Proposal for determining the energy content of gravitational waves by using approximate symmetries of differential equations

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Since gravitational wave spacetimes are time-varying vacuum solutions of Einstein's field equations, there is no unambiguous means to define their energy content. However, Weber and Wheeler had demonstrated that they do impart energy to test particles. There have been various proposals to define the energy content, but they have not met with great success. Here we propose a definition using "slightly broken" Noether symmetries. We check whether this definition is physically acceptable. The procedure adopted is to appeal to "approximate symmetries" as defined in Lie analysis and use them in the limit of the exact symmetry holding. A problem is noted with the use of the proposal for plane-fronted gravitational waves. To attain a better understanding of the implications of this proposal we also use an artificially constructed time-varying nonvacuum metric and evaluate its Weyl and stress-energy tensors so as to obtain the gravitational and matter components separately and compare them with the energy content obtained by our proposal. The procedure is also used for cylindrical gravitational wave solutions. The usefulness of the definition is demonstrated by the fact that it leads to a result on whether gravitational waves suffer self-damping.

I. INTRODUCTION

Gravitational wave spacetimes are nonstatic vacuum solutions of the Einstein Field Equations (EFEs), i.e., they have no timelike Killing vectors (KVs) [1]. This creates a problem with the definition of energy for gravitational waves in general relativity (GR), as energy conservation is guaranteed for spacetimes that admit timelike KVs. Since for these waves the stress-energy tensor is zero, there was a debate whether they *really* exist [2,3]. To demonstrate their reality, Weber and Wheeler obtained (first- and second-order) approximate formulae for the momentum imparted to test particles in the path of cylindrical gravitational waves [2,4]. Later Ehlers and Kundt did the same for plane gravitational waves [3]. Using the pseudo-Newtonian formalism Qadir and Sharif [5] obtained a general closed form expression for the momentum imparted to test particles in an arbitrary spacetime, which gave the Weber-Wheeler approximation for cylindrical waves.

To define the energy content of gravitational waves, different people have attempted different *approximate symmetry* approaches. One such attempt was to assume that conservation of energy holds *asymptotically* and examine whether it would work for gravitational waves assuming a positive definite energy [6]. An altogether

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different idea was adopted to provide a measure of the extent of breakdown of symmetry by the integral of the square of the symmetrized derivative of a vector field divided by its mean square norm [7,8]. This led to what was called an *almost symmetric space* and the corresponding vector field an *almost Killing vector* [9]. This measure of "nonsymmetry" in a given direction was applied to the Taub cosmological solution [10] and to study gravitational radiation. It provides a choice of gauge that makes calculations simpler and was used for this purpose [11]. Essentially based on the almost symmetry, the concept of an "approximate symmetry group" was presented [12]. In [13] a method for computing approximate KVs on closed 2-surfaces was established and used to study the spin of black holes. This latter work was related to the earlier proposal of Matzner [7] to calculate the approximate Killing fields using an eigenvalue approach, so as to define a meaningful spin for nonsymmetric black holes in GR [14]. However none of all the above discussed attempts seem unequivocally successful.

Many relativists (notably including Roger Penrose [15]) believe that the invariants of the Weyl tensor should give the gravitational radiation field. It is not, *a priori*, so clear *which* (or which combination) of the scalar invariants should be used. A proposal for a "radiation scalar" was provided [16] and used to extract gauge-independent (or coordinate-independent) information particularly characterizing gravitational fields for numerical relativity by encoding it in the numerical variables [17–20]. However, this radiation scalar is physically meaningful only in spe-

cial regions of spacetime and may not have general applicability. Further, the actual energy has not been evaluated and there are no unambiguous physical predictions coming from it. Until these are extracted one cannot be sure that this proposal will *actually* give such results.

The approach of a "slightly broken symmetry" seems promising but merely providing simplicity of calculations is not physically convincing. Other approaches need to be tried to find one that is significantly better than the others, in that it is consistent with other physical concepts and leads to new physical insights. It was speculated that the use of approximate Lie symmetry methods for differential equations (DEs) [21,22] may give a resolution to the problem of energy in nonstatic spacetimes [23,24]. In this article we apply these methods to propose a resolution of the problem of the energy content of gravitational wave spacetimes. The approximation, or the breaking, involves a small parameter whose powers, higher than some chosen value, are neglected. The scaling factors (discussed in Secs. III and V), are independent of the strength of the perturbation parameter, which means that we can take the limit as it goes to zero. This is reminiscent of the d'Alembert principle of virtual work to obtain results for statics from dynamical considerations [25].

By virtue of Noether's theorem [26] for every infinitesimal generator of symmetry of a Lagrangian (called a *Noether symmetry*), there is a conserved quantity. For time translational invariance it is energy that is conserved. It is for this reason that it was hoped that the symmetry approach could prove fruitful for defining the energy content of gravitational waves. Reducing from the maximally symmetric Minkowski spacetime (10 KVs) to the Schwarzschild [27] and Reissner-Nordström (RN) [23] spacetimes linear and spin angular momentum conservation are lost. Using approximate Lie symmetry methods for DEs these conservation laws were recovered as trivial first-order and second-order approximate conservation laws, respectively. Reducing from the Schwarzschild spacetime to the Kerr (or charged-Kerr) spacetime we lose angular momentum conservation. Going directly from the Minkowski spacetime to the Kerr (or charged-Kerr) spacetime we lose linear and spin angular momentum conservation as well. These lost conservation laws were recovered as "trivial" first-order and second-order approximate conservation laws for this spacetime [24] (in that there is no exact symmetry part mixed in with the approximate symmetry infinitesimal generator).

The trivial first-order approximate symmetries did not provide any new insights. However, for the second-order approximate symmetries of the geodesic equations for the RN [23] and charged-Kerr [24] spacetimes the time translational approximate symmetry generator had to pick up a rescaling factor to provide a necessary cancellation. This corresponds to rescaling the energy of test particles and hence could possibly lead to a definition of energy in the

spacetime. Clearly, the energy *content* would then be defined as the scaling factor.

To check the proposal for defining energy in gravitational wave spacetimes by using *slightly broken symmetry* as defined in Lie analysis [21], plane-fronted parallel-rays (pp) gravitational waves [28] are first investigated. For the approximate symmetries of pp-waves first the t-dependent part of this spacetime is removed to make it static and taken as the unperturbed spacetime. Then the exact pp-wave is taken as a perturbation on this static spacetime by considering the arbitrary amplitude of the wave as a small parameter, ϵ. Since ϵ^2 does not appear in the geodesic equations for perturbed pp-waves, there is a problem in applying the definition of second-order approximate symmetries of ordinary differential equations (ODEs), which gives the scaling factor mentioned earlier, to them. It can also be seen from the geometry of pp-waves in which the wave fronts are like moving parallel planes and the curvature of the spacetime is absolutely zero before the pp-wave pulse arrives and after it has passed [1]. There is no region where there is a *slight* shift from the flat geometry as required for obtaining an approximate symmetry. Thus the proposal for determining the energy content of pp-waves cannot be checked. The conformally invariant Weyl tensor [29] represents a pure gravitational field. In some sense it tells us about the gravitational energy of the spacetime, but it does not give a direct measure of the gravitational energy. The stress-energy tensor gives the matter content of the spacetime [1]. For this perturbed spacetime the stress-energy tensor is zero while the Weyl tensor is nonzero. To obtain a better understanding of the energy rescaling in plane gravitational waves, the artificially constructed example of a plane symmetric "wavelike" spacetime [30], which represents a gravitational wave interacting with matter, is investigated. Here we *do* obtain a scaling factor which gives the rescaling of energy in the spacetime field.

For the plane wavelike spacetime [30], along with trivial approximate symmetries, a nontrivial first-order approximate symmetry was found. The first-order approximate (stable) first integral corresponding to the nontrivial first-order approximate symmetry of this wavelike spacetime is calculated here. The first-order nontrivial approximate timelike Noether symmetry is used with the momentum vector to obtain a conserved quantity which gives the energy nonconservation due to time variation. To check this quantity first-order approximate Noether symmetries of pp-wave spacetime are investigated. Only the exact symmetries are recovered as trivial first-order approximate Noether symmetries, which gives the exact conservation laws as trivial first-order approximate conservation laws.

Next cylindrically symmetric exact gravitational waves [31], which are physically easier to understand, are considered. To study their approximate symmetries the t-dependent part is first removed to define a static space-

time and the exact wave is dealt with as a perturbation of this static spacetime, taking the strength of the wave as a small parameter, ϵ. A scaling factor is obtained for this spacetime which gives the rescaling of energy. Noether symmetries for the cylindrically symmetric case are also considered to look at the conserved quantities. First a cylindrically symmetric wavelike spacetime is investigated which has a nontrivial first-order approximate Noether symmetry that gives the conserved quantity like the plane symmetric case. There is no nontrivial approximate symmetry for the perturbed cylindrical wave spacetime. The approximate Weyl and stress-energy tensors (up to first order in ϵ) for the cylindrical wave spacetimes are nonzero. Electromagnetic waves in their interaction with matter get damped. This is known as Landau damping [32]. Since GR is a nonlinear theory, gravitational waves have self-interactions. The question arises whether we should expect Landau self-damping. With our proposal, a self-damping is seen for cylindrical waves.

The plan of the paper is as follows. The next section briefly reviews the mathematical formalism to be used. In Sec. III, second-order approximate symmetries of the geodesic equations for the plane wave spacetimes are discussed and graphs of the scaling factor for them are given. In the same section the Weyl and stress-energy tensors for plane wave spacetimes are also discussed. Noether symmetries of pp-waves are studied and a review of the approximate Noether symmetries for the plane symmetric wavelike spacetime is given in Sec. IV. In the next section, second-order approximate symmetries of the geodesic equations for the cylindrically symmetric case are provided and the graphs of the scaling factor for them are given. The Weyl and stress-energy tensors for the cylindrical waves are also discussed in the same section. In Sec. VI approximate Noether symmetries for the cylindrical wave spacetimes are investigated. Finally a summary and discussion are presented in Sec. VII.

II. REVIEW OF MATHEMATICAL FORMALISM USED

We first review the definition of the second-order approximate symmetries of a system of ODEs under point symmetries. A vector field

$$\mathbf{X} = \mathbf{X}_0 + \epsilon \mathbf{X}_1 + \epsilon^2 \mathbf{X}_2 + O(\epsilon^3), \tag{1}$$

is called a second-order approximate symmetry of the system of perturbed ODEs

$$\mathbf{E} = \mathbf{E}_0 + \epsilon \mathbf{E}_1 + \epsilon^2 \mathbf{E}_2 + O(\epsilon^3), \tag{2}$$

if the following condition ([23] and references given there in) holds

$$(\mathbf{X}_0 + \epsilon \mathbf{X}_1 + \epsilon^2 \mathbf{X}_2)(\mathbf{E}_0 + \epsilon \mathbf{E}_1 + \epsilon^2 \mathbf{E}_2)|_{\mathbf{E}_0 + \epsilon \mathbf{E}_1 + \epsilon^2 \mathbf{E}_2 = O(\epsilon^3)}$$

$$= O(\epsilon^3), \tag{3}$$

where $\mathbf{X}_0$ is the exact symmetry generator of the system of ODEs $\mathbf{E}_0$, i.e.,

$$(\mathbf{X}_0)(\mathbf{E}_0)|_{\mathbf{E}_0 = 0} = 0, \tag{4}$$

$\mathbf{X}_1$, $\mathbf{X}_2$ are the first-order and second-order approximate parts of the approximate symmetry generator, respectively, $\mathbf{E}_1$ is the first-order perturbed part, and $\mathbf{E}_2$ is the second-order perturbed part of the system of ODEs, respectively. It should be noted that the scaling factor comes from the applications of the perturbed system of DEs in subscript of (3), as required.

Noether symmetries are those infinitesimal symmetry generators that leave a Lagrangian $L(s, x^\mu, \dot{x}^\mu)$ invariant. They form a Lie algebra that contains the isometries for the Lagrangian that minimizes arc length, with at least one extra symmetry, $\partial/\partial s$, [33]. It is defined as a vector field [22,34,35]

$$\mathbf{X} = \xi(s, x^\mu)\frac{\partial}{\partial s} + \eta^\nu(s, x^\mu)\frac{\partial}{\partial x^\nu}, \tag{5}$$

where μ, $\nu = 0, 1, 2, 3$, such that

$$\mathbf{X}^{[1]}L + (D_s\xi)L = D_sA, \tag{6}$$

where $A(s, x^\mu)$, is a gauge function. The total derivative operator D_s and the first prolongation $\mathbf{X}^{[1]}$ of the vector field $\mathbf{X}$ given by (5) are

$$D_s = \frac{\partial}{\partial s} + \dot{x}^\mu \frac{\partial}{\partial x^\mu}, \tag{7}$$

and

$$\mathbf{X}^{[1]} = \mathbf{X} + (\eta^\nu_{,s} + \eta^\nu_{,\mu}\dot{x}^\mu - \xi_{,s}\dot{x}^\nu - \xi_{,\mu}\dot{x}^\mu\dot{x}^\nu)\frac{\partial}{\partial \dot{x}^\nu}. \tag{8}$$

For more general considerations see [22]. The significance of Noether symmetries is clear from the following theorem [26], proved in [36].

Theorem 1. If $\mathbf{X}$ is a Noether point symmetry corresponding to a Lagrangian $L(s, x^\mu, \dot{x}^\mu)$ of a second-order ODE $\ddot{x}^\mu = g(s, x, \dot{x}^\mu)$, then

$$I = \xi L + (\eta^\mu - \dot{x}^\mu \xi)\frac{\partial L}{\partial \dot{x}^\mu} - A, \tag{9}$$

is a first integral of the ODE associated with $\mathbf{X}$.

First-order approximate symmetries of a Lagrangian (or first-order approximate Noether symmetries) [35,37] are defined as follows. For a first-order perturbed system of ODEs

$$\mathbf{E} = \mathbf{E}_0 + \epsilon \mathbf{E}_1 = O(\epsilon^2), \tag{10}$$

corresponding to a first-order Lagrangian, which is perturbed up to first-order in ϵ,

$$L(s, x^\mu, \dot{x}^\mu, \epsilon) = L_0(s, x^\mu, \dot{x}^\mu) + \epsilon L_1(s, x^\mu, \dot{x}^\mu) + O(\epsilon^2), \tag{11}$$

the functional $\int_V L ds$ is invariant under the one-parameter

group of transformations with approximate Lie symmetry generator

$$\mathbf{X} = \mathbf{X}_0 + \epsilon\mathbf{X}_1 + O(\epsilon^2), \tag{12}$$

up to gauge

$$A = A_0 + \epsilon A_1, \tag{13}$$

where

$$\mathbf{X}_j = \xi_j \frac{\partial}{\partial s} + \eta_j^\mu \frac{\partial}{\partial x^\mu}, \qquad (j = 0, 1), \tag{14}$$

if

$$\mathbf{X}_0^{[1]}L_0 + (D_s\xi_0)L_0 = D_sA_0, \tag{15}$$

and

$$\mathbf{X}_1^{[1]}L_0 + \mathbf{X}_0^{[1]}L_1 + (D_s\xi_1)L_0 + (D_s\xi_0)L_1 = D_sA_1. \tag{16}$$

Here L_0 is the exact Lagrangian corresponding to the exact equations and $L_0 + \epsilon L_1$ the first-order perturbed Lagrangian corresponding to the first-order perturbed equations. The perturbed Eqs. (3) and (16) always have the approximate symmetry generators $\epsilon\mathbf{X}_0$ which are known as trivial approximate symmetries, and $\mathbf{X}$ given by (1) and (12) with $\mathbf{X}_0 \neq 0$ is called a nontrivial approximate symmetry. These approximate symmetries of a manifold form an approximate Lie algebra [38].

The first-order approximate first integrals are defined by setting $I = I_0 + \epsilon I_1$, $L = L_0 + \epsilon L_1$, $\xi = \xi_0 + \epsilon\xi_1$, $\eta = \eta_0 + \epsilon\eta_1$, and $A = A_0 + \epsilon A_1$ in (9) and equating the coefficients of like powers of ϵ on both sides. This gives the zeroth (exact part) and first-order approximate part of the first-order approximate first integrals

$$I_0 = \xi_0 L_0 + (\eta_0^\mu - \dot{x}^\mu\xi_0)\frac{\partial L_0}{\partial\dot{x}^\mu} - A_0, \tag{17}$$

$$I_1 = \xi_0 L_1 + \xi_1 L_0 + (\eta_0^\mu - \dot{x}^\mu\xi_0)\frac{\partial L_1}{\partial\dot{x}^\mu}$$
$$+ (\eta_1^\mu - \dot{x}^\mu\xi_1)\frac{\partial L_0}{\partial\dot{x}^\mu} - A_1. \tag{18}$$

If I_0 vanishes, then I is called an unstable approximate first integral and is otherwise called stable. A detailed discussion on the approximate first integrals for Hamiltonian dynamical systems is given in [39].

The Weyl tensor in component form is given by

$$C^a{}_{bcd} = R^a{}_{bcd} - \frac{1}{2}(\delta^a_c R_{bd} - \delta^a_d R_{bc} + g_{bd}R^a{}_c - g_{bc}R^a{}_d)$$
$$+ \frac{1}{6}R(\delta^a_d g_{bc} - \delta^a_c g_{bd}). \tag{19}$$

Here $R^a{}_{bcd}$ is the Riemann curvature tensor, R_{ab} is the Ricci tensor, R is the Ricci scalar, g_{ab} is the metric tensor and δ^a_b is the Kroneker delta. For a four-dimensional space-

time the Weyl tensor has 10 independent components [40]. If the Weyl tensor vanishes in a neighborhood of a space-time, the neighborhood is locally conformally equivalent to the Minkowski spacetime. Thus the Weyl tensor has geometric meaning independent of any physical interpretation.

The stress-energy tensor can be calculated from the EFEs

$$T_{ab} = \frac{1}{\kappa}\left(R_{ab} - \frac{1}{2}Rg_{ab}\right), \tag{20}$$

where κ is the gravitational coupling. For a four-dimensional spacetime this tensor has 10 independent components. At each event of the spacetime this tensor gives the energy density, momentum density and stress as measured by any and all observers at that event. Since for gravitational wave spacetimes T_{ab} is always zero and $C^a{}_{bcd}$ is nonzero, there is no matter or energy or momentum however there is the Weyl curvature. If there is no mass or energy at a given event, the Ricci tensor vanishes. If it were not for the Weyl tensor, this would mean that matter at one place could not have gravitational influence on distant matter separated by a void. Thus the Weyl tensor represents that part of spacetime curvature which can propagate across and curve up a void.

III. SECOND-ORDER APPROXIMATE SYMMETRIES AND ENERGY RESCALING: PLANE WAVE SPACETIMES

The line element for pp-waves [28] is

$$ds^2 = h\omega^2[(x^2 - y^2)\sin(\omega(t - z)) + 2xy\cos(\omega(t - z))]$$
$$\times (dt^2 + dz^2 - 2dtdz) + dt^2 - dx^2 - dy^2 - dz^2, \tag{21}$$

where h is the amplitude of the wave and ω is the frequency.

Now we remove the t-dependent part of (21) and putting $h = 1$ to define a static spacetime

$$ds^2 = \omega^2[(x^2 - y^2) + 2xy](dt^2 + dz^2 - 2dtdz) + dt^2$$
$$- dx^2 - dy^2 - dz^2. \tag{22}$$

To obtain the approximate symmetries of pp-waves the exact pp-waves are considered as a perturbation on the static spacetime (22). For this purpose the amplitude $h = \epsilon$, is taken as a small parameter and the line element of the perturbed pp-waves is

$$ds^2 = \omega^2[x^2 - y^2 + 2xy + \epsilon\{(x^2 - y^2)\sin(\omega(t - z))$$
$$+ 2xy\cos(\omega(t - z))\}](dt^2 + dz^2 - 2dtdz)$$
$$+ dt^2 - dx^2 - dy^2 - dz^2. \tag{23}$$

For this perturbed pp-wave spacetime (23) we have the system of first-order perturbed geodesic equations and there do not appear ϵ^2,

$$\ddot{t} + \omega^2(\dot{t} - \dot{z})\{(x+y)\dot{x} + (x-y)\dot{y}\} + \epsilon\left[\frac{\omega^3}{2}\{(x^2-y^2)\cos\omega(z-t) + 2xy\sin\omega(z-t)\}(\dot{t}^2 + \dot{z}^2 - \dot{t}\dot{z}) - \omega^2(\dot{t}-\dot{z})\right.$$

$$\left. \times\{x\sin\omega(z-t) - y\cos\omega(z-t)\}\dot{x} + \omega^2\{y\sin\omega(z-t) + x\cos\omega(z-t)\}\dot{y}\right] = 0, \quad (24)$$

$$\ddot{x} + [\omega^2(x+y) - \epsilon\omega^2\{x\sin\omega(z-t) - y\cos\omega(z-t)\}](\dot{t}^2 + \dot{z}^2 - \dot{t}\dot{z}) = 0, \tag{25}$$

$$\ddot{y} + [\omega^2(x+y) - \epsilon\omega^2\{x\cos\omega(z-t) - y\sin\omega(z-t)\}](\dot{t}^2 + \dot{z}^2 - \dot{t}\dot{z}) = 0, \tag{26}$$

$$\ddot{z} + \omega^2(\dot{t} - \dot{z})\{(x+y)\dot{x} + (x-y)\dot{y}\} + \epsilon\left[\frac{\omega^3}{2}\{(x^2-y^2)\cos\omega(z-t) + 2xy\sin\omega(z-t)\}(\dot{t}^2 + \dot{z}^2 - \dot{t}\dot{z}) - \omega^2(\dot{t}-\dot{z})\right.$$

$$\left. \times\{x\sin\omega(z-t) - y\cos\omega(z-t)\}\dot{x} + \omega^2\{y\sin\omega(z-t) + x\cos\omega(z-t)\}\dot{y}\right] = 0. \quad (27)$$

Since there is no quadratic term in ϵ, in the above geodesic equations, we cannot apply the definition of second-order approximate symmetries, which gives us the energy rescaling factor to them. This behavior is consistent with the pp-wave geometry in which the wave front moves as parallel planes and the spacetime curvature is zero before and after the pp-wave pulse [1].

To obtain a better understanding of the energy rescaling in plane gravitational waves we apply the definition of second-order approximate symmetries of ODEs to the second-order perturbed geodesic Eqs. (29)–(32) for the artificially constructed example of plane symmetric wavelike spacetime [30]. For this purpose a nonflat plane symmetric static spacetime [41] was considered with $\mu(x) = \nu^2(x) = (x/X)^2$,

$$ds^2 = e^{2\nu(x)}dt^2 - dx^2 - e^{2\mu(x)}(dy^2 + dz^2), \tag{28}$$

where X is a constant having the same dimensions as x. Since gravitational waves are nonstatic spacetimes therefore the static spacetime (28) was perturbed with a time-dependent small parameter (for definiteness by ϵt) to make it slightly nonstatic. For this the metric (28) was taken with $\nu(x) = (x/X + \epsilon t/T)$ and $\mu(x) = (x^2/X^2 + \epsilon t/T)$, where T is a constant having dimensions of t. Retaining ϵ^2 and neglecting its higher powers second-order perturbed geodesic equations are obtained:

$$\ddot{t} + \frac{2}{X}\dot{t}\dot{x} - \frac{\epsilon}{T}[\dot{t}^2 - (\dot{y}^2 + \dot{z}^2)e^{2((x/X)^2 - x/X)}]$$

$$+ \frac{t\epsilon^2}{T^2}[\dot{t}^2 + (\dot{y}^2 + \dot{z}^2)e^{2((x/X)^2 - x/X)}] + O(\epsilon^3) = 0, \quad (29)$$

$$\ddot{x} + \frac{\dot{t}^2}{X}e^{2x/X} - \frac{2x}{X^2}(\dot{y}^2 + \dot{z}^2)e^{2(x/X)^2}$$

$$+ \frac{2t\epsilon}{TX}\left[\dot{t}^2 e^{2x/X} - \frac{2x}{X}(\dot{y}^2 + \dot{z}^2)e^{2(x/X)^2}\right]$$

$$+ \frac{t^2\epsilon^2}{XT^2}\left[\dot{t}^2 e^{2x/X} - \frac{2x}{X}(\dot{y}^2 + \dot{z}^2)e^{2(x/X)^2}\right] + O(\epsilon^3) = 0,$$

$$\tag{30}$$

$$\ddot{y} + \frac{4x}{X^2}\dot{x}\dot{y} + \frac{2\epsilon}{T}\dot{t}\dot{y} - \frac{2t\epsilon^2}{T^2}\dot{t}\dot{y} + O(\epsilon^3) = 0, \tag{31}$$

$$\ddot{z} + \frac{4x}{X^2}\dot{x}\dot{z} + \frac{2\epsilon}{T}\dot{t}\dot{z} - \frac{2t\epsilon^2}{T^2}\dot{t}\dot{z} + O(\epsilon^3) = 0. \tag{32}$$

The Lie symmetry algebra of the exact or unperturbed geodesic equations (i.e., when $\epsilon = 0$, in (29)–(32)) includes the generators of the dilation algebra $\partial/\partial s$, $s\partial/\partial s$, corresponding to

$$\xi(s) = c_0 s + c_1. \tag{33}$$

In the determining equations for the first-order approximate symmetries of the geodesic equations for the Schwarzschild spacetime [27] the terms involving $\xi_s = c_0$ cancel out. Here the terms involving $\xi_s = c_0$ do not cancel automatically but, like the RN [23] and charged-Kerr [24] spacetimes, collect a scaling factor to cancel out. In this case the scaling factor is

$$\frac{t}{T^2}[\dot{t}^2 + (\dot{y}^2 + \dot{z}^2)e^{2((x/X)^2 - x/X)}]. \tag{34}$$

Energy conservation is related with time translation and ξ is the coefficient of $\partial/\partial s$ in the point transformation given by (5), where s is the proper time. The scaling factor (34) corresponds to the rescaling of energy of a test particle in this wavelike spacetime field. Since the scaling factor (34) involves the derivatives of the coordinates and the derivatives only apply to the paths of the particles, to get energy in the spacetime field the derivatives of the coordinates are replaced by the first integrals. Therefore we get

$$\frac{t}{4T^2}[e^{-4x/X} + 2e^{-2(x/X)(x/X+1)}]. \tag{35}$$

This energy expression is plotted below for different values of t and x, using Mathematica 5.0. The values of X and T are arbitrary. The above scaling factor for this wavelike spacetime depends linearly on t, and in both diagrams below the energy in the gravitational field increases linearly with time. In Fig. 1 the energy is seen to decrease along x and disappear sharply close to $x = 0$. To see the
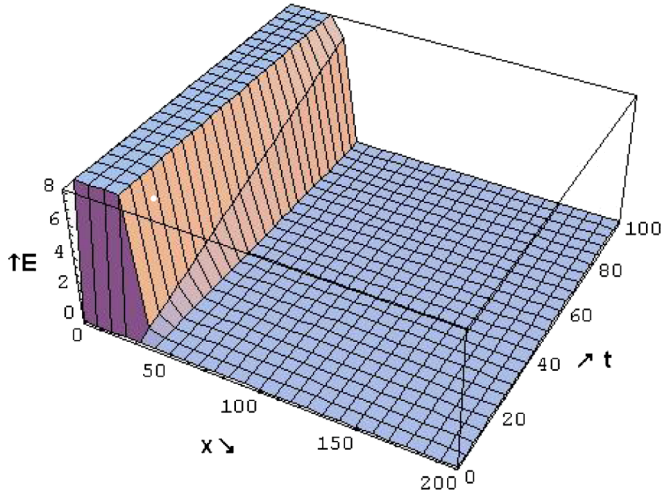
FIG. 1 (color online). Plane symmetric gravitational wavelike spacetime. The energy increases indefinitely in time close to $x = 0$ and then disappears suddenly after some distance. The small parameter ϵ, (considered as strength of the wave) is arbitrary in all the spacetimes discussed in this article. Thus the units of energy are chosen arbitrarily. Throughout this paper gravitational units are adopted and space, time, and mass are given in seconds.

variation with x we enlarge the diagram by reducing the range of x in Fig. 2. As we move along x the increase in energy with time becomes gradual. Since the small parameter ϵ, (which is considered as the strength of the wave) is arbitrary the units of energy are arbitrarily chosen. Throughout this article gravitational units are used and space, time, and mass are given in seconds.

Though the Weyl tensor gives information about the gravitational energy of the spacetime, it is not clear how to obtain a *measure* of the energy in it. For the pure
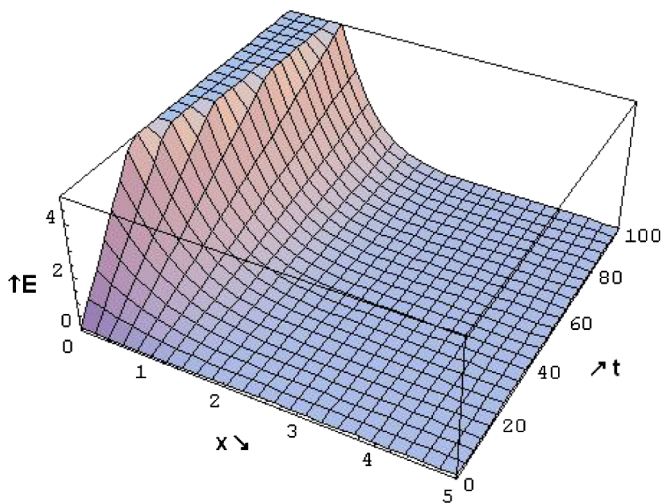

FIG. 2 (color online). This is an expanded version of Fig. 1. Here the range of x is shrunk and it is seen that the energy decreases smoothly with distance.

gravitational part of the perturbed pp-wave spacetime the independent nonzero components of the Weyl tensor are

$$
\begin{aligned}
C^0{}_{101} &= -\omega^2[\omega^2(x^2 - y^2 - 2xy) + 1 \\
&\quad + \epsilon\{2\omega^2(x^2 - y^2 - xy)\sin\omega(z - t) \\
&\quad - 2\omega^2 xy\cos\omega(z - t) - \sin\omega(z - t)\}] + O(\epsilon^2) \\
&= C^0{}_{113} = C^1{}_{313} = -C^0{}_{202} = C^0{}_{223} = C^2{}_{323}, \\
C^0{}_{102} &= -\omega^2[\omega^2(y^2 - x^2 - 2xy) + 1 \\
&\quad + \epsilon\{\omega^2(x^2 - y^2)\sin\omega(z - t) \\
&\quad - \omega^2(x^2 - y^2 + 4xy)\cos\omega(z - t) \\
&\quad + \cos\omega(z - t)\}] + O(\epsilon^2) \\
&= C^0{}_{123} = C^0{}_{213} = C^1{}_{323}.
\end{aligned}
\tag{36}
$$

In the literature [42] the Weyl tensor is usually defined with valence (1, 3). In spinors it is naturally given as a tensor of valence (0, 4) [43]. For usual purposes the form does not matter, but for differential symmetries of the tensor the form is crucial [44]. In covariant form the components of the Weyl tensor are

$$
\begin{aligned}
C_{0101} &= -\omega^2[1 - \epsilon\sin\omega(z - t)] = C_{0113} = C_{1313} \\
&= -C_{0202} = C_{0223} = C_{2323}, \\
C_{0102} &= -\omega^2[1 + \epsilon\sin\omega(z - t)] = C_{0123} = C_{0213} \\
&= C_{1323}.
\end{aligned}
\tag{37}
$$

From here it appears that the (0, 4) form may give the physically relevant quantities as the space dependence in (36) does not seem to correspond to the geometry of the pp-wave, while (37) does. Here the pure gravitational field which "curves up the void" seems to be sinusoidal. For this spacetime there is obviously no nonzero component of the stress-energy tensor.

There are six nonzero components (up to first order in ϵ) of the Weyl tensor for the above plane symmetric wavelike spacetime

$$
\begin{aligned}
C^0{}_{101} &= \frac{1}{3X^3}(2x + X) + O(\epsilon^2), \\
C^0{}_{202} &= C^0{}_{303} = \frac{e^{2x^2/X^2}}{6X^3}\left(1 + \epsilon\frac{2t}{T}\right)(2x + X) + O(\epsilon^2), \\
C^1{}_{212} &= C^1{}_{313} = -C^0{}_{202}, \qquad C^2{}_{323} = -2C^0{}_{202}.
\end{aligned}
\tag{38}
$$

From Figs. 1 and 2 (where the wave is along the x direction), it is clear that the energy in the gravitational field of the plane wavelike spacetime increases with time. Therefore the first component of the Weyl tensor must depend on t linearly which corresponds to the covariant form (given below) and not the mixed form.

$$C_{0101} = \frac{e^{2x/X}}{3X^3}\left(1 + \epsilon\frac{2t}{T}\right)(2x + X) + O(\epsilon^2),$$

$$C_{0202} = C_{0303}$$

$$= \frac{e^{2x^2/X^2 + 2x/X}}{6X^3}\left(1 + \epsilon\frac{4t}{T}\right)(2x + X) + O(\epsilon^2),$$ (39)

$$C_{1212} = C_{1313} = -C_{0202}, \qquad C_{2323} = -2C_{0202}.$$

The nonzero components of the stress-energy tensor for this wavelike spacetime are

$$T_{00} = \frac{4e^{2x/X}}{\kappa X^4}\left(1 + \epsilon\frac{2t}{T}\right)\left(1 + \frac{3x^2}{X^2}\right) + O(\epsilon^2),$$

$$T_{11} = \frac{-8x}{\kappa X^4}(x + X) + O(\epsilon^2),$$ (40)

$$T_{22} = T_{33} = \frac{e^{2x^2/X^2}}{\kappa X^4}\left(1 + \epsilon\frac{2t}{T}\right)(2xX + 4x^2 + 3X^2) + O(\epsilon^2),$$

$$T_{01} = \frac{\epsilon}{\kappa TX^2}(x + X) + O(\epsilon^2).$$

It is worth noting that the x-direction stress has no approximate part of the first order and the approximate part of the energy increases linearly with time and quadratically at large distances. More interestingly there is an approximate momentum in the x-direction that increases linearly with the value of x. This linear increase in energy was built into the metric and it entails the momentum in the x-direction.

We give the ratio of energy density imparted to the matter field

$$E_{\text{imp}} = \frac{(T_{00})_P}{(T_{00})_E},$$ (41)

where $(T_{00})_E$ and $(T_{00})_P$ are the energy densities of the exact (i.e., when $\epsilon = 0$) and first-order approximate spacetimes, respectively. For the plane symmetric wavelike spacetime we have

$$E_{\text{imp}} = \epsilon\frac{2t}{T}.$$ (42)

IV. APPROXIMATE NOETHER SYMMETRIES OF PLANE WAVE SPACETIMES

The Lagrangian defined for (28) is [30] (throughout this paper the Lagrangian of a spacetime would mean the Lagrangian for the geodesic equations of the spacetime)

$$L = e^{2x/X}\dot{t}^2 - \dot{x}^2 - e^{2x^2/X^2}(\dot{y}^2 + \dot{z}^2).$$ (43)

Its symmetry generators are

$$\mathbf{X}_0 = \frac{\partial}{\partial t}, \qquad \mathbf{X}_1 = \frac{\partial}{\partial y}, \qquad \mathbf{X}_2 = \frac{\partial}{\partial z},$$

$$\mathbf{X}_3 = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}, \qquad \mathbf{Y}_0 = \frac{\partial}{\partial s}, \qquad A = c,$$ (44)

where c is a constant, $\mathbf{X}_0$ corresponds to energy conservation, $\mathbf{X}_1$ and $\mathbf{X}_2$ correspond to linear momentum conservation along y and z, while $\mathbf{X}_3$ corresponds to angular momentum conservation in the yz plane [45].

The first-order perturbed Lagrangian is

$$L = e^{2x/X}\dot{t}^2 - \dot{x}^2 - e^{2x^2/X^2}(\dot{y}^2 + \dot{z}^2)$$

$$+ \frac{2\epsilon t}{T}[e^{2x/X}\dot{t}^2 - e^{2x^2/X^2}(\dot{y}^2 + \dot{z}^2)] + O(\epsilon^2), \quad (45)$$

yielding the nontrivial approximate symmetry

$$\mathbf{X}_a = \frac{\partial}{\partial t} - \epsilon\frac{1}{T}\left(t\frac{\partial}{\partial t} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right),$$ (46)

along with the trivial symmetries, and the gauge function A_1 is again a constant. The stable first integral for the symmetry generator (46) is

$$I = 2e^{2x/X}\dot{t} + \frac{2\epsilon}{T}[e^{2x/X}t\dot{t} + e^{2(x/X)^2}(y\dot{y} + z\dot{z})].$$ (47)

Contract the energy-momentum vector with the timelike approximate Noether symmetry generator (46), to obtain the conserved quantity

$$Q = E - \frac{\epsilon}{T}(t\dot{E} + y\dot{p}_y + z\dot{p}_z),$$ (48)

where E is the energy and p is the momentum. This gives the energy nonconservation due to time variation. That is the energy imparted to the test particles with energy and momentum given by (48). However this does not give the energy in the spacetime field.

To check the conserved quantities in the pp-wave spacetime, we investigate the first-order approximate Noether symmetries for this spacetime. The Lagrangian for the pp-wave spacetime (21) is

$$L = h\omega^2[(x^2 - y^2)\sin(\omega(t - z)) + 2xy\cos(\omega(t - z))]$$

$$\times (\dot{t}^2 + \dot{z}^2 - 2\dot{t}\dot{z}) + \dot{t}^2 - \dot{x}^2 - \dot{y}^2 - \dot{z}^2.$$ (49)

This Lagrangian admits the following symmetry generators:

$$\mathbf{X}_0 = \frac{\partial}{\partial t} + \frac{\partial}{\partial z},$$ (50)

$$\mathbf{Y}_0 = \frac{\partial}{\partial s} \quad \text{and} \quad A = c \quad \text{(constant)}.$$

The Lagrangian for the static spacetime (22)

$$L = \omega^2[(x^2 - y^2) + 2xy](\dot{t}^2 + \dot{z}^2 - 2\dot{t}\dot{z}) + \dot{t}^2 - \dot{x}^2$$

$$- \dot{y}^2 - \dot{z}^2,$$ (51)

has three symmetry generators

$$\mathbf{X}_0 = \frac{\partial}{\partial t}, \qquad \mathbf{X}_1 = \frac{\partial}{\partial z}, \qquad \mathbf{Y}_0 = \frac{\partial}{\partial s}, \qquad (52)$$

and the gauge function is a constant.

The Lagrangian for the perturbed pp-wave spacetime (23) is

$$
\begin{aligned}
L = {} & \omega^2[x^2 - y^2 + 2xy + \epsilon\{(x^2 - y^2)\sin(\omega(t - z)) \\
& + 2xy\cos(\omega(t - z))\}](\dot{t}^2 + \dot{z}^2 - 2\dot{t}\,\dot{z}) + \dot{t}^2 - \dot{x}^2 - \dot{y}^2 \\
& - \dot{z}^2.
\end{aligned}
\tag{53}
$$

For $\epsilon = 0$, the above Lagrangian (53) reduces to (51). Using this first-order perturbed Lagrangian and the three exact symmetry generators given by (52) in (16) in the resulting system of determining equations two constants corresponding to the exact symmetry generators appear. These two generators have to be eliminated for consistency of the determining equations, making them homogeneous. The resulting system is the same as for the static (exact) spacetime (23). Thus there is no nontrivial approximate symmetry for this perturbed Lagrangian and the gauge function is a constant. Hence we cannot obtain the conserved quantity in the case of perturbed pp-wave spacetime. Only the three exact symmetry generators are recovered as trivial first-order approximate Noether symmetries which give trivial first-order approximate conservation laws for energy and linear momentum along z.

V. SECOND-ORDER APPROXIMATE SYMMETRIES AND ENERGY RESCALING: CYLINDRICAL WAVE SPACETIMES

The line element of the cylindrically symmetric exact waves [31] is

$$ds^2 = e^{2(\gamma - \psi)}(dt^2 - d\rho^2) - \rho^2 e^{-2\psi}d\phi^2 - e^{2\psi}dz^2, \tag{54}$$

where γ and ψ are arbitrary functions of t and ρ, subject to the vacuum EFEs

$$\psi'' + \frac{1}{\rho}\psi' - \ddot{\psi} = 0, \qquad \gamma' = \rho(\psi'^2 + \dot{\psi}^2),$$
$$\dot{\gamma} = 2\rho\dot{\psi}\psi', \tag{55}$$

where a dot denotes differentiation with respect to t and the prime with respect to ρ. The solution of (55) is given by [4]

$$\psi = AJ_0(\omega\rho)\cos(\omega t) + BY_0(\omega\rho)\sin(\omega t), \tag{56}$$

$$
\begin{aligned}
\gamma = {} & \frac{1}{2}\omega\rho[(A^2 J_0 J_0' - B^2 Y_0 Y_0')\cos(2\omega t) \\
& - AB\{(J_0 Y_0' + Y_0 J_0')\sin(2\omega t) - 2(J_0 Y_0' - Y_0 J_0')\omega t\}].
\end{aligned}
\tag{57}
$$

This metric has two KVs $\partial/\partial\phi$ and $\partial/\partial z$ [40]; this means that there is only azimuthal angular momentum conservation and linear momentum conservation along z.

To discuss the approximate symmetries of cylindrical waves first a static spacetime is defined as follows. We remove the t-dependent part in (54) and put the strength of the wave $A = 1$. Since Y_0 is badly behaved at $\rho = 0$, we choose $B = 0$ (as the units of the strength are arbitrary).

$$ds^2 = e^{2(\gamma_0 - \psi_0)}(dt^2 - d\rho^2) - \rho^2 e^{-2\psi_0}d\phi^2 - e^{2\psi_0}dz^2, \tag{58}$$

where

$$\psi_0 = J_0(\omega\rho), \qquad \gamma_0 = \frac{\omega\rho}{2}J_0(\omega\rho)J_0'(\omega\rho). \tag{59}$$

For the approximate case we put the strength of the wave as a small parameter, i.e., $A = \epsilon$ and take the exact wave as a perturbation on the static metric (58) in the following way.

$$\psi = J_0(\omega\rho)(1 + \epsilon\cos(\omega t)) = \psi_0 + \epsilon\psi_1 \tag{60}$$
$$\gamma = \frac{\omega\rho}{2}J_0(\omega\rho)J_0'(\omega\rho)(1 + \epsilon^2\cos(2\omega t)) = \gamma_0 + \epsilon^2\gamma_1.$$

Thus the second-order perturbed geodesic equations are

$$
\begin{aligned}
& \ddot{t} + 2(\gamma_0' - \psi_0')\dot{t}\,\dot{\rho} - \epsilon[\dot{\psi}_1(\dot{t}^2 + \dot{\rho}^2) + \rho^2 e^{-2\gamma_0}\dot{\psi}_1\dot{\phi}^2 \\
& \quad - e^{2(2\psi_0 - \gamma_0)}\dot{\psi}_1\dot{z}^2 + 2\psi_1'\dot{t}\,\dot{\rho}] + \epsilon^2[\dot{\gamma}_1(\dot{t}^2 + \dot{\rho}^2) \\
& \quad + 4e^{2(2\psi_0 - \gamma_0)}\psi_1\dot{\psi}_1\dot{z}^2 - 2\gamma_1'\dot{t}\,\dot{\rho}] + O(\epsilon^3) = 0, \tag{61}
\end{aligned}
$$

$$
\begin{aligned}
& \ddot{\rho} + (\gamma_0' - \psi_0')(\dot{t}^2 + \dot{\rho}^2 + \dot{t}\,\dot{\rho}) + \rho e^{-2\gamma_0}(\psi_0' - 1)\dot{\phi}^2 \\
& \quad + e^{2(2\psi_0 - \gamma_0)}\psi_0'\dot{z}^2 - \epsilon[\psi_1'(\dot{t}^2 + \dot{\rho}^2) - \rho^2 e^{-2\gamma_0}\dot{\psi}_1\dot{\phi}^2 \\
& \quad - e^{2(2\psi_0 - \gamma_0)}(4\psi_0'\psi_1 - \psi_1')\dot{z}^2 + \psi_1'\dot{t}\,\dot{\rho}] \\
& \quad + \epsilon^2[\gamma_1'(\dot{t}^2 + \dot{\rho}^2) - \rho\gamma_1 e^{-2\gamma_0}(\psi_0' - 1)\dot{\phi}^2 \\
& \quad + 2e^{2(2\psi_0 - \gamma_0)}(4\psi_0'\psi_1^2 - \gamma_1\psi_0' - \psi_1\psi_1')\dot{z}^2 + \gamma_1'\dot{t}\,\dot{\rho}] \\
& \quad + O(\epsilon^3) = 0, \tag{62}
\end{aligned}
$$

$$
\begin{aligned}
& \ddot{\phi} + \frac{1}{\rho}(1 - \psi_0')\dot{\rho}\,\dot{\phi} - \epsilon(\psi_1'\dot{\rho} + \dot{\psi}_1\dot{t})\dot{\phi} \\
& \quad - \frac{\epsilon^2}{\rho}(1 - \psi_0')\dot{\rho}\,\dot{\phi} + O(\epsilon^3) = 0, \tag{63}
\end{aligned}
$$

$$\ddot{z} + \psi_0'\dot{\rho}\,\dot{z} + \epsilon(\psi_1'\dot{\rho} + \dot{\psi}_1\dot{t})\dot{z} - 2\epsilon^2\psi_0'\psi_1^2\dot{\rho}\,\dot{z} + O(\epsilon^3) = 0. \tag{64}$$

The dot and prime over γ_0, ψ_0, γ_1, and ψ_1 denote derivatives with respect to ωt and $\omega\rho$, respectively. For this perturbed wave spacetime the scaling factor is

$$\dot{\gamma}_1(\dot{t}^2 + \dot{\rho}^2) + 4e^{2(2\psi_0 - \gamma_0)}\psi_1\dot{\psi}_1\dot{z}^2 - 2\gamma_1'\dot{t}\dot{\rho}. \quad (65)$$

To replace the derivative of the coordinates t, z, and ρ the exact first integrals and the metric (58) are used. Further it is assumed that there is no initial velocity in the z and ϕ directions. Hence $\dot{z}$ and $\dot{\phi}$ vanishes and the following scaling factor is obtained:

$$\dot{\gamma}_1 e^{2(\psi_0 - \gamma_0)}[e^{2(\psi_0 - \gamma_0)} + e^{3(\psi_0 - \gamma_0)} - 1]$$
$$- 2\gamma_1' e^{3(\psi_0 - \gamma_0)}(e^{3(\psi_0 - \gamma_0)} - 1)^{1/2}, \quad (66)$$

where γ_1 is given in (60). This scaling factor involves the Bessel function of first kind and its derivatives. The asymptotic representation of the Bessel function of first kind for large value of the argument is given in [46]. Using this asymptotic representation of the Bessel function in (66), we obtain an asymptotic representation of it as follows:

$$\frac{3 \times 2^{11/4}}{\pi^{3/2}}[(|\cos(\omega\rho)|)^{3/2}\sin(2\omega t)](\omega\rho)^{-1/2}$$
$$+ O([\omega\rho]^{-3/2}). \quad (67)$$

Thus the energy in this perturbed spacetime field is rescaled by the factor (67). It is plotted below for different values of t, ρ, and ω (in radians per second), in which the energy oscillates between positive and negative values and goes to zero as ρ tends to infinity. Here the behavior is much more recognizably wavelike. Since the strength of the wave, $A = \epsilon$, is arbitrary the energy is given in arbitrarily chosen units.

For completeness we also investigate second-order approximate symmetries of the geodesic equations for the cylindrically symmetric wavelike spacetime. For this purpose a cylindrically symmetric static metric is taken [47]

$$ds^2 = e^{\nu(\rho)}dt^2 - d\rho^2 - e^{\mu(\rho)}(a^2 d\phi^2 + dz^2), \quad (68)$$

with $\nu(\rho) = (\rho/R)^2$ and $\mu(\rho) = (\rho/R)^3$, where R is a constant having the same dimensions as ρ. For the approximate symmetries of this cylindrical wavelike spacetime, $\nu(x) = (\rho/R)^2 + 2\epsilon t/T$ and $\mu(x) = (\rho/R)^3 + 2\epsilon t/T$ are taken in the metric (68), where T is a constant having dimensions of t. Like the plane symmetric case the scaling factor for this cylindrical wavelike spacetime is

$$\frac{t}{4T^2}[e^{-2(\rho/R)^2} + 2e^{-(\rho/R)^2(\rho/R+1)}]. \quad (69)$$

Thus the energy in the field of this wavelike spacetime is rescaled by the factor (69). The plots for this case are similar to those for the plane wavelike case discussed in Sec. II, with x replaced by ρ.

The nonzero components of the Weyl tensor for the cylindrically symmetric perturbed wave spacetime are

$$C^0{}_{101} = -\frac{1}{3}[\psi_0'' - \gamma_0'' + 2\psi_0'^2 - 2\psi_0' + \epsilon(4\psi_0'\psi_1' + \psi_1''$$
$$- \ddot{\psi}_1 - 2\psi_1' + 4\psi_0'\psi_1 - 4\psi_0'^2\psi_1 - 2\psi_0''\psi_1$$
$$+ 2\gamma_0''\psi_1)] + O(\epsilon^2),$$

$$C^0{}_{202} = \frac{\rho^2}{6}[4\psi_0'' - \gamma_0'' + 8\psi_0'^2 + 3\gamma_0' - 2\psi_0' - 6\psi_0'\gamma_0'$$
$$+ 2\epsilon(2\psi_1'' + \ddot{\psi}_1 + 8\psi_0'\psi_1' - \psi_1' - 3\gamma_0'\psi_1')]$$
$$+ O(\epsilon^2),$$

$$C^0{}_{303} = \frac{e^{2(2\psi_0 - \gamma_0)}\rho}{6}[2\psi_0'' + \gamma_0'' + 4\psi_0'^2 + 3\gamma_0' + 2\psi_0'$$
$$- 6\psi_0'\gamma_0' + 2\epsilon(\psi_1'' + 2\ddot{\psi}_1 + 4\psi_0'\psi_1' + \psi_1'$$
$$- 3\gamma_0'\psi_1')] + O(\epsilon^2),$$

$$C^1{}_{212} = -e^{-4\psi_0}C^0{}_{303},$$

$$C^1{}_{313} = \frac{1}{\rho}e^{2(2\psi_0 - \gamma_0)}C^0{}_{202}$$

$$C^2{}_{323} = \frac{e^{2(2\psi_0 - \gamma_0)}}{3}[\gamma_0'' - \psi_0'' + \psi_0' - 2\psi_0'^2$$
$$- 2\epsilon(\psi_1'' - \ddot{\psi}_1 + 2\psi_0'\psi_1' - \frac{1}{\rho}\psi_1')] + O(\epsilon^2),$$

$$C^0{}_{212} = -\epsilon\rho^2 e^{-2\gamma_0}[(2\psi_0' + \gamma_0')\dot{\psi}_1 + \dot{\psi}_1'] + O(\epsilon^2),$$

$$C^0{}_{313} = \epsilon e^{2(2\psi_0 - \gamma_0)}[(2\psi_0' - \gamma_0')\dot{\psi}_1 + \dot{\psi}_1'] + O(\epsilon^2). \quad (70)$$

This yields the pure gravitational field for this cylindrically symmetric perturbed wave spacetime. As it is evident from Figs. 3–6 that the energy in the gravitational field oscillates and then vanishes for large ρ, here all the components of the Weyl tensor also depend on the Bessel function of the first kind and its derivatives which oscillates and goes to zero as ρ approaches very large value. The last two com-
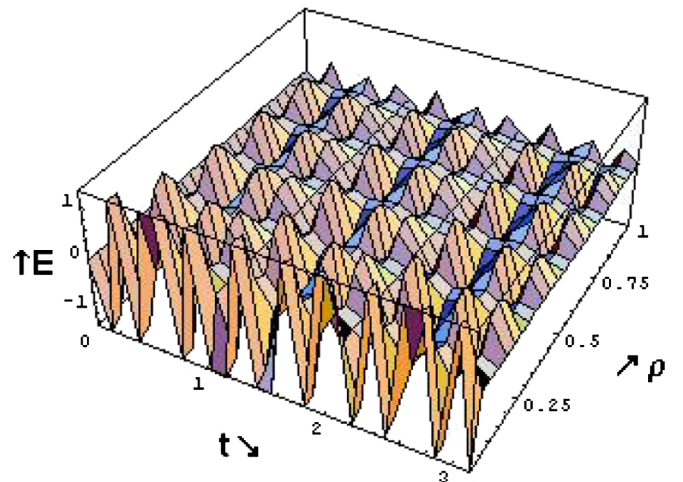

FIG. 3 (color online). Cylindrically symmetric gravitational waves with $\omega = 15$. The gravitational energy oscillates between positive and negative values and disappears as ρ approaches very large value. The units of energy are arbitrary in all diagrams.
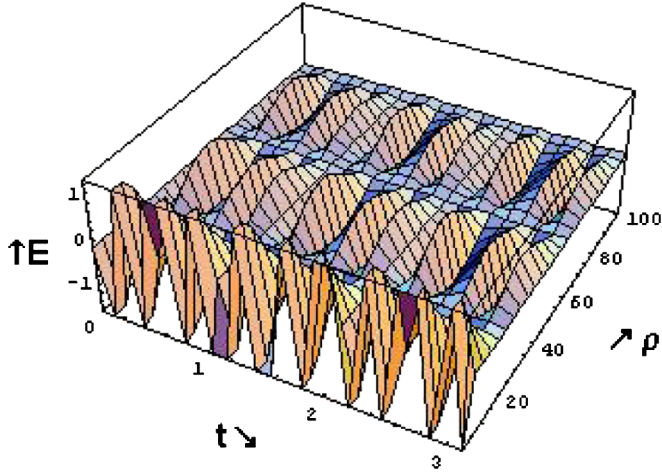
FIG. 4 (color online). To see the behavior of energy for comparatively larger distance, therefore the range of ρ is extended to 100 units.
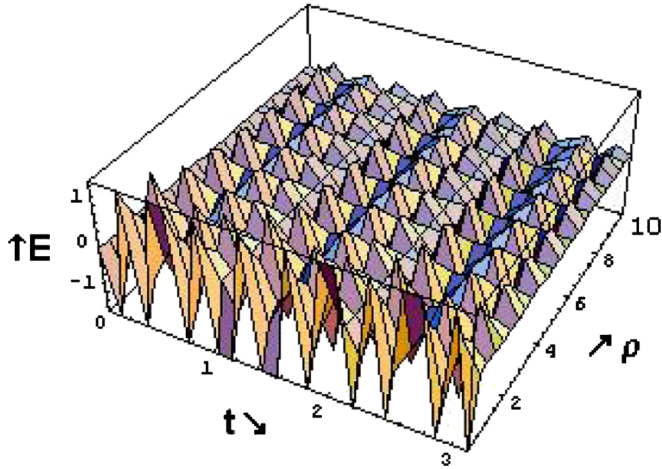

FIG. 5 (color online). To see a further extended version of the above Fig. 4, the range of ρ is given in units of 10^5.
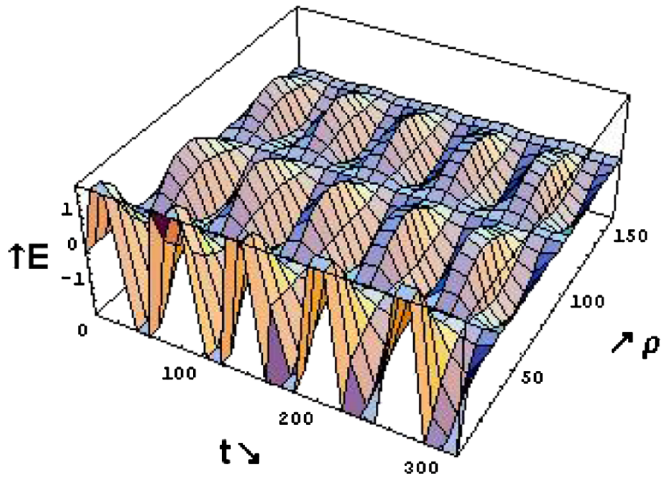

FIG. 6 (color online). Here the value of the frequency is comparatively small, i.e., $\omega = 0.05$. To see the variation along time, therefore the range of t is kept larger.

ponents only appear for the approximate part of the spacetime. In this case the components of the Weyl tensor in the covariant form are not very different from those in the mixed form given above and therefore we do not give them separately.

The nonvanishing components of the stress-energy tensor are

$$T_{00} = T_{11} = \frac{1}{\kappa}(\psi_0'^2 - \gamma_0' + 2\epsilon\psi_0'\psi_1') + O(\epsilon^2),$$

$$T_{22} = \rho^2\kappa e^{2(\gamma_0-2\psi_0)}[\psi_0'^2 + \gamma_0'' + 2\epsilon\psi_0'\psi_1'] + O(\epsilon^2),$$

$$T_{33} = \frac{1}{\kappa}e^{-2(\gamma_0-2\psi_0)}\left[2\psi_0'' - \gamma_0'' - \psi_0'^2 + 2\psi_0'\right.$$

$$+ \frac{2}{\rho}\epsilon\{\psi_1'' - \ddot{\psi}_1 + 2\psi_1(\psi_0'' - \gamma_0'' - \psi_0'^2 + 3\psi_0'$$

$$\left. - \psi_0'\psi_1' + 2(\psi_1' - 2\psi_0'\psi_1))\}\right] + O(\epsilon^2),$$

$$T_{01} = \frac{1}{\kappa}\epsilon\psi_0'\dot{\psi}_1 + O(\epsilon^2). \tag{71}$$

Like the components of the Weyl tensor the above components of the stress-energy tensor also depend on the Bessel function of the first kind and its derivatives. In this case of cylindrical perturbed waves the fraction of energy density imparted to the matter field is

$$E_{\text{imp}} = 2\epsilon\frac{\psi_0'\psi_1'}{\psi_0'^2 - \gamma_0'}. \tag{72}$$

The nonvanishing components of the Weyl tensor for the cylindrical wavelike spacetime are

$$C^0{}_{101} = \frac{1}{3R^5}(3R^2\rho - 2R\rho^2 + 3\rho^3 - R^3) + O(\epsilon^2),$$

$$C^0{}_{202} = a^2C^0{}_{303}$$

$$= -\left(1 + \epsilon\frac{2t}{T}\right)\frac{a^2e^{\rho^3/R^3}}{6R^5}(3R^2\rho - 2R\rho^2 + 3\rho^3 - R^3)$$

$$+ O(\epsilon^2),$$

$$C^1{}_{212} = a^2C^1{}_{313} = -C^0{}_{202},$$

$$C^2{}_{323} = -2C^0{}_{202}. \tag{73}$$

In covariant form the components of the Weyl tensor are

$$C_{0101} = \frac{1}{3R^5}\left(1 + \epsilon\frac{2t}{T}\right)(3R^2\rho - 2R\rho^2 + 3\rho^3 - R^3)$$

$$+ O(\epsilon^2),$$

$$C_{0202} = a^2C_{0303} = -\frac{a^2e^{\rho^3/R^3 + \rho^2/R^2}}{6R^5}\left(1 + \epsilon\frac{4t}{T}\right)$$

$$\times (3R^2\rho - 2R\rho^2 + 3\rho^3 - R^3) + O(\epsilon^2),$$

$$C_{1212} = a^2C_{1313} = -C_{0202},$$

$$C_{2323} = -2C_{0202}. \tag{74}$$

The components in the covariant form are physically rea-

sonable as they follow the geometry of the constructed metric and the energy defined by approximate symmetry.

The nonzero components of stress-energy tensor are

$$T_{00} = \frac{3e^{\rho^2/R^2}\rho}{2\kappa R^6}\left(1 + \epsilon\frac{2t}{T}\right)(4R^3 + 9\rho^3) + O(\epsilon^2),$$

$$T_{11} = \frac{3}{2\kappa R^6}(3\rho + 4R) + O(\epsilon^2),$$

$$T_{22} = a^2 T_{33} = \frac{-a^2 e^{\rho^3/R^3}\rho}{2\kappa R^6}\left(1 + \epsilon\frac{2t}{T}\right)(4\rho^2 R^2 + 6R^3\rho$$
$$+ 9\rho^4 + 6R\rho^3 + 6R^4) + O(\epsilon^2),$$

$$T_{01} = \epsilon\frac{\rho}{\kappa T R^3}(3\rho - 2R) + O(\epsilon^2). \qquad (75)$$

Here the momentum density is along radius of the cylinder. For this case we have the same relative energy density imparted to the matter field, as given by (42).

VI. APPROXIMATE NOETHER SYMMETRIES OF THE CYLINDRICAL WAVE SPACETIMES

The Lagrangian of the spacetime (68) is

$$L = e^{(\rho/R)^2}\dot{t}^2 - \dot{\rho}^2 - e^{(\rho/R)^3}(a^2\dot{\phi}^2 + \dot{z}^2), \qquad (76)$$

which yields the following symmetry generators:

$$\mathbf{X}_0 = \frac{\partial}{\partial t}, \qquad \mathbf{X}_1 = \frac{\partial}{\partial \phi}, \qquad \mathbf{X}_2 = \frac{\partial}{\partial z},$$
$$\mathbf{X}_3 = z\frac{\partial}{\partial \phi} - a^2\phi\frac{\partial}{\partial z}, \qquad \mathbf{Y}_0 = \frac{\partial}{\partial s}, \qquad A = c, \qquad (77)$$

where c is a constant, $\mathbf{X}_0$ corresponds to energy conservation, $\mathbf{X}_1$ corresponds to azimuthal angular momentum conservation, and $\mathbf{X}_2$ to linear momentum conservation along z, while $\mathbf{X}_3$ corresponds to angular momentum conservation.

The first-order perturbed Lagrangian for the cylindrical wavelike spacetime is

$$L = e^{(\rho/R)^2}\dot{t}^2 - \dot{\rho}^2 - e^{(\rho/R)^3}(a^2\dot{\phi}^2 + \dot{z}^2)$$
$$+ \frac{2\epsilon t}{T}[e^{(\rho/R)^2}\dot{t}^2 - e^{(\rho/R)^3}(a^2\dot{\phi}^2 + \dot{z}^2)] + O(\epsilon^2). \quad (78)$$

For this Lagrangian along with the exact symmetry generators given by (77), the nontrivial approximate symmetry $\mathbf{X}_a$ given by (79) is obtained. The gauge function A_1 is again a constant,

$$\mathbf{X}_a = \frac{\partial}{\partial t} - \epsilon\frac{1}{T}\left(t\frac{\partial}{\partial t} + \phi\frac{\partial}{\partial \phi} + z\frac{\partial}{\partial z}\right). \qquad (79)$$

The corresponding stable first integral is

$$I = 2e^{(\rho/R)^2}\dot{t} + \frac{2\epsilon}{T}[e^{(\rho/R)^2}t\dot{t} + e^{(\rho/R)^3}(a^2\phi\dot{\phi} + z\dot{z})].$$
$$(80)$$

As for the plane wavelike spacetime the following con-

served quantity corresponding to (79) is calculated:

$$Q = E - \frac{\epsilon}{T}(t\dot{E} + \phi\dot{p}_\phi + z\dot{p}_z). \qquad (81)$$

Using the Lagrangian for the spacetime (54)

$$L = e^{2(\gamma-\psi)}(\dot{t}^2 - \dot{\rho}^2) - \rho^2 e^{-2\psi}\dot{\phi}^2 - e^{2\psi}\dot{z}^2, \qquad (82)$$

in (6) and solving the resulting system of determining equations we obtain the symmetry generator $\partial/\partial s$ along with the two KVs $\partial/\partial\phi$, $\partial/\partial z$ and the gauge function A is a constant.

The Lagrangian for the spacetime (58) is

$$L = e^{2(\gamma_0-\psi_0)}(\dot{t}^2 - \dot{\rho}^2) - \rho^2 e^{-2\psi_0}\dot{\phi}^2 - e^{2\psi_0}\dot{z}^2, \quad (83)$$

which admits the following four symmetry generators along with the gauge function as a constant

$$\mathbf{X}_0 = \frac{\partial}{\partial t}, \qquad \mathbf{X}_1 = \frac{\partial}{\partial \phi}, \qquad \mathbf{X}_2 = \frac{\partial}{\partial z}, \qquad \mathbf{Y}_0 = \frac{\partial}{\partial s}. \qquad (84)$$

The first-order perturbed Lagrangian for the cylindrical wave spacetime (54) with ψ and γ defined by (60) is given by

$$L = e^{2(\gamma_0-\psi_0)}(\dot{t}^2 - \dot{\rho}^2) - \rho^2 e^{-2\psi_0}\dot{\phi}^2 - e^{2\psi_0}\dot{z}^2$$
$$- 2\epsilon\psi_1[e^{2(\gamma_0-\psi_0)}(\dot{t}^2 - \dot{\rho}^2) - \rho^2 e^{-2\psi_0}\dot{\phi}^2 + e^{2\psi_0}\dot{z}^2]$$
$$+ O(\epsilon^2). \qquad (85)$$

For $\epsilon = 0$ this Lagrangian reduces to the Lagrangian (83). Using this perturbed Lagrangian and the four-dimensional exact symmetry algebra of the static spacetime (58) in (16), we get a set of determining equations in which only one constant corresponding to the exact symmetry generator appears. This exact symmetry generator has to be eliminated for consistency of these determining equations, making them homogeneous. The resulting system is once more the same as for the spacetime (58), yielding first-order trivial approximate symmetry generators. Thus there is no nontrivial approximate symmetry for this perturbed Lagrangian and the gauge function A is a constant. Hence energy conservation, azimuthal angular momentum conservation, and linear momentum conservation along the axis of the cylinder are obtained as trivial first-order approximate conservation laws. (Note that the technically trivial law may be physically nontrivial.)

VII. SUMMARY AND DISCUSSION

The problem of energy in gravitational wave spacetimes using approximate Lie symmetry methods for DEs is addressed. To resolve this problem we used the second-order approximate symmetries of the geodesic equations for perturbed gravitational wave spacetimes discussed here. First the pp-wave spacetime is investigated. Since there is no ϵ^2 in the geodesic equations for the perturbed pp-

waves, the definition of second-order approximate symmetries of ODEs which gives the scaling factor cannot be applied to them. This is similar to the result of Qadir and Sharif's work [5] using the pseudo-Newtonian formalism, which just gave a constant momentum imparted to test particles in the path of the waves and no determinable value for it. For a better understanding of the implication of the definition of second-order approximate symmetries of ODEs in plane symmetric waves this definition has applied to the artificially constructed time-varying non-vacuum plane symmetric spacetime [30] for which the scaling factor (35) is obtained. It is seen from the plots of the plane symmetric wavelike spacetime that the energy increases with time close to the origin for x and then disappears. Then we investigated the second-order approximate symmetries of the geodesic equations for the cylindrical wave spacetimes. The scaling factors (67) and (69) are obtained for these spacetimes. In the factor (67) the magnitude of the coefficient of $(\omega\rho)^{-1/2}$ is greater than the magnitude of the coefficient of $(\omega\rho)^{-3/2}$, therefore, the contribution of the second term is very small and is neglected. This factor is plotted for different values of t, ρ, and ω. It shows a behavior much more recognizably wavelike. In Figs. 3–6 the energy oscillates between positive and negative values along t and ρ. It disappears as ρ tends to infinity.

To obtain the pure gravitational field and the matter field the approximate Weyl and stress-energy tensors for the gravitational wave spacetimes are calculated. The components of the Weyl tensor are given in the (0, 4) (covariant) form as well. For the perturbed pp-wave spacetime it appears that the (0, 4) form gives the physically relevant quantities as the space dependence in the (1, 3) (mixed) form of the Weyl tensor does not seem to correspond to the geometry of the pp-wave, while the covariant form does. For the wavelike spacetimes the components in the covariant form are physically reasonable as they follow the geometry of the constructed metrics and the energy defined by approximate symmetry. The stress-energy tensor density imparted to the matter field in the wavelike and perturbed cylindrical wave spacetimes was obtained.

In GR, different people have tried to introduce the concept of a pseudotensor, to define energy and momentum. In this regard first Einstein obtained a pseudotensor to define energy in GR [48]. Following Einstein's idea, Landau-Lifshitz [49], Papapetrou [50], and Weinberg [51] gave different pseudotensors to represent the energy and momentum of the gravitational field. The idea of introducing a pseudotensor has been criticized because all the pseudotensors are coordinate dependent and hence *nontensorial*. This violates the basic principles of GR. Because of the coordinate dependence, many others, including Möller [52,53], Komar [54], Ashtekar-Hansen [55], and Penrose [56], have proposed coordinate-independent definitions. Möller realized that the use of a tetrad as the field variable,

instead of a metric, makes it possible to introduce a first-order scalar Lagrangian for the EFEs. Komar introduced a tensorial superpotential which is independent of any background structure and has a uniqueness property. Ashtekar and Hansen defined the angular momentum in their specific conformal model of the spatial infinity as a certain 2-surface integral near infinity. Penrose defined quasilocal energy-momentum and angular momentum using twistor-theoretical idea. However, each of these has its own drawbacks [53,57,58]. A detailed discussion on different definitions of gravitational energy is available in "Quasi-local Energy-momentum and Angular Momentum in GR: A Review Article" [59] and "Energy and Momentum in GR" [60]. Using the idea of pseudotensors different people claimed that the gravitational energy should be positive at large scales as well as at small scales [61–65]. The positivity of gravitational energy does not seem convincing because the total energy of the Universe is zero [66], which suggests that the gravitational energy must fluctuate between positive and negative values to give the net energy of a spacetime zero.

Our definition of gravitational energy, obtained from approximate Lie symmetries, avoids the pseudotensor and hence does not violate the spirit of GR. The radiation scalar [16] does not violate the spirit of GR either. However, nor does it give a measure of the energy of GWs. Our expression of energy *does* give such a measure and is also reasonable as the gravitational energy oscillates over positive and negative values, as it should. Admittedly, in the artificial example we constructed the energy increased linearly without limit. This was because of the (nonphysical) choice of a linearly increasing component of the metric tensor for convenience of computation, leading to a corresponding increase in the scaling factors (35) and (69). For the physical example of cylindrical exact waves, the Bessel function of the first kind goes to zero asymptotically for large values of the argument [46]. Correspondingly, our scaling factor (67) for cylindrical waves dies out asymptotically, giving a net zero energy.

One of us (A. Q.) was drawn to study the energy of gravitational waves when a question was posed [67] whether there is the analogue of Landau-damping of electromagnetic waves for gravitational waves. Since Maxwell's theory of electromagnetism is linear, electromagnetic waves do not interact with the field but are damped due to their interaction with matter. On the other hand GR is nonlinear and so gravitational waves undergo self-interaction. This gives rise to the possibility of "Landau self-damping" of gravitational waves. On the other hand, the Khan-Penrose [68] and Szekeres [69] solutions of colliding plane gravitational waves suggest that there could even be *enhancement* of the waves, as they lead to curvature singularities after the collision. The problem of definition of energy in GR makes it very difficult to answer the question posed. Using Wheeler's "poor man's

approach,'' we can ask whether ''the mass equivalent to the energy of the gravitational wave attracts and hence damps the waves,'' or like the black hole, ''the energy enhances the mass and hence the energy equivalent to it in the wave.'' With our present proposal the question seems to be answerable. Classically the energy density in cylindrical waves reduces by the factor $1/(2\pi\rho)$. From (67) the energy density decreases by a further factor of $(3 \times 2^{11/4})/\sqrt{\pi^3 \times (\omega\rho)}$. Hence for sufficiently large ρ the scaling factor $\sim 1/\sqrt{\omega\rho^3}$ which is a significant *self-damping* of the waves! This enhanced asymptotic attenuation of gravitational waves will obviously have profound observational significance.

It would be of great interest to apply this approximate symmetry analysis to the Khan-Penrose and Szekeres solutions to see whether they suffer self-damping or enhancement according to our definition. Of course, it may be that the procedure will be inapplicable for those plane wave solutions as well. Also, the analysis should be applied to ''spherical solutions'' like those of Nutku [70].

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