z$ which give $w = -z$ and $x = y = z$ in their notation. Note also that their notation for θ functions is $\theta_r(z) = \theta_r(z|\tau)$ with $r = 1, 2, 3, 4$. We also use the facts that $\theta_1(0|\tau) = 0$ and $\theta_1(-z|\tau) = -\theta_1(z|\tau)$.

$$\begin{aligned}\Gamma &= \frac{2n_1n_2V_{p+1}C_F\sin\pi\nu}{(8\pi^2\alpha')^{(1+p)/2}}\int_0^\infty\frac{dt}{t}e^{-Y^2/2\pi\alpha' t}t^{-(7-p)/2}\frac{1}{\eta^9(it)}\frac{\theta_1^4(\frac{\nu}{2}|it)}{\theta_1(\nu|it)}\\ &= \frac{U(m_1,n_1;m_2,n_2)V_{p+1}}{2(8\pi^2\alpha')^{(1+p)/2}}\frac{\sin\pi\nu}{\sin^4\frac{\pi\nu}{2}}\int_0^\infty\frac{dt}{t}e^{-Y^2/2\pi\alpha' t}t^{-(7-p)/2}\frac{1}{\eta^9(it)}\frac{\theta_1^4(\frac{\nu}{2}|it)}{\theta_1(\nu|it)}\\ &= \frac{4U(m_1,n_1;m_2,n_2)V_{p+1}}{(8\pi^2\alpha')^{(1+p)/2}}\int_0^\infty\frac{dt}{t}e^{-Y^2/2\pi\alpha' t}t^{-(7-p)/2}\prod_{n=1}^\infty\frac{(1-e^{i\pi\nu}|z|^{2n})^4(1-e^{-i\pi\nu}|z|^{2n})^4}{(1-|z|^{2n})^6(1-e^{2i\pi\nu}|z|^{2n})(1-e^{-2i\pi\nu}|z|^{2n})},\end{aligned}\tag{73}$$

where in the second equality we have made use of

$$\begin{aligned}\sin^4\frac{\pi\nu}{2} &= \frac{1}{4}(\cos\pi\nu-1)^2 = \frac{1}{4}(D_F-1)^2,\\ n_1n_2C_F(D_F-1)^2 &= U(m_1,n_1;m_2,n_2),\end{aligned}\tag{74}$$

with $U(m_1,n_1;m_2,n_2) = U_{\rm NS}(m_1,n_1;m_2,n_2) - U_{\rm R}(m_1,n_1;m_2,n_2)$ as given by (45) for either case under consideration and with the respective quantization for f_i as given previously, and in the third equality we have made use of explicit expressions for the Dedekind η function and the theta function θ_1.

One can check now that our above amplitude in the present various forms does agree with the calculations given in [21] for the $p=1$ case in the light-cone approach up to an overall constant factor[6] of $1/(8\pi^6)$. In making the comparison, we need also to consider that in their calculations they chose $\alpha'=2$ and their parameter α is related to our ν as $\alpha = 2\pi\nu$.

We now consider the large Y limit of the amplitude (73). This amounts to accounting for the massless-mode contribution of the closed string, and therefore the result should agree with our low-energy effective field theory calculations performed in the previous section. We will find that this is indeed true.[7] For large Y, the separation-dependent exponential suppression factor in (73) implies that the contribution to the amplitude comes from the large t integration. Note that, for large t, $|z| = e^{-\pi t}\to 0$ and

$$\begin{aligned}\theta_1(\nu|it) &\to 2e^{-\pi t/4}\sin\pi\nu,\\ \theta_1\left(\frac{\nu}{2}\,\middle|\,it\right) &\to 2e^{-\pi t/4}\sin\frac{\pi\nu}{2},\\ \eta(it) &\to e^{-\pi t/12}.\end{aligned}\tag{75}$$

So

$$\begin{aligned}\Gamma &\to \frac{U(m_1,n_1;m_2,n_2)V_{p+1}}{2(8\pi^2\alpha')^{(1+p)/2}}\frac{\sin\pi\nu}{\sin^4\frac{\pi\nu}{2}}\int_0^\infty\frac{dt}{t}e^{-Y^2/2\pi\alpha' t}t^{-(7-p)/2}\frac{1}{e^{-3\pi t/4}}\frac{2^4e^{-\pi t}\sin^4\frac{\pi\nu}{2}}{2e^{-\pi t/4}\sin\pi\nu}\\ &= \frac{4U(m_1,n_1;m_2,n_2)V_{p+1}}{(8\pi^2\alpha')^{(1+p)/2}}\int_0^\infty\frac{dt}{t}e^{-Y^2/2\pi\alpha' t}t^{-(7-p)/2} = \frac{4U(m_1,n_1;m_2,n_2)V_{p+1}}{(8\pi^2\alpha')^{(1+p)/2}}\left(\frac{2\pi\alpha'}{Y^2}\right)^{(7-p)/2}\Gamma\left(\frac{7-p}{2}\right)\\ &= \frac{C(m_1,n_1;m_2,n_2)}{Y^{7-p}},\end{aligned}\tag{76}$$

where $C(m_1,n_1;m_2,n_2)$ is given by (44). So this is in complete agreement with our low-energy result (43), as expected, which in turn shows that even our normalization constant is also correct. In reaching the last equality, we have made use of (68) and $(7-p)\Omega_{8-p} = 4\pi\pi^{(7-p)/2}/\Gamma((7-p)/2)$ with Ω_q the volume of the unit q sphere.

The interaction amplitude (73) vanishes when $U(m_1,n_1;m_2,n_2)=0$ which gives $m_1/n_1 = m_2/n_2$ (note that $n_1n_2>0$) as shown in the previous section (now $\nu = 0$ since $f_1=f_2$), reflecting the BPS property of the system. If we take one pair of integers, say, the pair (m_2,n_2), as coprime, then the vanishing amplitude would need $(m_1,n_1) = k(m_2,n_2)$ with k a positive integer. Note that, unlike the single brane case, the nonthreshold bound states have infinitely many stable fundamental states with each characterized by a different pair of coprime integers (m,n). When placing a brane with a pair of integers $k(m,n)$ parallel to one with its pair of integers $k'(m,n)$, we have

[6] In making the comparison, we have considered both the zero-mode contribution (77) and the oscillator contribution (82) in [21] for the magnetic flux. For the case of electric flux, one should send $f_1\to if_1$ and $f_2\to if_2$ as well as $\alpha\to i\alpha$ as mentioned there. In their calculation, the volume factor was not considered, and the overall constant factor difference mentioned in the text should not be of concern here since it is well known that the light-cone calculations alone cannot fix the overall constant.

[7] One can also show that $\Gamma_{\rm NS}$ (65) and $\Gamma_{\rm R}$ (67) give also their corresponding low-energy limits as discussed in the previous section in a similar fashion.

the system breaking no supersymmetry and being BPS if $kk' > 0$; i.e., integer k and integer k' have the same sign. When (m_1, n_1) and (m_2, n_2) are both coprime, the interaction vanishes only if $(m_1, n_1) = (m_2, n_2)$. Further, when none of the above is satisfied, we have $U(m_1, n_1; m_2, n_2) > 0$. Note that each numerator in the infinite product in the integrand of (73)

$$(1 - e^{i\pi\nu}|z|^{2n})^4(1 - e^{-i\pi\nu}|z|^{2n})^4 = (1 - 2\cos\pi\nu|z|^{2n} + |z|^{4n})^4 > 0, \qquad (77)$$

so the sign of the interaction amplitude will depend on that of the factor in each denominator in the infinite product in the integrand

$$(1 - e^{2i\pi\nu}|z|^{2n})(1 - e^{-2i\pi\nu}|z|^{2n}) = (1 - 2\cos 2\pi\nu|z|^{2n} + |z|^{4n}), \qquad (78)$$

which is always positive for the case of (D_{p-2}, D_p) (now ν is real) while it is positive for large t, but it can be negative for small t for the case of (F, D_p) for which ν is purely imaginary. So for the case of (D_{p-2}, D_p), the interaction amplitude is now greater than zero and is solely determined by the positiveness of $U(m_1, n_1; m_2, n_2)$. In this aspect it shares the same feature as its long-distance interaction shown in the previous section, reflecting the attractive nature of the interaction. For the case of (F, D_p), while the long-distance interaction amplitude is again now greater than zero (implying attractive interaction) and is also solely determined by the positiveness of the corresponding $U(m_1, n_1; m_2, n_2)$ as shown in the previous section, the sign of the small separation amplitude (corresponding to small t contribution) is uncertain in the present representation of integration variable t since, even with the factor in (77) less than zero, the sign of the product of infinitely such factors in the integrand remains indefinite. So one would expect some interesting physics to appear in this case for small t.

The small t contribution to the amplitude mainly concerns the physics for small separation Y. The appropriate frame for describing the underlying physics as well as the analytic structure as a function of the separation in the short cylinder limit $t \to 0$ is in terms of an annulus, which can be achieved by the Jacobi transformation $t \to t' = 1/t$. This is also stressed in [24] that the lightest open string modes now contribute most and the open string description is most relevant. So in terms of the annulus variable t', noting that

$$\eta(\tau) = \frac{1}{(-i\tau)^{1/2}}\eta\left(-\frac{1}{\tau}\right),$$
$$\theta_1(\nu|\tau) = i\frac{e^{-i\pi\nu^2/\tau}}{(-i\tau)^{1/2}}\theta_1\left(\frac{\nu}{\tau}\,\middle|\,-\frac{1}{\tau}\right), \qquad (79)$$

the second equality in (73) now becomes

$$\begin{aligned}\Gamma &= -i\frac{U(m_1, n_1; m_2, n_2)V_{p+1}}{2(8\pi^2\alpha')^{(1+p)/2}}\frac{\sin\pi\nu}{\sin^4\frac{\pi\nu}{2}} \times \int_0^\infty \frac{dt'}{t'} e^{-Y^2 t'/2\pi\alpha'} t'^{(1-p)/2}\frac{1}{\eta^9(it')}\frac{\theta_1^4(\frac{-i\nu t'}{2}|it')}{\theta_1(-i\nu t'|it')} \\ &= -i\frac{4U(m_1, n_1; m_2, n_2)V_{p+1}}{(8\pi^2\alpha')^{(1+p)/2}}\frac{\sin\pi\nu}{\sin^4\frac{\pi\nu}{2}} \times \int_0^\infty \frac{dt'}{t'} e^{-Y^2 t'/2\pi\alpha'} t'^{(1-p)/2}\frac{\sin^4(\frac{-i\pi\nu t'}{2})}{\sin(-i\pi\nu t')} \\ &\quad \times \prod_{n=1}^\infty \frac{(1 - e^{\pi\nu t'}|z|^{2n})^4(1 - e^{-\pi\nu t'}|z|^{2n})^4}{(1 - |z|^{2n})^6(1 - e^{2\pi\nu t'}|z|^{2n})(1 - e^{-2\pi\nu t'}|z|^{2n})},\end{aligned} \qquad (80)$$

with now $|z| = e^{-\pi t'}$. We follow [21] to discuss the underlying analytic structure and the possible associated physics of the amplitude of (80). For the case of (D_{p-2}, D_p), we limit ourselves to the interesting non-BPS amplitude, i.e., $\nu = \nu_0$ with $0 < \nu_0 < 1$, and for this the above amplitude is real and has no singularities unless $Y \le 2\pi\sqrt{\nu\alpha'}$, i.e., on the order of the string scale, for which the integrand is dominated by, in the short cylinder limit $t' \to \infty$,

$$\begin{aligned}&\lim_{t'\to\infty}\frac{e^{-Y^2 t'/2\pi\alpha'}\theta_1^4(-i\pi\nu t'/2|it')}{i\eta(it')\theta_1(-i\pi\nu t'|it')} \\ &\sim \lim_{t'\to\infty}\frac{e^{-Y^2 t'/2\pi\alpha'}\sin^4(-i\pi\nu t'/2)}{i\sin(-i\pi\nu t')} \\ &\sim \lim_{t'\to\infty} e^{-(t'/2\pi\alpha')(Y^2 - 2\pi^2\nu\alpha')}.\end{aligned} \qquad (81)$$

The contribution of the annulus to the vacuum amplitude (free energy) should be real if the integrand in (80) has no simple poles on the positive t' axis since the imaginary part of the amplitude is given by the sum of residues at the poles times π due to the integration contour passing to the right of all poles as dictated by the proper definition of the Feynman propagator [25]. In the present case, the amplitude appears purely real, but there are no simple poles on the positive t' axis, therefore giving zero imaginary amplitude, i.e., zero pair-production (absorptive) rate, which is consistent with the conclusion reached in [26] in the quantum field theory context and also pointed out in a similar context in [27]. When $Y \le \pi\sqrt{2\nu_0\alpha'}$, i.e., on the order of the string scale, the integration in (80) diverges, and this therefore gives a divergent amplitude which indicates the breakdown of the calculations and behaves similarly to the situation of brane/antibrane systems as studied in [28,29], signalling the possible onset of tachyonic instability now caused instead by the magnetic fluxes[8] and the relaxation of the system to form a new nonthreshold bound state. However, the detail of this requires further dynamical understanding.

Let us move to the case of (F, D_p). We have now $\nu = i\nu_0$ with $0 < \nu_0 < \infty$ ($\nu_0 = 0$ corresponds to the BPS case and is not considered here). The amplitude (80) is now

$$\Gamma = \frac{4U(m_1, n_1; m_2, n_2)V_{p+1}}{(8\pi^2\alpha')^{(1+p)/2}} \frac{\sinh\pi\nu_0}{\sinh^4\frac{\pi\nu_0}{2}} \times \int_0^\infty \frac{dt'}{t'} e^{-Y^2t'/2\pi\alpha'} t'^{(1-p)/2} \frac{\sin^4(\frac{\pi\nu_0 t'}{2})}{\sin(\pi\nu_0 t')} \times \prod_{n=1}^\infty \frac{(1 - e^{i\pi\nu_0 t'}|z|^{2n})^4(1 - e^{-i\pi\nu_0 t'}|z|^{2n})^4}{(1 - |z|^{2n})^6(1 - e^{2i\pi\nu_0 t'}|z|^{2n})(1 - e^{-2i\pi\nu_0 t'}|z|^{2n})}. \tag{82}$$

Exactly the same as the $p = 1$ case given in [21], the above integrand has also an infinite number of simple poles on the positive real t' axis at $t' = (2k+1)/\nu_0$ with $k = 0, 1, 2, \ldots$. This leads to an imaginary part of the amplitude, which is given as the sum over the residues of the poles as described in [25,30]. Therefore the rate of pair production of open strings per unit world volume in a constant electric flux in the present context is

$$\mathcal{W} \equiv -\frac{2\,\mathrm{Im}\Gamma}{V_{p+1}} = \frac{8U(m_1, n_1; m_2, n_2)}{\nu_0(8\pi^2\alpha')^{(1+p)/2}} \frac{\sinh\pi\nu_0}{\sinh^4\frac{\pi\nu_0}{2}} \sum_{k=0}^\infty \left(\frac{\nu_0}{2k+1}\right)^{(1+p)/2} e^{-((2k+1)Y^2)/2\pi\nu_0\alpha'} \prod_{n=1}^\infty \left(\frac{1 + e^{-2n\pi(2k+1)/\nu_0}}{1 - e^{-2n(2k+1)\pi/\nu_0}}\right)^8$$
$$= \frac{32n_1n_2\left|\frac{m_1}{\triangle^{1/2}_{(m_1,n_1)}} - \frac{m_2}{\triangle^{1/2}_{(m_2,n_2)}}\right|}{\nu_0(8\pi^2\alpha')^{(1+p)/2}} \sum_{k=0}^\infty \left(\frac{\nu_0}{2k+1}\right)^{(1+p)/2} e^{-((2k+1)Y^2)/2\pi\nu_0\alpha'} \prod_{n=1}^\infty \left(\frac{1 + e^{-2n(2k+1)\pi/\nu_0}}{1 - e^{-2n(2k+1)\pi/\nu_0}}\right)^8, \tag{83}$$

where $\Delta_{(m,n)}$ is defined in (3) and ν_0 can be determined from

$$\cosh\pi\nu_0 = \frac{g_s^2(\Omega - m_1m_2)}{n_1n_2} \tag{84}$$

with Ω defined in (11). This rate has been calculated in different context before [25,30–32], but, as stressed in [21] for the $p = 1$ case, the rather complicated sum over spin structures obtained in those papers reduces to our simple expression of (80), (82), or (83). Note that the above rate is suppressed by the brane separation and the integer k but increases with the value of ν_0 which is expected. Let us consider $\nu_0 \to 0$ and $\nu_0 \to \infty$ limits for the above rate, respectively. The former limit corresponds to the near extremal limit for which we can set $f_1 = f_2 + \epsilon$ with $|\epsilon| \ll 1$, while the latter corresponds to the critical field limit for which one can set either $|f_i| \to 1$ while keeping the other less than unity (but fixed) or set $f_1 = -f_2$ with both $|f_i| \to 1$ as mentioned earlier. The definition for f_i with $i = 1, 2$ is given in (63). For the near extremal limit, we have, to leading order,

$$\nu_0 \approx \frac{|\epsilon|}{\pi(1 - f_2^2)}, \tag{85}$$

and the rate (83) is now well approximated by the $k = 0$ term as

$$\mathcal{W} \approx \frac{32n_1n_2|\epsilon|}{(8\pi^2\alpha')^{(1+p)/2}} \times \left(\frac{|\epsilon|}{\pi(1 - f_2^2)}\right)^{(p-1)/2} e^{-(Y^2(1-f_2^2))/2\alpha'|\epsilon|}, \tag{86}$$

very tiny as expected. For the critical field limit mentioned above, now $\nu_0 \to \infty$, and it is easy to see that each term in the summation of (83) diverges and so does the rate, signalling also an instability as mentioned in a similar context in [33].

IV. SUMMARY

In this paper, we calculate explicitly the interaction amplitude between two (F, D_p) or $(D_{(p-2)}, D_p)$ nonthreshold bound states with a separation. In doing so, we make use of their respective boundary state representation with a quantized world-volume electric (or magnetic) flux. Each such nonthreshold bound state is therefore characterized by a pair of integers (m_i, n_i) with $i = 1, 2$. When the two

[8]Without the presence of the magnetic flux, the system is a BPS one and the amplitude vanishes. With the presence of the magnetic flux, in addition to the evidence given in the text, that the open string tachyon mode appears to arise is also indicated from the leading term $e^{\pi\nu t'}$, which diverges in the short cylinder limit $t' \to \infty$, in the expansion of the θ functions and η function in (80) in the open string channel. The occurrence of an open string tachyon mode at this distance can also be checked explicitly by looking at the quantization of the twisted open strings stretched between the two D-branes. We thank the anonymous referee for bringing the last point to our attention.

bound states are (D_{p-2}, D_p), the interaction is, in general, attractive, but this remains so and can be certain only at large brane separation when the two states are (F, D_p). In both cases, the interaction vanishes only if $m_1/n_1 = m_2/n_2$ and $n_1 n_2 > 0$. We also calculate the respective long-distance interaction independently from the low-energy field theory approach, and each agrees with the long-distance part of the corresponding general string amplitude. We also study the analytic structure of the amplitude, and, in particular, we calculate the rate of pair production of open strings for the case of (F, D_p). In general, one expects that the interacting system is unstable and will relax itself by releasing the excess energy via so-called tachyonic condensation [34] to form eventually a BPS nonthreshold bound state, characterized by a pair of integers $(m_1 + m_2, n_1 + n_2)$. If $m_1 + m_2$ and $n_1 + n_2$ are coprime, this state will be stable; otherwise, it will be marginally unstable. Similar to the brane/antibrane systems studied in [28,29], the open string tachyonic condensation manifests itself for the case of (D_{p-2}, D_p) by showing a divergent amplitude but now caused by the presence of magnetic fluxes when the brane separation is on the order of the string scale. However, for the case of (F, D_p), this manifests itself by the pair production of open strings which takes the excess energy away so that the system can lower its energy and relax itself to form the final BPS bound state. By all means, what has been said here is just an indication being responsible for forming the final BPS states of the systems under consideration. To determine whether this actually leads to the formation of final BPS states requires a more detailed dynamical understanding, which is beyond the scope of this paper.

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