Dirac quantization condition for monopole in noncommutative space-time

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Since the structure of space-time at very short distances is believed to get modified possibly due to noncommutativity effects and as the Dirac quantization condition, $\mu e = \frac{N}{2}\hbar c$, probes the magnetic field point singularity, a natural question arises whether the same condition will still survive. We show that the Dirac quantization condition on a noncommutative space in a model of dynamical noncommutative quantum mechanics remains the same as in the commutative case to first order in the noncommutativity parameter θ, leading to the conjecture that the condition will not alter in higher orders.

I. INTRODUCTION

The idea of magnetic monopoles—the so-far hypothetical particles carrying magnetic charge—is one of the most influential in modern theoretical physics. The first effective theoretical proposal that magnetic charge should exist was made by Dirac [1], who argued that in quantum mechanics the unobservability of phase permits singularities that manifest themselves as sources of magnetic fields. The Dirac quantization condition (DQC) $\mu e = \frac{N}{2}\hbar c$ is a topological property, independent of space-time points, that tells us that the mere existence of a single magnetic monopole would imply that the electric charge is quantized. Before Dirac, the surprising asymmetry in Maxwell's equations made Poincaré and Thompson introduce the magnetic charge in the theory as an artefact for simplifying the computation, while Curie suggested even the actual existence of magnetic charge [2]. The idea of magnetic monopoles was later extended by the discovery of monopole solutions of classical non-Abelian theories [3,4] and the introduction of the concept of dyons—particles carrying both electric and magnetic charge [5], eventually leading to the concept of duality [6], with its significant influence on the string theory.

In recent years magnetic monopole structures have created a lot of interest in condensed matter physics. In studying the anomalous Hall effect, magnetic monopole structures in *momentum space* have been experimentally verified and theoretically explained in [7,8].

However, to date no magnetic monopole has been found (for a comprehensive review of magnetic monopole searches, see, e.g. [9]), but the theoretical interest has stayed, as nothing in the theory has ever been found to contradict the DQC.

In the recent decade there has been a growing interest in the research concerning noncommutative spaces mainly due to the results in [10,11]. In [11] it was shown that string theory in a constant background field leads to a noncommutative field theory as a low energy limit.

Moreover, the result of [10] has encouraged many to believe that noncommutative field theory is a step toward a more complete description of physics. In the "gedanken" experiment of [10], it was argued that in the process of measurement of space points, as the energy grows, eventually black holes are formed and consequently objects of smaller extent than the diameter of the black holes cannot be observed and one can think of space-time "points" as operators obeying a Heisenberg-like uncertainty principle from which it follows that space-time is homogeneous and can be interpreted as being noncommutative.

Although the main interest in this field lies in the formulation of a consistent field theory on a noncommutative space-time, it is also interesting to apply the noncommutative space to pure quantum mechanics to see whether it is possible to extend ordinary quantum mechanics to the noncommutative case. Specifically, the result [10] is quantum mechanical in nature and some results such as the DQC [1], which we will be exploring in this paper, are not obtained directly from field theory.

In the noncommutative case, the space-time is particularly sensitive to the short-distance effects. Since the DQC in its essence probes the singularity structure of the magnetic field, one would think that this condition could no longer remain valid in the noncommutative case. This is the main motivation for the present work.

We shall start by briefly reviewing one known method for deriving the DQC in the commutative case. Then, starting from a classical Lagrangian corresponding to a dynamical model of noncommutative quantum mechanics, we shall derive the DQC to first order in the noncommutativity parameter θ, and finally we shall discuss the result and its possible generalizations.

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II. ONE WAY OF DERIVING THE DQC

In the commutative case, there is an ingenious way to derive the DQC, first introduced by Jackiw [12], which uses a gauge-invariant algebra, dependent only on the magnetic field. The derivation is an example of the three-cocycles, which appear when a representation of a transformation group is nonassociative; in particular, when the translations group is represented by gauge-invariant operators in the presence of a magnetic monopole, the Jacobi identity among the translation generators fails. The restoration of the associativity of *finite* translations leads to the DQC. A sketch of the derivation [12] will be quite illuminating: for a nonrelativistic particle with the electric charge e, moving in a magnetic field $\boldsymbol{B}(\boldsymbol{x})$, one starts by finding the noncanonical quantum brackets

$$[x_i, x_j] = 0, \qquad [x_i, \pi_j] = i\hbar\delta_{ij}, \qquad (1)$$

$$[\pi_i, \pi_j] = ie\frac{\hbar}{c}\epsilon_{ijk}B_k(\boldsymbol{x}), \qquad (2)$$

where one defines the operators π_i in the x representation as

$$\pi_i = -i\hbar\partial_i - \frac{e}{c}A_i(\boldsymbol{x}), \qquad (3)$$

with $A_i(\boldsymbol{x})$ being the vector potential [13]. These commutation relations, along with the Hamiltonian

$$H = \frac{\boldsymbol{\pi}^2}{2m}, \qquad \boldsymbol{\pi} = m\dot{\boldsymbol{x}}, \qquad (4)$$

yield the well-known Lorentz-Heisenberg equations of motion

$$\dot{\boldsymbol{x}} = \frac{i}{\hbar}[H, \boldsymbol{x}] = \frac{\boldsymbol{\pi}}{m}, \qquad (5)$$

$$\dot{\boldsymbol{\pi}} = \frac{i}{\hbar}[H, \boldsymbol{\pi}] = \frac{e}{2mc}[\boldsymbol{\pi} \times \boldsymbol{B} - \boldsymbol{B} \times \boldsymbol{\pi}], \qquad (6)$$

where $\boldsymbol{\pi}$ is the gauge-invariant mechanical momentum. So far there is no restriction on $\boldsymbol{B}$ but the following Jacobi identity violation

$$\frac{1}{2}\epsilon_{ijk}[[\pi_i, \pi_j], \pi_k] = \frac{e\hbar^2}{c}\nabla \cdot \boldsymbol{B} \qquad (7)$$

indicates that the magnetic field has to be source free. Otherwise, $\nabla \cdot \boldsymbol{B} \neq 0$ will lead to a loss of associativity of the translation operators $T(\boldsymbol{a}) \equiv \exp(-\frac{i}{\hbar}\boldsymbol{a} \cdot \boldsymbol{\pi})$,

$$(T(\boldsymbol{a}_1)T(\boldsymbol{a}_2))T(\boldsymbol{a}_3) = \exp\left(-\frac{ie}{\hbar c}\omega(\boldsymbol{x}; \boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3)\right)$$
$$\times T(\boldsymbol{a}_1)(T(\boldsymbol{a}_2)T(\boldsymbol{a}_3)). \qquad (8)$$

Here, $\boldsymbol{a}_i$ are constant vectors and the nontrivial phase factor turns out to be the magnetic flux coming out of the tetrahedron formed by $\boldsymbol{a}_i$:

$$\exp\left(-\frac{ie}{\hbar c}\omega(\boldsymbol{x}; \boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3)\right), \qquad (9)$$

which is nonzero if a magnetic monopole is enclosed by the tetrahedron. The phase factor (9) becomes unity and thus the associativity of *finite* translations in the presence of the magnetic monopoles can be re-established for

$$\int d^3x\nabla \cdot \boldsymbol{B} = 2\pi\frac{\hbar c}{e}N, \qquad (10)$$

where N is an integer. This condition, together with the Gauss equation for a monopole of magnetic charge μ, $\nabla \cdot \boldsymbol{B} = 4\pi\mu\delta^3(\boldsymbol{x})$, yields the celebrated DQC

$$\mu e = \frac{1}{2}N\hbar c. \qquad (11)$$

Note that the Jacobi identity is still violated at the location of each monopole and these points are conventionally excluded from the manifold.

III. THE NONCOMMUTATIVE DQC

The extension of the approach in [12] to the noncommutative case can be achieved once one finds the algebra of coordinate and gauge-invariant momentum operators for a charged quantum mechanical particle in motion in a magnetic field in the noncommutative space-time, i.e. the analogue of the algebra (1) and (2). It is expected that the noncommutativity of space-time coordinates would change the dynamics of the charged particle in the magnetic field (i.e. the Lorentz force), and this in turn will require a change in the commutation relations (2). However, we can find the new noncommutative algebra by starting from a classical Lagrangian, for example, the one for the model [14], and deriving the corresponding Dirac brackets and then quantizing them. We therefore consider a Lagrangian of the form

$$L = \left(P_i + \frac{e}{c}A_i\right)\dot{X}_i + \frac{1}{2}\epsilon_{ijk}P_i\dot{P}_j\theta_k - \frac{1}{2m}P^2 + eA_0, \qquad (12)$$

where P_i is the momentum, θ_k—the noncommutativity parameter, of dimension (length)^2/action, and A_i, A_0—the magnetic and electric potential, respectively. The Lagrangian (12) is a straightforward generalization to three dimensions of the one considered in [15], which is a Lagrangian for the model [14].

The Lagrangian (12) is a Lagrangian of dynamical non-commutativity of the space coordinates. This statement is better understood once we derive the Dirac brackets from this Lagrangian. For this, we need the canonical momenta, which are given by

$$\pi_i = \frac{\partial L}{\partial \dot{X}_i} = P_i + \frac{e}{c}A_i, \qquad \pi_i^P = \frac{\partial L}{\partial \dot{P}_i} = -\frac{1}{2}\epsilon_{ijk}P_j\theta_k.$$

These lead to the constraints

$$\eta_i \equiv \pi_i - P_i - \frac{e}{c} A_i, \qquad \psi_i \equiv \pi_i^P + \frac{1}{2} \epsilon_{ijk} P_j \theta_k.$$

In the classical framework, with $\{X_i, \pi_j\} = \delta_{ij}, \{P_i, \pi_j^P\} = \delta_{ij}$, we calculate the constraint algebra

$$\{\eta_i, \eta_j\} = \frac{e}{c} (\partial_i A_j - \partial_j A_i) = \frac{e}{c} F_{ij} = \frac{e}{c} \epsilon_{ijk} B_k,$$

$$\{\psi_i, \psi_j\} = \epsilon_{ijk} \theta_k, \qquad \{\eta_i, \psi_j\} = -\delta_{ij}. \tag{13}$$

From this algebra we find that the constraints are second class and, performing the Dirac constraint analysis [16,17], we obtain the classical Dirac brackets as

$$\{X_i, X_j\} = \frac{\epsilon_{ijk} \theta_k}{1 - \frac{e}{c} \boldsymbol{\theta} \cdot \boldsymbol{B}}, \qquad \{X_i, P_j\} = \frac{\delta_{ij} - \frac{e}{c} B_i \theta_j}{1 - \frac{e}{c} \boldsymbol{\theta} \cdot \boldsymbol{B}},$$

$$\{P_i, P_j\} = \frac{\epsilon_{ijk} \frac{e}{c} B_k}{1 - \frac{e}{c} \boldsymbol{\theta} \cdot \boldsymbol{B}}. \tag{14}$$

These brackets have also been obtained in [18,19].

This is exactly how interactions have been introduced in the model of [15]. In this model the noncommutativity of coordinate operators is dynamical in the sense that it is generated within the system. Thus, the gauge field cannot be affected by the noncommutativity, which emerges upon quantization. Therefore, the field A_i is the Abelian $U(1)$ gauge field in this model of noncommutativity.

Our next step is to quantize the brackets. We do this by promoting the classical variables X_i and P_i in the Dirac brackets (14) to the status of operators $\hat{X}_i$, $\hat{P}_i$ and multiplying the right-hand side of the Dirac brackets by $i\hbar$. This is the standard procedure [16,17]. We consider the Dirac brackets (14) expanded to first order in θ and hereafter we perform all our calculations to this order only. We resort to this approximation because we shall need to find representations for our operators in order to have a well-defined quantum theory [17], and this is a task that is difficult to do exactly for the algebra (14). The quantization of (14) gives

$$[\hat{X}_i, \hat{X}_j] = i\hbar \epsilon_{ijk} \theta_k + \mathcal{O}(\theta^2),$$

$$[\hat{X}_i, \hat{P}_j] = i\hbar \left[\delta_{ij} - \frac{e}{c} B_i(\hat{X}) \theta_j + \frac{e}{c} \delta_{ij} \boldsymbol{\theta} \cdot \boldsymbol{B}(\hat{X}) \right] + \mathcal{O}(\theta^2),$$

$$[\hat{P}_i, \hat{P}_j] = i\hbar \frac{e}{c} \epsilon_{ijk} B_k(\hat{X}) \left[1 + \frac{e}{c} \boldsymbol{\theta} \cdot \boldsymbol{B}(\hat{X}) \right] + \mathcal{O}(\theta^2). \tag{15}$$

The algebra (15) poses a twofold problem. Firstly, the operator $\hat{P}_j$ in (15) does not represent the translation generator, since there are extra terms on the right-hand side of $[\hat{X}_i, \hat{P}_j]$, other than $i\hbar \delta_{ij}$. Secondly, we face the problem of how to represent the operators $\hat{X}_i$, since they do not commute to first order in θ. This problem becomes much simpler if we are able to define some new operators x_i, in terms of the old ones $\hat{X}_i$ and $\hat{P}_i$, such that they commute to first order in θ. An appropriate definition for

our purpose is

$$x_i = \hat{X}_i + \frac{1}{2} \epsilon_{ijk} \hat{P}_j \theta_k. \tag{16}$$

Then the functions of the operator $\hat{X}_i$ can be expanded in terms of the new coordinate operator x_i as, e.g.,

$$B_i(\hat{x}) = B_i(\boldsymbol{x}) - \frac{1}{2} \epsilon_{njk} \theta_k \hat{P}_j \partial_n B_i(\boldsymbol{x}) + \mathcal{O}(\theta^2). \tag{17}$$

We use the operator x_i (16) and the expansion (17) to obtain an intermediate algebra of x_i and $\hat{P}_j$, and further define the generator of translations corresponding to x_i:

$$p_j = \hat{P}_j - \frac{1}{2} \frac{e}{c} (\hat{P}_j (\boldsymbol{B} \cdot \boldsymbol{\theta}) - \hat{\boldsymbol{P}} \cdot \boldsymbol{B} \theta_j). \tag{18}$$

The newly defined operators p_i and x_i obey the algebra

$$[x_i, x_j] = 0 + \mathcal{O}(\theta^2),$$

$$[x_i, p_j] = i\hbar \delta_{ij} + \mathcal{O}(\theta^2),$$

$$[p_i, p_j] = i\hbar \frac{e}{c} \epsilon_{ijk} B_k - \frac{e}{2c} [i\hbar (p_{[j} \partial_{i]} (\boldsymbol{B} \cdot \boldsymbol{\theta}) + p_{[i} \theta_{j]} \nabla \cdot \boldsymbol{B}$$

$$+ \boldsymbol{p} \cdot \theta_{[i} \partial_{j]} \boldsymbol{B}) + p_{[j} [p_{i]}, \boldsymbol{B}] \cdot \boldsymbol{\theta} + \boldsymbol{p} \cdot [\boldsymbol{B}, p_{[i]} \theta_{j]}]$$

$$+ \mathcal{O}(\theta^2), \tag{19}$$

where the indices in brackets are antisymmetrized.

To have properly quantized the algebra (19), we need a representation of its operators. From the similarity of the algebras (2) and (19), we infer that in the x representation, we can realize the translation generators as (3) plus an extra term involving the first order noncommutativity contribution. Explicitly,

$$p_i = -i\hbar \partial_i - \frac{e}{c} A_i(\boldsymbol{x}) + T_i(\theta, \boldsymbol{x}) + \mathcal{O}(\theta^2). \tag{20}$$

Inserting (20) into the right-hand side of the commutator $[p_i, p_j]$ of the algebra (19), it simplifies to

$$[p_i, p_j] = i\hbar \frac{e}{c} \epsilon_{ijk} B_k + \frac{1}{2} \frac{e}{c} \left[\left(i\hbar \partial_{[i} + \frac{e}{c} A_{[i} \right) \theta_{j]} \nabla \cdot \boldsymbol{B} \right]$$

$$+ \mathcal{O}(\theta^2). \tag{21}$$

By directly computing the commutator of the operators p_i in the representation (20), we have to reproduce the result (21), which holds true if we set

$$T_i(\theta, \boldsymbol{x}) = -\frac{1}{2} \frac{e}{c} \theta_i \nabla \cdot \boldsymbol{B} + G_i, \tag{22}$$

where

$$\partial_j G_i = \frac{1}{2\hbar} \left(\frac{e}{c} \right)^2 A_j \theta_i \nabla \cdot \boldsymbol{B}. \tag{23}$$

Thus, the quantized algebra (19) is given by

$$[x_i, x_j] = 0 + \mathcal{O}(\theta^2),$$

$$[x_i, p_j] = i\hbar\delta_{ij} + \mathcal{O}(\theta^2),$$

$$[p_i, p_j] = i\hbar\frac{e}{c}\epsilon_{ijk}B_k + \frac{e}{2c}\left[\left(i\hbar\partial_{[i} + \frac{e}{c}A_{[i}\right)\theta_{j]}\nabla \cdot \boldsymbol{B}\right]$$
$$+ \mathcal{O}(\theta^2) \qquad (24)$$

in the x representation.

We can now calculate the Jacobi identities of the algebra (24), and find that the only nonvanishing one is

$$\frac{1}{2}\epsilon_{ijk}[[p_i, p_j], p_k] = -\hbar^2\frac{e}{c}\nabla \cdot \boldsymbol{B}$$
$$+ \frac{i\hbar}{2}\left(\frac{e}{c}\right)^2\epsilon_{ijk}\partial_k(A_i\theta_j\nabla \cdot \boldsymbol{B})$$
$$+ \mathcal{O}(\theta^2). \qquad (25)$$

Since the nonvanishing terms in the right-hand side of (25) are proportional to $\nabla \cdot \boldsymbol{B}$, for a divergenceless magnetic field there are no Jacobi indentity violations. However, if the magnetic field is produced by monopoles $\nabla \cdot \boldsymbol{B} = 4\pi\mu\delta^3(\boldsymbol{x})$, the Jacobi identity (25) is violated, meaning nonassociativity of the translation generators p_i.

We would like to remark at this point that although the Lagrangian (12) contains no magnetic sources, the algebra (24) is valid whether the magnetic field is source free or not. The reason is simply that the Lorentz force describes the movement of electrically charged particles in a magnetic field, but does not set any requirement on how the magnetic field is produced.

Thus, in the noncommutative space we end up with a Jacobi identity violation consisting of the original commutative space term plus a θ-dependent total-derivative term. Let us recall that the DQC appears in the commutative case [12] through a volume integration [see (9)] over the tetrahedron formed by the three translation vectors $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, $\boldsymbol{a}_3$. Now, the θ term in (25), being a total derivative, should contribute at the boundary of the tetrahedron. However, this contribution will be necessarily zero, because the integrand contains the δ function coming from $\nabla \cdot \boldsymbol{B} = 4\pi\mu\delta^3(\boldsymbol{x})$, which has support only at the origin, i.e. on the monopole. Hence, these two features conspire to cancel the effect of the θ term. The DQC remains unchanged in the presence of spatial noncommutativity, since the argument for restoring the associativity of the noncommutative translation operators goes through in the same manner as in the commutative case [12], but now with the translation operators

$$T_{\text{NC}}(\boldsymbol{a}) = \exp\left(-\frac{i}{\hbar}\boldsymbol{a} \cdot \boldsymbol{p}\right), \qquad (26)$$

generated by $\boldsymbol{p}$ as the element of the algebra (24), valid to first order in θ and with the x representation (20).

IV. SUMMARY AND DISCUSSION

We have explicitly shown that the DQC (11) remains unaltered in noncommutative space to first order in θ. Based on the structure of the classical algebra (14) and the representation of the quantum algebra (20), and also considering the fact that the form of any topological correction is strongly constrained, we conjecture that the DQC will hold true in all orders in the noncommutative spacetime. We intend to elucidate this issue in the future.

We would also like to emphasize that, had we obtained a correction to the DQC involving θ, it would immediately invalidate the DQC as a topological result in the noncommutative case. This is simply due to the fact that the θ parameter is dimensionful, and any correction involving it has to necessarily introduce model-dependent factors, such as the mass of the electron or the length of the trajectory of the electron around the monopole. Since the DQC must remain dimensionless to be a topological result, any correction to it involving θ is not permissible.

It would be interesting to obtain the same kind of indication of a DQC to first order in θ using a noncommutative non-Abelian vector potential [11,20], especially since a gauge-covariant noncommutative Aharonov-Bohm effect has been formulated in [21]. However, this formulation gives the required phase factor with the help of non-Abelian noncommutative Wilson lines, which are notoriously tedious to work with even to first order in θ, due to the path ordering appearing in the Wilson line. Therefore, obtaining a possible DQC in this approach stands as a challenge for the future.

Our conclusion is that the DQC remains unchanged in the noncommutative case, to the first order in θ and expectably to all orders. This is of significance, since a vast amount of work has been devoted to studying various effects of noncommutative space only to the lowest order in θ. Finally, we would like to mention that our work reinforces similar topological results in the noncommutative case for other nonperturbative monopole, soliton, and dyon solutions [22], although these solutions are purely noncommutative in origin and have no commutative counterpart.

ACKNOWLEDGMENTS

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