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Building the full fermion-photon vertex of QED by imposing multiplicative renormalizability of the Schwinger-Dyson equations for the fermion and photon propagators

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In principle, calculation of a full Green's function in any field theory requires knowledge of the infinite set of multipoint Green's functions, unless one can find some way of truncating the corresponding Schwinger-Dyson equations. For the fermion and boson propagators in QED this requires an *ansatz* for the full 3-point vertex. Here we illustrate how the properties of gauge invariance, gauge covariance and multiplicative renormalizability impose severe constraints on this fermion-boson interaction, allowing a consistent truncation of the propagator equations. We demonstrate how these conditions imply that the 3-point vertex *in the propagator equations* is largely determined by the behavior of the fermion propagator itself and not by knowledge of the many higher-point functions. We give an explicit form for the fermion-photon vertex, which in the fermion and photon propagator fulfills these constraints to all orders in leading logarithms for massless QED, and accords with the weak coupling limit in perturbation theory at $\mathcal{O}(\alpha)$. This provides the first attempt to deduce nonperturbative Feynman rules for strong physics calculations of propagators in massless QED that ensure a more consistent truncation of the 2-point Schwinger-Dyson equations. The generalization to next-to-leading order and masses will be described in a longer publication.

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I. INTRODUCTION

Solution of the Schwinger-Dyson equations (SDE) for any field theory would constitute the complete determination of that theory and every possible measurable quantity would be known. Even though it is nearly 60 years since these field equations were first derived [1–5], we are far from obtaining their solution even for a relatively simple theory like QED. Progress has been hampered by the very structure that makes field theory interesting, namely, that the Schwinger-Dyson equations form an infinite nested set. Each n-point function must be multiplicatively renormalizable and, in a gauge theory, respects gauge invariance. To achieve this, the solution even for the 2-point functions (the propagators) appears to require knowledge of all the other n-point functions. Consequently, studies in gauge theories have resorted foremostly to a perturbative approximation, in which each Green's function is expanded to a given order in the coupling squared. Or as an approximation to nonperturbative physics, simple (even simplistic) ansatz have been used for the 3-point function to allow the fermion propagator to be investigated. In return dynamical mass generation has been studied in the *rainbow approximation* [6–12] and some level of understanding of when chiral symmetry breaking can occur has been reached. While valuable for gaining intuition, this is no substitute for a genuine nonperturbative study. While formal results on gauge invariance and multiplicative renormalizability (MR) have long been known using the gauge technique of Salam, Delbourgo [13–15] and others, this method has not proved useful for providing equations that can be readily solved either analytically or numerically. Here, an alternative approach, an attempt to develop nonperturbative Feynman rules, has proved more fruitful. The aim is to write down explicit representations for the effective n-point functions, in particular, for the 3-point function, which ensures that the solutions of the Schwinger-Dyson equations for the 2-point functions respect gauge invariance and are multiplicatively renormalizable [16,17].

What has previously impeded the practical study of the Schwinger-Dyson equations has been the need to handle overlapping divergences that dramatically complicate the renormalization of the equations. The present approach overcomes this difficulty by requiring that the 2-point functions must be multiplicatively renormalizable and no overlapping divergences can thereby occur. This procedure is genuinely nonperturbative and is not readily relatable to attempts at summing subsets of Feynman graphs with these same properties [11,18–23].

The first of such nonperturbative studies has been in the case of quenched QED [12,23–35]—that is, QED in which

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the explicit factor of N_F multiplying the fermion loop corrections to the photon propagator is set equal to zero. Then a form for the fermion-boson vertex that satisfies the Ward identity, the Ward-Green-Takahashi identity [36] and renders the fermion propagator multiplicatively renormalizable, has been written down explicitly [18,37]. While the form is nonperturbative, the fact that it must agree with perturbation theory in the weak coupling regime is a key pointer to the ultraviolet structure, expressed in terms of logarithms of momenta. The purpose of the present paper is to extend this study by developing the constraints that have to be fulfilled in the case of *massless unquenched* QED to ensure *both* the fermion and photon propagators are multiplicatively renormalizable (at least as far as leading logarithms are concerned).

In general, the full fermion-boson vertex has 12 components, all of which are in principle independent, though one is forced to be zero by gauge invariance. The fermion and photon propagators do not require complete knowledge of the full complexity of this structure, but just two projections that arise in the Schwinger-Dyson equations for these 2-point functions. We present a simple solution to the constraints from multiplicative renormalizability. While the general structure of the full vertex is not complete, the projections within the SDEs for the 2-point functions have no freedom.

While it is clear that the full 3-point function must involve knowledge of the 4-point kernel and higher-point functions, as far as its role in the equations for the propagators is concerned, this is not the case. Thus it can be that the *effective* 3-point function involves only the full 2-point functions. A clue to this is provided by the Ward-Green-Takahashi [38–40] identity, which tells us that part of the 3-point vertex (often called the longitudinal part) is precisely fixed by the fermion propagator alone. Moreover, a hint that the remaining transverse part may be similarly con-

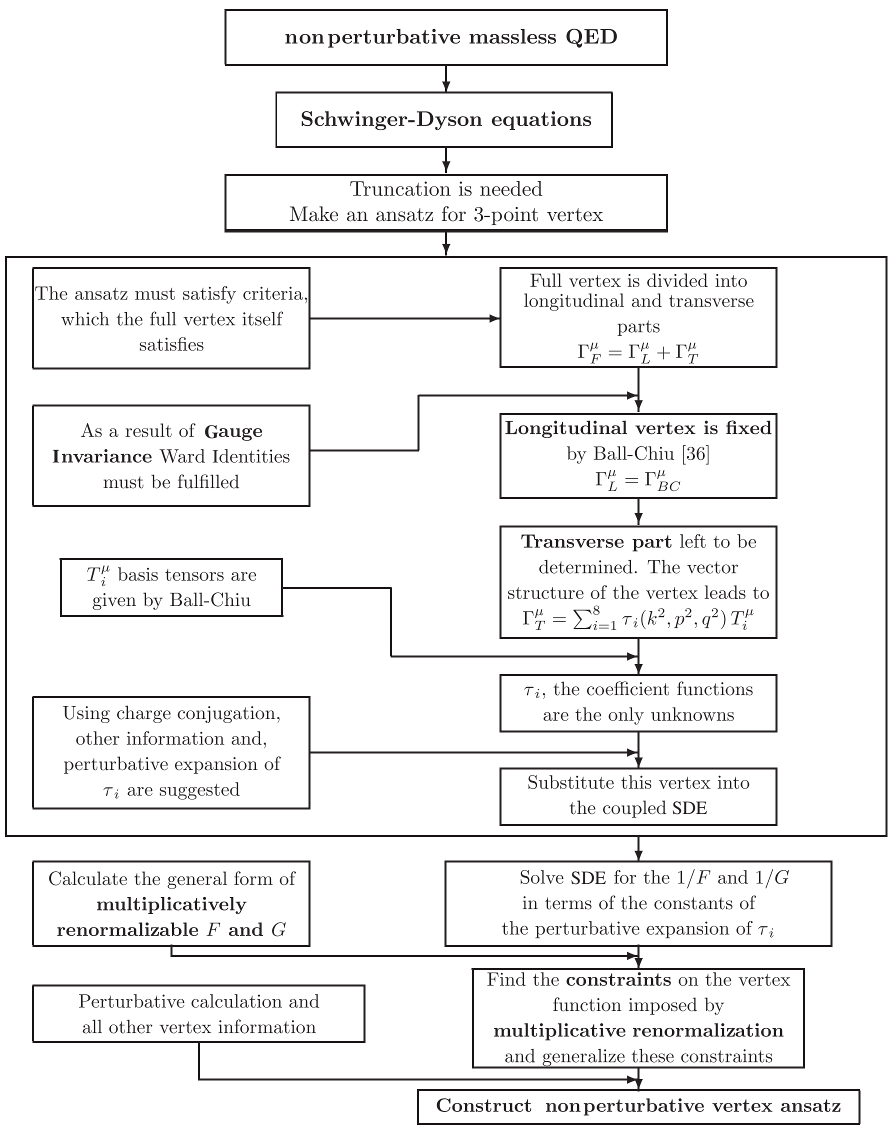

FIG. 1. Flow diagram of the Schwinger-Dyson calculation presented here.

strained is the fact that the vertex and fermion wave function renormalization have common renormalization factors ($Z_1 = Z_2$) as a consequence of gauge invariance. Thus the transverse part must know about the fermion propagator functions too. How, this works in full QED is what we investigate in this paper.

In Sec. II we consider the structure of the fermion-boson vertex and its ultraviolet behavior. In Sec. III we compute the Schwinger-Dyson equations for the fermion and boson propagators. In Sec. IV we deduce the ultraviolet structure imposed by multiplicative renormalizability. Section V gives the constraints on the vertex imposed by MR conditions. The pattern of constraints indicates a general analytic form for the transverse part of the vertex structure. In Sec. VI we deduce a solution to these constraints involving the full fermion wave function renormalization. The vertex in the weak coupling limit is studied in Sec. VII and the restrictions it imposes derived. In Sec. VIII we conclude and outline a program for future work. Since this procedure is rather complicated, we show in Fig. 1 a flow diagram of this calculation.

II. VERTEX AND PROPAGATORS AND THEIR RENORMALIZATIONS DEFINED

The two key constraints on the fermion-boson vertex are provided by the gauge invariance of the theory and by multiplicative renormalizability. Here we begin with the first of these and describe the importance of the Ward-Green-Takahashi identity [38–40]. Though this is well known, it forms the essential background allowing us to establish our notation.

The vertex, displayed in Fig. 2, is a function of the two independent momenta flowing through the vertex. We take these to be the fermion momenta, k and p. The vertex function is $\Gamma^\mu(k, p; q)$ with $q = k - p$. It is well known that the coupling of two spin-1/2 particles to a spin-1 boson involves 12 independent vectors; of these, eight are transverse to the boson momentum q. The structure of the four (longitudinal) components are constrained by the Ward-Green-Takahashi identity (WGTI)

$$q^\mu \Gamma_\mu(k, p; q) = S_F^{-1}(k) - S_F^{-1}(p), \qquad (1)$$

where $S_F(p)$ is the full fermion propagator carrying momentum p. In general

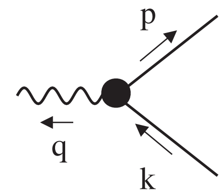

FIG. 2. Fermion-boson vertex.

$$iS_F(p) = i\frac{F(p^2)}{\not p - \mathcal{M}(p^2)} = i\frac{1}{A(p^2)\not p - \mathcal{B}(p^2)}, \qquad (2)$$

where $F(p^2)$ [or $A(p^2) = 1/F(p^2)$] is the fermion wave function renormalization and $\mathcal{M}(p^2)$ [or $B(p^2) = M(p^2)/F(p^2)$] is its mass function. The bare fermion propagator is just $S_F^0(p) = 1/(\not p - m_0)$. From the form of this propagator, the Ward-Green-Takahashi identity, Eq. (1), contains terms with both one and no gamma matrices, so that the vertex component involving two through $\sigma^{\mu\nu} \equiv \frac{1}{2}[\gamma^\mu, \gamma^\nu]$ must be zero. Thus in a gauge theory there are in fact 11 independent nonzero vectors in terms of which to decompose $\Gamma^\mu(p, k; q)$. Of these, six occur if the fermions are massless as we consider here, i.e. $\mathcal{M}(p^2) = 0$. Equation (1) has a well-known zero photon momentum limit; the Ward identity:

$$\Gamma_\mu(p, p; 0) = \lim_{k\to p} \Gamma_\mu(p, k; q) = \frac{\partial S_F^{-1}(p)}{\partial p_\mu}. \qquad (3)$$

The full vertex can be divided into longitudinal and transverse components

$$\Gamma^\mu(p, k; q) = \Gamma_L^\mu(p, k; q) + \Gamma_T^\mu(p, k; q), \qquad (4)$$

where

$$q_\mu \Gamma_T^\mu(p, k; q) = 0. \qquad (5)$$

We demand that the longitudinal part *alone* is responsible for the vertex satisfying both Eqs. (1) and (3). This means that each component must be separately free of kinematic singularities, so that

$$\Gamma_T^\mu(p, p; 0) = 0. \qquad (6)$$

The longitudinal part is then defined, following Ball-Chiu [36], to be

$$\Gamma_L^\mu(p, k, q) \equiv \Gamma_{BC}^\mu(p, k, q), = \sum_{i=1}^{4} \lambda_i(p^2, k^2, q^2) L_i^\mu(p, k; q), \qquad (7)$$

where

$$\lambda_1(p^2, k^2, q^2) = \frac{1}{2}\left(\frac{1}{F(k^2)} + \frac{1}{F(p^2)}\right), \qquad L_1^\mu(p, k; q) = \gamma^\mu,$$

$$\lambda_2(p^2, k^2, q^2) = \frac{1}{2}\frac{1}{(k^2 - p^2)}\left(\frac{1}{F(k^2)} - \frac{1}{F(p^2)}\right), \qquad L_2^\mu(p, k; q) = (k^\mu + p^\mu)(\not k + \not p),$$

$$\lambda_3(p^2, k^2, q^2) = -\frac{1}{(k^2 - p^2)}\left(\frac{M(k^2)}{F(k^2)} - \frac{M(p^2)}{F(p^2)}\right), \qquad L_3^\mu(p, k; q) = (k^\mu + p^\mu),$$

$$\lambda_4(p^2, k^2, q^2) = 0, \qquad L_4^\mu(p, k; q) = \sigma^{\mu\nu}(k_\nu + p_\nu). \qquad (8)$$

Crucially, because of gauge invariance, this longitudinal component of the vertex is wholly determined by the fermion propagator. Moreover, it is this longitudinal component that gives the dominant ultraviolet behavior of the vertex [41].

Quite generally, the transverse vertex can be decomposed in the *massless* fermion case in terms of the remaining four basis vectors as

$$\Gamma_T^\mu(p,k;q) = \sum_{i=2,3,6,8} \tau_i(p^2,k^2,q^2) T_i^\mu(p,k;q), \tag{9}$$

where the τ_i are coefficient functions depending on momenta k^2, p^2 and q^2, which are as yet undetermined, and the T_i are the basis tensors defined by Ball and Chiu [36]—the modification of this basis by Kızılersü *et al.* [41] does not affect these four vectors:

$$\begin{aligned} T_2^\mu(p,k;q) &= (p^\mu(k\cdot q) - k^\mu(p\cdot q))(\not k + \not p),\\ T_3^\mu(p,k;q) &= q^2\gamma^\mu - q^\mu \not q,\\ T_6^\mu(p,k;q) &= \gamma^\mu(p^2-k^2) + (p+k)^\mu \not q,\\ T_8^\mu(p,k;q) &= -\gamma^\mu k^\lambda p^\nu \sigma_{\lambda\nu} + k^\mu \not p - p^\mu \not k. \end{aligned} \tag{10}$$

With these basis vectors, the $\tau_i (i = 2, 3, 6, 8)$ are individually free of kinematic singularities at $\mathcal{O}(\alpha)$ in perturbation theory in any covariant gauge as shown in Ref. [41]. It is these τ_i's that are constrained by multiplicative renormalizability [18]. It is our key presumption that this will *force* these transverse components (or at least their projections in the Schwinger-Dyson equations for the 2-point functions) to depend only on propagator functions just like the longitudinal part of Eqs. (7) and (8).

What we can say about these coefficients? Here we discuss the fundamental constraints on the transverse vertex that follow from (i) dimensional analysis, (ii) symmetry properties, (iii) order of perturbation theory, (iv) gauge dependence and (v) renormalization:

(i) The transverse vertex is dimensionless. Knowing the dimensions of the basis vectors from Eq. (10) tells us the dimensions of the τ_i's. With $d =$ momentum^2, then

$$\begin{aligned} &\text{dim of } T_2^\mu\text{: } d^2 \to \text{dim of } \tau_2\text{: } \frac{1}{d^2},\\ &\text{dim of } T_3^\mu\text{: } d \to \text{dim of } \tau_3\text{: } \frac{1}{d},\\ &\text{dim of } T_6^\mu\text{: } d \to \text{dim of } \tau_6\text{: } \frac{1}{d},\\ &\text{dim of } T_8^\mu\text{: } d \to \text{dim of } \tau_8\text{: } \frac{1}{d}. \end{aligned} \tag{11}$$

(ii) The C-parity operation [24,42] on Eqs. (7) and (9) requires

$$\begin{aligned} \tau_2(k^2,p^2,q^2) &= \tau_2(p^2,k^2,q^2), & \lambda_1(k^2,p^2,q^2) &= \lambda_1(p^2,k^2,q^2),\\ \tau_3(k^2,p^2,q^2) &= \tau_3(p^2,k^2,q^2), & \lambda_2(k^2,p^2,q^2) &= \lambda_2(p^2,k^2,q^2),\\ \tau_6(k^2,p^2,q^2) &= -\tau_6(p^2,k^2,q^2), & \lambda_3(k^2,p^2,q^2) &= \lambda_3(p^2,k^2,q^2),\\ \tau_8(k^2,p^2,q^2) &= \tau_8(p^2,k^2,q^2), & \lambda_4(k^2,p^2,q^2) &= -\lambda_4(p^2,k^2,q^2). \end{aligned} \tag{12}$$

(iii) At zeroth order in perturbation theory the full vertex is γ^μ. Since at this order $F = 1$, we see from Eqs. (7) and (8) that $\Gamma_L^\mu = \gamma^\mu$, consequently, $\Gamma_T^\mu = 0$. Thus the $\tau_i = \mathcal{O}(\alpha)$ in perturbation theory.

(iv) The propagator for the photon carrying momentum q is

$$\begin{aligned} i\Delta_{\mu\nu}(q) &= -i\left[\frac{G(q^2)}{q^2}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) + \xi\frac{q_\mu q_\nu}{q^4}\right],\\ &= -i\left[\Delta_{\mu\nu}^T + \xi\frac{q_\mu q_\nu}{q^4}\right], \end{aligned} \tag{13}$$

where $G(q^2)$ is the photon renormalization function, ξ is the covariant gauge parameter and the $\Delta_{\mu\nu}^T$ is the transverse part of the photon propagator. The bare photon propagator, $\Delta_{\mu\nu}^0$, has $G(q^2) \equiv 1$ in Eq. (13). Gauge covariance is expressed through the Landau-Khalatnikov-Fradkin (LKF) transformations [43,44]. These mean that once a Green's function is known in some gauge, then its form in all other gauges is determined. In general, this is, of course, only useful if we know the relevant Green's function precisely in some gauge. Nevertheless, the LKF transformations provides two key results we shall use. The first concerns the fermion wave function renormalization, $F(p^2)$, which can only depend on the covariant gauge through a unique factor of ξ in its anomalous dimension. The second fact is that the photon wave function renormalization, $G(q^2)$, must be gauge independent. Both of these requirements place restrictions on the form of the nonperturbative interactions.

(v) In QED the full propagators and the vertex function are all divergent. However, as is well known [16,17,45], one can define finite (renormalized) propagators and vertex function by absorbing these divergences into functions, Z_i ($i = 1, 2, 3$). As usual we introduce field renormalizations:

$$\Psi_R = Z_2^{-1/2}\Psi_0, \qquad A_R^\mu = Z_3^{-1/2}A_0^\mu, \tag{14}$$

where the subscripts R and 0 denote renormalized and bare quantities, respectively. The latter are conveniently made finite by introducing an ultraviolet momentum cutoff Λ and the former renormalized quantities depend on the momentum scale μ at which we choose to renormalize. The divergence of the fermion propagator is absorbed into Z_2, the fermion renormalization function, by

$$S_R(p, \mu) = Z_2^{-1}(\mu, \Lambda)S_0(p, \Lambda), \tag{15}$$

and similarly for the photon,

$$\Delta^R_{\mu\nu}(p, \mu) = Z_3^{-1}(\mu, \Lambda)\Delta^0_{\mu\nu}(p, \Lambda). \tag{16}$$

The gauge covariance of the photon propagator requires that the covariant gauge parameter is similarly renormalized:

$$\xi_R = Z_3^{-1}\xi. \tag{17}$$

The divergence of the vertex function is canceled by the factor Z_1:

$$\Gamma^R_\mu(p, \mu) = Z_1(\mu, \Lambda)\Gamma^0_\mu(p, \Lambda), \tag{18}$$

with the above definitions, the coupling constant is renormalized according to

$$e_R = \frac{Z_2}{Z_1}\sqrt{Z_3 e}. \tag{19}$$

Making use of the Ward-Green-Takahashi identity [38–40],

$$Z_1 = Z_2, \tag{20}$$

the coupling constant renormalization becomes

$$e_R = Z_3^{1/2}e.$$

As usual, we define $\alpha = e^2/(4\pi)$, where $e_R = Z_3^{1/2}e$, α_0, α_R denote the bare and renormalized couplings related to e and e_R, respectively.

What we want to determine are the constraints these renormalizations of the fermion and photon propagators impose on the transverse part of the fermion-boson vertex. The renormalization of the 3-point vertex is proportional to fermion renormalization constant Z_2^{-1}.

This can be seen already in the longitudinal vertex from the WGTI [36]. Consequently, the nonperturbative structure of the transverse component, and hence the τ_i's, must be proportional to the inverse of the fermion wave function renormalization, i.e. $\tau_i(F, G) \sim 1/F$, just as the longitudinal λ_i's of Eq. (8) are.

To go further, the basic idea is easily explained by considering the fermion propagator in quenched massless QED. The nonperturbative quantity is the fermion wave function renormalization $F(p^2, \Lambda^2)$. Let us imagine expanding this perturbatively and just keeping leading logarithms, so that we have

$$F(p^2, \Lambda^2) = 1 + \alpha_0 A_1 \ln\frac{p^2}{\Lambda^2} + \alpha_0^2 A_2 \ln^2\frac{p^2}{\Lambda^2} + \alpha_0^3 A_3 \ln^3\frac{p^2}{\Lambda^2} + \ldots, \tag{22}$$

then inserting such a form in the loop integral of Fig. 3. For this to be a solution of the Schwinger-Dyson equation, the equation has to deliver $F(p^2, \Lambda^2)$ with the same perturbative expansion as output. However, to be multiplicatively renormalizable, the coefficients A_n cannot be independent, but related by $A_2 = A_1^2/2$, $A_3 = A_1^3/6$ and finally $A_n = A_1^n/n!$. This requirement places a severe constraint on the fermion-boson vertex. Since its longitudinal part is known, it is its transverse components that are constrained. The aim of this paper is to determine these conditions on the τ_i of Eq. (9) for full QED. In general, these $\tau_i(p^2, k^2q^2)$ functions can be written as a sum of terms, each with the correct dimensions, Eq. (11), symmetry properties, Eq. (12), and renormalization requirements, as

$$\tau_i(p^2, k^2, q^2) = \sum_j f_{ij}(p^2, k^2, q^2)\bar{\tau}_i^{(j)}(F, G). \tag{23}$$

Each of these $\tau_i' s$ has been divided into two parts: *a kinematic part* encoded in f_{ij}, giving the right dimensions, Eq. (11), which depends on momenta squared, and *a functional part*, $\bar{\tau}_i^{(j)}$, that is *assumed* only to know about the fermion and photon renormalization functions F and G at k^2, p^2 or q^2. Such a form would provide a genuine nonperturbative construction,

$$\begin{aligned}\tau_i^{\rm sym}(p^2, k^2, q^2) &= \sum_j [f_{ij}^{\rm anti}(p^2, k^2, q^2)\tau_i^{{\rm anti}(j)}(F, G) + f_{ij}^{\rm sym}(p^2, k^2, q^2)\tau_i^{{\rm sym}(j)}(F, G)],\\ \tau_i^{\rm anti}(p^2, k^2, q^2) &= \sum_j [f_{ij}^{\rm sym}(p^2, k^2, q^2)\tau_i^{{\rm anti}(j)}(F, G) + f_{ij}^{\rm anti}(p^2, k^2, q^2)\tau_i^{{\rm sym}(j)}(F, G)].\end{aligned} \tag{24}$$

The forms of the τ_i's are structured such that the integrals are soluble. First we deal with the kinematic factors for each τ_i's, which are included in the following way:

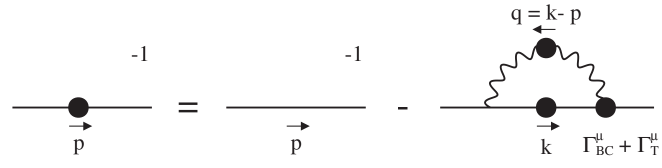

FIG. 3. Unquenched Schwinger-Dyson equation for fermion propagator.

$$\begin{aligned}\tau_2^M(p^2,k^2,q^2) &= \frac{2}{(k^4-p^4)}\left[\beta_2+\gamma_2\frac{2k\cdot p}{k^2+p^2}\right]\tau_2^{\rm anti}(p^2,k^2,q^2)+\frac{2}{(k^2+p^2)^2}\left[\delta_2+\epsilon_2\frac{2k\cdot p}{k^2+p^2}\right]\tau_2^{\rm sym}(p^2,k^2,q^2),\\ \tau_3^M(p^2,k^2,q^2) &= \frac{1}{(k^2-p^2)}\left[\beta_3+\gamma_3\frac{2k\cdot p}{k^2+p^2}\right]\tau_3^{\rm anti}(p^2,k^2,q^2)+\frac{1}{(k^2+p^2)}\left[\delta_3+\epsilon_3\frac{2k\cdot p}{k^2+p^2}\right]\tau_3^{\rm sym}(p^2,k^2,q^2),\\ \tau_6^M(p^2,k^2,q^2) &= \frac{1}{(k^2+p^2)}\left[\beta_6+\gamma_6\frac{2k\cdot p}{k^2+p^2}\right]\tau_6^{\rm anti}(p^2,k^2,q^2)+\frac{(k^2-p^2)}{(k^2+p^2)^2}\left[\delta_6+\epsilon_6\frac{2k\cdot p}{k^2+p^2}\right]\tau_6^{\rm sym}(p^2,k^2,q^2),\\ \tau_8^M(p^2,k^2,q^2) &= \frac{1}{(k^2-p^2)}\left[\beta_8+\gamma_8\frac{2k\cdot p}{k^2+p^2}\right]\tau_8^{\rm anti}(p^2,k^2,q^2)+\frac{1}{(k^2+p^2)}\left[\delta_8+\epsilon_8\frac{2k\cdot p}{k^2+p^2}\right]\tau_8^{\rm sym}(p^2,k^2,q^2).\end{aligned} \tag{25}$$

The factor 2 in the numerator of τ_2 is merely for later convenience and superscript M stands for Minkowski space. The kinematic factors, $f_{ij}^{\rm sym,anti}$ play two roles: first to ensure that each of $\tau_i^{\rm sym,anti}$ is dimensionless, and to define the appropriate symmetry of these functions under the interchange of k, p. To make the problem tractable we do not include q^2 dependence in the denominator factors. However, the dimensions and symmetry of the $\tau_i^{\rm sym,anti}$ is, of course, maintained by multiplying by a factor of $q^2/(k^2+p^2)$. Such a factor can be rewritten as $1-2k\cdot p/(k^2+p^2)$, and this is the origin of the inclusion of the β_i, γ_i, δ_i, ϵ_i terms in Eq. (25).

The $\tau_i^{\rm anti}$ and $\tau_i^{\rm sym}$ are antisymmetric and symmetric under $k^2\leftrightarrow p^2$, respectively. The $\tau_i^{\rm sym,anti}$ are assumed to be solely functions of the fermion and boson renormalization functions F and G, with consequently simplified dependence on k^2, p^2 and q^2. Since here we expand these functions in terms of leading logarithms, it is helpful to note that combinations like $\log(k^2/p^2)$ are antisymmetric, while $\log(q^4/k^2p^2)$ is clearly symmetric under the interchange of k and p, with each power of a "log" being multiplied by a factor of α_0. Such forms are the basis for the leading logarithmic expansion of the $\tau_i^{\rm sym,anti}$. Before renormalization, these will depend on the ultraviolet cutoff Λ, and we can represent the $\tau_i^{\rm sym,anti}$ by

$$\begin{aligned}\tau_i^{\rm anti}(p^2,k^2,q^2) = \sum_{m=1}^{\infty}\sum_{n,r=0}^{\infty}\mathcal{A}_{mnrr}\Bigg[\left(\alpha_0\ln\frac{k^2}{\Lambda^2}\right)^m \\ -\left(\alpha_0\ln\frac{p^2}{\Lambda^2}\right)^m\Bigg]\left(\alpha_0\ln\frac{q^2}{\Lambda^2}\right)^n \\ \times\left(\alpha_0^2\ln\frac{k^2}{\Lambda^2}\ln\frac{p^2}{\Lambda^2}\right)^r,\end{aligned} \tag{26}$$

$$\begin{aligned}\tau_i^{\rm sym}(p^2,k^2,q^2) = \sum_{m=0}^{\infty}\sum_{n,r=0}^{\infty}\mathcal{S}_{mnrr}\Bigg[\left(\alpha_0\ln\frac{k^2}{\Lambda^2}\right)^m \\ +\left(\alpha_0\ln\frac{p^2}{\Lambda^2}\right)^m\Bigg]\left(\alpha_0\ln\frac{q^2}{\Lambda^2}\right)^n \\ \times\left(\alpha_0^2\ln\frac{k^2}{\Lambda^2}\ln\frac{p^2}{\Lambda^2}\right)^r.\end{aligned} \tag{27}$$

The fact mentioned earlier that the zeroth order vertex contribution comes from the longitudinal component, γ^μ, imposes the condition that there can be no leading order term in any transverse component. Consequently $\mathcal{S}_{0000}=0$. It is important to note that the coefficients $\mathcal{A}$ and $\mathcal{S}$ are constants in the above expressions and these depend on indices m, n, r. These are labeled by $mnrr$ to make it easy to read off that such a term contributes at $\mathcal{O}(\alpha_0^{m+n+r+r})$. Expanding Eqs. (26) and (27) to $\mathcal{O}(\alpha_0^3)$:

$$\begin{aligned}\tau_i^{\rm anti}(p^2,k^2,q^2) &= \alpha_0\mathcal{A}_{1000}\left(\ln\frac{k^2}{\Lambda^2}-\ln\frac{p^2}{\Lambda^2}\right)+\alpha_0^2\left\{\mathcal{A}_{2000}\left(\ln^2\frac{k^2}{\Lambda^2}-\ln^2\frac{p^2}{\Lambda^2}\right)+\mathcal{A}_{1100}\left(\ln\frac{k^2}{\Lambda^2}-\ln\frac{p^2}{\Lambda^2}\right)\ln\frac{q^2}{\Lambda^2}\right\}\\ &\quad+\alpha_0^3\left\{\mathcal{A}_{3000}\left(\ln^3\frac{k^2}{\Lambda^2}-\ln^3\frac{p^2}{\Lambda^2}\right)+\mathcal{A}_{2100}\left(\ln^2\frac{k^2}{\Lambda^2}-\ln^2\frac{p^2}{\Lambda^2}\right)\ln\frac{q2}{\Lambda^2}+\mathcal{A}_{1200}\left(\ln\frac{k^2}{\Lambda^2}-\ln\frac{p^2}{\Lambda^2}\right)\ln^2\frac{q^2}{\Lambda^2}\right.\\ &\quad\left.+\mathcal{A}_{1011}\left(\ln\frac{k^2}{\Lambda^2}-\ln\frac{p^2}{\Lambda^2}\right)\ln\frac{k^2}{\Lambda^2}\ln\frac{p^2}{\Lambda^2}\right\}+\mathcal{O}(\alpha^4),\end{aligned} \tag{28}$$

$$\begin{aligned}\tau_i^{\rm sym}(p^2,k^2,q^2) &= \alpha_0\left\{\mathcal{S}_{1000}\left(\ln\frac{k^2}{\Lambda^2}\right)+2\mathcal{S}_{0100}\ln\frac{q^2}{\Lambda^2}\right\}+\alpha_0^2\left\{\mathcal{S}_{2000}\left(\ln^2\frac{k^2}{\Lambda^2}+\ln^2\frac{p^2}{\Lambda^2}\right)+2\mathcal{S}_{0200}\ln^2\frac{q^2}{\Lambda^2}+\mathcal{S}_{1100}\left(\ln\frac{k^2}{\Lambda^2}+\ln\frac{p^2}{\Lambda^2}\right)\right.\\ &\quad\left.\times\ln\frac{q^2}{\Lambda^2}+2\mathcal{S}_{0011}\ln\frac{k^2}{\Lambda^2}\ln\frac{p^2}{\Lambda^2}\right\}+\alpha_0^3\left\{\mathcal{S}_{3000}\left(\ln^3\frac{k^2}{\Lambda^2}+\ln^3\frac{p^2}{\Lambda^2}\right)+2\mathcal{S}_{0300}\ln^3\frac{q^2}{\Lambda^2}+\mathcal{S}_{2100}\left(\ln^2\frac{k^2}{\Lambda^2}+\ln^2\frac{p^2}{\Lambda^2}\right)\right.\\ &\quad\left.\times\ln\frac{q^2}{\Lambda^2}+\mathcal{S}_{1200}\left(\ln\frac{k^2}{\Lambda^2}+\ln\frac{p^2}{\Lambda^2}\right)+\mathcal{S}_{1011}\left(\ln\frac{k^2}{\Lambda^2}+\ln\frac{p^2}{\Lambda^2}\right)\ln\frac{k^2}{\Lambda^2}\ln\frac{p^2}{\Lambda^2}+2\mathcal{S}_{0111}\ln\frac{k^2}{\Lambda^2}\ln\frac{p^2}{\Lambda^2}\ln\frac{q^2}{\Lambda^2}\right\}+\mathcal{O}(\alpha_0^4).\end{aligned} \tag{29}$$

One should keep in mind in the rest of this section that the sum of m, n, r, r adds up to the order of the expansion. Thus, for example, at $\mathcal{O}(\alpha_0^2)$ one only has coefficients ($\mathcal{A}_{2000}$, $\mathcal{A}_{1100}$) in $\tau_i^{\rm anti}$ and ($\mathcal{S}_{2000}$, $\mathcal{S}_{0200}$, $\mathcal{S}_{1100}$, $\mathcal{S}_{0011}$) in $\tau_i^{\rm sym}$. In turn, the dependence of $\mathcal{A}_{mnrr}$ and $\mathcal{S}_{mnrr}$ on ξ and N_F can only happen such that the maximum power of each of them is $m+n+2r$, i.e. the order of α_0 too.

As mentioned earlier the dominant ultraviolet behavior of the vertex to $\mathcal{O}(\alpha_0)$ is given by the longitudinal component [41], Eq. (7), the transverse vertex has no leading logarithms, i.e. ($\alpha_0^n \ln^n \Lambda^2$) terms must vanish. Consequently, in Eqs. (27) and (29) the relation at $\mathcal{O}(\alpha_0 \ln \Lambda^2)$:

$$\mathcal{S}^i_{1000} + \mathcal{S}^i_{0100} = 0; \tag{30}$$

at $\mathcal{O}(\alpha_0^2 \ln^2 \Lambda^2)$:

$$\mathcal{S}^i_{2000} + \mathcal{S}^i_{0200} + \mathcal{S}^i_{0011} + \mathcal{S}^i_{1100} = 0; \tag{31}$$

at $\mathcal{O}(\alpha_0^3 \ln^3 \Lambda^2)$:

$$\mathcal{S}^i_{2100} + \mathcal{S}^i_{3000} + \mathcal{S}^i_{0300} + \mathcal{S}^i_{1011} + \mathcal{S}^i_{0111} + \mathcal{S}^i_{1200} = 0; \tag{32}$$

and, in general, at $\mathcal{O}(\alpha_0^u \ln^u \Lambda^2)$:

$$\sum_{nr=0}^{u} \mathcal{S}^i_{m=[u-n-2r]nrr} = 0 \tag{33}$$

must hold.

Our aim is to determine the conditions on the constants $\mathcal{A}^i_{mnrr}$ and $\mathcal{S}^i_{mnrr}$ for $i = 2, 3, 6, 8$ imposed by the fact that the fermion and photon propagators satisfy the appropriate Schwinger-Dyson equations and that these must be multiplicatively renormalizable. These constraints must, of course, be fulfilled by the full 3-point vertex. In the weak coupling limit, perturbative calculation of the relevant Feynman graphs will give explicit values for these constants. However, the τ_i's that enter here determine not the full vertex, but projections defined by the Schwinger-Dyson equations of the next section.

III. UNQUENCHED SCHWINGER-DYSON CALCULATIONS

A. Fermion propagator

The Schwinger-Dyson equation for the fermion propagator displayed in Fig. 3 can be written as

$$-iS_F^{-1}(p) = -iS_F^{0^{-1}}(p) - \int_M \frac{d^4k}{(2\pi)^4} \times (-ie\Gamma^\mu(p,k;q))iS_F(k)(-ie\gamma^\nu)i\Delta_{\mu\nu}(q). \tag{34}$$

Substituting the form of the longitudinal part of the photon propagator from Eq. (13) and using the Ward-Green-Takahashi identity of Eq. (1), we can rewrite Eq. (34) as

$$\begin{aligned} iS_F^{-1}(p) &= iS_F^{0^{-1}}(p) - e^2 \int_M \frac{d^4k}{(2\pi)^4}\Big\{\Gamma^\mu(p,k;q)S_F(k)\gamma^\nu \\ &\quad \times \Delta^T_{\mu\nu}(q) + \xi(S_F^{-1}(k) - S_F^{-1}(p))S_F(k)\frac{\not q}{q^4}\Big\}, \\ &= iS_F^{0^{-1}}(p) - e^2 \int_M \frac{d^4k}{(2\pi)^4}\Big\{\Gamma^\mu(p,k;q)S_F(k)\gamma^\nu \\ &\quad \times \Delta^T_{\mu\nu}(q) + \xi\Big(\frac{\not q}{q^4} - S_F^{-1}(p)S_F(k)\frac{\not q}{q^4}\Big)\Big\}. \end{aligned} \tag{35}$$

The second term in the integrand being an odd integral gives zero:

$$\int \frac{d^4k}{(2\pi)^4}\frac{\not q}{q^4} = 0, \tag{36}$$

if a translation invariant regularization is employed [24]. After substituting the fermion and photon propagators, Eqs. (2) and (13), explicitly in Eq. (35), we obtain

$$\begin{aligned} \frac{\not p}{F(p^2,\Lambda^2)} &= \not p + \frac{ie^2}{(2\pi)^4}\int_M d^4k\Big\{\Gamma^\mu(p,k;q)\frac{F(k^2)}{\not k} \\ &\quad \times \gamma^\nu \frac{G(q^2)}{q^2}\Big(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\Big) \\ &\quad - \xi\frac{\not p}{F(p^2)}\frac{F(k^2)}{\not k}\frac{\not q}{q^4}\Big\}. \end{aligned} \tag{37}$$

Multiplying this equation by $\not p/4$, taking its trace and rearranging, we arrive at the following equation for the fermion wave function renormalization:

$$\begin{aligned} \frac{1}{F(p^2,\Lambda^2)} &= 1 + \frac{ie^2}{4p^2(2\pi)^4}\int_M \frac{d^4k}{k^2q^2} \\ &\quad \times {\rm Tr}\not p\Big\{\Gamma^\mu(p,k;q)\not k\gamma^\nu F(k^2)G(q^2) \\ &\quad \times \Big(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\Big) - \frac{\xi}{q^2}, \not p\not k\not q\frac{F(k^2)}{F(p^2)}\Big\}. \end{aligned} \tag{38}$$

We see this equation involves a particular projection of the full vertex Γ^μ. To make this explicit we substitute into this equation the general form given by the Ball-Chiu longitudinal part, Eq. (7), and the transverse component, Eq. (7):

$$\frac{1}{F(p^2,\Lambda^2)}=1+\frac{ie^2}{4p^2(2\pi)^4}\int_M\frac{d^4k}{k^2q^2}F(k^2)\bigg\{-\frac{\xi}{q^2}\frac{1}{F(p^2)}\mathrm{Tr}(\not p\not p\not k\not q)+\lambda_1(p^2,k^2,q^2)G(q^2)\mathrm{Tr}\bigg[\not pL_1^\mu(p,k,q)\not k\gamma^\nu\bigg(g_{\mu\nu}-\frac{q_\mu q_\nu}{q^2}\bigg)\bigg]$$
$$+\lambda_2(p^2,k^2,q^2)G(q^2)\mathrm{Tr}\bigg[\not pL_2^\mu(p,k,q)\not k\gamma^\nu\bigg(g_{\mu\nu}-\frac{q_\mu q_\nu}{q^2}\bigg)\bigg]+G(q^2)\mathrm{Tr}\bigg[\not p\Gamma_T^\mu(p,k,q)\not k\gamma^\nu\bigg(g_{\mu\nu}-\frac{q_\mu q_\nu}{q^2}\bigg)\bigg]\bigg\}, \quad (39)$$

where $d^4k=2\pi k^2dk^2\sin^2\Psi d\Psi$ and Ψ is the angle between the 4-vectors k and p. To perform these integrals, we move to Euclidean space using the Wick rotation ($k_0\to ik_0$, $k_i\to k_i$). After performing an explicit trace algebra in Eq. (39) and inserting the transverse vertex, Γ_T^μ, Eqs. (9) and (10), with its undetermined τ_i's, we obtain

$$\frac{1}{F(p^2,\Lambda^2)}=1-\frac{e^2}{(2\pi)^3p^2}\int_E k^2dk^2\int_0^\pi\sin^2\Psi d\Psi\frac{1}{k^2q^2}\bigg\{-\xi\frac{F(k^2)}{F(p^2)}\frac{p^2}{q^2}(k^2-k\cdot p)+F(k^2)G(q^2)\bigg[\lambda_1^E(p^2,k^2,q^2)$$
$$\times\bigg\{\frac{1}{q^2}[-2\Delta^2-3q^2k\cdot p]\bigg\}+\lambda_2^E(p^2,k^2,q^2)\bigg\{\frac{1}{q^2}[2(k^2+p^2)\Delta^2]\bigg\}+\tau_2^E(p^2,k^2,q^2)\{-(k^2+p^2)\Delta^2\}$$
$$+\tau_3^E(p^2,k^2,q^2)\{2\Delta^2+3q^2k\cdot p\}+\tau_6^E(p^2,k^2,q^2)\{-3(k^2-p^2)k\cdot p\}+\tau_8^E(p^2,k^2,q^2)\{-2\Delta^2\}\bigg]\bigg\}, \quad (40)$$

where $\Delta^2=(k\cdot p)^2-k^2p^2$.

Since multiplicative renormalizabilty is closely related to the ultraviolet behavior of the Green's functions, we make a general perturbative expansion of the nonperturbative fermion and photon wave function renormalizations in powers of leading logarithms as follows:

$$F(p^2,\Lambda^2)=\sum_{u=0}^\infty\alpha_0^uA_u\ln^u\frac{p^2}{\Lambda^2}, \quad (41)$$

$$G(q^2,\Lambda^2)=\sum_{u=0}^\infty\alpha_0^uB_u\ln^u\frac{q^2}{\Lambda^2}. \quad (42)$$

In this paper we will consider leading logarithms only in order to present the ideas and techniques and postpone to a future paper the more involved next-to-leading order. Of course, in perturbation theory the coefficients A_u, B_u have definite values. However, it is the general structure that multiplicative renormalizability determines. We substitute these expansions into Eq. (40) in order to calculate this. The photon wave function renormalization $G(q^2)$ depends on the momentum $q^2=k^2+p^2-2k\cdot p$ therefore it has both a radial and an angular component. However, the angular dependent part of this quantity only contributes to $1/F(p^2)$ beyond the leading order, and so here we can simply approximate $G(q^2)$ with $G(k^2)$. We can then carry out the angular integration in Eq. (40) after inserting the coefficients of the basis tensors, i.e. λ_i's and τ_i's from Eqs. (8) and (25):

$$\frac{1}{F(p^2,\Lambda^2)}=1+\frac{\alpha_0\xi}{4\pi}\int_{p^2}^{\Lambda^2}\frac{dk^2}{k^2}\frac{F(k^2)}{F(p^2)}-\frac{3\alpha_0}{8\pi}\int_{p^2}^{\Lambda^2}\frac{dk^2}{k^2}F(k^2)G(k^2)\bigg[\frac{1}{2}\bigg(\frac{1}{F(k^2)}-\frac{1}{F(p^2)}\bigg)+(\bar\tau_f^{\mathrm{anti}}+\bar\tau_f^{\mathrm{sym}})\bigg], \quad (43)$$

where

$$\bar\tau_f^{\mathrm{anti}}\equiv\beta_2\tau_2^{\mathrm{anti}}+(\beta_3-\gamma_3)\tau_3^{\mathrm{anti}}+(\beta_6+\gamma_6)\tau_6^{\mathrm{anti}}-\beta_8\tau_8^{\mathrm{anti}},$$
$$\bar\tau_f^{\mathrm{sym}}\equiv\delta_2\tau_2^{\mathrm{sym}}+(\delta_3-\varepsilon_3)\tau_3^{\mathrm{sym}}+(\delta_6+\varepsilon_6)\tau_6^{\mathrm{sym}}-\delta_8\tau_8^{\mathrm{sym}}. \quad (44)$$

To evaluate this expression, we have to insert the coefficients of the basis tensors, i.e. the $\tau_i^{\mathrm{anti,sym}}$ from Eqs. (26) and (27) into Eq. (43). Λ is the ultraviolet cutoff for the momentum k introduced in Eq. (3) in accord with Eqs. (15), (16), (18), (41), and (42). One observes from Eq. (43) that there is no contribution to $1/F(p^2,\Lambda^2)$ from the λ_1 part of the longitudinal vertex, Eq. (7), but only from λ_2. On laboriously integrating Eq. (43) and using Eqs. (41) and (42) we arrive at

$$\frac{1}{F(p^2,\Lambda^2)} = 1 - \left\{\frac{\xi}{4\pi}\frac{1}{F(p^2)}\sum_{u=0}^{\infty}\alpha_0^{(u+1)}\frac{A_u}{(u+1)}\ln^{u+1}\frac{p^2}{\Lambda^2} + \frac{3}{16\pi}\left[\frac{1}{F(p^2)}\sum_{u=0}^{\infty}\sum_{t=0}^{\infty}\alpha_0^{u+t+1}\frac{A_uB_t}{(u+t+1)}\ln^{u+t+1}\frac{p^2}{\Lambda^2} - \sum_{t=0}^{\infty}\alpha_0^{t+1}\frac{B_t}{(t+1)}\ln^{t+1}\frac{p^2}{\Lambda^2}\right] - \frac{3}{8\pi}\sum_{u=0}^{\infty}\sum_{t=0}^{\infty}A_uB_t\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\alpha_0^{u+t+m+n+2r+1}\ln^{u+t+m+n+2r+1}\frac{p^2}{\Lambda^2}\bar{\mathcal{A}}^f_{mnrr} \times\left[\frac{1}{(u+t+m+n+r+1)} - \frac{1}{(u+t+n+r+1)}\right] - \frac{3}{8\pi}\sum_{u=0}^{\infty}\sum_{t=0}^{\infty}A_uB_t\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\alpha_0^{u+t+m+n+2r+1} \times\ln^{u+t+m+n+2r+1}\frac{p^2}{\Lambda^2}\bar{\mathcal{S}}^f_{mnrr}\left[\frac{1}{(u+t+m+n+r+1)} + \frac{1}{(u+t+n+r+1)}\right]\right\}, \tag{45}$$

where

$$\bar{\mathcal{A}}^f_{mnrr} \equiv \beta_2\mathcal{A}^2_{mnrr} + (\beta_3-\gamma_3)\mathcal{A}^3_{mnrr} + (\beta_6+\gamma_6)\mathcal{A}^6_{mnrr} - \beta_8\mathcal{A}^8_{mnrr},$$
$$\bar{\mathcal{S}}^f_{mnrr} \equiv \delta_2\mathcal{S}^2_{mnrr} + (\delta_3-\varepsilon_3)\mathcal{S}^3_{mnrr} + (\delta_6+\varepsilon_6)\mathcal{S}^6_{mnrr} - \delta_8\mathcal{S}^8_{mnrr}. \tag{46}$$

In order to rearrange the infinite sums in Eq. (45) in terms of powers of α_0, we convert some of the infinite sums to finite sums:

$$\frac{1}{F(p^2,\Lambda^2)} = 1 - \left\{\frac{\xi}{4\pi}\frac{1}{F(p^2)}\sum_{u=0}^{\infty}\frac{A_u}{(u+1)}\alpha_0^{u+1}\ln^{u+1}\frac{p^2}{\Lambda^2} - \frac{3}{16\pi}\sum_{u=1}^{\infty}\alpha_0^{u+1}\ln^{u+1}\frac{p^2}{\Lambda^2}\left\{\sum_{a=1}^{u}A_aB_{u-a}\frac{1}{(u+1)} + \sum_{b=1}^{u}\sum_{a=0}^{u-b}(-1)^bA_bA_aB_{u-b-a}\frac{1}{(u-b+1)}\right\} + \frac{3}{8\pi}\sum_{u=1}^{\infty}\alpha_0^{u+1}\ln^{u+1}\frac{p^2}{\Lambda^2}(H_u+\bar{H}_u)\right\}, \tag{47}$$

where

$$H_u = \sum_{b=1}^{u}\sum_{c=1}^{b}\sum_{d=0}^{b-c}\sum_{a=1}^{c}A_dB_{b-c-d}R_{u-b}\left\{\frac{1}{[\frac{1}{2}(u+b)+1]} - \frac{1}{[\frac{1}{2}(u+b)-a+1]}\right\}\bar{\mathcal{A}}^f_{a(c-a)((u-b)/2)((u-b)/2)}, \tag{48}$$

$$\bar{H}_u = \sum_{b=0}^{u}\sum_{c=0}^{b}\sum_{d=0}^{b-c}\sum_{a=0}^{c}A_dB_{b-c-d}R_{u-b}\left\{\frac{1}{[\frac{1}{2}(u+b)+1]} + \frac{1}{[\frac{1}{2}(u+b)-a+1]}\right\}\bar{\mathcal{S}}^f_{a(c-a)((u-b)/2)((u-b)/2)}, \tag{49}$$

with

$$R_j = \begin{cases}1 & \text{if } j \text{ is even}\\ 0 & \text{if } j \text{ is bold.}\end{cases} \tag{50}$$

The above expression for the fermion wave function renormalization, $1/F(p^2,\Lambda^2)$, is the exact nonpertubative calculation for the massless fermions in a general covariant gauge at leading logarithmic order. In this equation the $\mathcal{A}^i_{mnrr}$'s and $\mathcal{S}^i_{mnrr}$'s are the constants to be constrained by multiplicative renormalization. For the purpose of explaining how this works, we will first implement it order-by-order, then, we generalize. To do this, we expand the fermion wave function renormalization, Eq. (47), in $\mathcal{O}(\alpha^4)$:

$$\frac{1}{F(p^2,\Lambda^2)} = 1 + \frac{1}{4\pi}\left\{-\boldsymbol{\alpha_0}\xi\,\mathbf{ln}\frac{\mathbf{p^2}}{\boldsymbol{\Lambda^2}} - \boldsymbol{\alpha_0^2}\,\mathbf{ln^2}\frac{\mathbf{p^2}}{\boldsymbol{\Lambda^2}}\left[-\left(\frac{\xi}{2}+\frac{3}{8}\right)A_1 + \frac{3}{4}(\bar{\mathcal{A}}^f_{1000} - \bar{\mathcal{S}}^f_{1000})\right] - \boldsymbol{\alpha_0^3}\,\mathbf{ln^3}\frac{\mathbf{p^2}}{\boldsymbol{\Lambda^2}}\left[-\left(\frac{\xi}{2}+\frac{3}{8}\right)A_1^2 + \left(\frac{4\xi}{3}+1\right)A_2 - \frac{A_1B_1}{8} + \frac{1}{4}(A_1+B_1)(\bar{\mathcal{A}}^f_{1000} - \bar{\mathcal{S}}^f_{1000}) + \bar{\mathcal{A}}^f_{2000} + \frac{1}{4}\bar{\mathcal{A}}^f_{1100} - \frac{3}{4}\bar{\mathcal{S}}^f_{2000} + \frac{1}{4}\bar{\mathcal{S}}^f_{0200} - \frac{1}{4}\bar{\mathcal{S}}^f_{0011}\right] - \boldsymbol{\alpha_0^4}\,\mathbf{ln^4}\frac{\mathbf{p^2}}{\boldsymbol{\Lambda^2}}\left[-A_3\left(\frac{9}{16}+\frac{3}{4}\xi\right) + A_1A_2\left(\frac{1}{8}+\frac{1}{6}\xi\right) - \frac{1}{4}A_1^2B_1 + \frac{9}{16}A_2B_1 - \frac{1}{16}A_1B_2 + \frac{1}{8}(A_2+B_2)(\bar{\mathcal{A}}^f_{1000} - \bar{\mathcal{S}}^f_{1000}) + \frac{1}{8}A_1B_1(\bar{\mathcal{A}}^f_{1000} - \bar{\mathcal{S}}^f_{1000}) + (A_1+B_1)\left(+\frac{3}{8}\bar{\mathcal{A}}^f_{2000} + \frac{1}{8}\bar{\mathcal{A}}^f_{1100} - \frac{1}{4}\bar{\mathcal{S}}^f_{2000} - \frac{1}{8}\bar{\mathcal{S}}^f_{0011} + \frac{1}{8}\bar{\mathcal{S}}^f_{0200}\right) + \frac{1}{4}\bar{\mathcal{A}}^f_{1011} + \frac{1}{8}\bar{\mathcal{A}}^f_{1200} + \frac{3}{8}\bar{\mathcal{A}}^f_{2100} + \frac{9}{8}\bar{\mathcal{A}}^f_{3000} + \frac{1}{8}\bar{\mathcal{S}}^f_{0111} + \frac{3}{8}\bar{\mathcal{S}}^f_{0300} - \frac{1}{8}\bar{\mathcal{S}}^f_{1011} + \frac{1}{4}\bar{\mathcal{S}}^f_{1200} - \frac{3}{4}\bar{\mathcal{S}}^f_{3000}\right] - \mathcal{O}(\boldsymbol{\alpha_0^5})\right\}. \tag{51}$$

Equations (30)–(32) have been input to obtain this expression. Equations (45), (47), and (51) illustrate how the fermion wave function renormalization depends on the explicit form of the full 3-point vertex. As we shall see in Sec. IV, the expansion to $\mathcal{O}(\alpha^4 \ln^4)$ is the minimum order at which we can recognize the pattern of constraints.

B. Photon propagator

Next we discuss the Schwinger-Dyson equation for the gauge boson. This equation has some different features from the fermion SDE. Now, the two fermion legs have to be treated equally. We can ensure this symmetry property by dividing the external momentum flow equally in the loop as shown in Fig. 4

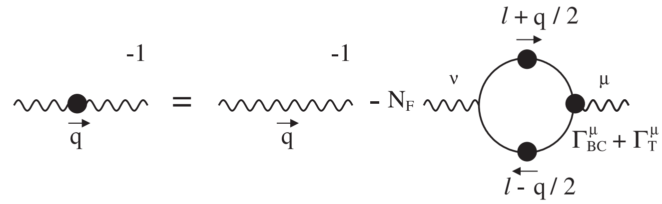

FIG. 4. Unquenched Schwinger-Dyson equation for photon propagator.

Using the Feynman rules, Fig. 4 can be expressed as

$$
\begin{aligned}
-i\Delta^{-1}_{\mu\nu}(q) = -i\Delta^{0}_{\mu\nu}{}^{-1}(q) - (-1)N_F \int_M \frac{d^4\ell}{(2\pi)^4} \\
\times \mathrm{Tr}[(-ie\Gamma^\mu(\ell_-, \ell_+; q))iS_F(\ell_+) \\
\times (-ie\gamma^\nu)iS_F(\ell_-)],
\end{aligned} \tag{52}
$$

which can be symbolically written as $\Delta^{-1}_{\mu\nu}(q) = \Delta^{0}_{\mu\nu}{}^{-1}(q) + \Pi_{\mu\nu}(q)$, where $\Pi_{\mu\nu}$ is the photon self-energy and $\ell_+ \equiv (\ell + q/2)$, $\ell_- \equiv (\ell - q/2)$.

The definitions of the fermion and photon propagators[1] are given already in Sec. II,

$$
\begin{aligned}
iS_F(\ell_+) &= iF(\ell_+)/\not{\ell}_+, \\
i\Delta^{\mu\nu}(q) &= -\frac{i}{q^2}\left[G(q)\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right) + \xi\frac{q^\mu q^\nu}{q^2}\right].
\end{aligned}
$$

Equation (52) must satisfy the photon Ward identity, $q^\mu \Delta^{-1}_{\mu\nu} = q_\nu q^2/\xi$, which is, of course, fulfilled by the bare propagator in Eq. (52). Consequently, the loop graph of Fig. 4 must be transverse. Contracting Eq. (52) with q^μ and using the Ward-Green-Takahashi identity of Eq. (1), this transversality requires

$$
q_\mu \Pi^{\mu\nu} = \frac{iN_F e^2}{q^2(2\pi)^4}\int_M d^4\ell\, \mathrm{Tr}[\gamma^\nu\{S_F(\ell_+) - S_F(\ell_-)\}] = 0. \tag{53}
$$

If dimensional regularization is used, then this integral is automatically zero. However with cutoff regularization, this is not the case. Then $\Pi_{\mu\nu}$ is not entirely transverse. To extract the correct component, we introduce the following tensor [46,47]:

$$
P_{\mu\nu} = \frac{1}{3q^4}(4q_\mu q_\nu - q^2 g_{\mu\nu}). \tag{54}
$$

Projecting Eq. (52) with $P_{\mu\nu}$ allows us to remove the potentially quadratically divergent term in four-dimensions, and project out the required ultraviolet logarithmically divergent terms. It is easy to check that this leaves the correct leading logarithms. We then have a scalar equation for the photon wave function renormalization:

$$
\begin{aligned}
\frac{1}{G(q^2, \Lambda^2)} = 1 + N_F\frac{i\alpha_0}{4\pi^3}\int_M \frac{d^4\ell}{\ell_+^2\ell_-^2}F(\ell_-)F(\ell_+)P_{\mu\nu} \\
\times\ \mathrm{Tr}[\Gamma^\mu_F(\ell_-, \ell_+, q))\not{\ell}_+\gamma^\nu\not{\ell}_-].
\end{aligned} \tag{55}
$$

Recalling the definition of the vertex of Eqs. (7)–(10), we obtain

$$
\begin{aligned}
\frac{1}{G(q^2, \Lambda^2)} = 1 + N_F\frac{i\alpha_0}{4\pi^3}\int_M \frac{d^4\ell}{\ell_+^2\ell_-^2}F(\ell_-)F(\ell_+)P_{\mu\nu} \\
\times\{\lambda_1^M(\ell_-^2, \ell_+^2, q^2)\,\mathrm{Tr}(\gamma^\mu\not{\ell}_+\gamma^\nu\not{\ell}_-) \\
+ \lambda_2^M(\ell_-^2, \ell_+^2, q^2)\,\mathrm{Tr}(4\gamma^\mu\not{\ell}_+\not{\ell}\ell^\nu\not{\ell}_-) \\
+ \mathrm{Tr}(\Gamma^\mu_T\not{\ell}_+\gamma^\nu\not{\ell}_-)\}.
\end{aligned} \tag{56}
$$

Moving to Euclidean space, we perform a Wick rotation. Substituting $d^4\ell = 2\pi\ell^2 d\ell^2 d\psi \sin^2\psi$ and the form of the transverse vertex from Eqs. (9) and (10), and then taking the traces leads to

[1]Where appropriate, we denote the fermion and photon wave function renormalization functions as $F(p)$ or $F(p^2)$ and $G(p)$ or $G(p^2)$, respectively. Where we wish to emphasize that the quantities are unrenormalized, Λ^2 will be added to the list of arguments—with similar conventions for the renormalized quantities, for instance $F_R(p)$ and $G_R(p)$.

$$\frac{1}{G(q^2,\Lambda^2)} = 1 - \frac{\alpha_0 N_F}{6\pi^2 q^2}\int_E \frac{\ell^2 d\ell^2}{\ell_+^2\ell_-^2}\int_0^\pi \sin^2\psi d\psi F(\ell_+)F(\ell_-)\Big\{2\lambda_1^E(\ell_-^2,\ell_+^2,q^2)\Big\{16\frac{(\ell\cdot q)^2}{q^2} - 3q^2 - 4\ell^2\Big\} + 2\lambda_2^E(\ell_-^2,\ell_+^2,q^2)\Big\{-\Big(16\frac{\ell^2}{q^2}-2\Big)(\ell\cdot q)^2 + 4\ell^4 + q^2\ell^2\Big\} + \tau_2^E(\ell_-^2,\ell_+^2,q^2)\{2(4\ell^2+q^2)\Delta^2\} + \tau_3^E(\ell_-^2,\ell_+^2,q^2)\{-8\Delta^2 - 3q^2(4\ell^2-q^2)\} + \tau_6^E(\ell_-^2,\ell_+^2,q^2)\{6\ell\cdot q(4\ell^2-q^2)\} + \tau_8^E(\ell_-^2,\ell_+^2,q^2)\{8\Delta^2\}\Big\}, \quad (57)$$

where $\Delta^2 = (\ell\cdot q)^2 - \ell^2 q^2$ and the photon Schwinger-Dyson equation picks out loop momentum regions where $\ell_+^2 \sim \ell_-^2 \sim \ell^2 \gg q^2$. This allows us to carry out the angular integrals in Eq. (57) for the leading log terms after inserting λ_i's and τ_i's from Eqs. (8) and (25). This gives

$$\frac{1}{G(q^2,\Lambda^2)} = 1 + \frac{\alpha_0 N_F}{3\pi}\int_{q^2}^{\Lambda^2}\frac{d\ell^2}{\ell^2}F^2(\ell)\Big\{\frac{1}{F(\ell)} + \frac{3}{4}\bar{\tau}_\gamma^{\rm sym}\Big\}, \quad (58)$$

where

$$\bar{\tau}_\gamma^{\rm sym} \equiv (\delta_2+\varepsilon_2)\tau_2^{\rm sym} - (\delta_3+\varepsilon_3)\tau_3^{\rm sym} + (\delta_6+\varepsilon_6)\tau_6^{\rm sym} - (\delta_8+\varepsilon_8)\tau_8^{\rm sym}. \quad (59)$$

This time the explicit longitudinal contribution comes from λ_1; λ_2 does not contribute to the leading log's. Using Eq. (41) and performing the radial integration, Eq. (58) yields

$$\frac{1}{G(q^2,\Lambda^2)} = 1 - \frac{N_F}{3\pi}\alpha_0\ln\frac{q^2}{\Lambda^2} + \frac{N_F}{3\pi}\times\Big\{-\sum_{u=1}^\infty \alpha_0^{u+1}\frac{A_u}{(u+1)}\ln^{u+1}\frac{q^2}{\Lambda^2} - \frac{3}{2}\sum_{u=0}^\infty A_u' \times\sum_{m=0}^\infty\sum_{n=0}^\infty\sum_{r=0}^\infty \alpha_0^{u+n+m+2r+1}\ln^{u+n+m+2r+1}\frac{q^2}{\Lambda^2}\frac{\bar{\mathcal{S}}^\gamma_{mnrr}}{(u+n+2r+1)}\Big\}, \quad (60)$$

where

$$\bar{\mathcal{S}}^\gamma_{mnrr} \equiv (\delta_2+\varepsilon_2)\mathcal{S}^2_{mnrr} - (\delta_3+\varepsilon_3)\mathcal{S}^3_{mnrr} + (\delta_6+\varepsilon_6)\mathcal{S}^6_{mnrr} - (\delta_8+\varepsilon_8)\mathcal{S}^8_{mnrr}, \quad (61)$$

$$A_u' \equiv \sum_{d=1}^u \frac{2d}{u}A_d A_{u-d}. \quad (62)$$

Evaluating the multiple sums using the symmetries and rearranging terms with respect to powers of α_0 yields

$$\frac{1}{G(q^2,\Lambda^2)} = 1 - \frac{N_F}{3\pi}\alpha_0\ln\frac{q^2}{\Lambda^2} - \frac{N_F}{3\pi}\sum_{u=1}^\infty \alpha_0^{u+1}\ln^{u+1}\frac{q^2}{\Lambda^2}\times\Big\{\frac{A_u}{(u+1)} + \frac{3}{2}K_u\Big\},$$

where

$$K_u = \sum_{b=0}^u \frac{R_{u-b}}{(u+a-c+1)}\sum_{c=0}^b\sum_{a=0}^c\Big(\sum_{d=1}^{(b-c)}\frac{2d}{(b-c)}A_d A_{(b-c-d)}\Big)\times\bar{\mathcal{S}}_{a(c-a)((u-b)/2)((u-b)/2)},$$
$$K_0 = 0, \quad (63)$$

with R_j defined by Eq. (50). Employing the expansion of the transverse vector coefficients introduced in Eqs. (25)–(27), we can then write $1/G(q^2)$ analogous to the fermion result for $1/F(p^2)$ of Eq. (51), after performing the many integrals:

$$\frac{1}{G(q^2,\Lambda^2)} = 1 + \frac{N_F}{3\pi}\Big\{-\boldsymbol{\alpha_0}\ln\frac{\mathbf{q^2}}{\mathbf{\Lambda^2}} - \boldsymbol{\alpha_0^2}\ln^2\frac{\mathbf{q^2}}{\mathbf{\Lambda^2}}\Big[\frac{A_1}{2} - \frac{3}{4}\bar{\mathcal{S}}^\gamma_{1000}\Big] - \boldsymbol{\alpha_0^3}\ln^3\frac{\mathbf{q^2}}{\mathbf{\Lambda^2}}\Big[\frac{A_2}{3} - \frac{1}{2}A_1\bar{\mathcal{S}}^\gamma_{1000} - \frac{1}{4}\bar{\mathcal{S}}^\gamma_{2000} + \frac{3}{4}\bar{\mathcal{S}}^\gamma_{0200} - \frac{1}{4}\bar{\mathcal{S}}^\gamma_{0011}\Big] - \boldsymbol{\alpha_0^4}\ln^4\frac{\mathbf{q^2}}{\mathbf{\Lambda^2}}\Big[\frac{A_3}{4} - \frac{1}{4}A_1^2\bar{\mathcal{S}}^\gamma_{1000} - \frac{1}{4}A_1\bar{\mathcal{S}}^\gamma_{0011} + \frac{1}{2}A_1\bar{\mathcal{S}}^\gamma_{0200} - \frac{1}{4}A_1\bar{\mathcal{S}}^\gamma_{2000} + \bar{\mathcal{S}}^\gamma_{0300} - \frac{1}{8}\bar{\mathcal{S}}^\gamma_{1011} + \frac{1}{4}\bar{\mathcal{S}}^\gamma_{1200} - \frac{1}{8}\bar{\mathcal{S}}^\gamma_{3000}\Big] + \mathcal{O}(\boldsymbol{\alpha_0^5})\Big\}. \quad (64)$$

We have already made use of Eqs. (30)–(32) in the above expression. Now the transverse vertex must have the right structure, i.e. the right coefficients $\mathcal{A}^i_{mnrr}$, $\mathcal{S}^i_{mnrr}$, so that the solution of the Schwinger-Dyson equations for $1/F(p^2)$ and $1/G(q^2)$, Eqs. (47), (51), (63), and (64), are multiplicatively renormalizable.

IV. MULTIPLICATIVELY RENORMALIZABLE $F(p^2)$ AND $G(q^2)$

A. The photon propagator

We shall first look for the most general form of the multiplicatively renormalizable photon wave function renormalization. In order to do so, the renormalized G_R can be written in the following form by using Eq. (16):

$$G_R(q^2,\mu^2) = Z_3^{-1}(\mu^2,\Lambda^2)G(q^2,\Lambda^2). \quad (65)$$

We define the most general leading logarithmic expansion of the unrenormalized photon wave function renormalization by

$$G(q^2, \Lambda^2) = \sum_{u=0}^{\infty} \alpha_0^u B_u \ln^u \frac{q^2}{\Lambda^2} = 1 + \alpha_0 B_1 \ln\frac{q^2}{\Lambda^2} + \alpha_0^2 B_2 \ln^2 \frac{q^2}{\Lambda^2} + \alpha_0^3 B_3 \ln^3 \frac{q^2}{\Lambda^2} + \mathcal{O}(\alpha_0^4). \tag{66}$$

We impose the renormalization condition that $G_R(q^2 = \mu^2) = 1$. The coefficients $B_i (i > 2)$ are then constrained by multiplicative renormalizability, i.e., $B_2 = B_1^2$, $B_n = (B_1)^n$ so that the renormalized photon wave function renormalization can be written as

$$G_R(q^2, \mu^2) = \sum_{u=0}^{\infty} \alpha_R^u (B_1)^u \ln^u \frac{q^2}{\Lambda^2} = 1 + \alpha_R B_1 \ln\frac{q^2}{\mu^2} + \alpha_R^2 B_1^2 \ln^2 \frac{q^2}{\mu^2} + \alpha_R^3 B_1^3 \ln^3 \frac{q^2}{\mu^2} + \mathcal{O}(\alpha_R^4). \tag{67}$$

Then, as we shall use later, the inverse of G and G_R are

$$\frac{1}{G_R(q^2, \mu^2)} = 1 - \boldsymbol{\alpha}_R B_1 \ln\frac{\mathbf{q^2}}{\boldsymbol{\mu^2}}, \qquad \frac{1}{G(q^2, \Lambda^2)} = 1 - \boldsymbol{\alpha}_0 B_1 \ln\frac{\mathbf{q^2}}{\boldsymbol{\Lambda^2}}, \tag{68}$$

where, as is well known, $1/G(q^2)$ in QED only has a leading logarithm at one loop order, just like $1/\alpha(q^2)$ and being related to this physical quantity is independent of the gauge.

B. The fermion propagator

Analogously to the previous section, we deal with the fermion wave function renormalization. We similarly define the general leading logarithmic expansion of the unrenormalized F as

$$F(p^2, \Lambda^2) = \sum_{u=0}^{\infty} \alpha_0^u A_u \ln^u \frac{q^2}{\Lambda^2} = 1 + \alpha_0 A_1 \ln\frac{p^2}{\Lambda^2} + \alpha_0^2 A_2 \ln^2 \frac{p^2}{\Lambda^2} + \alpha_0^3 A_3 \ln^3 \frac{p^2}{\Lambda^2} + \mathcal{O}(\alpha_0^4). \tag{69}$$

Since not only the coupling, but the gauge parameter have to be renormalized, we need to make the dependence of the A_u on ξ explicit. As gauge dependence in the coefficients arises from photon propagators, any A_u cannot have a higher power of ξ than ξ^u. Consequently, $F(p^2, \Lambda^2)$ can be written as

$$F(p^2, \Lambda^2) = 1 + \alpha_0(a_1\xi + b_1)\ln\frac{p^2}{\Lambda^2} + \alpha_0^2(a_2\xi^2 + b_2\xi + c_2)\ln^2\frac{p^2}{\Lambda^2} + \alpha_0^3(a_3\xi^3 + b_3\xi^2 + c_3\xi + d_3)\ln^3\frac{p^2}{\Lambda^2} + \mathcal{O}(\alpha_0^4), \tag{70}$$

where a_i, b_i, c_i, d_i are constants related to the A_u by comparing Eqs. (69) and (70). Recalling Eqs. (17) and (19), $\xi_0 = Z_3\xi_R$, $\alpha_0 = Z_3^{-1}\alpha_R$ we note that

$$\alpha_0\xi = \alpha_R\xi_R, \quad \text{and}$$
$$F_R(p^2, \mu^2) = Z_2^{-1}(\mu^2/\Lambda^2)F(p^2, \Lambda^2), \tag{71}$$

with the renormalization condition for the fermion wave function renormalization $F_R(p^2 = \mu^2) = 1$. Equation (70) can then be inserted in this equation to give

$$F_R(p^2, \mu^2) = 1 + \alpha_R(a_1\xi_R + b_1)\ln\frac{p^2}{\mu^2} + \alpha_R^2(a_2\xi_R^2 + b_2\xi_R + c_2)\ln^2\frac{p^2}{\mu^2} + \alpha_R^3(a_3\xi_R^3 + b_3\xi_R^2 + c_3\xi_R + d_3)\ln^3\frac{p^2}{\mu^2} + \mathcal{O}(\alpha_R^4). \tag{72}$$

Multiplicative renormalizability requires that the inverse unrenormalized fermion wave function renormalization must have the following form keeping only the leading logarithms:

$$\frac{1}{F(p^2, \Lambda^2)} = 1 + \boldsymbol{\alpha_0} \ln\frac{\mathbf{p^2}}{\boldsymbol{\Lambda^2}}[-a_1\xi - b_1] + \boldsymbol{\alpha_0^2} \ln^2\frac{\mathbf{p^2}}{\boldsymbol{\Lambda^2}}\left[\frac{a_1^2}{2}\xi^2 + a_1b_1\xi + \frac{b_1}{2}(b_1 - B_1)\right] + \boldsymbol{\alpha_0^3} \ln^3\frac{\mathbf{p^2}}{\boldsymbol{\Lambda^2}}\left[-\frac{a_1^3}{6}\xi^3 - \frac{a_1^2b_1}{2}\xi^2 + \frac{a_1b_1}{2}(-b_1 + B_1)\xi - \frac{b_1^3}{6} - \frac{b_1B_1^2}{3} + \frac{b_1^2B_1}{2}\right] + \boldsymbol{\alpha_0^4} \ln^4\frac{\mathbf{p^2}}{\boldsymbol{\Lambda^2}}\left[\frac{a_1^4}{24}\xi^4 + \frac{a_1^3b_1}{6}\xi^3 + \frac{a_1^2b_1}{4}(b_1 - B_1)\xi^2 + \frac{a_1b_1}{2}\left(\frac{b_1^2}{3} - b_1B_1 + \frac{2}{3}B_1^2\right)\xi + \frac{b_1}{4}\left(\frac{b_1^3}{6} - b_1^2B_1 + \frac{11}{6}b_1B_1^2 - B_1^3\right)\right] + \mathcal{O}(\alpha_0^5). \tag{73}$$

The renormalized form of $1/F$ can be found by replacing $\alpha_0 \to \alpha_R$, $\xi \to \xi_R$ and $\Lambda \to \mu$ in the above expression.

V. MR CONSTRAINTS ON THE VERTEX

In Sec. III we have shown exactly how the full vertex contributes in the fermion and boson SDEs. In principle, truncation of the Schwinger-Dyson equations for the fermion and boson propagators requires knowledge of the complete structure of the vertex, all 12 independent components or, here in massless QED, all six. While two are fixed by the Ward-Green-Takahashi identity in terms of the fermion propagator, the four transverse components appear to embody information about all the higher-point Green's functions. Knowledge we do not have, unless we solve the theory completely. However, two simplifications have already occurred. First, the massless fermion and boson self-energies involve just two projections of the six independent vertex vectors, so we do not need to know their complete spin and Lorentz structure. This is helpful, since even at $\mathcal{O}(\alpha_0)$ in perturbation theory, this is of daunting complexity [41]. The second simplification is that multiplicative renormalizability is closely related to the ultraviolet behavior of the loops in Figs. 3 and 4. There not only is the structure of the vertex simpler, but importantly for the present study the two graphs explore the vertex in distinct kinematic regimes. For the fermion self-energy, the internal fermion momentum k and boson momentum q are very much larger than the external fermion momentum p, i.e. $k^2 \simeq q^2 \gg p^2$. In contrast, for the boson self-energy, it is the fermion momenta that are both large, i.e. $k^2 \simeq p^2 \gg q^2$. We shall see that this distinction plays a powerful role in our analysis.

First, in this section we combine the results of the previous two sections to find the constraints on the fermion-photon vertex imposed by multiplicative renormalizability.

A. MR constraints via fermion Schwinger-Dyson equation

In this and the next section, we apply the above strategy first to the fermion wave function renormalization in full massless QED. To do this, we start by comparing order-by-order the results fixed by multiplicatively renormalizable F, Eq. (73), with those found by solving the Schwinger-Dyson equation, Eq. (51). These comparisons will give what we refer to as the fermion conditions, labeled by $FC1$, $FC2$, *etc.*.

$\boldsymbol{\alpha_0 \ln p^2/\Lambda^2}$ *comparison*:

$$-A_1 \equiv -(a_1\xi + b_1) = -\frac{\xi}{4\pi},$$
$$\Downarrow$$
$$a_1 = \frac{1}{4\pi}, \qquad b_1 = 0. \tag{74}$$

In this first order comparison MR fixes the value of a_1 and b_1 and by that all leading order terms in $1/F$ or F, then Eq. (73) requires

$$FC1\colon A_1 = \frac{\xi}{4\pi}, \qquad A_2 = \frac{A_1^2}{2}. \tag{75}$$

$\boldsymbol{\alpha_0^2 \ln^2 p^2/\Lambda^2}$ *comparison*:

$$\frac{a_1^2}{2}\xi^2 + a_1 b_1 \xi + \frac{b_1^2}{2} = \frac{1}{4\pi}\bigg[\left(\frac{\xi}{2}+\frac{3}{8}\right)A_1 - \frac{3}{4}\bar{\mathcal{A}}^f_{1000} + \frac{3}{4}\bar{\mathcal{S}}^f_{1000}\bigg]. \tag{76}$$

Making use of Eqs. (74) and (75) and keeping in mind that $\bar{\mathcal{A}}^f_{1000}$ and $\bar{\mathcal{S}}^f_{1000}$ can be at most proportional to ξ or N_F from Eqs. (28) and (29), we immediately see that the ξ^2 term on both sides automatically matches and for the ξ term we must have

$$FC2\colon \frac{A_1}{2} = \bar{\mathcal{A}}^f_{1000} - \bar{\mathcal{S}}^f_{1000}. \tag{77}$$

$\boldsymbol{\alpha_0^3 \ln^3 p^2/\Lambda^2}$ *comparison*:

$$\begin{aligned} -\frac{a_1^3}{6}\xi^3 &= -\frac{A_1^3}{3!} \\ &= -\frac{1}{4\pi}\bigg\{-\left(\frac{\xi}{2}+\frac{3}{8}\right)A_1^2 + \left(\frac{4\xi}{3}+1\right)A_2 - \frac{A_1B_1}{8} \\ &\quad + \frac{1}{4}(A_1+B_1)(\bar{\mathcal{A}}^f_{1000} - \bar{\mathcal{S}}^f_{1000}) + \bar{\mathcal{A}}^f_{2000} \\ &\quad + \frac{1}{4}\bar{\mathcal{A}}^f_{1100} - \frac{3}{4}\bar{\mathcal{S}}^f_{2000} + \frac{1}{4}\bar{\mathcal{S}}^f_{0200} - \frac{1}{4}\bar{\mathcal{S}}^f_{0011}\bigg\}. \end{aligned} \tag{78}$$

The leading terms in ξ in Eq. (78) [i.e. $O(\xi^3)$] automatically match on the left- and right-hand sides. Imposing Eq. (77), the $O(\xi^2)$ terms require the transverse part to be fixed so that

$$FC3\colon \frac{A_1^2}{4} = -\bar{\mathcal{A}}^f_{2000} - \frac{1}{4}\bar{\mathcal{A}}^f_{1100} + \frac{3}{4}\bar{\mathcal{S}}^f_{2000} - \frac{1}{4}\bar{\mathcal{S}}^f_{0200} + \frac{1}{4}\bar{\mathcal{S}}^f_{0011}. \tag{79}$$

As one can see the B_1 term in Eq. (78) disappears from the above expression and this must repeat itself in every order, i.e. in leading order terms the photon contribution will be canceled out by the transverse vertex.

$\boldsymbol{\alpha_0^4 \ln^4 p^2/\Lambda^2}$ *comparison*:

$$\frac{a_1^4}{24}\xi^4 = \frac{A_1^4}{4!}$$
$$= -\frac{1}{4\pi}\Big\{-A_3\Big(\frac{9}{16}+\frac{3}{4}\xi\Big) + A_1A_2\Big(\frac{1}{8}+\frac{1}{6}\xi\Big) - \frac{1}{4}A_1^2B_1 + \frac{9}{16}A_2B_1 - \frac{1}{16}A_1B_2 + \frac{1}{8}(A_2+B_2)(\bar{\mathcal{A}}^f_{1000} - \bar{\mathcal{S}}^f_{1000})$$
$$+\frac{1}{8}A_1B_1(\bar{\mathcal{A}}^f_{1000} - \bar{\mathcal{S}}^f_{1000}) + (A_1+B_1)\Big(\frac{3}{8}\bar{\mathcal{A}}^f_{2000} + \frac{1}{8}\bar{\mathcal{A}}^f_{1100} - \frac{1}{4}\bar{\mathcal{S}}^f_{2000} - \frac{1}{8}\bar{\mathcal{S}}^f_{0011} + \frac{1}{8}\bar{\mathcal{S}}^f_{0200}\Big) + \frac{1}{4}\bar{\mathcal{A}}^f_{1011}$$
$$+\frac{1}{8}\bar{\mathcal{A}}^f_{1200} + \frac{3}{8}\bar{\mathcal{A}}^f_{2100} + \frac{9}{8}\bar{\mathcal{A}}^f_{3000} + \frac{1}{8}\bar{\mathcal{S}}^f_{0111} + \frac{3}{8}\bar{\mathcal{S}}^f_{0300} - \frac{1}{8}\bar{\mathcal{S}}^f_{1011} + \frac{1}{4}\bar{\mathcal{S}}^f_{1200} - \frac{3}{4}\bar{\mathcal{S}}^f_{3000}\Big\}. \quad (80)$$

Once again in above expression the leading terms in ξ [i.e. $\mathcal{O}(\xi^4)$] terms match on both sides. After substituting the $FC2$ and $FC3$ conditions in Eq. (80), we have the following combined constraints on ξ^3 and $\xi^2 N_F$ terms:

$$FC4:\ \frac{A_1^3}{16} = \frac{(A_1+B_1)}{48}\{\bar{\mathcal{A}}^f_{1100} - \bar{\mathcal{S}}^f_{0011} + \bar{\mathcal{S}}^f_{2000} + \bar{\mathcal{S}}^f_{0200}\} + \frac{1}{6}\bar{\mathcal{A}}^f_{1011} + \frac{1}{12}\bar{\mathcal{A}}^f_{1200} + \frac{1}{4}\bar{\mathcal{A}}^f_{2100} + \frac{3}{4}\bar{\mathcal{A}}^f_{3000} + \frac{1}{12}\bar{\mathcal{S}}^f_{0111} + \frac{1}{4}\bar{\mathcal{S}}^f_{0300} - \frac{1}{12}\bar{\mathcal{S}}^f_{1011} + \frac{1}{6}\bar{\mathcal{S}}^f_{1200} - \frac{1}{2}\bar{\mathcal{S}}^f_{3000}. \quad (81)$$

The Schwinger-Dyson Equation for the fermion propagator involves corrections from photon emission and absorption as displayed in Fig. 4. This requires the fermion renormalization function to depend on the photon renormalization function, which in turn depends on the number of fermions N_F. Therefore in general $\{\bar{\mathcal{A}}^f_{1100}, \bar{\mathcal{S}}^f_{0011}, \bar{\mathcal{S}}^f_{2000}, \bar{\mathcal{S}}^f_{0200}\}$ and $\{\bar{\mathcal{A}}^f_{1011}, \bar{\mathcal{A}}^f_{1200}, \bar{\mathcal{A}}^f_{2100}, \bar{\mathcal{A}}^f_{3000}, \bar{\mathcal{S}}^f_{0111}, \bar{\mathcal{S}}^f_{0300}, \bar{\mathcal{S}}^f_{1011}, \bar{\mathcal{S}}^f_{1200}\bar{\mathcal{S}}^f_{3000}\}$ terms in Eq. (81) can be proportional to (ξ^2 or N_F^2 or ξN_F) and (ξ^3 or $\xi^2 N_F$ or ξN_F^2 or N_F^3), respectively. Remarkably, the matching required by multiplicatively renormalizability of these renormalization functions is automatically satisfied if the transverse fermion-boson vertex is independent of the photon renormalization function at leading logarithmic order. Therefore $\{\bar{\mathcal{A}}^f_{1100}, \bar{\mathcal{S}}^f_{0011}, \bar{\mathcal{S}}^f_{2000}, \bar{\mathcal{S}}^f_{0200}\}$ and $\{\bar{\mathcal{A}}^f_{1011}, \bar{\mathcal{A}}^f_{1200}, \bar{\mathcal{A}}^f_{2100}, \bar{\mathcal{A}}^f_{3000}, \bar{\mathcal{S}}^f_{0111}, \bar{\mathcal{S}}^f_{0300}, \bar{\mathcal{S}}^f_{1011}, \bar{\mathcal{S}}^f_{1200}\bar{\mathcal{S}}^f_{3000}\}$ terms would be proportional to only ξ^2 and ξ^3 terms, respectively. This will clearly constrain the nonperturbative forms of the transverse vertex that we wish to determine. In other words constraint $FC4$ of Eq. (81), will divide into two separate conditions for $\xi^2 N_F$ and ξ^3 comparisons:

$$FC41:\ 0 = \bar{\mathcal{A}}^f_{1100} - \bar{\mathcal{S}}^f_{0011} + \bar{\mathcal{S}}^f_{2000} + \bar{\mathcal{S}}^f_{0200},$$
$$FC42:\ \frac{A_1^3}{16} = \frac{1}{6}\bar{\mathcal{A}}^f_{1011} + \frac{1}{12}\bar{\mathcal{A}}^f_{1200} + \frac{1}{4}\bar{\mathcal{A}}^f_{2100} + \frac{3}{4}\bar{\mathcal{A}}^f_{3000} + \frac{1}{12}\bar{\mathcal{S}}^f_{0111} + \frac{1}{4}\bar{\mathcal{S}}^f_{0300} - \frac{1}{12}\bar{\mathcal{S}}^f_{1011} + \frac{1}{6}\bar{\mathcal{S}}^f_{1200} - \frac{1}{2}\bar{\mathcal{S}}^f_{3000}. \quad (82)$$

The idea is then to find a nonperturbative structure for the transverse pieces that delivers such relations. This we do in the next section. However, first we determine the conditions imposed by multiplicative renormalizability for the photon wave function renormalization.

B. MR constraints via photon Schwinger-Dyson equation

We now repeat the previous steps for the photon wave function renormalization. Comparison takes place between Eq. (64) and (68) order-by-order for $1/G$. Obviously, this time instead of looking at the terms depending on the gauge parameter ξ, we compare the dependence on N_F, the number of flavours hidden in the B_i terms. These give what we refer to as the photon conditions labeled $PC1$, $PC2$, *etc.*. Then,

$\boldsymbol{\alpha_0 \ln p^2/\Lambda^2}$ *comparison*:

$$PC1:\ B_1 = \frac{N_F}{3\pi}, \qquad B_n = B_1^n = \Big(\frac{N_F}{3\pi}\Big)^n. \quad (83)$$

First order comparison defines the value of B_1 in terms of N_F and as given in Eq. (67) fixes all the higher order terms.

$\boldsymbol{\alpha_0^2 \ln^2 p^2/\Lambda^2}$ *comparison*:

$$PC2:\ \frac{2}{3}A_1 = \bar{S}^\gamma_{1000}. \quad (84)$$

As we see above the second order comparison imposes this condition on the symmetric part of the transverse vertex.

$\boldsymbol{\alpha_0^3 \ln^3 p^2/\Lambda^2}$ *comparison*:

$$\frac{A_1^2}{6} = \frac{A_1}{2}\bar{\mathcal{S}}^\gamma_{1000} + \frac{1}{4}\bar{\mathcal{S}}^\gamma_{2000} - \frac{3}{4}\bar{\mathcal{S}}^\gamma_{0200} + \frac{1}{4}\bar{\mathcal{S}}^\gamma_{0011}. \quad (85)$$

Substituting Eq. (84) in above condition yields

$$PC3:\ \frac{A_1^2}{3} = -\frac{1}{2}\bar{\mathcal{S}}^\gamma_{0011} + \frac{3}{2}\bar{\mathcal{S}}^\gamma_{0200} - \frac{1}{2}\bar{\mathcal{S}}^\gamma_{2000}, \quad (86)$$

where every term is proportional to ξ^2.

$\boldsymbol{\alpha_0^4 \ln^4 p^2/\Lambda^2}$ *comparison*:

$$\frac{A_1^3}{24} = \frac{A_1}{4}\bar{\mathcal{S}}^\gamma_{0011} - \frac{A_1}{2}\bar{\mathcal{S}}^\gamma_{0200} + \frac{A_1^2}{4}\bar{\mathcal{S}}^\gamma_{1000} + \frac{A_1}{4}\bar{\mathcal{S}}^\gamma_{2000} + \frac{1}{8}\bar{\mathcal{S}}^\gamma_{3000} - \bar{\mathcal{S}}^\gamma_{0300} + \frac{1}{8}\bar{\mathcal{S}}^\gamma_{1011} - \frac{1}{4}\bar{\mathcal{S}}^\gamma_{1200}. \quad (87)$$

Making use of Eqs. (84) and (86), the above expression becomes

$$PC4\colon\ \frac{A_1^3}{24} = \frac{A_1}{4}\bar{\mathcal{S}}^\gamma_{0200} - \bar{\mathcal{S}}^\gamma_{0300} + \frac{1}{8}\bar{\mathcal{S}}^\gamma_{1011} - \frac{1}{4}\bar{\mathcal{S}}^\gamma_{1200} + \frac{1}{8}\bar{\mathcal{S}}^\gamma_{3000}, \tag{88}$$

where every term is proportional to ξ^3. So far we have expressed the general multiplicative renormalizability constraints on the 3-point vertex function in terms of the constants $\bar{\mathcal{A}}^\gamma_{mnrr}$ and $\bar{\mathcal{S}}^\gamma_{mnrr}$ up to $\mathcal{O}(\alpha^4)$.

C. Generalized fermion and photon MR constraints

Let us first look at the general picture. First, a_1 and b_1 being fixed by Eq. (74), allows the expansion coefficients A_u in Eq. (69) to be fixed in all orders:

$$A_1 = \frac{\xi}{4\pi}, \qquad A_2 = \frac{A_1^2}{2!}, \qquad A_3 = \frac{A_1^3}{3!}, \cdots, \qquad A_u = \frac{A_1^u}{u!}, \tag{89}$$

and in turn the infinite leading log series of $F(p^2)$ in Eq. (69) can be summed up as a power series:

$$F(p^2,\Lambda^2) = \sum_{u=0}^\infty \alpha_0^u \frac{A_1^u}{u!}\ln^u\frac{p^2}{\Lambda^2} = \left(\frac{p^2}{\Lambda^2}\right)^{\alpha A_1}. \tag{90}$$

This is the nonperturbative expression for the unquenched (full) fermion wave function renormalization. Moreover, it has exactly the same form as in the quenched QED [12,18,19,21,32,33,48,49]. Second, the relation between the photon coefficients are also found through $PC1$:

$$B_1 = \frac{N_F}{3\pi}, \qquad B_n = B_1^n = \left(\frac{N_F}{3\pi}\right)^n, \tag{91}$$

hence the infinite series of $G(p^2,\Lambda^2)$, Eq. (66) can also be summed up as

$$G(q^2,\Lambda^2) = \sum_{u=0}^\infty \alpha_0^u B_1^u \ln^u\frac{q^2}{\Lambda^2} = \frac{1}{1-\alpha_0 B_1\ln q^2/\Lambda^2}. \tag{92}$$

1. Generalized MR constraints from fermion SDE

Making use of Eqs. (89)–(92), we can then rewrite the inverse fermion wave function renormalization calculated from SDE, Eq. (47) as

$$\frac{1}{F(p^2,\Lambda^2)} = 1 - \Bigg\{ -\frac{1}{F(p^2,\Lambda^2)} + 1 + \frac{3}{8\pi}\sum_{u=1}^\infty \alpha_0^{u+1}\ln^{u+1}\frac{p^2}{\Lambda^2} \times\Bigg[-\frac{1}{2}\sum_{a=1}^u \frac{A_1^a}{a!}B_1^{u-a}\frac{1}{u+1} - \frac{1}{2}\sum_{b=1}^u\sum_{a=0}^{u-b}(-1)^b \times\frac{A_1^b}{b!}\frac{A_1^a}{a!}B_1^{u-b-a}\frac{1}{(u-b+1)}\Bigg] + \frac{3}{8\pi}\sum_{u=1}^\infty \alpha_0^{u+1} \times\ln^{u+1}\frac{p^2}{\Lambda^2}(H_u+\bar{H}_u)\Bigg\}, \tag{93}$$

and as a consequence of equating the multiplicatively renormalized F, Eq. (90) to Eq. (93) we can extract the generalized MR constraints to all orders, which, of course, reproduces $FC1$ to $FC4$:

$$0 = \sum_{u=1}^\infty \alpha_0^{u+1}\ln^{u+1}\frac{p^2}{\Lambda^2}(H_u+\bar{H}_u) + \sum_{u=1}^\infty \alpha_0^{u+1}\ln^{u+1}\frac{p^2}{\Lambda^2} \times\Bigg\{-\frac{1}{2}\sum_{a=1}^u A_aB_{u-a}\frac{1}{(n+1)} - \frac{1}{2}\sum_{b=1}^u\sum_{a=0}^{u-b}(-1)^b A_bA_aB_{u-b-a}\frac{1}{(u-b+1)}\Bigg\}, \tag{94}$$

where in Eqs. (48) and (49) for the H_u and $\bar{H}_u$, one can now substitute for the A_n, B_n from Eqs. (89) and (91).

2. Generalized MR constraints from photon SDE

Making use of Eqs. (89)–(92), we repeat the above procedure for photons, which is analogous to the fermion case above, in order to rewrite the inverse photon wave function renormalization, Eq. (63):

$$\frac{1}{G(q^2,\Lambda^2)} = 1 - \frac{N_F}{3\pi}\alpha_0\ln\frac{q^2}{\Lambda^2} - \frac{N_F}{3\pi}\sum_{u=1}^\infty \alpha_0^{u+1}\ln^{u+1}\frac{q^2}{\Lambda^2} \times\left\{\frac{A_1^u}{(u+1)!} + \frac{3}{2}K_u\right\}, \tag{95}$$

where in the expression for K_u of Eq. (63) we can substitute the conditions for A_n from Eq. (89). The generalized MR photon constraints can then be written as

$$\sum_{u=1}^\infty \alpha_0^{u+1}\ln^{u+1}\frac{q^2}{\Lambda^2}\left\{\frac{A_1^u}{(u+1)!} + \frac{3}{2}K_u\right\} = 0, \tag{96}$$

automatically satisfying $PC1 - PC4$.

D. Nonperturbative fermion and photon MR constraints

1. Nonperturbative MR constraints on transverse vertex from photon SDE

To understand the above conditions in full generality (i.e. beyond their expansion in leading logarithms) we first turn our attention to photon equation. Starting from Eq. (68) for multiplicatively renormalizable $1/G(q^2,\Lambda^2)$, we see that multiplicative renormalizability for leading logs is given by just the $O(\alpha_0)$ term, B_1, which is gauge independent. Equating the photon Schwinger-Dyson equation at the leading logarithmic order, Eq. (58), with the multiplicatively renormalizable $G(q^2)$, Eq. (68):

$$\frac{1}{G(q^2, \Lambda^2)} = 1 + \frac{\alpha_0 N_F}{3\pi} \int_{q^2}^{\Lambda^2} \frac{d\ell^2}{\ell^2} F^2(\ell) \left\{ \frac{1}{F(\ell)} + \frac{3}{4} \bar{\tau}_\gamma^{\rm sym} \right\} = 1 - \frac{\alpha_0 N_F}{3\pi} \ln\left(\frac{q^2}{\Lambda^2}\right). \tag{97}$$

We observe that the λ_1 term of the Ball-Chiu longitudinal vertex generates this. However, importantly for the present purpose this is part of a whole series:

$$\frac{\alpha_0 N_F}{3\pi} \int_{q^2}^{\Lambda^2} \frac{d\ell^2}{\ell^2} F(\ell) = \frac{N_F}{3\pi} \alpha_0 \ln\frac{q^2}{\Lambda^2} \left\{ -1 - \left[\frac{1}{2} X + \frac{1}{6} X^2 + \frac{1}{24} X^3 + \frac{1}{120} X^4 + \frac{1}{720} X^5 + \frac{1}{5040} X^6 + \frac{1}{40320} X^7 + \mathcal{O}(\alpha_0^8) \right] \right\}, \tag{98}$$

where $X = \alpha_0 A_1 \ln\frac{q^2}{\Lambda^2}$. Beyond $O(\alpha_0)$, this series (i.e. terms inside the square bracket) has to be canceled exactly by the contribution from the vertex components. Since the λ_2 term in the Ball-Chiu longitudinal component only contributes at nonleading order, it is the symmetric part of the transverse vertex, $\bar{\tau}_\gamma^{\rm sym}$, with its implicit gauge dependence that has to provide this cancellation. $PC2$ to $PC4$ in Eqs. (84)–(88) give the conditions for this cancellation to be achieved at $\mathcal{O}(\alpha_0^2)$, $\mathcal{O}(\alpha_0^3)$ and $\mathcal{O}(\alpha_0^4)$ and the general condition in Eqs. (96) and (97) for all orders. To go further, we note that multiplicative renormalizability of the photon Schwinger-Dyson equation, Eq. (97), picks out loop momentum regions where $\ell_+^2 \simeq \ell_-^2 \sim \ell^2 \gg q^2$. The second term in Eq. (97) must give the following result:

$$\frac{\alpha_0 N_F}{4\pi} \int_{q^2}^{\Lambda^2} \frac{d\ell^2}{\ell^2} F^2(\ell) \bar{\tau}_\gamma^{\rm sym} = \frac{N_F}{3\pi} \alpha_0 \ln\frac{q^2}{\Lambda^2} \left[\frac{1}{2} X + \frac{1}{6} X^2 + \frac{1}{24} X^3 + \frac{1}{120} X^4 + \frac{1}{720} X^5 + \frac{1}{5040} X^6 + \frac{1}{40320} X^7 + \mathcal{O}(\alpha_0^8) \right]. \tag{99}$$

This surely determines the structure of the $\bar{\tau}_\gamma^{\rm sym}$'s for this to happen. The dependence on the fermion wave function renormalization must be more complicated than $1/F$ times a kinematic factor. It must be proportional to a function of a function of F's so let us write

$$\bar{\tau}_\gamma^{\rm sym} \sim \frac{1}{F(q)} h(Y). \tag{100}$$

In keeping with the ethos of this work, we assume that Y is determined by the fermion wave function renormalization. Since the renormalization of the τ_i's is replicated wholly by the factor of $1/F$, Y must be renormalization independent. As an example let us choose it to be

$$Y = \frac{F(q^2)}{F(\ell^2)} - 1, \tag{101}$$

where the factor of -1 ensures that the leading logarithm expansion of Y begins at $\mathcal{O}(\alpha_0 \ln)$ as required by Eq. (99). Can we find what function $h(Y)$ is to satisfy Eqs. (97) and (99)? Let us assume we can expand $h(Y)$ as a power series in Y, and in turn expand this in leading logs of momenta. Then to produce the cancellation required, we deduce

$$h(Y) = Y + \frac{1}{2} Y^2 - \frac{1}{6} Y^3 + \frac{1}{12} Y^4 - \frac{1}{20} Y^5 + \frac{1}{30} Y^6 - \frac{1}{42} Y^7 + \frac{1}{56} Y^8 + \mathcal{O}(Y^9). \tag{102}$$

We recognize this as

$$h(Y) = Y\left(1 - \sum_{n=1}^{\infty} \frac{(-Y)^n}{n(n+1)}\right), \qquad = (1+Y)\ln(1+Y), \tag{103}$$

substituting Y from Eq. (101), $h(Y)$ becomes

$$h(Y) = \frac{F(q^2)}{F(\ell^2)} \ln\frac{F(q^2)}{F(\ell^2)}. \tag{104}$$

Since a form like $\bar{\tau}_\gamma^{\rm sym} \sim 1/F(\ell^2) \ln(F(q^2)/F(\ell^2))$ in Eq. (100) is the $k \to p = \ell$ limit of the evolving structure, this naturally generalizes to the $k \neq p$ configuration as

$$\bar{\tau}_\gamma^{\rm sym}(p^2, k^2, q^2) \sim \frac{1}{2}\left(\frac{1}{F(k^2)} + \frac{1}{F(p^2)}\right) \ln\frac{F(q^2)}{2} \times \left(\frac{1}{F(k^2)} + \frac{1}{F(p^2)}\right). \tag{105}$$

While the form in the photon limit is determined, the structure in general momentum configurations is not unique and there are several possibilities differing only beyond leading logarithmic order. Three of these are

$$\mathcal{S}^{(1)} = \frac{1}{2}\left(\frac{1}{F(k^2)} + \frac{1}{F(p^2)}\right) \ln\frac{F(q^2)}{2} \left(\frac{1}{F(k^2)} + \frac{1}{F(p^2)}\right),$$
$$\mathcal{S}^{(2)} = \frac{1}{2} \frac{1}{(F(k^2)F(p^2))^{1/2}} \ln\frac{F(q^2)^2}{F(k^2)F(p^2)},$$
$$\mathcal{S}^{(3)} = \frac{1}{4}\left(\frac{1}{F(k^2)} + \frac{1}{F(p^2)}\right) \ln\frac{F(q^2)^2}{F(k^2)F(p^2)}, \tag{106}$$

all of which give the same $h(Y)$ of Eq. (104) in the photon limit of $k^2 \simeq p^2 \gg q^2$.

2. Nonperturbative MR constraints on transverse vertex from fermion SDE

Similarly, for the multiplicatively renormalizable $1/F(q^2, \Lambda^2)$, the result at leading logarithmic order is given by the leading ξ dependent piece, as required by the Landau-Khalatnikov-Fradkin transformation [43,44]. This leading term is provided by the first term in the integrals of Eqs. (43), (45), and (47). Let us recall Eq. (43) and in this equation we perform both the radial and angular integration for the first term, but only the angular integration for the second term, then we find

$$\frac{1}{F(p^2)} = 1 + \left\{\frac{1}{F(p^2)} - 1 - \frac{3\alpha_0}{8\pi}\int_{p^2}^{\Lambda^2}\frac{dk^2}{k^2}F(k^2)G(q^2) \times \left[\frac{1}{2}\left(\frac{1}{F(k^2)} - \frac{1}{F(p^2)}\right) + (\bar{\tau}_f^{\text{anti}} + \bar{\tau}_f^{\text{sym}})\right]\right\}. \quad (107)$$

Imposing the MR fermion condition, Eq. (90), on this expression yields the following constraint on the transverse vertex:

$$-\frac{3\alpha_0}{8\pi}\int_{p^2}^{\Lambda^2}\frac{dk^2}{k^2}F(k^2)G(q^2) \times \left[\frac{1}{2}\left(\frac{1}{F(k^2)} - \frac{1}{F(p^2)}\right) + (\bar{\tau}_f^{\text{anti}} + \bar{\tau}_f^{\text{sym}})\right] = 0. \quad (108)$$

This cancellation involves both the longitudinal and transverse pieces together. At leading logarithmic order the longitudinal contribution comes from just the λ_2 term in the Ball-Chiu vertex.

While antisymmetric forms do not contribute to the leading logarithmic behavior of the photon Schwinger-Dyson equation, this is not the case for the fermion equation. Indeed, here the distinction between symmetric and antisymmetric disappears when $k^2 \simeq q^2 \gg p^2$. Thus, a seemingly symmetric form like

$$\ln\frac{F(q^2)}{2}\left(\frac{1}{F(k^2)} + \frac{1}{F(p^2)}\right) \sim \ln\left(\frac{F(k^2)}{F(p^2)}\right) = \alpha_0 A_1 \ln\frac{k^2}{p^2} + O(\alpha_0^2), \quad (109)$$

is antisymmetric in k and p. Such a form contributes equally to the antisymmetric terms like

$$\frac{1}{F(k^2)} - \frac{1}{F(p^2)} = -\alpha_0 A_1 \ln\frac{k^2}{p^2} + O(\alpha_0^2). \quad (110)$$

The $\mathcal{O}(\alpha_0^2)$, $\mathcal{O}(\alpha_0^3)$ and $\mathcal{O}(\alpha_0^4)$ conditions of Eqs. (84)–(88), which embody the gauge independence of the photon wave function renormalization and the known gauge dependence of the fermion function, require the transverse vertex to deliver a very particular gauge dependence itself. Our aim is to reproduce this by constructing the nonperturbative transverse vertex from the fermion wave function renormalization. This means from Eq. (108) that

$$\bar{\tau}_f^{\text{anti}} + \bar{\tau}_f^{\text{sym}} = -\frac{1}{2}\left(\frac{1}{F(k^2)} - \frac{1}{F(p^2)}\right). \quad (111)$$

Hence this expression tells us that the total transverse vertex, i.e. combination of antisymmetric and symmetric parts, must be proportional to antisymmetric form in the limit $k^2 \simeq q^2 \gg p^2$. These considerations suggest particular antisymmetric and symmetric vertex forms. In Table I, we give the specific coefficients $\bar{\mathcal{A}}_{mnrr}$ and $\bar{\mathcal{S}}_{mnrr}$ at $\mathcal{O}(\alpha_0^3)$ for these examples.

VI. APPLICATION

The next step is to make use of all the examples in Table I as inputs to the multiplicative renormalizability constraints. In order to satisfy these, we have a set of equations to solve. As a first step the coefficient functions, τ_i's, can in general be written as a sum of different nonperturbative forms of F and G using the above examples. Hence, an antisymmetric and symmetric combination of F and G in τ_i^{anti} and τ_i^{sym}, respectively, become

$$\tau_i^{\text{anti}} = (f^{(1)}\mathcal{A}^{(1)})_i + (f^{(2)}\mathcal{A}^{(2)})_i + \cdots + (f^{(n)}\mathcal{A}^{(n)})_i,$$
$$\tau_i^{\text{sym}} = (\tilde{f}^{(1)}\mathcal{S}^{(1)})_i + (\tilde{f}^{(2)}\mathcal{S}^{(2)})_i + \cdots + (\tilde{f}^{(n)}\mathcal{S}^{(n)})_i, \quad (112)$$

where $\mathcal{A}^{(n)}$ and $\mathcal{S}^{(n)}$ refer to the relevant expressions in the left hand column of Table I. In general, the number of constants needed to solve these equations is proportional to the number n of various combinations of the F and G. These combinations will appear in the ansatz for the nonperturbative transverse vertex. We then try to solve these equations by choosing a minimal number of combinations, in order to find the simplest possible vertex ansatz.

From Eqs. (46) and (61), we see that the coefficients β_i, γ_i, δ_i, ϵ_i, defined in Eqs. (25) appear in the fermion and photon conditions in rather specific combinations. To make this explicit and simplify the notation, it is useful to define

$$\beta_f \equiv (\beta_2 + \beta_3 + \beta_6 - \beta_8), \qquad \gamma_f \equiv (-\gamma_3 + \gamma_6),$$
$$\delta_f \equiv (\delta_2 + \delta_3 + \delta_6 - \delta_8), \qquad \epsilon_f \equiv (-\epsilon_3 - \epsilon_6),$$
$$\delta_\gamma \equiv (\delta_2 - \delta_3 + \delta_6 - \delta_8), \qquad \epsilon_\gamma \equiv (\epsilon_2 - \epsilon_3 + \epsilon_6 - \epsilon_8). \quad (113)$$

Recall that antisymmetric forms for the τ_i's do not contribute to the photon renormalization at leading logarithmic order, and so we have no corresponding combinations of β_γ and γ_γ.

A. Fermion constraints

We now wish to write down the fermion constraints $FC1 - FC4$, Eqs. (71) and (81), which we obtained in the previous section for the specific choices for $\bar{\tau}_f^{\text{anti}}$ and $\bar{\tau}_f^{\text{sym}}$, namely, $\mathcal{A}^{(1)}$ as the antisymmetric form of the transverse vertex and $\mathcal{S}^{(1)}$ as the symmetric one in the Table I:

TABLE I. Antisymmetric combinations of F and G.

$\mathcal{A}^{(1)}$	$\frac{1}{F(k)} - \frac{1}{F(p)}$	$\mathcal{A}_{1000} = -A_1$, $\mathcal{A}_{2000}\frac{A_1^2}{2!}$, $\mathcal{A}_{1100} = 0$, $\mathcal{A}_{3000} - \frac{A_1^3}{3!}$, $\mathcal{A}_{2100} = \mathcal{A}_{1011} = \mathcal{A}_{1200} = 0$, $\mathcal{A}_{4000} = -\frac{A_1^4}{4!}$, $\mathcal{A}_{3100} = \mathcal{A}_{2200} = \mathcal{A}_{1300} = \mathcal{A}_{2011} = \mathcal{A}_{1111} = 0$
$\mathcal{S}^{(1)}$	$\frac{1}{2}(\frac{1}{F(k)} + \frac{1}{F(p)}) \ln\frac{F(q)}{2}(\frac{1}{F(k)} + \frac{1}{F(p)})$	$\mathcal{S}_{1000} = -\frac{A_1}{2}$, $\mathcal{S}_{0100} = \frac{A_1}{2}$, $\mathcal{S}_{2000} = \frac{3}{8}A_1^2$, $\mathcal{S}_{1100} = -\frac{A_1^2}{2}$, $\mathcal{S}_{0011} = \frac{A_1^2}{8}$, $\mathcal{S}_{0200} = 0$, $\mathcal{S}_{3000} = -\frac{3}{16}A_1^3$, $\mathcal{S}_{2100} = \frac{A_1^3}{4}$, $\mathcal{S}_{1011} = -\frac{A_1^3}{16}$, $\mathcal{S}_{0300} = \mathcal{S}_{1200} = \mathcal{S}_{0111} = 0$
$\mathcal{S}^{(2)}$	$\frac{1}{2}(\frac{1}{F(k)F(p)})^{1/2} \ln\frac{F(q)^2}{F(k)F(p)}$	$\mathcal{S}_{1000} = -\frac{A_1}{2}$, $\mathcal{S}_{0100} = \frac{A_1}{2}$, $\mathcal{S}_{2000} = \frac{A_1^2}{4}$, $\mathcal{S}_{1100} = -\frac{A_1^2}{2}$, $\mathcal{S}_{0011} = \frac{A_1^2}{4}$, $\mathcal{S}_{0200} = 0$, $\mathcal{S}_{3000} = -\frac{A_1^3}{16}$, $\mathcal{S}_{2100} = \frac{A_1^3}{8}$, $\mathcal{S}_{1011} = -\frac{3}{16}A_1^3$, $\mathcal{S}_{0300} = \mathcal{S}_{1200} = 0$ $\mathcal{S}_{0111} = \frac{A_1^3}{8}$
$\mathcal{S}^{(3)}$	$\frac{1}{4}(\frac{1}{F(k)} + \frac{1}{F(p)}) \ln\frac{F(q)^2}{F(k)F(p)}$	$\mathcal{S}_{1000} = -\frac{A_1}{2}$, $\mathcal{S}_{0100} = \frac{A_1}{2}$, $\mathcal{S}_{2000} = \frac{A_1^2}{4}$, $\mathcal{S}_{1100} = -\frac{A_1^2}{2}$, $\mathcal{S}_{0011} = \frac{A_1^2}{4}$, $\mathcal{S}_{0200} = 0$, $\mathcal{S}_{3000} = -\frac{A_1^3}{8}$, $\mathcal{S}_{2100} = \frac{A_1^3}{4}$, $\mathcal{S}_{1011} = -\frac{A_1^3}{8}$, $\mathcal{S}_{0300} = \mathcal{S}_{1200} = \mathcal{S}_{0111} = 0$

$$\mathcal{A}^{(1)} = \left(\frac{1}{F(k^2)} - \frac{1}{F(p^2)}\right),$$
$$\mathcal{S}^{(1)} = \frac{1}{2}\left(\frac{1}{F(k^2)} + \frac{1}{F(p^2)}\right) \ln\frac{F(q^2)}{2}\left(\frac{1}{F(k^2)} + \frac{1}{F(p^2)}\right). \tag{114}$$

After recalling the definition of $\bar{\mathcal{A}}^f_{mnrr}$ and $\bar{\mathcal{S}}^f_{mnrr}$ from Eq. (46) and reading off the specific values of $\mathcal{A}_{mnrr}$ and $\mathcal{S}_{mnrr}$'s from the Table I, the MR constraint $FC2$, Eq. (77), which comes from $\alpha_0^2 \ln^2 \mathbf{p}^2/\Lambda^2$ order comparison together with Eq. (113) gives the following condition:

$$-(\beta_f + \gamma_f) + \frac{1}{2}(\delta_f + \varepsilon_f) = \frac{1}{2}. \tag{115}$$

The $\alpha_0^3 \ln^3 \mathbf{p}^2/\Lambda^2$ order constraint $FC3$, Eq. (79), splits the combined β_f, γ_f, δ_f, ε_f form of previous constraint into two separate ones:

$$(\delta_f + \varepsilon_f) = 0, \qquad (\beta_f + \gamma_f) = -\frac{1}{2}. \tag{116}$$

The $\alpha_0^4 \ln^4 \mathbf{p}^2/\Lambda^2$ order constraint $FC4$, Eq. (81) does not give further new information, but again yields Eq. (116).

B. Photon constraints

We repeat this procedure for the photon constraints $PC2 - PC4$ for the same choices of $\mathcal{A}^{(1)}$ and $\mathcal{S}^{(1)}$ in Table I. All the MR constraints $PC2$ to $PC4$, Eqs. (84) and (86), which follow from $\alpha_0^2 \ln^2$ to $\alpha_0^4 \ln^4$ comparisons give the same condition and that is

$$(\delta_\gamma + \varepsilon_\gamma) = -\frac{4}{3}. \tag{117}$$

Since this condition repeats itself at every order, this means we have the exact solutions. There are 14 constants to be fixed, and Eqs. (116) and (117) can only fix three of them in terms of the others, for instance,

$$\beta_2 = -\frac{1}{2} - \beta_3 - \beta_6 + \beta_8 + \gamma_3 - \gamma_6,$$
$$\delta_2 = -\frac{2}{3} - \delta_6 + \delta_8 - \frac{\varepsilon_2}{2} + \varepsilon_3 - \varepsilon_6 + \frac{\varepsilon_8}{2}, \tag{118}$$
$$\delta_3 = \frac{2}{3} + \frac{\varepsilon_2}{2} - \frac{\varepsilon_8}{2}.$$

Substituting these constants into Eq. (25), we can write the nonperturbative coefficient functions τ_i's as

$$\tau_2^M(p^2, k^2, q^2) = \frac{2}{(k^4 - p^4)}\left[\left(\frac{1}{2} - \beta_3 - \beta_6 + \beta_8 + \gamma_3 - \gamma_6\right) + \gamma_2 \frac{2k\cdot p}{k^2 + p^2}\right]\tau_2^{\rm anti}(p^2, k^2, q^2)$$
$$+ \frac{2}{(k^2 + p^2)^2}\left[\left(-\frac{2}{3} - \delta_6 + \delta_8 - \frac{\varepsilon_2}{2} + \varepsilon_3 - \varepsilon_6 + \frac{\varepsilon_8}{2}\right) + \epsilon_2 \frac{2k\cdot p}{k^2 + p^2}\right]\tau_2^{\rm sym}(p^2, k^2, q^2),$$
$$\tau_3^M(p^2, k^2, q^2) = \frac{1}{(k^2 - p^2)}\left[\beta_3 + \gamma_3 \frac{2k\cdot p}{k^2 + p^2}\right]\tau_3^{\rm anti}(p^2, k^2, q^2)$$
$$+ \frac{1}{(k^2 + p^2)}\left[\left(\frac{2}{3} + \frac{\varepsilon_2}{2} - \frac{\varepsilon_8}{2}\right) + \epsilon_3 \frac{2k\cdot p}{k^2 + p^2}\right]\tau_3^{\rm sym}(p^2, k^2, q^2),$$
$$\tau_6^M(p^2, k^2, q^2) = \frac{1}{(k^2 + p^2)}\left[\beta_6 + \gamma_6 \frac{2k\cdot p}{k^2 + p^2}\right]\tau_6^{\rm anti}(p^2, k^2, q^2) + \frac{(k^2 - p^2)}{(k^2 + p^2)^2}\left[\delta_6 + \epsilon_6 \frac{2k\cdot p}{k^2 + p^2}\right]\tau_6^{\rm sym}(p^2, k^2, q^2),$$
$$\tau_8^M(p^2, k^2, q^2) = \frac{1}{(k^2 - p^2)}\left[\beta_8 + \gamma_8 \frac{2k\cdot p}{k^2 + p^2}\right]\tau_8^{\rm anti}(p^2, k^2, q^2) + \frac{1}{(k^2 + p^2)}\left[\delta_8 + \epsilon_8 \frac{2k\cdot p}{k^2 + p^2}\right]\tau_8^{\rm sym}(p^2, k^2, q^2), \tag{119}$$

with $\tau_i^{\rm anti,sym}$ having specific forms such as those determined in Sec. V, Eqs. (106) and (111), examples of which are given in Table I.

Multiplicative renormalizability relates the coefficients at order $(\alpha_0 \ln)^n$ to that at $n = 1$. This lowest leading logarithm coefficient is fixed by the longitudinal component of the fermion-boson vertex. Transverse components only enter at $n = 2$.

Remarkably, once the MR conditions at this first nontrivial order are satisfied, the conditions at all orders in leading logarithms for both the fermion and photon Schwinger-Dyson equations are fulfilled.

As far as the leading terms are concerned, the above constraints ensure that both fermion and photon propagators are multiplicatively renormalizable in massless unquenched QED_4. These constraints impose conditions on the transverse part of the vertex. The 3-point vertex calculated at $\mathcal{O}(\alpha_0)$ and the coefficient constants, τ_i's, at one loop order [41] will be very helpful in fixing some of these constants.

VII. PERTURBATION THEORY

The vertex coefficients τ_i's were calculated exactly in $\mathcal{O}(\alpha_0)$ for the massive fermions in a general covariant gauge [41] and for our purpose their massless limits are given in Appendix A.

We observe in Eqs. (A1)–(A4) that all the four τ_i's ($i = 2, 3, 6, 8$) contain four different structures in general. The first one is the J_0 dependent part, which contains Spence functions (or dilogarithms) of momenta p^2, k^2, q^2 in Eq. (A5). The second part is proportional to $\ln k^2/p^2$ which is the perturbative expansion of the asymmetric combination of F and G in first order, and the third part is proportional to $\ln q^4/(k^2p^2)$, which is the perturbative expansion of the symmetric combination of F and G, and the final one is the kinematical term dependent on k^2, p^2, q^2.

In order to fix some of the individual constants β_i, γ_i, δ_i, ϵ_i's appearing in Eqs. (116) and (117) we need to make a comparison between perturbative transverse vertex coefficients τ_i^{pert} of Eqs. (A1)–(A4) and the nonperturbative ones we used in fermion and photon SDE, $\tau_i^{\text{non-pert}}$ of Eqs. (26) and (27) in the previous sections. However this comparison has to be made in a particular way in order to be meaningful. There are two points to be considered. The first is how these τ_i coefficients behave inside the fermion and photon SDEs, since these equations project out different parts of the vertex. Recall, that with this in mind we started with a simplified ansatz for the explicit kinematic factors in the $\tau_i^{\text{non-pert}}$, Eq. (25), and assumed their denominators did not depend on $k \cdot p$. We therefore need to take the corresponding limits of both pure perturbative τ_i^{pert}'s, Eq. (A1)–(A4), and the $\tau_i^{\text{non-pert}}$'s, Eqs. (26) and (27) which we inserted into SDE. While for the fermion SDE the relevant limit would be where either of the fermion momenta are large, e.g. $k^2 \simeq q^2 \gg k \cdot p \gg p^2$, for the photon SDE the relevant one is where the both internal fermion momenta are same and much greater than the photon momentum, e.g. $k^2 \simeq p^2 \gg q^2$.

The second point is that the real τ_i functions depend on the angle between momenta k and p. This means that when we obtained MR constraints, Eqs. (115) and (117), on the vertex, i.e. on τ_i functions, their angular dependences were already integrated out. These angular averaged functions we call *effective* τ_i's [50]. It is these that we have to compare with perturbation theory.

A. $k^2 \simeq q^2 \gg p^2$: The fermion limit

Let us take the fermion limit of the perturbative τ_i^{pert}'s, Eqs. (A1)–(A4) in Euclidean space. In order to do this, J_0 of Eqs. (A5) and (B1) has to be expanded up to $\mathcal{O}(1/k^7)$ to ensure we keep all the terms of the required order. As shown in Appendix B, these results are

$$\begin{aligned}
(\tau_2^E)^{\text{pert}}_{\text{Real}}(p^2, k^2, q^2) &= \frac{\alpha_0 \xi}{8\pi k^4} \ln\frac{k^2}{p^2}\left\{\frac{4}{3} + 2\frac{k\cdot p}{k^2} + \frac{14}{15}\frac{p^2}{k^2}\right\},\\
(\tau_3^E)^{\text{pert}}_{\text{Real}}(p^2, k^2, q^2) &= \frac{\alpha_0 \xi}{8\pi k^2} \ln\frac{k^2}{p^2}\left\{\frac{2}{3} + \frac{k\cdot p}{k^2} + \frac{2}{15}\frac{p^2}{k^2}\right\},\\
(\tau_6^E)^{\text{pert}}_{\text{Real}}(p^2, k^2, q^2) &= \frac{\alpha_0 \xi}{8\pi k^2} \ln\frac{k^2}{p^2}\left\{-\frac{1}{3} - \frac{1}{3}\frac{k\cdot p}{k^2} - \frac{1}{5}\frac{p^2}{k^2}\right\},\\
(\tau_8^E)^{\text{pert}}_{\text{Real}}(p^2, k^2, q^2) &= 0.
\end{aligned} \tag{120}$$

In this limit one observes that both J_0 and $\ln(q^4/k^2p^2)$ behave like $\ln(k^2/p^2)$. Therefore all four coefficient functions become proportional to $\ln(k^2/p^2)$ signaling that the structure of nonperturbative transverse vertex consists of purely asymmetric combination of F or G. Next we expand the nonperturbative $\tau_i^{\text{non-pert}}$'s, Eq. (25), using Eqs. (28) and (29) at the order $\mathcal{O}(\alpha_0)$:

$$\begin{aligned}
(\tau_2^E)^{\text{non-pert}}(p^2, k^2, q^2) &= \frac{2}{k^4}\left(\beta_2 + \gamma_2\frac{2k\cdot p}{k^2}\right)\left[\alpha_0 \mathcal{A}^2_{1000}\ln\frac{k^2}{p^2}\right] + \frac{2}{k^4}\left(\delta_2 + \epsilon_2\frac{2k\cdot p}{k^2}\right)\left[-\alpha_0 \mathcal{S}^2_{1000}\ln\frac{k^2}{p^2}\right] + \mathcal{O}(\alpha_0^2),\\
(\tau_3^E)^{\text{non-pert}}(p^2, k^2, q^2) &= \frac{1}{k^2}\left(\beta_3 + \gamma_3\frac{2k\cdot p}{k^2}\right)\left[-\alpha_0 \mathcal{A}^3_{1000}\ln\frac{k^2}{p^2}\right] + \frac{1}{k^2}\left(\delta_3 + \epsilon_3\frac{2k\cdot p}{k^2}\right)\left[\alpha_0 \mathcal{S}^3_{1000}\ln\frac{k^2}{p^2}\right] + \mathcal{O}(\alpha_0^2),\\
(\tau_6^E)^{\text{non-pert}}(p^2, k^2, q^2) &= \frac{1}{k^2}\left(\beta_6 + \gamma_6\frac{2k\cdot p}{k^2}\right)\left[-\alpha_0 \mathcal{A}^6_{1000}\ln\frac{k^2}{p^2}\right] + \frac{1}{k^2}\left(\delta_6 + \epsilon_6\frac{2k\cdot p}{k^2}\right)\left[\alpha_0 \mathcal{S}^6_{1000}\ln\frac{k^2}{p^2}\right] + \mathcal{O}(\alpha_0^2),\\
(\tau_8^E)^{\text{non-pert}}(p^2, k^2, q^2) &= \frac{1}{k^2}\left(\beta_8 + \gamma_8\frac{2k\cdot p}{k^2}\right)\left[-\alpha_0 \mathcal{A}^8_{1000}\ln\frac{k^2}{p^2}\right] + \frac{1}{k^2}\left(\delta_8 + \epsilon_8\frac{2k\cdot p}{k^2}\right)\left[\alpha_0 \mathcal{S}^8_{1000}\ln\frac{k^2}{p^2}\right] + \mathcal{O}(\alpha_0^2).
\end{aligned} \tag{121}$$

As we mentioned earlier, during the process of finding MR constraints in Eq. (116) from the fermion SDE we performed both radial and angular integrations therefore these constraints on the vertex are for the τ_i's whose angular dependence has been integrated out, *viz*. they are the effective $\tau_i^{\text{non-pert}}$'s. To make consistent comparison between the Eqs. (120) and (121), we must integrate out the angular dependence of both τ_i^{pert} and $\tau_i^{\text{non-pert}}$. The details of this procedure can be found in Appendix C. Following this, the effective coefficient functions can be found from $\tau_{\text{Real}}^{\text{pert}}$'s in Eq. (120):

$$\begin{aligned}(\tau_2^E)_{\text{eff}}^{\text{pert}}(p^2,k^2) &= \frac{\alpha_0\xi}{8\pi k^4}\ln\frac{k^2}{p^2}\left(\frac{4}{3}\right), \qquad (\tau_3^E)_{\text{eff}}^{\text{pert}}(p^2,k^2) = \frac{\alpha_0\xi}{8\pi k^2}\ln\frac{k^2}{p^2}\left(\frac{1}{6}\right),\\ (\tau_6^E)_{\text{eff}}^{\text{pert}}(p^2,k^2) &= \frac{\alpha_0\xi}{8\pi k^2}\ln\frac{k^2}{p^2}\left(-\frac{1}{2}\right), \qquad (\tau_8^E)_{\text{eff}}^{\text{pert}}(p^2,k^2) = 0.\end{aligned} \tag{122}$$

We repeat the same procedure for the first order expansion of the nonperturbative coefficients $\tau_{\text{Real}}^{\text{non-pert}}$'s in Eq. (121) to give

$$\begin{aligned}(\tau_2^E)_{\text{eff}}^{\text{non-pert}}(p^2,k^2) &= \frac{2}{k^4}\beta_2\left(\alpha_0\mathcal{A}_{1000}^2\ln\frac{k^2}{p^2}\right) + \frac{2}{k^4}\delta_2\left(-\alpha_0\mathcal{S}_{1000}^2\ln\frac{k^2}{p^2}\right) + \mathcal{O}(\alpha^2),\\ (\tau_3^E)_{\text{eff}}^{\text{non-pert}}(p^2,k^2) &= \frac{1}{k^2}(\beta_3-\gamma_3)\left(-\alpha_0\mathcal{A}_{1000}^3\ln\frac{k^2}{p^2}\right) + \frac{1}{k^2}(\delta_3-\epsilon_3)\left(\alpha_0\mathcal{S}_{1000}^3\ln\frac{k^2}{p^2}\right) + \mathcal{O}(\alpha^2),\\ (\tau_6^E)_{\text{eff}}^{\text{non-pert}}(p^2,k^2) &= \frac{1}{k^2}(\beta_6+\gamma_6)\left(-\alpha_0\mathcal{A}_{1000}^6\ln\frac{k^2}{p^2}\right) + \frac{1}{k^2}(\delta_6+\epsilon_6)\left(\alpha_0\mathcal{S}_{1000}^6\ln\frac{k^2}{p^2}\right) + \mathcal{O}(\alpha^2),\\ (\tau_8^E)_{\text{eff}}^{\text{non-pert}}(p^2,k^2) &= \frac{1}{k^2}\beta_8\left(-\alpha_0\mathcal{A}_{1000}^8\ln\frac{k^2}{p^2}\right) + \frac{1}{k^2}\delta_8\left(\alpha_0\mathcal{S}_{1000}^8\ln\frac{k^2}{p^2}\right) + \mathcal{O}(\alpha^2).\end{aligned} \tag{123}$$

The constants β_i's, δ_i's, γ_i's and ε_i's appearing in Eq. (123) are the ones which must satisfy the MR constraints, Eqs. (116) and (117). Let us check we have obtained the correct result in three key situations.

First we compare Eq. (122) with Eq. (123) to read off the constraints on $\mathcal{A}_{1000}^i$ and $\mathcal{S}_{1000}^i$ for $i = 2, 3, 6, 8$:

$$\begin{aligned}\beta_2\mathcal{A}_{1000}^2 - \delta_2\mathcal{S}_{1000}^2 &= \frac{A_1}{3},\\ (\beta_3-\gamma_3)\mathcal{A}_{1000}^3 - (\delta_3-\varepsilon_3)\mathcal{S}_{1000}^3 &= -\frac{A_1}{12},\\ (\beta_6+\gamma_6)\mathcal{A}_{1000}^6 - (\delta_6+\varepsilon_6)\mathcal{S}_{1000}^6 &= \frac{A_1}{4},\\ \beta_8\mathcal{A}_{1000}^8 - \delta_8\mathcal{S}_{1000}^8 &= 0.\end{aligned} \tag{124}$$

(1a) General case at $\mathcal{O}(\alpha_0)$: Recall the definition of $\bar{\mathcal{A}}_{1000}^f$ and $\bar{\mathcal{S}}_{1000}^f$, Eq. (46), in order to form the $FC2$ constraint in Eq. (77) using above expressions by adding them up appropriately:

$$\bar{\mathcal{A}}_{1000}^{\,f} - \bar{\mathcal{S}}_{1000}^f = \left(\frac{1}{3} - \frac{1}{12} + \frac{1}{4}\right)A_1 = \frac{A_1}{2}. \tag{125}$$

(1b) For the special vertex ($\mathcal{A}^{(1)}$ and $\mathcal{S}^{(1)}$) at $\mathcal{O}(\alpha_0)$: Making use of Table I we can read off the values of $\mathcal{A}_{1000}^i$ and $\mathcal{S}_{1000}^i$ and insert them into Eq. (124) to see whether we can satisfy the fermion MR constraint of Eq. (115) by using Eq. (113):

$$\begin{aligned}&[(\beta_2+\beta_3+\beta_6-\beta_8)+(-\gamma_3+\gamma_6)](-A_1)\\ &\quad -[(\delta_2+\delta_3+\delta_6-\delta_8)+(-\varepsilon_3+\varepsilon_6)]\left(-\frac{A_1}{2}\right),\\ &= \left[-(\beta_f+\gamma_f)+\frac{1}{2}(\delta_f+\varepsilon_f)\right]A_1,\\ &= \left(\frac{1}{3}-\frac{1}{12}+\frac{1}{4}\right)A_1, \qquad = \frac{A_1}{2}.\end{aligned} \tag{126}$$

As we see, all effective τ_{eff}^i's, Eq. (122) add up to $A_1/2$, as required.

(2) Nonperturbative check: If we trace back the MR constraint in fermion SDE equation, Eq. (108), we have already observed that the ξ dependent part will give the right equality and the rest must be zero to give the fermion MR condition. Hence this MR constraint for the effective τ_i's after the angular and before the radial integration was performed can be written as

$$\frac{3\alpha_0}{8\pi}\int\frac{dk^2}{k^2}F(k^2)G(k^2)\left[\frac{1}{2}\left(\frac{1}{F(k^2)}-\frac{1}{F(p^2)}\right) + k^2\sum(\tau_i^E)_{\text{eff}}^{\text{non-pert}}(p^2,k^2)\right] = 0, \tag{127}$$

where

$$\sum(\tau_i^E)_{\text{eff}}^{\text{non-pert}}(p^2,k^2) = \frac{1}{2}k^2(\tau_2^E)_{\text{eff}} - (\tau_3^E)_{\text{eff}} - (\tau_6^E)_{\text{eff}} + (\tau_8^E)_{\text{eff}}. \tag{128}$$

At $\mathcal{O}(\alpha_0)$

$$\frac{1}{2}\left(\frac{1}{F(k^2)}-\frac{1}{F(p^2)}\right) = -\frac{A_1}{2}\alpha_0\ln\frac{k^2}{p^2} + \mathcal{O}(\alpha_0^2). \tag{129}$$

Making use of $\tau^i_{\rm eff}$'s in Eq. (122) to form Eq. (128) gives

$$k^2 \sum(\tau_i^E)_{\rm eff}(p^2,k^2) = \alpha_0 \ln\frac{k^2}{p^2}\left[\frac{2}{3}-\frac{1}{6}+\frac{1}{2}\right]\frac{A_1}{2}, \\ = \frac{A_1}{2}\alpha_0 \ln\frac{k^2}{p^2} + \mathcal{O}(\alpha_0^2). \quad (130)$$

Since Eq. (129) cancels out in Eq. (130), Eq. (127) is satisfied.

B. $k^2 \simeq p^2 \gg q^2$: The photon limit

Let us turn our attention now to the photon limit of the perturbative $\tau_i^{\rm pert}$'s, Eqs. (A1)–(A4) in Euclidean space. The technical details of this limit can be found in Appendix B 2 and then we have

$$(\tau_2^E)^{\rm pert}_{\rm real}(\ell^2,q^2) = \frac{\alpha_0\xi}{12\pi\ell^4}\ln\frac{\ell^2}{q^2} + \mathcal{O}(\alpha_0^2), \\ (\tau_3^E)^{\rm pert}_{\rm real}(\ell^2,q^2) = \frac{\alpha_0\xi}{12\pi\ell^2}\ln\frac{\ell^2}{q^2} + \mathcal{O}(\alpha_0^2), \quad (131) \\ (\tau_6^E)^{\rm pert}_{\rm real}(\ell^2,q^2) = 0 + \mathcal{O}(\alpha_0^2), \\ (\tau_8^E)^{\rm pert}_{\rm real}(\ell^2,q^2) = 0 + \mathcal{O}(\alpha_0^2),$$

since in this limit $\ln(k^2/p^2)$ approaches 1 and $\ln(q^4/(k^2p^2))$ approaches $\ln(q^4/\ell^4)$. Therefore all four coefficient functions become proportional to $\ln(q^2/\ell^2)$. This signals that the structure of the nonperturbative transverse vertex consists of purely symmetric combination of F or G. We expand the nonperturbative $\tau_i^{\rm non\text{-}pert}$'s, Eq. (25), using Eqs. (28) and (29) at the order $\mathcal{O}(\alpha_0)$:

$$(\tau_2^E)^{\rm non\text{-}pert}_{\rm real}(\ell^2,q^2) = \frac{1}{\ell^4}(\delta_2+\varepsilon_2)\alpha_0\mathcal{S}^2_{1000}\ln\frac{\ell^2}{q^2} + \mathcal{O}(\alpha_0^2), \\ (\tau_3^E)^{\rm non\text{-}pert}_{\rm real}(\ell^2,q^2) = -\frac{1}{\ell^2}(\delta_3+\varepsilon_3)\alpha_0\mathcal{S}^3_{1000}\ln\frac{\ell^2}{q^2} + \mathcal{O}(\alpha_0^2), \\ (\tau_6^E)^{\rm non\text{-}pert}_{\rm real}(\ell^2,q^2) = -\frac{\ell\cdot q}{\ell^2}(\delta_6+\varepsilon_6)\alpha_0\mathcal{S}^6_{1000}\ln\frac{\ell^2}{q^2} + \mathcal{O}(\alpha_0^2), \\ (\tau_8^E)^{\rm non\text{-}pert}_{\rm real}(\ell^2,q^2) = -\frac{1}{\ell^2}(\delta_8+\varepsilon_8)\alpha_0\mathcal{S}^8_{1000}\ln\frac{\ell^2}{q^2} + \mathcal{O}(\alpha_0^2). \quad (132)$$

Comparing Eqs. (131) and (132) one can read off the symmetric coefficients as

$$(\delta_2+\varepsilon_2)\mathcal{S}^2_{1000} = \frac{A_1}{3}, \qquad (\delta_3+\varepsilon_3)\mathcal{S}^3_{1000} = -\frac{A_1}{3}, \\ (\delta_6+\varepsilon_6)\mathcal{S}^6_{1000} = 0, \qquad (\delta_8+\varepsilon_8)\mathcal{S}^8_{1000} = 0. \quad (133)$$

Analogously to the fermion case, we now perform similar checks for the photon constraints in the same three situations:

(1a) General case at $\mathcal{O}(\alpha_0)$: Recalling Eq. (61) let us check whether the photon MR constraint $PC2$, Eq. (84), at $\mathcal{O}(\alpha_0)$ is satisfied by Eq. (133) after adding them appropriately:

$$(\delta_2+\varepsilon_2)\mathcal{S}^2_{1000} - (\delta_3+\varepsilon_3)\mathcal{S}^3_{1000} \\ + (\delta_6+\varepsilon_6)\mathcal{S}^6_{1000} - (\delta_8+\varepsilon_8)\mathcal{S}^8_{1000} \\ = \left(\frac{A_1}{3} - \left(-\frac{A_1}{3}\right)\right), \\ \text{i.e. } \bar{\mathcal{S}}^\gamma_{1000} = \frac{2}{3}A_1. \quad (134)$$

(1b) For the special vertex ($\mathcal{A}^{(1)}$ and $\mathcal{S}^{(1)}$) at $\mathcal{O}(\alpha_0)$: We also check if the photon MR constraint, Eq. (117) at $\mathcal{O}(\alpha_0)$ is satisfied for this special choice of the vertex:

$$((\delta_2+\varepsilon_2) - (\delta_3+\varepsilon_3) + (\delta_6+\varepsilon_6) \\ -(\delta_8+\varepsilon_8))\left(\frac{-A_1}{2}\right) = \frac{2}{3}A_1, \\ \text{i.e. } \delta_\gamma + \varepsilon_\gamma = -\frac{4}{3}. \quad (135)$$

As we can see from both results, Eqs. (134) and (135), the effective $\tau^i_{\rm eff}$'s satisfy the photon MR constraint.

(2) Nonperturbative check: Recalling Eq. (97) and after extracting the nonperturbative MR constraints, we can usefully rewrite this as

$$\frac{\alpha N_F}{3\pi}\int_{q^2}^{\Lambda^2}\frac{d\ell^2}{\ell^2}\left\{[F(\ell)-1] + \frac{3}{2}\ell^2F^2(\ell)\sum(\tau_i^E)^{\rm non\text{-}pert}_{\rm eff} \times(\ell^2,q^2)\right\} = 0, \quad (136)$$

where

$$\sum(\tau_i^E)^{\rm non\text{-}pert}_{\rm eff}(\ell^2,q^2) = \ell^2(\tau_2^E)_{\rm eff} + (\tau_3^E)_{\rm eff} + (\tau_8^E)_{\rm eff}. \quad (137)$$

At $\mathcal{O}(\alpha_0)$

$$\int_{q^2}^{\Lambda^2}\frac{d\ell^2}{\ell^2}[F(\ell^2)-1] = -\frac{A_1}{2}\alpha_0\ln^2\frac{q^2}{\Lambda^2} + \mathcal{O}(\alpha_0^2). \quad (138)$$

Making use of Eq. (131) to form Eq. (137) we obtain

$$\int_{q^2}^{\Lambda^2}\frac{d\ell^2}{\ell^2}\left[\frac{3}{2}\ell^2F^2(\ell)\sum(\tau_i^E)_{\rm eff}(\ell^2,q^2)\right] \\ = \frac{A_1}{2}\alpha_0\ln^2\frac{q^2}{\Lambda^2} + \mathcal{O}(\alpha_0^2). \quad (139)$$

We see Eq. (138) cancels Eq. (139) and so Eq. (136) is satisfied.

C. Individual coefficients

With guidance from perturbation theory, we can now find further relations between the constants, Eq. (124) and (133). These eight equations fix eight of the 14 unknown constants (δ_2, δ_3, δ_6, δ_8, ...). In general these are

$$
\begin{aligned}
\delta_2 \mathcal{S}^2_{1000} &= \frac{A_1}{3} - \varepsilon_2 \mathcal{S}^2_{1000}, & \beta_2 \mathcal{A}^2_{1000} &= \frac{2}{3} A_1 - \varepsilon_2 \mathcal{S}^2_{1000},\\
\delta_3 \mathcal{S}^3_{1000} &= -\frac{A_1}{3} - \varepsilon_3 \mathcal{S}^3_{1000}, & (\beta_3 - \gamma_3) \mathcal{A}^3_{1000} &= -\frac{5}{12} A_1 - 2\varepsilon_3 \mathcal{S}^3_{1000},\\
\delta_6 \mathcal{S}^6_{1000} &= -\varepsilon_6 \mathcal{S}^6_{1000}, & (\beta_6 + \gamma_6) \mathcal{A}^6_{1000} &= \frac{1}{4} A_1,\\
\delta_8 \mathcal{S}^8_{1000} &= -\varepsilon_8 \mathcal{S}^8_{1000}, & \beta_8 \mathcal{A}^8_{1000} &= -\varepsilon_8 \mathcal{S}^8_{1000}.
\end{aligned}
\tag{140}
$$

For the specific choices of antisymmetric, $\mathcal{A}^{(1)}$ and symmetric $\mathcal{S}^{(1)}$ transverse vertex forms given in Table I, Eq. (140) becomes

$$
\begin{aligned}
\delta_2 &= -\frac{2}{3} + 2\varepsilon_3 - \varepsilon_8, & \beta_2 &= -\frac{2}{3} + \varepsilon_3 - \frac{1}{2}\varepsilon_8,\\
\delta_3 &= \frac{2}{3} - \varepsilon_3, & \beta_3 &= \frac{5}{12} + \gamma_3 - \varepsilon_3,\\
\delta_6 &= -\varepsilon_6, & \beta_6 &= -\frac{1}{4} - \gamma_6,\\
\delta_8 &= -\varepsilon_8, & \beta_8 &= -\frac{1}{2}\varepsilon_8,\\
\varepsilon_2 &= -2\varepsilon_3 + \varepsilon_8.
\end{aligned}
\tag{141}
$$

As we can see the unknown constraints in τ_i's, Eqs. (25) and (119), have now been fixed to match with perturbation theory. If we insert these constants in Eqs. (25) and (119), we can write the coefficient functions, τ_i's, in Euclidean space to obtain our final nonperturbative result:

$$
\begin{aligned}
\tau_2^E(p^2,k^2,q^2) &= \frac{2}{(k^4-p^4)}\left[\left(-\frac{2}{3}+\varepsilon_3-\frac{\varepsilon_8}{2}\right)+\gamma_2\frac{2k\cdot p}{k^2+p^2}\right]\tau_2^{\rm anti} + \frac{2}{(k^2+p^2)^2}\left[-\frac{2}{3}+(2\varepsilon_3-\varepsilon_8)\frac{q^2}{k^2+p^2}\right]\tau_2^{\rm sym},\\
\tau_3^E(p^2,k^2,q^2) &= -\frac{1}{(k^2-p^2)}\left[\left(\frac{5}{12}-\varepsilon_3\right)+\gamma_3\frac{(k+p)^2}{k^2+p^2}\right]\tau_3^{\rm anti} - \frac{1}{(k^2+p^2)}\left[\frac{2}{3}-\varepsilon_3\frac{q^2}{k^2+p^2}\right]\tau_3^{\rm sym},\\
\tau_6^E(p^2,k^2,q^2) &= -\frac{1}{(k^2+p^2)}\left[-\frac{1}{4}-\gamma_6\frac{q^2}{k^2+p^2}\right]\tau_6^{\rm anti} - \frac{(k^2-p^2)}{(k^2+p^2)^2}\left[-\varepsilon_6\frac{q^2}{k^2+p^2}\right]\tau_6^{\rm sym},\\
\tau_8^E(p^2,k^2,q^2) &= -\frac{1}{(k^2-p^2)}\left[-\frac{1}{2}\varepsilon_8+\gamma_8\frac{2k\cdot p}{k^2+p^2}\right]\tau_8^{\rm anti} - \frac{1}{(k^2+p^2)}\left[-\varepsilon_8\frac{q^2}{k^2+p^2}\right]\tau_8^{\rm sym},
\end{aligned}
$$

where

$$
\begin{aligned}
&\tau_i^{\rm anti} = \left(\frac{1}{F(k^2)}-\frac{1}{F(p^2)}\right), \quad \text{and} \quad \tau_i^{\rm sym} = \frac{1}{4}\left(\frac{1}{F(k^2)}+\frac{1}{F(p^2)}\right)\ln\left(\frac{F(q^2)^2}{F(k^2)F(p^2)}\right) \quad \text{OR}\\
&\tau_i^{\rm sym} = \frac{1}{2}\left(\frac{1}{F(k^2)}+\frac{1}{F(p^2)}\right)\ln\left[\frac{1}{2}\left(\frac{F(q^2)}{F(k^2)}+\frac{F(q^2)}{F(p^2)}\right)\right].
\end{aligned}
\tag{142}
$$

The fermion and photon SDE's at leading log order do not fix the constants γ_i, ε_i, Eq. (142). As the simplest example for later exploration we choose $\gamma_i = \varepsilon_i = 0$ in the above expressions and insert the second form of $\tau_i^{\rm sym}$ in Eq. (142), we then have:

$$
\begin{aligned}
\tau_2^E(p^2,k^2,q^2) &= \frac{1}{(k^4-p^4)}\left(-\frac{4}{3}\right)\left(\frac{1}{F(k^2)}-\frac{1}{F(p^2)}\right) + \frac{1}{(k^2+p^2)^2}\left(-\frac{2}{3}\right)\left(\frac{1}{F(k^2)}+\frac{1}{F(p^2)}\right)\ln\left[\frac{1}{2}\left(\frac{F(q^2)}{F(k^2)}+\frac{F(q^2)}{F(p^2)}\right)\right],\\
\tau_3^E(p^2,k^2,q^2) &= -\frac{1}{(k^2-p^2)}\left(\frac{5}{12}\right)\left(\frac{1}{F(k^2)}-\frac{1}{F(p^2)}\right) - \frac{1}{(k^2+p^2)}\left(\frac{1}{3}\right)\left(\frac{1}{F(k^2)}+\frac{1}{F(p^2)}\right)\ln\left[\frac{1}{2}\left(\frac{F(q^2)}{F(k^2)}+\frac{F(q^2)}{F(p^2)}\right)\right],\\
\tau_6^E(p^2,k^2,q^2) &= -\frac{1}{(k^2+p^2)}\left(-\frac{1}{4}\right)\left(\frac{1}{F(k^2)}-\frac{1}{F(p^2)}\right), \qquad \tau_8^E(p^2,k^2,q^2) = 0.
\end{aligned}
\tag{143}
$$

This is our simplest expression for the transverse part. We can then construct the full vertex from this using

$$
\Gamma^\mu(p,k;q) = \sum_{i=1}^{4}\lambda_i(p^2,k^2,q^2)L_i^\mu(p,k;q) + \sum_{j=2,3,6,8}\tau_i(p^2,k^2,q^2)T_i^\mu(p,k;q),
\tag{144}
$$

from Eqs. (7)–(10). This is our final result. Phenomenological studies of strong coupling QED with this vertex ansatz are presently underway [51,52].

VIII. CONCLUSIONS

The Schwinger-Dyson equations constitute the field equations of a theory. Being an infinite set of nested integral equations, they are in general intractable without some form of truncation. To date, the only known consistent truncation procedure is perturbation theory. This satisfies gauge invariance and multiplicative renormalizability order-by-order, and the meaning of any truncation is well defined. In the case of nonperturbative truncations, like the rainbow approximation, one has always been unsure as to how much physics has been encoded and how much lost. The calculation of dynamical mass generation nicely illustrates this. The properties of gauge invariance and multiplicative renormalizability are fundamental to our ability to calculate consistently in a gauge theory. It is thus natural that any truncation should respect these properties. They ensure not only the elimination of overlapping divergences that plague Schwinger-Dyson calculations, but allow all ultraviolet divergences to be handled appropriately. Here we have considered the fermion and boson propagators in four-dimensional massless QED. To be able to study these requires an *ansatz* for the full fermion-boson vertex. This interaction involves 11 nonzero components, three of which are fixed by the Ward-Green-Takahashi identity in terms of the fermion propagator functions. The other eight (transverse) components in principle require knowledge of the 4-, 5-, 6-,... point functions. However, very specific projections of this vertex appear in the fermion and boson self-energies. We have seen that these projections are strongly constrained by the multiplicative renormalizability of the fermion and boson propagators. At its simplest, multiplicative renormalzability is closely related to the ultraviolet behavior of loop integrals. This probes distinct limits for the fermion-boson vertex: one in the fermion equation and the other in the boson. In these two limits, the vertex has quite different structures. Such behavior ensures the multiplicative renormalizability of leading logarithms and shows that the 2-point Green's functions for both fermion and photon are wholly determined by the fermion wave function renormalization. This has enabled us to unravel for the first time the nonperturbative structure of the full vertex, Eqs. (143) and (144), at least as far as concerns the fermion and photon Schwinger-Dyson equations.

While the form of the 3-point vertex is determined in three kinematic limits, when k^2, $p^2 \gg q^2$, when k^2, $q^2 \gg p^2$ and when p^2, $q^2 \gg k^2$, its form at general momenta when all six vector structures of massless QED contribute involves free parameters. Imposing the known perturbative $\mathcal{O}(\alpha)$ result for the individual vertex components fixes these. This marks a significant step in the development of nonperturbative Feynman rules needed for realistic calculations in strong QED. There are many steps to go:

(i) to solve the extended constraints beyond leading logarithmic order and include masses [53],

(ii) to compute the Lamb shift of hydrogen and calculate the properties of positronium to asses how well our vertex ansatz automatically sums higher orders in α,

(iii) to explore strong physics with such a complete, unquenched vertex—extending the existing studies using bare, Ball-Chiu and *CP* vertices [11,19,20,23,29–34,48]. Such calculations are under way and will be reported elsewhere [52]

Eventually an extension to QCD will be our target.

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APPENDIX A: PERTURBATIVE τ'S

The vertex coefficients τ_i's given below are the massless limit of the exact $\mathcal{O}(\alpha)$ calculation for the massive fermions in general covariant gauge [41].

$$\tau_2^M(p^2,k^2,q^2) = \frac{\alpha_0}{8\pi\Delta^2}\bigg\{J_0\bigg[\bigg(\frac{k^2+p^2}{2}+\frac{3}{4\Delta^2}p^2k^2q^2\bigg)(\xi-2)+k\cdot p\bigg]+\ln\frac{k^2}{p^2}\bigg[\bigg(\frac{(k+p)^2}{2(p^2-k^2)}+\frac{3}{4\Delta^2}k\cdot p(p^2-k^2)\bigg)(\xi-2) + \frac{(p+k)^2}{(p^2-k^2)}\bigg]+\ln\frac{q^4}{k^2p^2}\bigg[\bigg(\frac{3}{4\Delta^2}k\cdot pq^2+1\bigg)(\xi-2)+1\bigg]+(\xi-2)\bigg\}, \tag{A1}$$

$$\tau_3^M(p^2,k^2,q^2) = \frac{\alpha_0}{8\pi\Delta^2}\bigg\{J_0\bigg[\bigg(\frac{(k^2+p^2)^2}{8}-\frac{3}{8\Delta^2}(k\cdot p)^2(k^2-p^2)^2\bigg)(\xi-2)-\Delta^2\bigg] + \ln\frac{k^2}{p^2}\bigg[\frac{(k^2-p^2)}{4}\bigg(-1+\frac{3}{2\Delta^2}k\cdot p(k+p)^2\bigg)(\xi-2)\bigg] + \ln\frac{q^4}{k^2p^2}\bigg[\frac{k\cdot p}{2}\bigg(1-\frac{3}{4\Delta^2}(k^2-p^2)^2\bigg)(\xi-2)\bigg]-\frac{(k+p)^2}{2}(\xi-2)\bigg\}, \tag{A2}$$

$$\tau_6^M(p^2,k^2,q^2) = \frac{\alpha_0}{8\pi\Delta^2}\frac{(p^2-k^2)}{2}\left\{J_0\left[\left(-\frac{q^2}{4}+\frac{3}{4\Delta^2}q^2(k\cdot p)^2\right)(\xi-2)\right]+\ln\frac{k^2}{p^2}\left[\left(\frac{3}{4\Delta^2}k\cdot p(p^2-k^2)-\frac{(p+k)^2}{2(p^2-k^2)}\right)\right.\right.$$
$$\left.\left.\times(\xi-2)\right]+\ln\frac{q^4}{k^2p^2}\left[\frac{3}{4\Delta^2}k\cdot pq^2(\xi-2)\right]+(\xi-2)\right\}, \tag{A3}$$

$$\tau_8^M(p^2,k^2,q^2) = \frac{\alpha_0}{8\pi\Delta^2}\left\{q^2\left[k\cdot pJ_0+\ln\frac{q^4}{k^2p^2}\right]+(p^2-k^2)\ln\left(\frac{k^2}{p^2}\right)\right\}, \tag{A4}$$

where

$$J_0 = \frac{2}{\Delta}\left[f\left(\frac{k\cdot p-\Delta}{p^2}\right)-f\left(\frac{k\cdot p+\Delta}{p^2}\right)+\frac{1}{2}\ln\left(\frac{q^2}{p^2}\right)\ln\left(\frac{k\cdot p-\Delta}{k\cdot p+\Delta}\right)\right], \tag{A5}$$

and

$$f(x) = Sp(1-x) = -\int_x^1 dy\frac{\ln y}{1-y}. \tag{A6}$$

APPENDIX B: LIMITS OF τ_i'S

1. Fermion limit

In order to take the $k^2 \simeq q^2 \gg p^2$ limit of the perturbative transverse vertex coefficients, namely, the τ_i functions, Eq. (A1)–(A4) we need to expand J_0 function, Eqs. (A5) and (B1), up to $\mathcal{O}(1/k^7)$:

$$J_0 = \frac{2}{k^2}\left\{1+\frac{1}{k^2}\left(k\cdot p-\frac{p^2}{3}\right)+\frac{1}{k^4}\left(\frac{4}{3}(k\cdot p)^2-(k\cdot p)p^2+\frac{1}{5}p^4\right)+\frac{1}{k^6}\left(2(k\cdot p)^3-\frac{12}{5}(k\cdot p)^2p^2+(k\cdot p)p^4-\frac{1}{7}p^6\right)\right.$$
$$+\frac{1}{k^8}\left(\frac{16}{5}(k\cdot p)^4-\frac{16}{3}(k\cdot p)^3p^2+\frac{24}{7}(k\cdot p)^2p^4-(k\cdot p)p^6+\frac{1}{9}p^8\right)+\frac{1}{k^{10}}\left(\frac{16}{3}(k\cdot p)^5-\frac{80}{7}(k\cdot p)^4p^2\right.$$
$$\left.\left.+10(k\cdot p)^3p^4-\frac{40}{9}(k\cdot p)^2p^6+(k\cdot p)p^8-\frac{p^{10}}{11}\right)+\mathcal{O}(1/k^7)\right\}\ln\left(\frac{k^2}{p^2}\right). \tag{B1}$$

2. Photon limit

In the photon limit, $k^2 \simeq p^2 \gg q^2$, J_0 behaves like

$$J_0 = \frac{2}{(p^2-k^2)}\left[\frac{2(p^2-k^2)}{p^2}+\frac{(p^2-k^2)^2}{p^4}+\frac{13}{18}\frac{(p^2-k^2)^3}{p^6}+\cdots\right]. \tag{B2}$$

APPENDIX C: EFFECTIVE τ'S

The connection between the effective and real τ_i functions are given below and the detail of this procedure can be found elsewhere [50]:

$$\begin{aligned}
(\tau_2^E)_{\rm eff}(p^2,k^2) &= \frac{1}{f(k^2,p^2)}\int_0^\pi d\psi\frac{\sin^2\psi}{q^2}(\tau_2^E)_{\rm Real}(p^2,k^2,q^2)\{-\Delta^2\},\\
(\tau_3^E)_{\rm eff}(p^2,k^2) &= \frac{1}{f(k^2,p^2)}\int_0^\pi d\psi\frac{\sin^2\psi}{q^2}(\tau_3^E)_{\rm Real}(p^2,k^2,q^2)\left\{-\Delta^2-\frac{3}{2}q^2k\cdot p\right\},\\
(\tau_6^E)_{\rm eff}(p^2,k^2) &= \frac{1}{f_6(k^2,p^2)}\int_0^\pi d\psi\frac{\sin^2\psi}{q^2}(\tau_6^E)_{\rm Real}(p^2,k^2,q^2)\{k\cdot p\},\\
(\tau_8^E)_{\rm eff}(p^2,k^2) &= \frac{1}{f(k^2,p^2)}\int_0^\pi d\psi\frac{\sin^2\psi}{q^2}(\tau_8^E)_{\rm Real}(p^2,k^2,q^2)\{-\Delta^2\},
\end{aligned} \tag{C1}$$

where

$$f(k^2, p^2) = \frac{\pi}{8} \frac{p^2}{k^2}(3k^2 - p^2), \qquad f_6(k^2, p^2) = \frac{\pi}{4} \frac{k^2}{k^2}. \tag{C2}$$

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