PHYSICAL REVIEW D **79,** 125012 (2009)

Non-Abelian duality and confinement in $\mathcal{N} = 2$ supersymmetric QCD

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(Received 15 April 2009; published 15 June 2009)

In $\mathcal{N} = 2$ supersymmetric QCD with the U(N) gauge group and $N_f > N$ we study the crossover transition from the weak coupling regime at large ξ to strong coupling at small ξ, where ξ is the Fayet-Iliopoulos parameter. We find that at strong coupling a dual non-Abelian weakly coupled $\mathcal{N} = 2$ theory exists, which describes low-energy physics at small ξ. The dual gauge group is U($N_f - N$), and the dual theory has N_f flavors of light dyons, to be compared with N_f quarks in the originalU(N) theory. Both, the original and dual theories are Higgsed and share the same global symmetry SU(N) $\times$ SU($N_f - N$) $\times$ U(1), albeit the physical meaning of the SU(N) and SU($N_f - N$) factors is different in the large- and small-ξ regimes. Both regimes support non-Abelian semilocal strings. In each of these two regimes particles that are in the adjoint representations with respect to one of the factor groups exist in two varieties: elementary fields and composite states bound by strings. These varieties interchange upon transition from one regime to the other. We conjecture that the composite stringy states can be related to Seiberg's M fields. The bulk duality that we observed translates into a two-dimensional duality on the world sheet of the non-Abelian strings. At large ξ the internal dynamics of the semilocal non-Abelian strings is described by the sigma model of N orientational and ($N_f - N$) size moduli, while at small ξ the roles of orientational and size moduli interchange. The Bogomol'nyi-Prasad-Sommerfield spectra of two dual sigma models (describing confined monopoles/dyons of the bulk theory) coincide. It would be interesting to trace parallels between the non-Abelian duality we found and string theory constructions.

DOI: 10.1103/PhysRevD.79.125012 PACS numbers: 12.38.Aw, 11.30.Pb, 12.60.Jv

I. INTRODUCTION

In this paper we continue studying transitions from weak to strong coupling in $\mathcal{N} = 2$ supersymmetric QCD induced by a change of parameters. The investigation began in [1] where we considered Yang-Mills theory with the gauge group U(N) and N matter hypermultiplets in the fundamental representation. The adjustable parameters in this theory are the Fayet-Iliopoulos (FI) [2] coefficient ξ and the quark mass differences described by a set of parameters Δm_{AB}. The overall scale is set by a dynamical parameter Λ. We started from $\xi \gg \Lambda$ while our task was to penetrate in domain $\xi \lesssim \Lambda$ and small (vanishing) Δm_{AB}. In the former limit the theory is weakly coupled, and one can obtain a reliable quasiclassical description of physics directly from the given microscopic theory. In particular, at $\Delta m_{AB} = 0$ there emerge non-Abelian strings [3–5] whose world-sheet dynamics is described by supersymmetric $CP(N-1)$ model (for reviews see [6–9]). These strings confine monopoles [5,10]. Nonperturbative light "mesonic" states are monopole-antimonopole pairs connected by two non-Abelian strings.

On the other hand, at $\xi \lesssim \Lambda$ our microscopic theory is strongly coupled. To develop an effective low-energy description of physics in this domain of small ξ (and small $|\Delta m_{AB}|$) we had to derive a dual weakly coupled theory. The dual theory turned out to be Abelian, based on $U(1)^{N-1}$. Moreover, we found that the light matter sector in this Abelian theory consisted of certain dyons, which condense in the vacuum resulting in Abelian strings of the Abrikosov-Nielsen-Olesen (ANO) [11] type. The light mesonic states built from the monopole-antimonopole pairs connected by two strings survive, albeit these strings are totally different from those in the large-ξ small-$|\Delta m_{AB}|$ domain. We came to the conclusion that the transition from the non-Abelian to Abelian (low-energy) regimes was of a crossover type rather than a phase transition.[1]

In this paper we extend the scope of our studies to cover the case of a larger number of the fundamental matter hypermultiplets, i.e. $N_f > N$, see Fig. 1. Other than that, the microscopic theory we work with is the same as in [1]. Namely, we deal with $\mathcal{N} = 2$ supersymmetric QCD with the gauge group U(N) and the Fayet-Iliopoulos term. Although $N_f > N$, we limit ourselves to $N_f < 2N$ to keep asymptotic freedom in our microscopic theory. The Fayet-Iliopoulos term $\xi \neq 0$ triggers condensation of N squark fields. The parameter space of this theory includes the FI parameter ξ and the squark mass differences

$$\Delta m_{AB} = m_A - m_B, \qquad A, B = 1, \ldots, N_f. \tag{1.1}$$

Various regimes of the theory in the $\{\xi, \Delta m\}$ plane are schematically shown in Fig. 2. The vertical axis in this figure denotes the values of the FI parameter ξ, while the

[1]It is worth adding that it does become a phase transition at $N = \infty$.

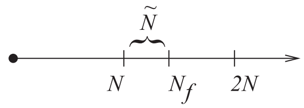

FIG. 1. The number of flavors exceeds the number of colors, $\tilde{N} \equiv N_f - N > 0$.

horizontal axis schematically represents all quark mass differences.

At $\xi \gg \Lambda^2$ the theory is at weak coupling. Perturbative and nonperturbative spectra, and all interactions can be exhaustively analyzed using quasiclassical methods. In the limit of degenerate quark masses $\Delta m_{AB} = 0$ the microscopic theory at hand has an unbroken global SU(N) symmetry, which is a diagonal combination of SU$(N)_{\rm color}$ and an SU(N) subgroup of the flavor SU(N_f) group acting in the theory. Thus, the color-flavor locking takes place, see Sec. II. All light states come in the adjoint and singlet representations of the unbroken SU$(N)_{\rm diag}$.

Much in the same way as in [1] the theory with $N_f > N$ supports non-Abelian flux tubes (strings) in the weak coupling domain I. In fact, at $N_f > N$ these strings are semilocal (for a review on Abelian semilocal strings see e.g. [12]). Internal dynamics of semilocal non-Abelian strings is described by two-dimensional $\mathcal{N} = 2$ supersymmetric sigma model with toric target space [3,10,13–15]. It contains N orientational and $\tilde{N}$ size moduli, where

$$\tilde{N} \equiv N_f - N. \tag{1.2}$$

Since the squark fields are condensed in domain I, and the theory is fully Higgsed, the monopoles are attached to strings. In fact, in the U(N) gauge theory the monopoles of the SU(N) sector represent junctions of two distinct degenerate strings and are seen as kinks in the world-sheet sigma model on the non-Abelian string [5,10,16], see also the review [6].

Domain II is that of the *Abelian* Higgs regime at weak coupling. As we increase Δm_{AB}, the (off diagonal) W bosons and their superpartners become exceedingly heavier and decouple from the low-energy spectrum. We are left with the photon (diagonal) gauge fields and their quark $\mathcal{N} = 2$ superpartners. Explicit breaking of the flavor symmetry by $\Delta m_{AB} \neq 0$ leads to the loss of non-Abelian nature of the string solutions; they become Abelian (the so-called Z_N) strings.

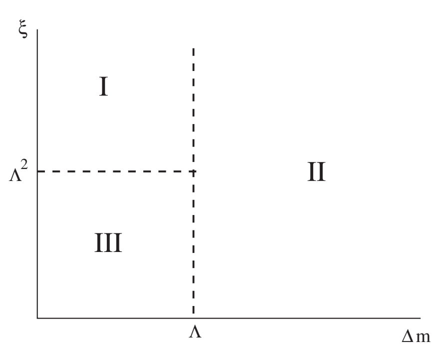

FIG. 2. Various regimes in $\mathcal{N} = 2$ QCD are separated by crossovers. The dynamical scale of our microscopic non-Abelian gauge theory is represented by the parameter Λ.

Finally, as we reduce ξ and $|\Delta m_{AB}|$ below Λ, we enter the strong-coupling domain III. Because of strong coupling the original microscopic theory is not directly analytically tractable here. Our task is to find a weakly coupled dual theory that describes physics in this domain. We show that at $N_f > N$ such a dual theory does exist and, moreover, it is non-Abelian, with the dual gauge group

$$\mathrm{U}(\tilde{N}) \times \mathrm{U}(1)^{N-\tilde{N}}, \tag{1.3}$$

and N_f flavors of charged non-Abelian dyons. The quarks we started from in domain I transform themselves into dyons due to monodromies as we reduce $|\Delta m_{AB}|$. In its gross features the dual $\mathcal{N} = 2$ theory we found is similar to Seiberg's dual [17] (for reviews see [18]) to our original microscopic theory. Because $N_f > 2\tilde{N}$, the dual theory is infrared (IR) free rather than asymptotically free. This result is in perfect match with the results obtained in [19] where the dual non-Abelian gauge group SU($\tilde{N}$) was identified at the root of a baryonic branch in the SU(N) gauge theory with massless quarks, see also [20]. In the limit of degenerate quark masses $\Delta m_{AB} = 0$ and small ξ, the dual theory has an unbroken global diagonal SU($\tilde{N}$) symmetry. It is obtained as a result of the spontaneous breaking of the gauge U($\tilde{N}$) group and an SU($\tilde{N}$) subgroup of the flavor SU(N_f) group. Thus, the color-flavor locking takes place in the dual theory as well, much in the same way as in the original microscopic theory in domain I, albeit the preserved diagonal symmetry is different. The light states come in adjoint and singlet representations of the global SU($\tilde{N}$). Thus, the low-energy spectrum of the theory in domain III is dramatically different from that of domain I. Excitation spectra are arranged in different representations of the global unbroken groups, SU(N) and SU($\tilde{N}$), respectively. Let us ask ourselves how this can happen in the absence of a phase transition?

Both, the original and dual theories are Higgsed and share the same global symmetry

$$\mathrm{SU}(N) \times \mathrm{SU}(\tilde{N}) \times \mathrm{U}(1).$$

To answer the above question we investigate how all states belonging to the adjoint representation of either SU(N) or SU($\tilde{N}$) evolve when we vary ξ and cross the boundary of the large- and small-ξ domains. In the large-ξ domain the original theory is at weak coupling, while the dual is at strong coupling and *vice versa*. It turns out that in both regimes we have particles that are adjoint with respect to SU(N) and SU($\tilde{N}$). They come in two varieties: as elementary fields and as composite mesons whose constituents are bound together by strings. For instance, at small ξ the adjoints in SU($\tilde{N}$) are elementary, while the adjoints in

SU(N) are composite. At large ξ their roles interchange. The spectrum as a whole is smooth. The phenomenon of level crossing takes place *en route*, at the crossover transition.

Next we show that monopoles are still attached to strings in domain III at small (but nonvanishing) ξ. They are represented by junctions of two different non-Abelian strings of the dual bulk theory and seen as kinks in the dual world-sheet theory on the string. However, since in domain III it is the condensation of dyons that ensures complete Higgsing of the gauge SU($\tilde{N}$) group, we in fact deal with oblique confinement [21]. This result provides a counterexample to a commonly accepted belief that if monopoles are confined in the original theory, then the quarks of the original theory should be confined in the dual one. We show that monopoles rather than quarks are confined in domain III. This observation presumably solves a paradox noted in [22]. Thus, the non-Abelian duality we found is *not* the electromagnetic duality. This should be contrasted with the Abelian Seiberg-Witten duality [23,24], which *is* the electromagnetic duality.

Three above-mentioned regimes of our microscopic theory—three domains shown in Fig. 2—are arguably separated by crossovers, much in the same way as it happens in the case $N_f = N$ [1]. In Ref. [1] we argued that the transitions between domains I, II, and III are crossovers rather than phase transitions. Now we will provide further evidence in favor of crossovers, which can be summarized as follows:

(i) In the equal quark mass limit domains I and III have Higgs branches of the same dimensions and the same pattern of global symmetry breaking, see Sec. III.

(ii) For generic masses $\Delta m_{AB} \neq 0$ all three regimes have the same number of isolated vacua at nonvanishing ξ, see Sec. V.

(iii) Each of these vacua has the same number ($= N$) of different elementary strings in all three domains. Moreover, Bogomol'nyi-Prasad-Sommerfield (BPS) spectra of excitations on the non-Abelian string coincide in domains I and III, see Sec. VII.

Still, as we will show in detail in the bulk of the paper, both the perturbative spectra and confining strings are dramatically different in domains I, II, and III. There are certain curves of marginal stability (CMS) separating these domains. Upon crossing these CMS, certain elementary particles (like W bosons) decay into magnetically charged states. At nonzero ξ these states are confined and cannot move far apart. They become mesons formed by (anti) monopoles and dyons bound together by confining strings. If, as was claimed above, we have a crossover rather than a phase transition between domains I and III, then adjoints in the global unbroken symmetry $\mathrm{SU}(N)_{C+F}$ (present in domain I) cannot just disappear upon passing in domain III. Although heavy and invisible in the low-energy effective action, they still must survive as particles in domain III. We identify these adjoints of $\mathrm{SU}(N)_{C+F}$ with composite mesons bound by strings.

Another issue to be discussed in the present paper is a possible origin of Seiberg's mesonic fields M [17], which appear in the dual bulk theory when we break $\mathcal{N} = 2$ supersymmetry by the superpotential mass term $\mu \mathcal{A}^2$ for the adjoint fields and take the limit $\mu \to \infty$ thus converting our theory into $\mathcal{N} = 1$ QCD. The composite mesons formed by (anti)monopoles and dyon bound by confining strings are good candidates for Seiberg's mesonic fields M. While they are heavy in the $\mathcal{N} = 2$ limit, they might well become light in the $\mathcal{N} = 1$ limit. Our arguments in favor of this conjecture are presented in Sec. III C.

Our results are in complete parallel with the situation in the special case $N_f = N$ analyzed in [1]. In this case, domain III is nothing but the Abelian Seiberg-Witten confinement [23,24]. The set of light surviving states includes photons and dyons with certain quantum numbers. The W bosons and their superpartners decay on the curves of the marginal stability as we move inside III. They are heavy and form (anti)monopole/dyon stringy mesons at nonzero ξ filling the adjoint representation of $\mathrm{SU}(N)_{C+F}$.

When we speak of dual pairs of theories, a clarifying remark is in order. There are two slightly different formulations of duality. In the first one we start from two different microscopic theories and show that both theories coincide in the infrared limit; the infrared description can be strongly coupled, as, say, in the middle of the conformal window [17]. In the second formulation, within the given microscopic theory, we identify two effective theories describing physics at large distances—one is weakly coupled in a certain domain of parameters where the other is strongly coupled and *vice versa*. This is the strategy of Seiberg and Witten [23] who, given the SU(2) Yang-Mills theory with $\mathcal{N} = 2$, identified a low-energy U(1) theory and then dualized it to demonstrate the dual Meissner effect upon a small $\mathcal{N} = 2$-breaking mass deformation of the original SU(2) theory. Our consideration follows the logic of that of [23].

Duality of the bulk theories translates into two-dimensional duality on the world sheet of the non-Abelian string. The dual SU($\tilde{N}$) bulk theory in the quasiclassical regime supports non-Abelian semilocal strings. Their internal dynamics is described by two-dimensional $\mathcal{N} = 2$ toric sigma model with $\tilde{N}$ orientational and N size moduli. Thus, the role of orientational and size moduli interchanges in domain III as compared with domain I. We demonstrate that the BPS spectra of two dual world-sheet theories are the same.

The general outline of the paper is as follows: In Sec. II, we review our basic microscopic theory, and discuss BPS-saturated flux tubes it supports in domain I. We outline the structure of the world-sheet theory on the strings, which, in the case at hand, is a toric $\mathcal{N} = 2$ sigma model. In Sec. III, we present a detailed consideration of the transition from domain I to III. We choose an instructive example $N = 3$ and $N_f = 5$ and trace the fate of the quarks in their evolution from domain I to III under a special choice of the quark masses. Deformations of the quark masses are studied in Sec. V. In Sec. IV, we consider monopoles attached to the strings. In Sec. VI, we address evolution and transmutations of the adjoint particles vs variation of ξ on the way from domain I to III, a question which is central for understanding consistency of our picture. Section VII is devoted to the evolution of the world-sheet theory on the way from domain I to III. Section VIII summarizes our conclusions.

II. LARGE VALUES OF THE FI PARAMETER (DOMAINS I AND II)

In this section we will briefly review main features of our basic theory—$\mathcal{N} = 2$ QCD with the gauge group U(N) and N_f quark flavors. As shown in Fig. 1, we assume $N_f > N$ but $N_f < 2N$. The latter inequality ensures asymptotic freedom of the original microscopic theory. Then we summarize main features of the non-Abelian strings in this theory [3–5,10,13,14].

A. Basic microscopic theory

The field content is as follows: The $\mathcal{N} = 2$ vector multiplet consists of the U(1) gauge field A_μ and the SU(N) gauge field A^a_μ, where $a = 1, \ldots, N^2 - 1$, and their Weyl fermion superpartners plus complex scalar fields a, and a^a and their Weyl superpartners. The N_f quark multiplets of the U(N) theory consist of the complex scalar fields q^{kA} and $\tilde{q}_{Ak}$ (squarks) and their fermion superpartners, all in the fundamental representation of the SU(N) gauge group. Here $k = 1, \ldots, N$ is the color index, while A is the flavor index $A = 1, \ldots, N_f$. We will treat q^{kA} and $\tilde{q}_{Ak}$ as rectangular matrices with N rows and N_f columns.

The bosonic part of our basic theory has the form (for details see the review paper [6])

$$S = \int d^4x \bigg[\frac{1}{4g_2^2}(F^a_{\mu\nu})^2 + \frac{1}{4g_1^2}(F_{\mu\nu})^2 + \frac{1}{g_2^2}|D_\mu a^a|^2 + \frac{1}{g_1^2}|\partial_\mu a|^2 + |\nabla_\mu q^A|^2 + |\nabla_\mu \bar{\tilde{q}}^A|^2 + V(q^A, \tilde{q}_A, a^a, a) \bigg]. \tag{2.1}$$

Here D_μ is the covariant derivative in the adjoint representation of SU(N), while

$$\nabla_\mu = \partial_\mu - \frac{i}{2}A_\mu - iA^a_\mu T^a. \tag{2.2}$$

We suppress the color SU(N) indices of the matter fields. The normalization of the SU(N) generators T^a is as follows:

$$\mathrm{Tr}\,(T^a T^b) = \tfrac{1}{2}\delta^{ab}.$$

The coupling constants g_1 and g_2 correspond to the U(1) and SU(N) sectors, respectively. With our conventions, the U(1) charges of the fundamental matter fields are $\pm 1/2$, see Eq. (2.2).

The scalar potential $V(q^A, \tilde{q}_A, a^a, a)$ in the action (2.1) is the sum of the D and F terms,

$$\begin{aligned} V(q^A, \tilde{q}_A, a^a, a) = {}& \frac{g_2^2}{2}\left(\frac{1}{g_2^2} f^{abc}\bar{a}^b a^c + \bar{q}_A T^a q^A - \tilde{q}_A T^a \bar{\tilde{q}}^A\right)^2 + \frac{g_1^2}{8}(\bar{q}_A q^A - \tilde{q}_A \bar{\tilde{q}}^A - N\xi)^2 \\ & + 2g_2^2 |\tilde{q}_A T^a q^A|^2 + \frac{g_1^2}{2}|\tilde{q}_A q^A|^2 \\ & + \frac{1}{2}\sum_{A=1}^{N_f}\{|(a + \sqrt{2}m_A + 2T^a a^a)q^A|^2 \\ & + |(a + \sqrt{2}m_A + 2T^a a^a)\bar{\tilde{q}}^A|^2\}. \end{aligned} \tag{2.3}$$

Here f^{abc} denote the structure constants of the SU(N) group, m_A is the mass term for the A-th flavor, and the sum over the repeated flavor indices A is implied. Above we introduced the FI D term for the U(1) gauge factor with the FI parameter ξ.

Now let us discuss the vacuum structure of this theory. The vacua of the theory (2.1) are determined by the zeros of the potential (2.3). At generic values of the quark masses we have

$$C^N_{N_f} = N_f!/N!\tilde{N}!$$

isolated r vacua, where $r = N$ quarks (out of N_f) develop vacuum expectation values (VEVs).

Consider, say, the $(1, 2, \ldots, N)$ vacuum in which the first N flavors develop VEVs. We can exploit gauge rotations to make all squark VEVs real. Then in the problem at hand they take the form

$$\begin{aligned} \langle q^{kA}\rangle &= \sqrt{\xi}\begin{pmatrix} 1 & \ldots & 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 1 & 0 & \ldots & 0 \end{pmatrix}, \\ \langle \bar{\tilde{q}}^{kA}\rangle &= 0, \qquad k = 1, \ldots, N, \qquad A = 1, \ldots, N_f, \end{aligned} \tag{2.4}$$

where we write down the quark fields as matrices in color and flavor indices. This particular form of the squark condensates is dictated by first two lines in Eq. (2.3). Note that the squark fields stabilize at nonvanishing values exclusively due to the U(1) factor represented by the term in the second line.

The FI term ξ singles $r = N$ vacua out of all set of r vacua, which are present in the theory if quadratic in the adjoint field superpotential deformation $\mu\mathcal{A}^2$ is added. In the vacuum under consideration the adjoint fields also develop VEVs, namely,

$$\left\langle \left(\frac{1}{2} a + T^a a^a \right) \right\rangle = -\frac{1}{\sqrt{2}} \begin{pmatrix} m_1 & \ldots & 0 \\ \ldots & \ldots & \ldots \\ 0 & \ldots & m_N \end{pmatrix}. \quad (2.5)$$

For generic values of the quark masses, the SU(N) subgroup of the gauge group is broken down to $\mathrm{U}(1)^{N-1}$. However, in the special limit

$$m_1 = m_2 = \ldots = m_{N_f}, \quad (2.6)$$

the SU(N) × U(1) gauge group remains unbroken by the adjoint field. In this limit the theory acquires a global flavor SU(N_f) symmetry.

While the adjoint VEVs do not break the SU(N) × U(1) gauge group in the limit (2.6), the quark condensate (2.4) results in the spontaneous breaking of both gauge and flavor symmetries. A diagonal global SU(N) combining the gauge SU(N) and an SU(N) subgroup of the flavor SU(N_f) group survives, however. Below we will refer to this diagonal global symmetry as to $\mathrm{SU}(N)_{C+F}$.

More exactly, the pattern of breaking of the color and flavor symmetry is as follows:

$$\mathrm{U}(N)_{\text{gauge}} \times \mathrm{SU}(N_f)_{\text{flavor}} \to \mathrm{SU}(N)_{C+F} \times \mathrm{SU}(\tilde{N})_F \times \mathrm{U}(1), \quad (2.7)$$

where $\tilde{N}$ is defined in (1.2). Here $\mathrm{SU}(N)_{C+F}$ is a global unbroken color-flavor rotation, which involves first N flavors, while the $\mathrm{SU}(\tilde{N})_F$ factor stands for the flavor rotation of the $\tilde{N}$ quarks. The phenomenon of color-flavor locking takes place in the vacuum, albeit in a slightly different way than in the case $N_f = N$ (or $\tilde{N} = 0$). The presence of the global $\mathrm{SU}(N)_{C+F}$ group is instrumental for formation of the non-Abelian strings (see below). For unequal quark masses the global symmetry (2.7) is broken down to $\mathrm{U}(1)^{N_f-1}$.

Now let us discuss the mass spectrum in the theory (2.1). Since both U(1) and SU(N) gauge groups are broken by squark condensation, all gauge bosons become massive. From (2.1) we get for the U(1) gauge boson mass

$$m_\gamma = g_1 \sqrt{\frac{N}{2} \xi}. \quad (2.8)$$

At the same time, ($N^2 - 1$) gauge bosons of the SU(N) group acquire one and the same mass

$$m_W = g_2 \sqrt{\xi}. \quad (2.9)$$

It is not difficult to see from (2.3) that the adjoint fields a and a^a as well as N^2 components of the quark matrix q acquire the same masses as the corresponding gauge bosons. Altogether we have one long $\mathcal{N} = 2$ massive vector multiplet (eight bosonic + eight fermionic states) with the mass (2.8) and ($N^2 - 1$) long $\mathcal{N} = 2$ massive vector multiplets with the mass (2.9). If the extra $\tilde{N}$ quark masses are different from those of the first N masses (i.e. $m_{1,\ldots,N}$), the extra quark flavors acquire masses determined by the mass differences $\Delta m_{PK} = m_P - m_K$, where $P = 1, \ldots, N$ numerates the quark flavors, which develop VEVs in the $(1, \ldots, N)$ vacuum, while $K = N + 1, \ldots, N_f$ numerates extra quark flavors. The extra flavors become massless in the limit (2.6), which we will consider momentarily.

Note that all states come in representations of the unbroken global group (2.7), namely, the singlet and adjoint representations of $\mathrm{SU}(N)_{C+F}$

$$(1, 1), \qquad (N^2 - 1, 1), \quad (2.10)$$

and bifundamentals

$$(\bar{N}, \tilde{N}), \qquad (N, \bar{\tilde{N}}), \quad (2.11)$$

where we mark representation with respect to two non-Abelian factors in (2.7).

If all quark mass terms are equal, then all $C_{N_f}^N$ isolated vacua we had in the case of unequal mass terms coalesce; a Higgs branch develops from the common root whose location on the Coulomb branch is given by Eq. (2.5) with $\Delta m_{AB} = 0$. The dimension of this branch is [19,25]

$$\dim \mathcal{H}|_{\xi \gg \Lambda} = 4NN_f - 2N^2 - N^2 - N^2 = 4\tilde{N}N, \quad (2.12)$$

where we take into account the fact that we have $4NN_f$ quark real degrees of freedom and subtracted $2N^2$ conditions due to F terms, N^2 conditions due to D terms and, finally, N^2 gauge phases eaten by the Higgs mechanism, see (2.3).

The Higgs branch is noncompact and is known to have a hyper-Kähler geometry [19,24]. At a generic point on the Higgs branch BPS-saturated string solutions do not exist [26]; strings become non-BPS if we move along noncompact directions [27]. However, the Higgs branch has a compact base manifold defined by the condition

$$\tilde{q}_{Ak} = 0, \qquad A = 1, \ldots, N_f. \quad (2.13)$$

The dimension of this manifold is $2N\tilde{N}$, twice less than the overall dimension of the Higgs branch. The BPS-saturated string solutions exist on the base manifold of the Higgs branch. As a result, the vacua belonging to the base manifold are our prime focus.

The base of the Higgs branch can be generated by flavor rotations of the $(1, \ldots, N)$ vacuum (2.4). The flavor rotations generate the manifold

$$\frac{\mathrm{SU}(N_f)}{\mathrm{SU}(N)_{C+F} \times \mathrm{SU}(\tilde{N}) \times \mathrm{U}(1)}, \quad (2.14)$$

see Eq. (2.7). We see that the number of broken generators of the global group is $2N\tilde{N}$. It coincides with the dimension of the base of the Higgs branch.

Since N different flavors develop VEVs on the Higgs branch it is a baryonic Higgs branch. It is a generalization of the baryonic Higgs branch [19] to the case of the U(N)

gauge group and nonvanishing masses. Note, however, that in the U(N) gauge theory the baryonic charge is gauged, in contradistinction with [19].

Now let us have a closer look at quantum effects in the theory (2.1). The SU(N) sector is asymptotically free. The semiclassical analysis outlined above is valid if the FI parameter ξ is large,

$$\xi \gg \Lambda, \tag{2.15}$$

where Λ is the dynamical scale of the SU(N) gauge theory. This condition ensures weak coupling in the SU(N) sector because the SU(N) gauge coupling does not run below the scale of the quark VEVs, which is determined by ξ. More explicitly,

$$\frac{8\pi^2}{g_2^2(\xi)} = (N - \tilde{N}) \ln \frac{g_2\sqrt{\xi}}{\Lambda} \gg 1. \tag{2.16}$$

Below we will see that if we pass to small ξ following the line $\Delta m_{A,B} = 0$, into the strong-coupling domain III, where the condition (2.15) is not met, the theory undergoes a crossover. In the case of $N_f = N$ studied in [1] this is a transition into the Seiberg-Witten Abelian regime. In this regime no non-Abelian strings develop. We will show below that if $N_f > N$ the theory at small ξ below the transition point at $\xi \sim \Lambda^2$ is still non-Abelian, with the dual gauge group U($\tilde{N}$). It supports non-Abelian semilocal strings for which the role of orientation and size moduli is interchanged.

To conclude this section we briefly recall the theory (2.1) at nonvanishing quark mass differences $m_A - m_B \neq 0$, see [5,6]. At $m_A - m_B \neq 0$ the global group (2.7) is explicitly broken down to $U(1)^{N_f - 1}$. The adjoint multiplet is split. The diagonal entries (photons and their $\mathcal{N} = 2$ quark superpartners) have masses given in (2.9), while the off-diagonal states (W bosons and the off-diagonal entries of the squark matrix q^{kA} with $A \neq k$) acquire additional contributions to their masses proportional to Δm_{AB}. In particular, $\tilde{N}$ "extra" quark flavors become massive, and the Higgs branch is lifted. As we make the mass differences larger, the W bosons become exceedingly heavier, decouple from the low-energy spectrum, and we are left with N photon states and N diagonal elements of the quark matrix with $A = k$. The low-energy spectrum becomes Abelian.

B. Non-Abelian strings at large ξ

Now we will briefly review non-Abelian strings in the theory (2.1), see [6] for details. Non-Abelian strings in $\mathcal{N} = 2$ QCD with

$$N_f = N,$$

where first found and studied in [3–5,10]. The Abelian Z_N-string solutions break the $\mathrm{SU}(N)_{C+F}$ global group. Therefore, strings have orientational zero modes, associated with rotations of their color flux inside the non-Abelian SU(N). This makes these strings non-Abelian. The global group is broken on the Z_N string solution down to SU($N-1$) × U(1). As a result, the moduli space of the non-Abelian string is described by the coset

$$\frac{\mathrm{SU}(N)}{\mathrm{SU}(N-1)\times \mathrm{U}(1)} \sim CP(N-1). \tag{2.17}$$

The $CP(N-1)$ space can be parametrized by a complex vector n^P in the fundamental representation of SU(N) subject to the constraint

$$n_P^* n^P = \mathrm{const},$$

where $P = 1, \ldots, N$. As we will show below, one U(1) phase will be gauged away in the low-energy sigma model. This gives the correct number of degrees of freedom, namely, $2(N-1)$.

Making the moduli vector n^P a slowly varying function of the string world-sheet coordinates x_α ($\alpha = 0, 3$), we can derive an effective low-energy theory on the string world sheet [4,5,28]. On topological grounds [see (2.17)] it is clear that we will get the two-dimensional $CP(N-1)$ model. The $\mathcal{N} = (2, 2)$ supersymmetric $CP(N-1)$ model can be understood as a strong-coupling limit of a U(1) gauge theory [29]. The bosonic part of the action of this model has the form

$$S_{CP(N-1)} = \int d^2x \Big\{ |\nabla_\alpha n^P|^2 + \frac{1}{4e^2} F^2_{\alpha\beta} + \frac{1}{e^2} |\partial_\alpha \sigma|^2 + 2 \left| \sigma + \frac{m_P}{\sqrt{2}} \right|^2 |n^P|^2 + \frac{e^2}{2} (|n^P|^2 - 2\beta)^2 \Big\}, \tag{2.18}$$

where $\nabla_\alpha = \partial_\alpha - iA_\alpha$, while σ is a complex scalar field, and summation over P is implied. The condition

$$n_P^* n^P = 2\beta \tag{2.19}$$

is implemented in the limit $e^2 \to \infty$. Moreover, in this limit the gauge field A_α and its $\mathcal{N} = 2$ bosonic superpartner σ become auxiliary and can be eliminated by virtue of the equations of motion

$$A_\alpha = -\frac{i}{4\beta} n_P^* \overset{\leftrightarrow}{\partial}_\alpha n^P, \qquad \sigma = 0. \tag{2.20}$$

The two-dimensional coupling constant β is determined by the four-dimensional non-Abelian coupling via the relation

$$\beta = \frac{2\pi}{g_2^2}. \tag{2.21}$$

In the limit of equal quark masses the global $\mathrm{SU}(N)_{C+F}$ symmetry is unbroken, and strings become non-Abelian. This is a strong-coupling quantum regime in the $CP(N-1)$ model (2.18). The vector n^P is smeared all over the entire $CP(N-1)$ space due to quantum fluctuations, and its average value vanishes [30]. The world-sheet theory develops a mass gap.

At small nonvanishing $|m_P - m_{P'}|$ the global $SU(N)_{C+F}$ symmetry is explicitly broken down to $U(1)^{(N-1)}$. A shallow potential is generated on the $CP(N-1)$ modular space as is seen from (2.18). As we increase $|m_P - m_{P'}|$ the strings become exceedingly more Abelian and eventually evolve into Abelian Z_N strings, which correspond to N classical vacua of the world-sheet model (2.18)

$$n^P = \sqrt{2\beta}\delta^{PP_0}, \qquad \sigma = -\frac{m_{P_0}}{\sqrt{2}}, \tag{2.22}$$

where P_0 can take any of N values, $P_0 = 1, \ldots, N$, see the review [6]. Note, that we should keep mass differences $(m_P - m_{P'})$ small as compared to the inverse string thickness,

$$|m_P - m_{P'}| \ll g\sqrt{\xi}, \tag{2.23}$$

where we assume that $g_1 \sim g_2 \sim g$.

The $CP(N-1)$ model is an effective low-energy description of the internal string dynamics, and the bulk mass scale $g\sqrt{\xi}$ plays the role of an ultraviolet (UV) cutoff in (2.18). The constraint (2.23) ensures that typical energies in the world-sheet theory are much lower then this UV cutoff.

Let us ask ourselves what happens if we add extra quark flavors with degenerate mass? Then the strings emerging in the theory with $N_f > N$ become semilocal. In particular, the string solutions on the Higgs branches (typical for multiflavor theories) usually are not fixed-radius strings, but, rather, semilocal strings, see the review paper [12] for a comprehensive survey of Abelian semilocal strings.

Let us start our discussion with such Abelian semilocal strings. The semilocal string interpolates between the Abrikosov-Nielsen-Olesen string [11] and two-dimensional sigma-model instanton lifted to four dimensions (this is referred to as lump). The relevance of instantons can be understood as follows: We can go to low energies [below the photon mass (2.8)] and then integrate out massive states. In this limit the theory reduces to a sigma model on the Higgs branch. If we stay at the base of the Higgs branch imposing condition (2.13), and this base has an S_2 cycle, the theory has lumps. Much in the same way as the instanton/lump, the semilocal string possesses additional zero modes associated with complexified string's transverse size ρ. At $\rho \to 0$ we have the ANO string, while at $\rho \to \infty$ it becomes a pure lump. At $\rho \neq 0$ the profile functions of the semilocal string falloff at infinity as inverse powers of the distance to the string axis, instead of the exponential falloff characteristic to the ANO strings at $\rho = 0$. This leads to a dramatic physical effect—semilocal strings, in contradistinction to the ANO ones, do not support linear confinement [13,27].

Non-Abelian semilocal strings in $\mathcal{N} = 2$ QCD with $N_f > N$ were studied in [3,10,13,14]. These strings have both types of moduli: orientational and size moduli. The orientational zero modes of the semilocal non-Abelian string are parametrized by the complex vector n^P, $P = 1, \ldots, N$, while its $\tilde{N}$ size moduli are parametrized by the complex vector ρ^K, $K = N+1, \ldots, N_f$. The effective two-dimensional theory that describes the internal dynamics of the non-Abelian semilocal string is an $\mathcal{N} = (2, 2)$ "toric" sigma model, which includes both types of fields. Its bosonic action in the gauge formulation (which assumes taking the limit $e^2 \to \infty$) has the form

$$\begin{aligned} S = \int d^2x \Big\{ &|\nabla_\alpha n^P|^2 + |\tilde{\nabla}_\alpha \rho^K|^2 + \frac{1}{4e^2}F^2_{\alpha\beta} + \frac{1}{e^2}|\partial_\alpha \sigma|^2 \\ &+ 2\left|\sigma + \frac{m_P}{\sqrt{2}}\right|^2 |n^P|^2 + 2\left|\sigma + \frac{m_K}{\sqrt{2}}\right|^2 |\rho^K|^2 \\ &+ \frac{e^2}{2}(|n^P|^2 - |\rho^K|^2 - 2\beta)^2 \Big\}, \end{aligned} \tag{2.24}$$

$$P = 1, \ldots, N, \qquad K = N+1, \ldots, N_f, \qquad \tilde{\nabla}_k = \partial_k + iA_k.$$

The fields n^P and ρ^K have charges $+1$ and -1 with respect to the U(1) gauge field, hence, the difference in the covariant derivatives ∇_α and $\tilde{\nabla}_\alpha$, respectively.

If only charge $+1$ fields were present, in the limit $e^2 \to \infty$ we would get a conventional twisted-mass deformed $CP(N-1)$ model. The presence of the charge -1 fields ρ^K converts the target space of the $CP(N-1)$ sigma model into a weighed $CP(N_f - 1)$ space. Like in the $CP(N-1)$ model (2.18), small mass differences $|m_A - m_B|$ lift orientational and size zero modes generating a shallow potential on the modular space.

The world-sheet theory (2.24) was argued to emerge as an effective low-energy theory on the world sheet of the semilocal non-Abelian string in [3,10]. The arguments were based on a D-brane construction. Later this result was confirmed by direct derivations from the bulk theory in [13,14]. These derivations have a subtle point, though. Both orientational and size moduli have a logarithmically divergent in the IR norm in the limit $\Delta m_{AB} = 0$. This divergence is cutoff by small mass differences $|m_P - m_K| \neq 0$ (here $P = 1, \ldots, N$ and $K = N+1, \ldots, N_f$). What counts is the difference between the masses of N quarks, which develop VEVs in the bulk vacuum and the masses of extra $\tilde{N}$ quarks. With this cutoff, The large logarithmic factor can be absorbed in the field definition [13]. The theory (2.24) emerges in a logarithmic approximation in which this logarithmic factor is large. This ensures that $|\rho| \ll g\sqrt{\xi}/|m_P - m_K|$.

The two-dimensional coupling constant β is related to the four-dimensional one via (2.21). This relation is obtained at the classical level [4,5]. In quantum theory both couplings run. In particular, the model (2.24) is asymptotically free [29] and develops its own scale Λ_σ. The ultraviolet cutoff in the sigma model on the string world sheet is determined by $g\sqrt{\xi}$. Equation (2.21) relating the two- and four-dimensional couplings is valid at this scale. At this scale the four-dimensional coupling is given by (2.16), while the two-dimensional one

$$4\pi\beta(\xi) = (N-\tilde{N})\ln\frac{g\sqrt{\xi}}{\Lambda} \gg 1. \tag{2.25}$$

Then Eq. (2.21) implies

$$\Lambda_\sigma = \Lambda. \tag{2.26}$$

Note that in the bulk theory *per se* the coupling constant is frozen at $g_2\sqrt{\xi}$, because of the VEVs of the squark fields. The logarithmic evolution of the coupling constant in the string world-sheet theory takes over. Moreover, the dynamical scales of the bulk and world-sheet theories turn out to be the same, much in the same way as in the $N_f = N$ theory [5].

III. THE BULK DUALITY

Our task in this section is to analyze the transition from domain I to III (see Fig. 2). This will be done in two steps. First we will take the quark mass differences to be large, passing to domain II. In this domain the theory stays at weak coupling, and we can safely diminish the value of the FI parameter ξ. Next, we will use the exact Seiberg-Witten solution of the theory on the Coulomb branch [23,24] (i.e. at $\xi = 0$) to perform the passage from domain II to III.

A. The dual gauge group

To begin with, let us identify the $r = N$ quark vacuum of the form $(1, \ldots, N)$, which was described above semiclassically. To this end we will use the exact Seiberg-Witten solution [23,24], more exactly, the SU(N) generalizations of the Seiberg-Witten solution [31–34].

Instead of considering generic quark masses we will make a representative (and convenient) choice. Then we will show that the low-energy effective theory at small ξ and small quark mass differences (in domain III) has the dual non-Abelian gauge group U($\tilde{N}$).

Our special choice of the quark masses ensures that this theory is not asymptotically free—in fact, it is IR free—and stays at weak coupling at small ξ, cf. Ref. [19]. The set of masses we will deal with in this section is as follows: the masses of the extra $\tilde{N}$ quark fields are to be set equal to the masses of the first $\tilde{N}$ quarks from those N squarks, which develop VEVs in the $(1, \ldots, N)$ vacuum. Namely, we set

$$m_1 = m_{N+1}, \qquad m_2 = m_{N+2}, \ldots, \qquad m_{\tilde{N}} = m_{N+\tilde{N}}. \tag{3.1}$$

Later on we will be able to relax these conditions. The Seiberg-Witten curve in the theory under consideration has the form [19]

$$y^2 = \prod_{k=1}^{N}(x-\phi_k)^2 - 4\left(\frac{\Lambda}{\sqrt{2}}\right)^{N-\tilde{N}}\prod_{A=1}^{N_f}\left(x+\frac{m_A}{\sqrt{2}}\right), \tag{3.2}$$

where ϕ_k are gauge invariant parameters on the Coulomb branch. Semiclassically, at large masses

$$\mathrm{diag}\left(\frac{1}{2}a + T^a a^a\right) \approx [\phi_1, \ldots, \phi_N]. \tag{3.3}$$

Therefore, in the $(1, \ldots, N)$ quark vacuum we have

$$\phi_P \approx -\frac{m_P}{\sqrt{2}}, \qquad P = 1, \ldots, N, \tag{3.4}$$

in the large m_A limit, see (2.5).

To identify this vacuum in terms of the curve (3.2) it is necessary to find such values of ϕ_P, which ensure that the curve has N double roots and ϕ_P's are determined by the quark masses in the semiclassical limit, see (3.4). For the mass choice (3.1) the solution can be easily obtained. Indeed, let us write the curve in the form

$$y^2 = \prod_{P=1}^{\tilde{N}}\left(x+\frac{m_P}{\sqrt{2}}\right)^2\left\{\prod_{k=\tilde{N}+1}^{N}(x-\phi_k)^2 - 4\left(\frac{\Lambda}{\sqrt{2}}\right)^{N-\tilde{N}}\prod_{P=\tilde{N}+1}^{N}\left(x+\frac{m_P}{\sqrt{2}}\right)\right\}, \tag{3.5}$$

where the first $\tilde{N}$ ϕ's are given by

$$\phi_P = -\frac{m_P}{\sqrt{2}}, \qquad P = 1, \ldots, \tilde{N}. \tag{3.6}$$

This curve has $\tilde{N}$ double roots located at

$$x_P = -\frac{m_P}{\sqrt{2}}, \qquad P = 1, \ldots, \tilde{N}. \tag{3.7}$$

Now to find other double roots and ϕ's we have to investigate the reduced curve in the curly brackets in (3.5). It corresponds to the U($N-\tilde{N}$) gauge theory with $(N-\tilde{N})$ flavors. This theory completely Abelianizes below the crossover transition (at small ξ) [1]. In other words, the corresponding ϕ's get shifts from their classical values (3.4) proportional to Λ. To see this explicitly let us consider the simplest special case with all extra $(N-\tilde{N})$ masses are equal,

$$m_P = m, \qquad P = (\tilde{N}+1), \ldots, N \tag{3.8}$$

and $(N-\tilde{N}) = 2^p$ (the latter condition is imposed for simplicity). Then the curve (3.5) reduces to a perfect square

$$y^2 = \prod_{P=1}^{\tilde{N}}\left(x+\frac{m_P}{\sqrt{2}}\right)^2\left\{\left(x+\frac{m}{\sqrt{2}}\right)^{N-\tilde{N}} - \left(\frac{\Lambda}{\sqrt{2}}\right)^{N-\tilde{N}}\right\}^2 \tag{3.9}$$

provided that

$$\phi_k = \frac{1}{\sqrt{2}}[-m_1, \ldots, -m_{\tilde{N}}, -m + \Lambda e^{(\pi i/N-\tilde{N})}, \ldots, -m + \Lambda e^{[2\pi i/(N-\tilde{N})](N-\tilde{N}-(1/2))}]. \tag{3.10}$$

The first $\tilde{N}$ double roots are given in Eq. (3.7), while the remaining $N-\tilde{N}$ double roots are

$$x_P = \frac{1}{\sqrt{2}}[\ldots, -m + \Lambda, \ldots, -m + \Lambda e^{[2\pi i/(N-\tilde{N})](N-\tilde{N}-1)}]. \tag{3.11}$$

The main feature of this solution is the absence of $\sim\Lambda$ corrections to the first $\tilde{N}$ ϕ's in (3.10). This means that in the equal mass limit these $\tilde{N}$ ϕ's become equal. This is a signal of restoration of the non-Abelian $U(\tilde{N})$ gauge group at the root of the Higgs branch (i.e. at $\xi = 0$). Namely, the gauge group at the root of the Higgs branch in the equal mass limit becomes

$$U(\tilde{N}) \times U(1)^{N-\tilde{N}}. \tag{3.12}$$

This is in perfect agreement with the results obtained in [19], where a dual non-Abelian gauge group was identified at the root of a baryonic Higgs branch in the SU(N) gauge theory with massless quarks. The novel element of our analysis in this section is that we started with the $r = N$ non-Abelian vacuum at large ξ and demonstrated that, as we reduce ξ, the theory in this vacuum undergoes crossover to another non-Abelian regime with the dual low-energy gauge group (1.3). As was already mentioned, the physical reason for the emergence of the non-Abelian gauge group is that the low-energy effective theory with the dual gauge group (1.3) is not asymptotically free in the equal mass limit and stays at weak coupling. Therefore, the classical analysis showing that the non-Abelian gauge group is restored at the root of the Higgs branch remains intact in quantum theory.

B. Monodromies

In this section we will study how quantum numbers of the massless quarks $q^{11}, \ldots, q^{NN}$ in the $(1, \ldots, N)$ vacuum change as we reduce Δm_{AB} to pass from domain II to III (along the Coulomb branch at $\xi = 0$), where the theory is at strong coupling. To simplify our subsequent discussion we will consider a particular case: the theory with

$$N = 3, \qquad N_f = 5$$

so that the dual group has the smallest nontrivial rank $\tilde{N} = 2$. We will consider the (1, 2, 3) vacuum. In addition, we will stick to a special choice of the quark masses (3.1), which in the case at hand implies

$$m_1 = m_4, \qquad m_2 = m_5. \tag{3.13}$$

The mass parameter m_3 remains unspecified for the time being.

The quark quantum numbers change due to monodromies with respect to $\Delta m_{PP'}$. The complex planes of $\Delta m_{PP'}$ have cuts, and when we cross these cuts, a and a_D fields acquire monodromies; the quantum numbers of the corresponding states change accordingly. Monodromies with respect to the quark masses were studied in [35] in the theory with the SU(2) gauge group through a monodromy matrix approach.

Here we will investigate the monodromies in the U(3) theory with five quark flavors using the approach of Ref. [1], which is similar to that of Ref. [20]. In the case $N_f = 2N - 1$, the Seiberg-Witten curve, instead of (3.2), is given by [19]

$$y^2 = \prod_{k=1}^{3}(x - \phi_k)^2 - 4\left(\frac{\Lambda}{\sqrt{2}}\right)\prod_{A=1}^{5}\left(x + \frac{\tilde{m}_A}{\sqrt{2}}\right), \tag{3.14}$$

where, according to [19], "shifted" masses

$$\tilde{m}_A \equiv m_A + \frac{\Lambda}{3} \tag{3.15}$$

replace m_A. Substituting (3.13) and

$$\phi_1 = -\frac{\tilde{m}_1}{\sqrt{2}}, \qquad \phi_2 = -\frac{\tilde{m}_2}{\sqrt{2}}, \tag{3.16}$$

we arrive at

$$y^2 = \left(x + \frac{\tilde{m}_1}{\sqrt{2}}\right)^2\left(x + \frac{\tilde{m}_2}{\sqrt{2}}\right)^2\left[(x - \phi_3)^2 - 4\frac{\Lambda}{\sqrt{2}}\left(x + \frac{\tilde{m}_3}{\sqrt{2}}\right)\right]. \tag{3.17}$$

The first two double roots of this curve are obviously located at

$$e_1 = e_2 = -\frac{\tilde{m}_1}{\sqrt{2}}, \qquad e_3 = e_4 = -\frac{\tilde{m}_2}{\sqrt{2}}, \tag{3.18}$$

cf. Eq. (3.7). The remaining two roots coincide provided we set

$$\phi_3 = -\frac{1}{\sqrt{2}}(\tilde{m}_3 + \Lambda). \tag{3.19}$$

If we do so, the last two coinciding roots are

$$e_5 = e_6 = -\frac{1}{\sqrt{2}}(\tilde{m}_3 - \Lambda), \tag{3.20}$$

cf. Eqs. (3.10) and (3.11).

If two roots of the Seiberg-Witten curve coincide, the contour that encircles these roots shrinks and produces a regular potential. We start from the quasiclassical regime at $\Delta m_{PP'} \gg \Lambda$. We have three double roots $e_1 = e_2$, $e_3 = e_4$, and $e_5 = e_6$ in the $r = 3$ vacuum. Thus, three contours α_1, α_2, and α_3 shrink (see Fig. 3), and the associated potentials a, a_3, and a_8 are regular. This is related to masslessness of three quarks q^{11}, q^{22}, and q^{33} (at $\xi = 0$),

$$\begin{aligned} \frac{1}{2}a + \frac{1}{2}a_3 + \frac{1}{2\sqrt{3}}a_8 + \frac{m_1}{\sqrt{2}} &= 0, \\ \frac{1}{2}a - \frac{1}{2}a_3 + \frac{1}{2\sqrt{3}}a_8 + \frac{m_2}{\sqrt{2}} &= 0, \\ \frac{1}{2}a - \frac{1}{\sqrt{3}}a_8 + \frac{m_3}{\sqrt{2}} &= 0. \end{aligned} \tag{3.21}$$

Here we exploit the fact that the charges of these three

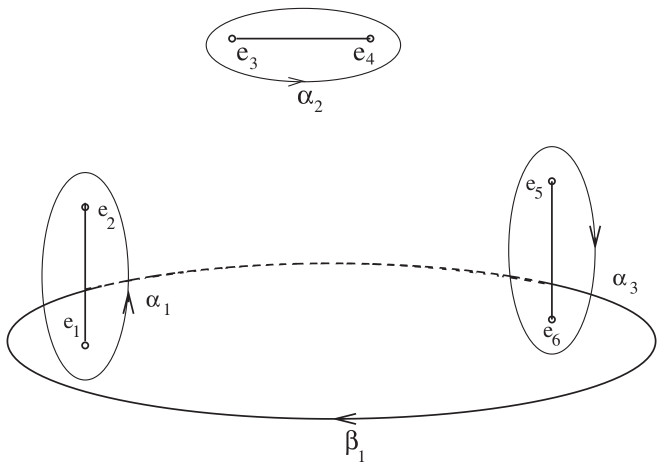

FIG. 3. Basis of α and β contours in U(3) gauge theory. Two roots e_3 and e_4 are far away near AD point (3.24).

quarks are as follows:

$$(n_e, n_m; n_e^3, n_m^3; n_e^8, n_m^8) = \left(\frac{1}{2}, 0; \frac{1}{2}, 0; \frac{1}{2\sqrt{3}}, 0\right),$$
$$(n_e, n_m; n_e^3, n_m^3; n_e^8, n_m^8) = \left(\frac{1}{2}, 0; -\frac{1}{2}, 0; \frac{1}{2\sqrt{3}}, 0\right), \quad (3.22)$$
$$(n_e, n_m; n_e^3, n_m^3; n_e^8, n_m^8) = \left(\frac{1}{2}, 0; 0, 0; -\frac{1}{\sqrt{3}}, 0\right),$$

respectively, where n_e and n_m denote electric and magnetic charges of a given state with respect to the U(1) gauge group, while n_e^3, n_m^3 and n_e^8, n_e^8 stand for the electric and magnetic charges with respect to the Cartan generators of the SU(3) gauge group (broken down to U(1) × U(1) by $\Delta m_{PP'}$).

In the monopole singularity certain other roots coincide. Say, $e_1 = e_6$, see Fig. 3. Thus, the β_1 contour shrinks producing a regular $1/2a_3^D + \frac{\sqrt{3}}{2}a_8^D$ potential. This is due to masslessness of the monopole (one of three SU(3) monopoles) with the charges[2]

$$(n_e, n_m; n_e^3, n_m^3; n_e^8, n_m^8) = \left(0, 0; 0, \frac{1}{2}; 0, \frac{\sqrt{3}}{2}\right). \quad (3.23)$$

If we decrease $|\Delta m_{PP'}|$ crossing cuts in the $\Delta m_{PP'}$ planes, the root pairing in the given vacuum may change. This would mean that other combinations of a's and a^D's become regular implying a change of the quantum numbers of the massless states in the given vacuum. To see how it works in our $r = 3$ vacuum let us go to the Argyres-Douglas (AD) point [36,37]. The AD point is a particular value of the quark mass parameters where more mutually nonlocal states become massless. In fact, we will study the collision of the $r = 3$ quark vacuum with monopole singularities. We approach the AD points from domain II at large $|\Delta m_{PP'}|$. We will show below that as we pass through the AD points the root pairings change in the $r = 3$ vacuum implying a change of the quantum numbers of the massless states. Three massless quarks transform into three massless dyons.

From (3.18) and (3.20) we see that there are two AD points where our (1, 2, 3) vacuum collides with the monopole singularities. The first one occurs at

$$\Delta m_{31} = \Lambda, \qquad e_1 = e_2 = e_5 = e_6 = -\frac{\tilde{m}_1}{\sqrt{2}}, \quad (3.24)$$

where four roots coincide, while the second is at

$$\Delta m_{32} = \Lambda, \qquad e_3 = e_4 = e_5 = e_6 = -\frac{\tilde{m}_2}{\sqrt{2}}, \quad (3.25)$$

where other four roots coincide.

We assume that m_1 and m_2 are real, $m_1 > m_2$, and consider the first AD point (3.24). At this point the (1, 2, 3) vacuum with three massless quarks (3.22) collide with the monopole singularity where the monopole (3.23) is massless. We will demonstrate below that as we reduce Δm_{31} along the real axis below the AD point (3.24) the root pairings change. The roots e_3 and e_4 are far away and, therefore, the charges of the q^{22} quark do not change. We focus on the colliding roots e_1, e_2, e_5, and e_6.

In order to see how the root pairings in the $r = 3$ vacuum change as we decrease Δm_{31} and pass through the AD point (3.24), we have to slightly split the roots by shifting ϕ_1 from its $r = 3$ solution (3.16). Let us parametrize the shift as

$$\phi_1 = -\frac{\tilde{m}_1}{\sqrt{2}} + \frac{\delta}{4\Lambda^2}, \quad (3.26)$$

where δ is a small deviation parameter of mass dimension three. Since we will consider x in the vicinity of $-\tilde{m}_1/\sqrt{2}$ we introduce another small parameter z,

$$z = x + \frac{\tilde{m}_1}{\sqrt{2}}. \quad (3.27)$$

Finally, we define

$$\varepsilon = \frac{\Delta m_{31} - \Lambda}{\sqrt{2}}. \quad (3.28)$$

The parameter ε is a small deviation from the AD point (3.24). Furthermore, we will expand (3.14) in δ, omitting terms $O(\delta^2)$, $O(\delta z^2)$, and $O(z\delta\varepsilon)$. The factor $(x + \frac{\tilde{m}_2}{\sqrt{2}})^2$ can and will be approximated by $\Delta m_{12}^2/2$. Then the curve (3.14) takes the form

$$y^2 \approx \frac{\Delta m_{12}^2}{2}[z^2(z + \varepsilon)^2 - z\delta]. \quad (3.29)$$

Above the AD point, at $\varepsilon > 0$ and $\delta \ll \varepsilon$, the roots of the curve (3.29) are split as follows:

[2] The charges of three elementary SU(3) monopoles are determined by the roots of the SU(3) Cartan subalgebra.

$$z_1 = 0, \qquad z_2 = \frac{\delta}{\varepsilon^2}, \\ z_5 = -\varepsilon - \sqrt{-\frac{\delta}{\varepsilon}}, \\ z_6 = -\varepsilon + \sqrt{-\frac{\delta}{\varepsilon}}, \tag{3.30}$$

where z_i are shifted e_i [see (3.27)], i.e.

$$z_i = e_i + (\tilde{m}_1/\sqrt{2}).$$

We take $-i\delta > 0$ and study the evolution of the roots of the curve (3.29) as a function of ε numerically. The results are schematically presented in Fig. 4. We see that the root pairings in the $r = 3$ vacuum change. Namely, at large Δm_{31} we have (at $\delta = 0$)

$$e_1 = e_2, \qquad e_5 = e_6, \qquad e_3 = e_4 \tag{3.31}$$

which, as was explained above, corresponds to shrinking of the α_1, α_2, and α_3 contours and masslessness of three quarks (3.22). Below the AD point (3.24), at small Δm_{31}, we have

$$e_1 = e_5, \qquad e_2 = e_6, \qquad e_3 = e_4, \tag{3.32}$$

which corresponds to shrinking of the contours

$$\alpha_1 + \beta_1 \to 0, \qquad \alpha_3 - \beta_1 \to 0 \qquad \alpha_2 \to 0, \tag{3.33}$$

see Fig. 3. This means that the massless quarks q^{11} and q^{33} in the $r = 3$ vacuum transform themselves into massless dyons D^{11} and $D^{33'}$, with the quantum numbers

$$D^{11}: \left(\frac{1}{2}, 0; \frac{1}{2}, \frac{1}{2}; \frac{1}{2\sqrt{3}}, \frac{\sqrt{3}}{2}\right), \\ D^{33'}: \left(\frac{1}{2}, 0; 0, -\frac{1}{2}; -\frac{1}{\sqrt{3}}, -\frac{\sqrt{3}}{2}\right), \tag{3.34}$$

while the charges of the quark q^{22} do not change. We see that the quantum numbers of the massless quarks q^{11} and q^{33} in the $r = 3$ vacuum, after the collision with the monopole singularity, get shifted, the shift being equal to $\pm$(monopole magnetic charge).

By the same token, we can analyze the second AD point (3.25), where the $r = 3$ vacuum collides with another monopole singularity in which the monopole with the charges

$$\left(0, 0; 0, -\frac{1}{2}; 0, \frac{\sqrt{3}}{2}\right) \tag{3.35}$$

is massless. The corresponding results are as follows: now D^{11} does not change its charges, while the charges of the quark q^{22} and dyon $D^{33'}$ get a shift by the $\pm$(charge of the monopole (3.35)). As a result, below the AD point (3.25) the charges of the massless dyons are

$$D^{11}: \left(\frac{1}{2}, 0; \frac{1}{2}, \frac{1}{2}; \frac{1}{2\sqrt{3}}, \frac{\sqrt{3}}{2}\right), \\ D^{22}: \left(\frac{1}{2}, 0; -\frac{1}{2}, -\frac{1}{2}; \frac{1}{2\sqrt{3}}, \frac{\sqrt{3}}{2}\right), \\ D^{33}: \left(\frac{1}{2}, 0; 0, 0; -\frac{1}{\sqrt{3}}, -\sqrt{3}\right). \tag{3.36}$$

The quark masslessness conditions (3.21) at small $\Delta m_{PP'}$, below two AD points, are replaced by dyon masslessness conditions, namely,

$$\frac{1}{2}a + \frac{1}{2}a_3 + \frac{1}{2}a_3^D + \frac{1}{2\sqrt{3}}a_8 + \frac{\sqrt{3}}{2}a_8^D + \frac{m_1}{\sqrt{2}} = 0, \\ \frac{1}{2}a - \frac{1}{2}a_3 - \frac{1}{2}a_3^D + \frac{1}{2\sqrt{3}}a_8 + \frac{\sqrt{3}}{2}a_8^D + \frac{m_2}{\sqrt{2}} = 0, \\ \frac{1}{2}a - \frac{1}{\sqrt{3}}a_8 - \sqrt{3}a_8^D + \frac{m_3}{\sqrt{2}} = 0. \tag{3.37}$$

Two remarks are in order here. First and foremost, it is crucially important to note that the massless dyons D^{11} and D^{22} have both electric and magnetic charges $1/2$ with respect to the T^3 generator of the dual $U(\tilde{N} = 2)$ gauge group. This means that they can fill the fundamental representation of this group. Moreover, all dyons D^{lA} ($l = 1, \ldots, \tilde{N} = 2$) can form color doublets. This is another confirmation of the conclusion of Sec. III A, that the non-Abelian factor $U(\tilde{N} = 2)$ of the dual gauge group gets restored in the equal mass limit.

A general reason ensuring that the dyons D^{lA} ($l = 1, \ldots, \tilde{N}$) fill the fundamental representation of $U(\tilde{N})$ group can be expressed as follows: due to monodromies the D^{lA} dyons pick up magnetic charges of particular monopoles of $SU(N)$. The magnetic charges of these particular monopoles are represented by weights rather than roots of the $U(\tilde{N})$ subgroup [$\pm 1/2$ for $U(\tilde{N} = 2)$, see (3.23) and (3.25)]. This is related to the absence of the AD points associated with collisions of first $\tilde{N}$ double roots, see (3.7), which, of course, is a consequence of the dual theory with the non-Abelian gauge factor $U(\tilde{N})$ being not asymptotically free. Say, in the case of the Abelian dual gauge group ($\tilde{N} = 0$) studied in Ref. [1], the massless dyons pick up integer magnetic charges and, therefore, cannot fill the fundamental representation of U(2).

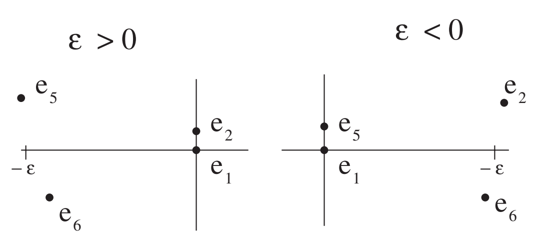

FIG. 4. As we decrease Δm_{31} and pass through the AD point (3.24), the pairing of roots $e_{1,2,5,6}$ in the x plane changes.

The second comment is that the dyon charges with respect to each U(1) generator are proportional to each other. This guarantees that these dyons are mutually local. Note also, that both the magnetic and electric charges of the dyon doublet D^{lA} with respect to the T^8 generator are $(-1/2)\times$ the charges of the D^{33} dyon. This is in accord with the result of Ref. [19], where the charges of the $\tilde{N}$ plet with respect to U(1) gauge factors of (1.3) were shown to be $(-1/\tilde{N})\times$ the singlet charges.

C. Low-energy effective action

In this section we present the low-energy theory in the $r=3$ vacuum in domain III, i.e. at small ξ and small $|\Delta m_{PP'}|$ (below the AD points).

As was shown above, the massless quarks q^{1A} and q^{2A} are transformed into the massless dyons D^{1A} and D^{2A}; the latter form a fundamental representation of the dual gauge group $\mathrm{U}(\tilde{N}=2)$. The D^{1A} and D^{2A} dyons interact with the U(1) gauge field

$$A_\mu, \tag{3.38}$$

and non-Abelian $\mathrm{SU}(\tilde{N}=2)$ gauge fields. According to the dyons charges (3.36), the third component of this SU(2) dual gauge field is the following linear combination:

$$B^3_\mu = \frac{1}{\sqrt{2}}(A^3_\mu + A^{3D}_\mu). \tag{3.39}$$

If the dual gauge group is restored, the $B^{1,2}_\mu$ components of the gauge field become massless at $m_1 = m_2$. Let us check this circumstance.

The electric and magnetic charges of the W bosons $B^1_\mu \mp iB^2_\mu$ coincide with the charges of the operators $\tilde{D}_{A2}D^{1A}$ and $\tilde{D}_{A1}D^{2A}$. From (3.36) we obtain for the W-boson charges

$$B^1_\mu \mp iB^2_\mu\colon\ (0,0;\pm 1,\pm 1;0,0). \tag{3.40}$$

These charges determine the mass of these states via the Seiberg-Witten mass formula [23]. We have

$$\sqrt{2}|a_3 + a^D_3| = |\Delta m_{12}|, \tag{3.41}$$

where the first two equations in (3.37) are used. We see that this mass tends to zero at $\Delta m_{12}\to 0$, i.e. $m_1 = m_2$, as was expected. Now it is clear that the gauge fields (3.39) and (3.40) fill the adjoint multiplet B^p_μ $(p=1,2,3)$ of the non-Abelian SU(2) factor of the dual gauge group.

Other light states include the D^{33} dyon and another photon associated with the T^8 generator of the underlying U(3) gauge group broken in the dual theory down to $\mathrm{U}(2)\times\mathrm{U}(1)$, see Eq. (1.3). According to the dyon charges this photon is presented by the following combination:

$$B^8_\mu = \frac{1}{\sqrt{10}}(A^8_\mu + 3A^{8D}_\mu). \tag{3.42}$$

In fact, the dyons D^{lA} $(l=1,2)$, D^{33} and the gauge fields A_μ, B^p_μ $(p=1,2,3)$, and B^8_μ, together with their superpartners, are the only light states to be included in the low-energy effective theory in domain III. All other states are either heavy (with masses of the order of Λ) or decay on curves of marginal stability [1,23,24,35,38,39]. In the case at hand, CMS are located around the origin in the $\Delta m_{PP'}$ complex planes and go through the AD point, cf. [39]. In fact, the W bosons of the underlying non-Abelian gauge theory, as well as the off-diagonal states of the quark matrix q^{kA}, decay on CMS. We discuss these decay processes in Sec. VI.

Taking this into account we can write the bosonic part of the effective low-energy action of the theory in domain III,

$$\begin{aligned} S_{\mathrm{III}} = \int d^4x\Big[&\frac{1}{4\tilde{g}_2^2}(F^p_{\mu\nu})^2 + \frac{1}{4g_1^2}(F_{\mu\nu})^2 + \frac{1}{4\tilde{g}_8^2}(F^8_{\mu\nu})^2 \\ &+ \frac{1}{\tilde{g}_2^2}|\partial_\mu b^p|^2 + \frac{1}{g_1^2}|\partial_\mu a|^2 + \frac{1}{\tilde{g}_8^2}|\partial_\mu b^8|^2 \\ &+ |\nabla^1_\mu D^A|^2 + |\nabla^1_\mu \tilde{D}_A|^2 + |\nabla^2_\mu D^3|^2 \\ &+ |\nabla^2_\mu \tilde{D}_3|^2 + V(D,\tilde{D},b^p,b^8,a)\Big], \end{aligned} \tag{3.43}$$

where b^p and b^8 are the scalar $\mathcal{N}=2$ superpartners of gauge fields B^p_μ and B^8_μ, while $F^p_{\mu\nu}$, $F^8_{\mu\nu}$ are their field strengths,

$$\begin{aligned} b^3 &= \frac{1}{\sqrt{2}}(a^3 + a^3_D) \quad \text{for } p=3, \\ b^8 &= \frac{1}{\sqrt{10}}(a^8 + 3a^8_D). \end{aligned} \tag{3.44}$$

Covariant derivatives are defined in accordance with the charges of the D^l and D^3 dyons. Namely,

$$\begin{aligned} \nabla^1_\mu &== \partial_\mu - i\Big(\frac{1}{2}A_\mu + \sqrt{2}B^p_\mu\frac{\tau^p}{2} + \frac{\sqrt{10}}{2\sqrt{3}}B^8_\mu\Big), \\ \nabla^2_\mu &== \partial_\mu - i\Big(\frac{1}{2}A_\mu - \frac{\sqrt{10}}{\sqrt{3}}B^8_\mu\Big). \end{aligned} \tag{3.45}$$

The coupling constants g_1, $\tilde{g}_8$, and $\tilde{g}_2$ correspond to two U (1) and the SU(2) gauge groups, respectively. The scalar potential $V(D,\tilde{D},b^p,b^8,a)$ in the action (3.43) is

$$V(D,\tilde{D},b^p,b^8,a)=\frac{\tilde{g}_2^2}{4}(\bar{D}_A\tau^p D_A-\tilde{D}_A\tau^p\bar{\tilde{D}}^A)^2+\frac{10}{3}\frac{\tilde{g}_8^2}{8}(|D^A|^2-|\tilde{D}_A|^2-2|D^3|^2+2|\tilde{D}_3|^2)^2+\frac{g_1^2}{8}(|D^A|^2-|\tilde{D}_A|^2$$
$$+|D^3|^2-|\tilde{D}_3|^2-3\xi)^2+\tilde{g}_2^2|\tilde{D}_A\tau^p D_A|^2+\frac{g_1^2}{2}|\tilde{D}_AD^A+\tilde{D}_3D_3|^2+\frac{10}{3}\frac{\tilde{g}_8^2}{2}|\tilde{D}_AD^A-2\tilde{D}_3D^3|^2$$
$$+\frac{1}{2}\bigg\{\bigg|a+\tau^p\sqrt{2}b^p+\sqrt{\frac{10}{3}}b^8+\sqrt{2}m_A\bigg|^2(|D^A|^2+|\tilde{D}_A|^2)$$
$$+\bigg|a-2\sqrt{\frac{10}{3}}b^8+\sqrt{2}m_3\bigg|^2(|D^3|^2+|\tilde{D}_3|^2)\bigg\}. \qquad (3.46)$$

Now we are ready move to the desired limit of the equal quark masses, $\Delta m_{PP'}=0$. The vacuum of the theory (3.43) is located at the following values of the scalars a, b^p and b^8:

$$\langle a\rangle=-\sqrt{2}m, \qquad \langle b^p\rangle=0, \qquad \langle b^8\rangle=0, \qquad (3.47)$$

while the VEVs of dyons are determined by the FI parameter ξ and can be chosen as

$$\langle D^{lA}\rangle=\sqrt{\xi}\begin{pmatrix}0&0&0&1&0\\0&0&0&0&1\end{pmatrix}, \qquad \langle\bar{\tilde{D}}^{lA}\rangle=0,$$
$$\langle D^{33}\rangle=\sqrt{\xi}, \qquad \langle\bar{\tilde{D}}^{33}\rangle=0. \qquad (3.48)$$

In fact, for the particular choice of quark masses (3.1) we deal with in this section it is impossible to see which particular flavors of dyons develop VEVs. In the equal mass limit all $r=3$ isolated vacua coalesce and become a root of the Higgs branch. In Sec. V, we will be able to relax the condition (3.1) and show that, in fact, the (1, 2, 3) vacuum we started from at large ξ transforms into the (4, 5, 3) vacuum of the dual theory at small ξ, as shown in (3.48).

Let us calculate the dimension of the Higgs branch, which emerges in the equal mass limit. We have

$$\dim\mathcal{H}|_{\xi\ll\Lambda}=4\tilde{N}N_f+4(N-\tilde{N})-2\tilde{N}^2-\tilde{N}^2-\tilde{N}^2$$
$$-2(N-\tilde{N})-(N-\tilde{N})-(N-\tilde{N})$$
$$=4\tilde{N}N, \qquad (3.49)$$

where we take into account that we have $4\tilde{N}N_f+4(N-\tilde{N})$ dyon real degrees of freedom and subtract $2\tilde{N}^2+2(N-\tilde{N})$ F-term conditions, $\tilde{N}^2+(N-\tilde{N})$ D-term conditions and $\tilde{N}^2+(N-\tilde{N})$ phases eaten by the Higgs mechanism, see (3.46).

Now we see that the dimension of the Higgs branch at small ξ coincides with the dimension of the Higgs branch (2.12) at large ξ. This strongly supports our arguments [1] that we have a crossover transition between two domains I and III, rather than a phase transition.

From Eqs. (3.47) and (3.48) we see that both, the gauge U(2) and flavor SU(5) groups, are broken in the vacuum. However, the color-flavor locked form of (3.48) guarantees that the diagonal global $\mathrm{SU}(\tilde{N}=2)_{C+F}$ survives. More exactly, the unbroken global group of the dual theory is

$$\mathrm{SU}(3)_F\times\mathrm{SU}(2)_{C+F}\times\mathrm{U}(1).$$

For generic N and $\tilde{N}$ the unbroken global group of the dual theory is

$$\mathrm{SU}(N)_F\times\mathrm{SU}(\tilde{N})_{C+F}\times\mathrm{U}(1). \qquad (3.50)$$

Here $\mathrm{SU}(\tilde{N})_{C+F}$ is a global unbroken color-flavor rotation, which involves the first $\tilde{N}$ flavors, while $\mathrm{SU}(N)_F$ factor stands for the flavor rotation of the remaining N dyons. Thus, a color-flavor locking takes place in the dual theory too. Much in the same way as in the original microscopic theory, the presence of the global $\mathrm{SU}(\tilde{N})_{C+F}$ group is the reason behind formation of the non-Abelian strings. For generic quark masses the global symmetry (2.7) is broken down to $\mathrm{U}(1)^{N_f-1}$. In parallel with the original microscopic theory, the dimension of the base of the Higgs branch $(2N\tilde{N})$ coincides with the number of the broken global generators for the symmetry breaking pattern (3.50), see (2.14).

Please, observe that in the equal mass limit the global unbroken symmetry (3.50) of the dual theory at small ξ coincides with the global group (2.7) present in the $r=N$ vacuum of the original microscopic theory at large ξ. This is, of course, expected and presents a check of our results. Note however, that this global symmetry is realized in two distinct ways in two dual theories. As was already mentioned, the quarks and U(N) gauge bosons of the original theory at large ξ come in the (1, 1), $(N^2-1, 1)$, $(\bar{N},\tilde{N})$, and $(N,\bar{\tilde{N}})$ representations of the global group (2.7), while the dyons and U($\tilde{N}$) gauge bosons form (1, 1), $(1,\tilde{N}^2-1)$, $(N,\bar{\tilde{N}})$, and $(\bar{N},\tilde{N})$ representations of (3.50). We see that adjoint representations of the $(C+F)$ subgroup are different in two theories. A similar phenomenon was detected in [1] for the Abelian dual theory in the case $\tilde{N}=0$.

We traced the evolution of light quarks from domain I to II and then back to the equal mass limit along the Coulomb branch at zero ξ. We demonstrated that quarks transform into dyons along the way, picking up magnetic charges. For consistency of our analysis it is instructive to consider another route from domain I to domain III, namely, the one along the line $\Delta m_{AB}=0$. On this line we keep the global group (3.50) unbroken. Then we obtain a surprising result: the quarks and gauge bosons that form the adjoint $(\tilde{N}^2-1)$ representation of SU(N) at large ξ and the dyons and gauge bosons that form the adjoint $(\tilde{N}^2-1)$ representation of SU($\tilde{N}$) at small ξ are, in fact, *distinct* states. How can this occur?

Since we have a crossover between domains I and III rather than a phase transition, this means that in the full microscopic theory the $(\tilde{N}^2 - 1)$ adjoints of SU(N) become heavy and decouple as we pass from domain I to III along the line $\Delta m_{AB} = 0$. Moreover, some composite $(\tilde{N}^2 - 1)$ adjoints of SU($\tilde{N}$), which are heavy and invisible in the low-energy description in domain I become light in domain III and form the D^{lK} dyons ($K = N+1, \ldots, N_f$) and gauge bosons B^p_μ. The phenomenon of level crossing takes place. Although this crossover is smooth in the full theory, from the standpoint of the low-energy description the passage from domain I to III means a dramatic change: the low-energy theories in these domains are completely different; in particular, the degrees of freedom in these theories are different.

This logic leads us to the following conclusion. In addition to light dyons and gauge bosons included in the low-energy theory (3.43), in domain III at small ξ, we have heavy fields (with masses of the order of Λ) that form the adjoint representation $(N^2 - 1, 1)$ of the global symmetry (3.50). These are screened (former) quarks and gauge bosons from domain I continued into III. Let us denote them as $M_P^{P'}$ ($P, P' = 1, \ldots, N$). In Sec. VI, we will discuss them in more detail and reveal their physical nature in domain III.

By the same token, it is seen that in domain I, in addition to the light quarks and gauge bosons, we have heavy fields $M_K^{K'}$ ($K, K' = N+1, \ldots, N_f$), which form the adjoint $(\tilde{N}^2 - 1)$ representation of SU($\tilde{N}$). This is schematically depicted in Fig. 5.

It is quite plausible to suggest that these fields $M_P^{P'}$ and $M_K^{K'}$ are Seiberg's mesonic fields [17,40], which occur in the dual theory upon breaking of $\mathcal{N} = 2$ supersymmetry by the mass-term superpotential $\mu[A^2 + (A^a)^2]$ for the adjoint fields when we take the limit $\mu \to \infty$. In this limit our theory becomes $\mathcal{N} = 1$ QCD. In the $\mathcal{N} = 2$ limit the $M_P^{P'}$ and $M_K^{K'}$ fields are heavy, with masses $\sim\Lambda$, and are absent in the low-energy action (3.43). However, in the

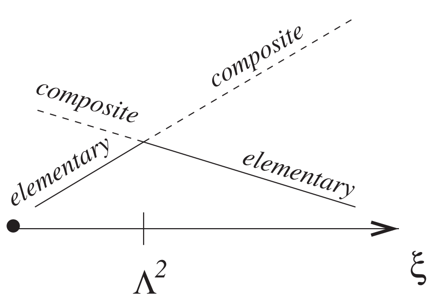

FIG. 5. Evolution of the SU(N) and SU($\tilde{N}$) W bosons vs ξ. On both sides of the level crossing at $\xi = \Lambda^2$ the global groups are SU(N) $\times$ SU($\tilde{N}$), however, above Λ^2 it is $\mathrm{SU}(N)_{C+F} \times \mathrm{SU}(\tilde{N})_F$, while below Λ^2 it is $\mathrm{SU}(N)_F \times \mathrm{SU}(\tilde{N})_{C+F}$.

$\mu \to \infty$ limit it is the $\mathcal{N} = 1$ scale $\Lambda_{\mathcal{N}=1}$ that is fixed,

$$\Lambda_{\mathcal{N}=1}^{2N-\tilde{N}} = \mu^N \Lambda^{N-\tilde{N}},$$

implying that $\Lambda \to 0$. The M_{AB} fields might become light in the limit of $\mathcal{N} = 1$ QCD. Previously, these M_{AB} fields were not identified in the $\mathcal{N} = 2$ theory.

IV. CONFINED MONOPOLES

Since the quarks are in the Higgs regime in the original microscopic theory, the monopoles are confined. It is known [5,10,16] that when we introduce a nonvanishing FI parameter ξ in $\mathcal{N} = 2$ QCD with the gauge group U(N), we confine the 't Hooft-Polyakov monopoles of the SU(N) subgroup to the string. In fact, they become string junctions of two elementary non-Abelian strings. They are seen as kinks in the world-sheet theory (2.24) at large ξ, and as kinks in the dual world-sheet theory in domain III at small ξ (see Sec. VII). In this domain it is dyons, rather than quarks, that condense. Therefore, here we deal with oblique confinement [21].

In this section we will determine the elementary string fluxes in the classical limit in domain III and show that the elementary monopole fluxes can be absorbed by two strings. Hence, the monopoles are indeed represented by junctions of different strings.

As a warm up example, we start from reviewing matching of the monopole and strings fluxes in domain I at large ξ. To this end we go to the quasiclassical limit in the world-sheet theory (2.24), i.e. $\Delta m_{PP'} \gg \Lambda$, where the non-Abelian strings become Abelian Z_N strings, see Ref. [6] for more details.

As in Sec. III B, we restrict ourselves to the simplest example $N = 3$, $\tilde{N} = 2$. Consider one of three Z_3 strings, which occur due to winding of the q^{11} quark at infinity,

$$q^{11}(r \to \infty) \sim \sqrt{\xi} e^{i\alpha}, \qquad q^{22}(r \to \infty) \sim q^{33}(r \to \infty) \sim \sqrt{\xi}, \tag{4.1}$$

see (2.4). Here r and α are the polar coordinates in the plane $i = 1, 2$ orthogonal to the string axis. This implies the following behavior of the gauge potentials at $r \to \infty$:

$$\begin{aligned} \frac{1}{2}A_i + \frac{1}{2}A_i^3 + \frac{1}{2\sqrt{3}}A_i^8 &\sim \partial_i \alpha, \\ \frac{1}{2}A_i - \frac{1}{2}A_i^3 + \frac{1}{2\sqrt{3}}A_i^8 &\sim 0, \\ \frac{1}{2}A_i - \frac{1}{\sqrt{3}}A_i^8 &\sim 0, \end{aligned} \tag{4.2}$$

see the quark charges in (3.22). The solution to these equations is

$$A_i \sim \frac{2}{3}\partial_i \alpha, \qquad A_i^3 \sim \partial_i \alpha, \qquad A_i^8 \sim \frac{1}{\sqrt{3}}\partial_i \alpha. \tag{4.3}$$

It determines the string gauge fluxes $\int dx_i A_i$, $\int dx_i A_i^3$ and $\int dx_i A_i^8$, respectively. The integration above is performed over a large circle in the (1, 2) plane. Let us call this string S_1.

Next, we define the string charges as

$$\int dx_i (A_i, A_i^D; A_i^3, A_i^{3D}; A_i^8, A_i^{8D}) = 4\pi(n_e, n_m; n_e^3, n_m^3; n_e^8, n_m^8). \tag{4.4}$$

This definition ensures that the string has the same charge as a trial monopole, which can be attached to the string endpoint. In other words, the flux of the given string is the flux of the trial monopole[3] sitting on string's end, with the charge defined by (4.4).

In particular, according to this definition, the charge of the string with the fluxes (4.3) is

$$\vec{n}_{S_1} = \left(0, \frac{1}{3}; 0, \frac{1}{2}; 0, \frac{1}{2\sqrt{3}}\right). \tag{4.5}$$

Since this string is formed through the quark condensation, it is magnetic; its charges with respect to the Cartan subalgebra of the SU(3) group are represented by the weight vector, as seen from (4.5).

There are other two elementary strings S_2 and S_3 in U (3), which arise due to winding of q^{22} and q^{33} quarks, respectively. Repeating the above procedure for these strings we get their charges,

$$\vec{n}_{S_2} = \left(0, \frac{1}{3}; 0, -\frac{1}{2}; 0, \frac{1}{2\sqrt{3}}\right), \qquad \vec{n}_{S_3} = \left(0, \frac{1}{3}; 0, 0; 0, -\frac{1}{\sqrt{3}}\right). \tag{4.6}$$

It is easy to check that each of three elementary SU(3) monopoles is confined by two elementary strings. Consider, say, the monopole with the charge (0, 0; 0, 1; 0, 0). This charge can be written as a difference of the charges of two elementary strings S_1 and S_2, namely,

$$(0, 0; 0, 1; 0, 0) = \vec{n}_{S_1} - \vec{n}_{S_2}. \tag{4.7}$$

This means that this monopole is a junction of these two strings at large ξ, with S_1 string having the outgoing flux, while S_2 the incoming flux.

Now we are ready to turn to the monopole confinement in domain III, described by the dual theory (3.43). Consider the $\tilde{S}_1$ string arising due to winding of the D^{14} dyon. At $r \to \infty$ we have

$$D^{14}(r\to\infty) \sim \sqrt{\xi} e^{i\alpha}, \qquad D^{25}(r\to\infty) \sim D^{33}(r\to\infty) \sim \sqrt{\xi}, \tag{4.8}$$

see (3.48). Taking into account the dyon charges quoted in Eq. (3.36) [the D^{14} and D^{25} dyons have the same electric and magnetic charges as D^{11} and D^{22}, respectively] we derive the behavior of the gauge potentials at infinity,

$$\begin{aligned} &\frac{1}{2}A_i + \frac{1}{2}A_i^3 + \frac{1}{2}A_i^{3D} + \frac{1}{2\sqrt{3}}A_i^8 + \frac{\sqrt{3}}{2}A_i^{8D} \sim \partial_i\alpha, \\ &\frac{1}{2}A_i - \frac{1}{2}A_i^3 - \frac{1}{2}A_i^{3D} + \frac{1}{2\sqrt{3}}A_i^8 + \frac{\sqrt{3}}{2}A_i^{8D} \sim 0, \\ &\frac{1}{2}A_i - \frac{1}{\sqrt{3}}A_i^8 - \sqrt{3}A_i^{8D} \sim 0, \end{aligned} \tag{4.9}$$

which, in turn, implies

$$A_i \sim \frac{2}{3}\partial_i\alpha, \qquad \frac{1}{2}A_i^3 + \frac{1}{2}A_i^{3D} \sim \frac{1}{2}\partial_i\alpha, \qquad \frac{1}{2\sqrt{3}}A_i^8 + \frac{\sqrt{3}}{2}A_i^{8D} \sim \frac{1}{6}\partial_i\alpha. \tag{4.10}$$

The combinations orthogonal to those which appear in (4.10) are required to tend to zero at infinity, namely, $A_i^3 - A_i^{3D} \sim 0$ and $A_i^{8D} - 3A_i^8 \sim 0$. As a result we get

$$A_i \sim \frac{2}{3}\partial_i\alpha, \qquad A_i^D \sim 0, \qquad A_i^3 \sim \frac{1}{2}\partial_i\alpha,$$
$$A_i^{3D} \sim \frac{1}{2}\partial_i\alpha, \qquad A_i^8 \sim \frac{1}{10\sqrt{3}}\partial_i\alpha, \qquad A_i^{8D} \sim \frac{\sqrt{3}}{10}\partial_i\alpha. \tag{4.11}$$

These expressions determine the charge of the $\tilde{S}_1$ string,

$$\vec{n}_{\tilde{S}_1} = \left(0, \frac{1}{3}; -\frac{1}{4}, \frac{1}{4}; -\frac{\sqrt{3}}{20}, \frac{1}{20\sqrt{3}}\right). \tag{4.12}$$

Paralleling the above analysis we determine the charges of two other Z_3 strings, which are due to windings of D^{25} and D^{33}, respectively. We get

$$\vec{n}_{\tilde{S}_2} = \left(0, \frac{1}{3}; \frac{1}{4}, -\frac{1}{4}; -\frac{\sqrt{3}}{20}, \frac{1}{20\sqrt{3}}\right), \qquad \vec{n}_{\tilde{S}_3} = \left(0, \frac{1}{3}; 0, 0; \frac{\sqrt{3}}{10}, -\frac{1}{10\sqrt{3}}\right). \tag{4.13}$$

Now we can check that each of three SU(3) monopoles can be confined by two strings. Say, for the monopole with the charge (0, 0; 0, 1; 0, 0) we have

$$(0, 0; 0, 1; 0, 0) = (\vec{n}_{\tilde{S}_1} - \vec{n}_{\tilde{S}_2}) + \tfrac{1}{2}(\vec{n}_{D^{14}} - \vec{n}_{D^{25}}), \tag{4.14}$$

where $\vec{n}_{D^{14}}$ and $\vec{n}_{D^{25}}$ are the charges of the D^{14} and D^{25} dyons given in (3.36). Only a part of the monopole flux is confined to the strings. The remainder of its flux is screened by the condensate of the D^{14} and D^{25} dyons. In

[3]This trial monopole does not necessarily exist in our theory. In fact, in U(N) theories we deal with here, the strings are stable and there are no monopoles in the theory *per se*, which could break these strings. The SU(N) monopoles are rather string junctions, so they are attached to *two* strings, as we will see below.

a similar manner we can check confinement of the other two SU(3) monopoles.

We see that although the quark charges change as we pass from domain I to III, and they become dyons, this does *not* happen with the monopoles. The monopole states do not change their charges. They are confined in both domains I and III, being junctions of two different elementary strings. In domain III in the dual theory there is a peculiarity: not all of the monopole flux is carried by two attached strings; a part of it is screened by dyon condensate.

Our result provides an explicit counterexample to the commonly accepted belief that if monopoles are confined in the original theory, then it is quarks that are confined in the dual theory. Above we demonstrated that monopoles rather than quarks are confined in domain III. The failure of this folklore belief eliminates a paradox mentioned in [22] where this folklore was tacitly assumed.

We can check that the dyons whose charges are the sum of the monopole and W-boson charges are also confined. As an example of such a state it is worth considering the dyon with the charge (0, 0; 1, 1; 0, 0) in domain II. Below the crossover, in domain III, its charge is shifted by the monopole charge due to monodromy. In domain III this dyon has the charge (0, 0; 1, 2; 0, 0). Therefore, we have

$$(0,0;1,2;0,0) = (\vec{n}_{\tilde{S}_1} - \vec{n}_{\tilde{S}_2}) + \frac{3}{2}(\vec{n}_{D^{14}} - \vec{n}_{D^{25}}), \quad (4.15)$$

which shows that this dyon is confined by two strings, $\tilde{S}_1$ and $\tilde{S}_2$, while the remainder of its flux is screened by condensation of the D^{14} and D^{25} dyons.

V. SPLITTING THE QUARK MASSES

In this section we relax the condition (3.1) and split the masses of the first $\tilde{N}$ quarks (out of N quarks, which develop VEVs at large ξ) and $\tilde{N}$ extra quarks. If all masses are generic, the Higgs branch disappears, and we have $C^N_{N_f}$ isolated $r = N$ vacua in the original theory (2.1) at large ξ in domain I. Again, we consider one of these vacua, namely, the $(1, \ldots, N)$ vacuum. We will show that in domain III at small ξ it converts[4] into the $(N+1, \ldots, N_f, \tilde{N}+1, \ldots, N)$ vacuum, as indicated in (3.48) for the case $N = 3$ and $\tilde{N} = 2$.

If the condition (3.1) is fulfilled the dual theory (3.43) is IR rather then asymptotically free. Once we relax this condition, it becomes asymptotically free at the scales below Δm_{PK} ($P = 1, \ldots, N$ and $K = N+1, \ldots, N_f$). We assume that all mass differences Δm_{PK} are of the same order. In fact, the theory generates its own low-energy scale

$$\tilde{\Lambda}^{\tilde{N}}_{\rm le} = \frac{\Delta m^N_{PK}}{\Lambda^{N-\tilde{N}}}. \quad (5.1)$$

In order to guarantee the weak coupling regime in the dual theory (3.43) we cannot choose ξ too small in domain III. We have assumed that

$$\tilde{\Lambda}_{\rm le} \ll \sqrt{\xi}. \quad (5.2)$$

Since $\Lambda \gg \sqrt{\xi}$ in domain III the above condition requires, in turn, that the mass splittings Δm_{PK} not to be too large. We impose the following constraint:

$$\Delta m_{PK} \ll \Lambda, \qquad P = 1, \ldots, N, \qquad K = N+1, \ldots, N_f. \quad (5.3)$$

In parallel with our discussion in Sec. III, we pass from domain I to II at weak coupling and then to domain III along the Coulomb branch (at $\xi = 0$), using the Seiberg-Witten exact solution of the theory. The role of the Δm variable in Fig. 2 is played by the mass differences $\Delta m_{PP'}$ and $\Delta m_{KK'}$, which we assume to be of the same order.

A. The Seiberg-Witten curve

To make our discussion simpler in this section we again consider the example of the U(3) gauge theory with $N_f = 5$. At first, we relax just the first of the conditions (3.13) and define

$$\Delta m_{14} \equiv m_1 - m_4, \qquad m_{14} \equiv \tfrac{1}{2}(m_1 + m_4), \quad (5.4)$$

keeping $m_5 = m_2$. Then the Seiberg-Witten curve takes the form

$$y^2 = \left(x + \frac{\tilde{m}_2}{\sqrt{2}}\right)^2 \Big[(x-\phi_1)^2(x-\phi_3)^2 - 4\frac{\Lambda}{\sqrt{2}}\left(x + \frac{\tilde{m}_{14} + \Delta m_{14}}{\sqrt{2}}\right)\left(x + \frac{\tilde{m}_{14} - \Delta m_{14}}{\sqrt{2}}\right) \times \left(x + \frac{\tilde{m}_5}{\sqrt{2}}\right)\Big], \quad (5.5)$$

where we substituted the solution (3.16) for ϕ_2. The double root $e_3 = e_4$ is given in the second equation in (3.18). Next we parametrize

$$\phi_1 = -\frac{\tilde{m}_{14}}{\sqrt{2}} + \chi, \quad (5.6)$$

where χ is small. Also we shift x,

$$x = -\frac{\tilde{m}_{14}}{\sqrt{2}} + z, \quad (5.7)$$

and arrive at

[4] Of course, the total number of vacua in the dual theory ($C^{\tilde{N}}_{N_f}$ with generic masses) matches the number of vacua in the original theory, $C^N_{N_f} = C^{\tilde{N}}_{N_f}$.

$$y^2 = \left(x + \frac{\tilde{m}_2}{\sqrt{2}}\right)^2 \left[z^2 \left(z + \frac{\Delta m_{31}}{\sqrt{2}} - \frac{\Lambda}{\sqrt{2}}\right)^2 + (-2\chi z + \chi^2) \times \left(z + \frac{\Delta m_{31}}{\sqrt{2}} + \frac{\Lambda}{\sqrt{2}}\right)^2 + 4\frac{\Lambda}{\sqrt{2}}\frac{\Delta m_{14}^2}{8}\left(z + \frac{\Delta m_{31}}{\sqrt{2}}\right)\right], \tag{5.8}$$

where $\Delta m_{31} = m_3 - m_{14}$. Here we use (approximately) the solution (3.19) for ϕ_3 obtained for unsplit masses.

Next, we look for roots of (5.8) located near the unperturbed values of e_1 and e_2 [see Eq. (3.18)], so that z is close to zero. The curve (5.8) approximately gives a quadratic equation for these roots,

$$z^2 - (2\chi z - \delta^2)\left(\frac{\Delta m_{31} + \Lambda}{\Delta m_{31} - \Lambda}\right)^2 + 4\Lambda\frac{\Delta m_{14}^2}{8}\frac{\Delta m_{31}}{(\Delta m_{31} - \Lambda)^2} = 0. \tag{5.9}$$

We need to find such χ that ensures that the two roots of this equation coincide. This is an easy exercise leading to

$$\chi = \pm\frac{\Delta m_{14}}{2\sqrt{2}}\left(\frac{\Delta m_{31} - \Lambda}{\Delta m_{31} + \Lambda}\right), \tag{5.10}$$

which gives, in turn,

$$\phi_1 = -\frac{\tilde{m}_{14}}{\sqrt{2}} - \frac{\Delta m_{14}}{2\sqrt{2}}\left(\frac{\Delta m_{31} - \Lambda}{\Delta m_{31} + \Lambda}\right). \tag{5.11}$$

The corrected roots e_1 and e_2 are

$$e_1 = e_2 = -\frac{\tilde{m}_{14}}{\sqrt{2}} - \frac{\Delta m_{14}}{2\sqrt{2}}\left(\frac{\Delta m_{31} + \Lambda}{\Delta m_{31} - \Lambda}\right). \tag{5.12}$$

Here we pick up only the solution with the minus sign for χ in (5.10). The reason is that in the quasiclassical regime of large Δm_{31} ($\Delta m_{31} \gg \Lambda$) the solution (5.11) is determined by m_1, see Eq. (3.4). This corresponds to the (1, 2, 3) vacuum we started from in domain I and II. The opposite sign would correspond to the (4, 2, 3) vacuum.

Please, observe that

$$\phi_1 = \begin{cases} -\frac{\tilde{m}_1}{\sqrt{2}}, & |m_{31}| \gg \Lambda, \\ -\frac{\tilde{m}_4}{\sqrt{2}}, & |m_{31}| \ll \Lambda. \end{cases} \tag{5.13}$$

We see that ϕ_1 evolves from m_1 to m_4 as we reduce Δm_{31} moving from domain II toward III and then inside III. By the same token, we can split the m_2 and m_5 masses and study the behavior of ϕ_2. In this way we get

$$\phi_2 = \begin{cases} -\frac{\tilde{m}_2}{\sqrt{2}}, & |m_{32}| \gg \Lambda, \\ -\frac{\tilde{m}_5}{\sqrt{2}}, & |m_{32}| \ll \Lambda. \end{cases} \tag{5.14}$$

These results demonstrate that the (1, 2, 3) vacuum of the original theory (2.1) in domains I and II converts into the (4, 5, 3) vacuum of the dual theory (3.43) as we go deep into domain III,

$$(1, 2, 3)|_{\mathrm{I,II}} \to (4, 5, 3)|_{\mathrm{III}}, \tag{5.15}$$

or, in the case of generic N and $\tilde{N}$,

$$(1, \ldots, N)|_{\mathrm{I,II}} \to (N+1, \ldots, N_f, \tilde{N}+1, \ldots, N)|_{\mathrm{III}}. \tag{5.16}$$

In other words, if we pick up the vacuum (2.4) and (2.5) in our theory (2.1) at large ξ in domain I and reduce ξ passing to domain III, the system goes through a crossover transition and ends up in the vacuum of the dual theory (3.43) with the following VEVs of the adjoint scalars:

$$\left\langle \frac{1}{2}a + \frac{\tau^p}{2}\sqrt{2}b^a + \frac{1}{2}\sqrt{\frac{10}{3}}b^8 \right\rangle = -\frac{1}{\sqrt{2}}\begin{pmatrix} m_4 & 0 \\ 0 & m_5 \end{pmatrix},$$
$$\left\langle \frac{1}{2}a - \sqrt{\frac{10}{3}}b^8 \right\rangle = -\frac{1}{\sqrt{2}}m_3, \tag{5.17}$$

while the VEVs of dyons are given in Eq. (3.48), where Eqs. (3.48) and (5.17) are specified for $N = 3$, $\tilde{N} = 2$.

Equation (5.17) ensures that the conditions for the massless dyons (3.37) are modified when $m_1 \neq m_4$ and $m_2 \neq m_5$ as follows:

$$\frac{1}{2}a + \frac{1}{2}a_3 + \frac{1}{2}a_3^D + \frac{1}{2\sqrt{3}}a_8 + \frac{\sqrt{3}}{2}a_8^D + \frac{m_4}{\sqrt{2}} = 0,$$
$$\frac{1}{2}a - \frac{1}{2}a_3 - \frac{1}{2}a_3^D + \frac{1}{2\sqrt{3}}a_8 + \frac{\sqrt{3}}{2}a_8^D + \frac{m_5}{\sqrt{2}} = 0, \tag{5.18}$$
$$\frac{1}{2}a - \frac{1}{\sqrt{3}}a_8 - \sqrt{3}a_8^D + \frac{m_3}{\sqrt{2}} = 0.$$

We pause here to make one last comment. The pole present in (5.11) at $\Delta m_{31} + \sqrt{2}\Lambda = 0$ has no physical meaning. It is canceled out in the expressions for the standard coordinates on the Coulomb branch

$$u_k = \phi_1^k + \phi_2^k + \phi_3^k.$$

To see that this is indeed the case one has to consider small deviations of ϕ_3 from its approximate solution (3.19).

B. The W-boson mass

In this section we will present another argument supporting our claim that as one passes through the crossover, the vacuum we had in domain III turns into a distinct $r = N$ vacuum, as shown in Eq. (5.16).

Consider again the already familiar example with $N = 3$ and $\tilde{N} = 2$. On the Coulomb branch in the (1, 2, 3) vacuum at weak coupling (in domain II at $\xi = 0$) the mass of the $A_\mu^{1,2}$ gauge fields is

$$m_W|_{\mathrm{II}} = \sqrt{2}|a_3| = |\Delta m_{12}|. \tag{5.19}$$

Below the crossover, in domain III, the charged components of the dual SU(2) gauge multiplet are the $B_\mu^{1,2}$ fields defined in (3.40). In Sec. III C we calculated the mass of these fields (the W-boson mass) in the limit of unsplit quark masses (3.13), see (3.41). In the limit (3.13) the W-boson mass coincides with the value (5.19). Now we will split quark masses and show that the W-boson mass experiences a jump as we pass from domain II to III.

Taking into account the charges of the $B^{1,2}_\mu$ fields—these fields will be referred to as the W^* bosons—quoted in Eq. (3.40) we arrive at the following expression for the W^*-boson masses in domain III at $\xi = 0$:

$$m_{W^*}|_{\rm III} = \sqrt{2}|a_3 + a_3^D| = |\Delta m_{45}|. \tag{5.20}$$

To derive (5.20) we take the difference of two first equations in (5.18). Note that both Eqs. (5.19) and (5.20) are exact. We see that, according to (5.16), the W-boson mass experiences a jump.

It is instructive to check this result by explicit calculation via the Seiberg-Witten curve. The mass of the SU(2) W boson coincides with the discontinuity of the following period integral:

$$m_W = \frac{\sqrt{2}}{2\pi}\left|\Delta\left[\sum_{P=1}^{N}\int_{e_3=e_4}^{e_1=e_2}\frac{xdx}{x+\frac{m_P}{\sqrt{2}}} - \sum_{K=\tilde{N}+1}^{N_f}\int_{e_3=e_4}^{e_1=e_2}\frac{xdx}{x+\frac{m_K}{\sqrt{2}}}\right]\right|, \tag{5.21}$$

where Δ means taking the discontinuity of the logarithmic function. Substituting here the expressions (5.12) for the $e_1 = e_2$ roots and similar expression for the $e_3 = e_4$ roots

$$e_3 = e_4 = -\frac{\tilde{m}_{25}}{\sqrt{2}} - \frac{\Delta m_{25}}{2\sqrt{2}}\left(\frac{\Delta m_{32}+\Lambda}{\Delta m_{32}-\Lambda}\right), \tag{5.22}$$

we obtain, with logarithmic accuracy,

$$m_W\bigg|_{\rm II} = \frac{1}{2\pi}\left|\Delta\left\{\Delta m_{12}\ln\frac{\Delta m}{\Lambda}\right\}\right| \tag{5.23}$$

at large Δm ($\Delta m \gg \Lambda$, where $\Delta m \equiv \Delta m_{31} \sim \Delta m_{32}$). On the other hand, at small Δm ($\Delta m \ll \Lambda$)

$$m_{W^*}\bigg|_{\rm III} = \frac{1}{2\pi}\left|\Delta\left\{\Delta m_{45}\ln\frac{\Lambda}{\Delta m}\right\}\right|. \tag{5.24}$$

Taking the discontinuity of logarithms we fully confirm the results presented in (5.19) and (5.20).

The key point of this calculation is Eq. (5.12) for the $e_1 = e_2$ roots and the companion expression (5.22) for the $e_3 = e_4$ roots. Say, the double root $e_1 = e_2$ tends to $\tilde{m}_1$ at $\Delta m \gg \Lambda$ and to $\tilde{m}_4$ at $\Delta m \ll \Lambda$. A more careful study of the integral in (5.21) shows that the two jumps occur precisely at two AD points (3.24) and (3.25).

Does the jump of the W-boson masses means that the *physical spectrum* has a genuine discontinuity at the AD points? Of course, not.

No real physical phase transitions are implied at these points. The physical spectrum is continuous. The apparent jump of the W-boson mass means that, in actuality, we have *two* W-boson-like states. Let us denote them as W and W^*, respectively. They have the same electric and magnetic charges [(0, 0; ±1, 0; 0, 0) above the crossover, and (0, 0; ±1, ±1; 0, 0) below the crossover, see (3.40)], but distinct global flavor U(1) charges. Note, that the global group (2.7) [or the dual global group (3.50)] is broken by mass differences down to

$$\mathrm{U}(1)^{N_f-1}. \tag{5.25}$$

All massive BPS states have nonvanishing charges with respect to this group. The W bosons acquire nonvanishing global charges due to the color-flavor locking.

Above the crossover (i.e. at large $|\Delta m|$) the W boson has mass (5.19), while that of W^* is

$$m^*_W|_{\rm II} = \sqrt{2}\left|a_3 + \frac{\Delta m_{14}}{\sqrt{2}} - \frac{\Delta m_{25}}{\sqrt{2}}\right| = |\Delta m_{45}|. \tag{5.26}$$

Below the crossover (i.e. at small $|\Delta m|$) the mass of the W^* boson is given by (5.20), while that of W is

$$m_W|_{\rm III} = \sqrt{2}\left|a_3 + a_3^D - \frac{\Delta m_{14}}{\sqrt{2}} + \frac{\Delta m_{25}}{\sqrt{2}}\right| = |\Delta m_{12}|. \tag{5.27}$$

We see that given two states, W and W^*, the physical spectrum *is* continuous, indeed.

VI. MORE ON PARTICLES IN THE ADJOINT REPRESENTATIONS OF SU(N) AND SU($\tilde{N}$): CROSSING THE BOUNDARIES

The problem of stability of massive BPS states on the Coulomb branch of our theory (i.e. at $\xi = 0$) needs additional studies. This is left for future work. Here we will make a few general comments following from consistency of our picture.

It is well known that the W bosons usually do not exist as localized states in the strong-coupling regime on the Coulomb branch (speaking in jargon, they "decay"). They split into antimonopoles and dyons on CMS on which the AD points lie [23,35].

In our theory this decay involves two steps. Consider the W boson associated with the T^3 generator (T^3 W boson for short) with the charge (0, 0; 1, 0; 0, 0) in domain II. As we approach the first AD point (3.24) from domain II, the T^3 W boson "emits" massless antimonopole with the charge opposite to the one in Eq. (3.23). After we pass by the second AD point (3.25) it emits massless monopole with the charge (3.35). The net effect is the decay of the W boson into the T^3 antimonopole and dyon with the charges (0, 0; 0, −1; 0, 0) and (0, 0; 1, 1; 0, 0), respectively. It means that the W boson is absent in domain III, in full accord with the analysis of the SU(2) theory in [35].

In our theory we have another T^3 W-boson-like state, namely, W^*. Clearly this state also can decay in the same T^3 antimonopole and a different dyon[5] as we pass through the crossover. In domain III the W^* state plays the role of

[5]This dyon has the same electric and magnetic charges ((0, 0; 1, 1; 0, 0) in domain II and the charge (0, 0; 1, 2; 0, 0) in domain III) as the dyon associated with the W state, but different global U(1) charges with respect to (5.25).

the gauge field of the dual theory. Therefore, we expect that it is stable in domain III and "decays" in domain II.

This picture is valid on the Coulomb branch at $\xi = 0$. As we switch on small $\xi \neq 0$ the monopoles and dyons become confined by strings. In fact, the elementary monopoles/dyons are represented by junctions of two different elementary non-Abelian strings [5,10,16], see also a detailed discussion of the monopole/dyon confinement in Sec. IV. This means that, as we move from domain II into III at small nonvanishing ξ the W boson decays into an antimonopole and dyon; however, these states cannot abandon each other and move far apart because they are confined. Therefore, the W boson evolves into a stringy meson formed by an antimonopole and dyon connected by two strings, as shown in Fig. 6, see [6] for a discussion of these stringy mesons.

These stringy mesons have nonvanishing U(1) global charges with respect to the Cartan generators of the SU(3) subgroup of the global group (2.7) (above we discussed only one W boson of this type, related to the T^3 generator, however, in fact, we have six different charged gauge boson/quark states of this type). In the equal mass limit these globally charged stringy mesons combine with neutral [with respect to the group (5.25)] stringy mesons formed by pairs of monopoles and antimonopoles (or dyons and antidyons) connected by two strings, to form the octet representation of the SU(3) subgroup of the global group (2.7) [in general, the adjoint representation of SU(N)]. They are heavy in domain III, with mass of the order of Λ.

We propose to identify these stringy mesons with ($N^2 - 1$) adjoints $M_P^{P'}$ ($P, P' = 1, \ldots, N$) of the SU(N) subgroup with which we have already had an encounter *en route* from domain I to III along the line $\Delta m_{AB} = 0$, see Sec. III C.

The same applies to the q^{kK} quarks ($K = N + 1, \ldots, N_f$) of domains I and II. As we go through the crossover into domain III at small ξ q^{kK} quarks evolve into stringy mesons formed by pairs of antimonopoles and dyons connected by two strings, see Fig. 6. However, these states are unstable. To see that this is indeed the case, please, observe that in the equal mass limit these stringy mesons fill the bifundamental representations $(N, \bar{\tilde{N}})$ and $(\bar{N}, \tilde{N})$ of the global group (3.50); hence, can decay into light dyons/dual gauge bosons with the same quantum numbers.

To summarize, in domain III we have the dyons and dual gauge fields in the (1, 1), $(1, \tilde{N}^2 - 1)$, $(N, \bar{\tilde{N}})$, and $(\bar{N}, \tilde{N})$ representations of the global group (3.50). They are light (with masses $\sim \tilde{g}\sqrt{\xi}$) and enter the low-energy effective action (3.43). In addition to these, we have stable neutral heavy (with masses $\sim\Lambda$) stringy mesonic M fields formed by pairs of (anti)monopoles and dyons connected by two strings, see Fig. 6. The set of stable states of this type forms the $(N^2 - 1, 1)$ representation of (3.50).

In domain I a reversed situation takes place: we have the quarks and gauge bosons in the (1, 1), $(N^2 - 1, 1)$, $(\bar{N}, \tilde{N})$ and $(N, \bar{\tilde{N}})$ representations of the global group (2.7). They have masses of the order of $g\sqrt{\xi}$. In addition to these "elementary" states, we also have stable neutral stringy mesonic M fields in the $(1, \tilde{N}^2 - 1)$ representation of (2.7). The latter mesons are heavier, they have masses of the order of $\sqrt{\xi}$, due to the presence of strings connecting the monopoles/dyons.

All other stringy mesons of the matrix M_A^B are metastable and decay into elementary excitations with the same global quantum numbers.

It is seen that non-Abelian confinement works in our theory as follows: It is a combined effect of the Higgs screening, decay process on CMS and confining strings formation. Strings always confine monopoles or dyons in both original and dual theories. These confined dyons have charges whose difference from the monopole charge can be screened in the given regime, for example, in the theory with $N = 3$, $\tilde{N} = 2$ the dyon charge (0, 0; 1, 1; 0, 0) in domain II and (0, 0; 1, 2; 0, 0) in domain III for the T^3 monopole (0, 0; 0, 1; 0, 0). As we pass from domain I to III, the screened quarks and gauge bosons decay into (anti) monopoles and dyons, which are still bound together in pairs by strings and form mesons. And *vice versa*, when we go from domain III to I, the screened dyons and dual gauge fields of the dual theory (3.43) decay into pairs of confined (anti)monopoles and dyons and form the corresponding stringy mesons. In other words, in both domains related by duality, I and III, the elementary excitations in the given region evolve into stringy composite mesons in the dual region and *vice versa*.

It is worth mentioning the $N_f = N$ theory studied in [1] is an important particular application of this picture. In this case, the dual theory in domain III is the Abelian $\mathrm{U}(1)^N$ gauge theory. It has N light Abelian dyons and photons. In addition to these states, it has $(N^2 - 1)$ heavy neutral mesonic $M_P^{P'}$ fields, which form the adjoint multiplet of the global $\mathrm{SU}(N)_{C+F}$ group. These states were identified in [1]. Here we reveal their physical nature. They are mesonic states formed by monopole/dyon pairs connected by two strings as shown in Fig. 6.

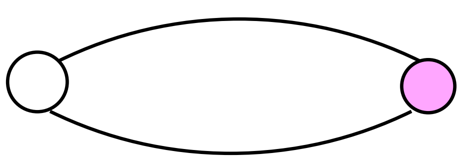

FIG. 6 (color online). Meson formed by antimonopole and dyon connected by two strings. Open and closed circles denote dyon and antimonopole, respectively.

VII. WORLD-SHEET DUALITY

In the previous sections we demonstrated that, as we reduce ξ below Λ^2 and enter domain III in Fig. 2, our original microscopic U(N) gauge theory with N_f flavors undergoes the crossover transition to the $U(\tilde{N}) \times U(1)^{N-\tilde{N}}$ gauge theory with N_f flavors. Now we show how this bulk duality is translated in the language of the world-sheet duality on the non-Abelian string.

A. Dual world-sheet theory

As was discussed in Sec. II B, if the quark mass differences are small, the $(1, \ldots, N)$ vacuum of the original microscopic $U(N)$ gauge theory supports non-Abelian semilocal strings. Their internal dynamics is described by the effective two-dimensional low-energy $\mathcal{N} = (2, 2)$ sigma model (2.24). The model has N orientational moduli n^P with the U(1) charge $+1$ and masses $m_P = \{m_1, \ldots, m_N\}$, plus $\tilde{N}$ size moduli ρ^K, with the U(1) charge -1 and masses $(-m_K) = -\{m_{N+1}, \ldots, m_{N_f}\}$.

Clearly, the dual bulk $U(\tilde{N})$ theory (3.43) in domain III also supports non-Abelian semilocal strings. We found that the $(1, \ldots, N)$ vacuum of the original theory transforms into the $(N+1, \ldots, N_f, \tilde{N}+1, \ldots, N)$ vacuum of the dual theory. Therefore, the internal string dynamics on the string world sheet is described by a similar $\mathcal{N} = (2, 2)$ sigma model. Now it has $\tilde{N}$ orientational moduli with the U(1) charge $+1$ and masses $m_K = \{m_{N+1}, \ldots, m_{N_f}\}$. To make contact with (2.24) let us call them $\tilde{\rho}^K$. In addition, it has N size moduli with the U(1) charge -1 and masses $(-m_P) = -\{m_1, \ldots, m_N\}$. We refer to these size moduli as $\tilde{n}^P$.

The bosonic part of the action of the world-sheet model in the gauge formulation (which assumes taking the limit $\tilde{e}^2 \to \infty$) has the form

$$S_{\rm dual} = \int d^2x \Big\{ |\nabla_\alpha \tilde{\rho}^K|^2 + |\tilde{\nabla}_\alpha \tilde{n}^P|^2 + \frac{1}{4e^2} F^2_{\alpha\beta} + \frac{1}{e^2} |\partial_\alpha \sigma|^2 + 2 \left| \sigma + \frac{m_P}{\sqrt{2}} \right|^2 |\tilde{n}^P|^2 + 2 \left| \sigma + \frac{m_K}{\sqrt{2}} \right|^2 |\tilde{\rho}^K|^2 + \frac{e^2}{2} (|\tilde{\rho}^K|^2 - |\tilde{n}^P|^2 - 2\tilde{\beta})^2 \Big\},$$
$$P = 1, \ldots, N, \qquad K = N+1, \ldots, N_f, \tag{7.1}$$

where

$$\nabla_\alpha = \partial_\alpha - iA_\alpha, \qquad \tilde{\nabla}_\alpha = \partial_\alpha + iA_\alpha. \tag{7.2}$$

We see that the roles of orientational and size moduli are interchanged in Eq. (7.1) compared with (2.24). As in the model (2.24), small mass differences $(m_A - m_B)$ lift orientational and size zero modes of the non-Abelian semilocal string generating a shallow potential on the moduli space. Much in the same way as in the model (2.24), the dual coupling constant $\tilde{\beta}$ is determined by the bulk dual coupling $\tilde{g}_2^2$,

$$4\pi \tilde{\beta}(\xi) = \frac{8\pi^2}{\tilde{g}_2^2}(\xi) = (N - \tilde{N}) \ln \frac{\Lambda}{\tilde{g}\sqrt{\xi}} \gg 1, \tag{7.3}$$

see Eqs. (2.21) and (2.25). Here we take into account the fact that both the bulk and world-sheet dual theories have identical β functions, with the first coefficient $(\tilde{N} - N) < 0$. They are both IR free; therefore, the coupling constant $\tilde{\beta}$ is positive at $\Lambda \gg \sqrt{\xi}$. As in the model (2.24), the coincidence of β functions of the bulk and world-sheet theories implies that the scale of the dual model (7.1) is equal to that of the bulk theory,

$$\tilde{\Lambda}_\sigma = \Lambda,$$

cf. (2.26). Comparing (7.3) with (2.25) we see that

$$\tilde{\beta} = -\beta. \tag{7.4}$$

Thus, the dual theory (7.1) can be interpreted as a continuation of the sigma model (2.24) to negative values of the coupling constant β.

Note also, that both dual world-sheet theories (2.24) and (7.1) give *effective low-energy* descriptions of string dynamics and are applicable only at scales well below $g\sqrt{\xi}$.

Concluding this section a comment is in order regarding the world-sheet duality between two-dimensional sigma models (2.24) and (7.1). It was previously noted in Ref. [14]. In this paper two bulk theories, with the $U(N)$ and $U(\tilde{N})$ gauge groups, were considered (these theories were referred to as a dual pair in [14]). Two-dimensional sigma models (2.24) and (7.1) were presented as effective low-energy descriptions of the non-Abelian strings for these two bulk theories.

B. The BPS spectrum

Dorey noted [38] that the exact BPS spectrum of two-dimensional $\mathcal{N} = (2, 2)$ $CP(N-1)$ model (2.18) coincides with the BPS spectrum of massive states in four-dimensional $\mathcal{N} = 2$ QCD (2.1) with the U(N) gauge group and $N_f = N$ flavors in the $r = N$ vacuum on the Coulomb branch (i.e. at $\xi = 0$). Later, this correspondence of the BPS spectra was generalized to cover the $N_f > N$ case [41]. Namely, it was shown that the BPS spectrum of kinks in the two-dimensional model (2.24) coincides with the BPS spectrum of massive monopoles and dyons in the $r = N$ vacuum on the Coulomb branch of the four-dimensional theory (2.1).

The reason for this amazing coincidence was understood and explained later in Refs. [5,10], for a review see [6]. Consider the bulk theory (2.1) at large ξ. As was discussed above, it is the monopoles that are confined by strings. Elementary monopoles are represented by string junctions of two different elementary non-Abelian strings [5,10,16]. Each string of the bulk theory corresponds to a particular vacuum of the world-sheet theory. In particular, the $\mathcal{N} = (2, 2)$ supersymmetric sigma model (2.24) on the string has N degenerate vacua and kinks interpolating between distinct vacua. These kinks are interpreted as confined monopoles of the bulk theory [5,10,16].

Please observe that the mass of the confined BPS monopole (a.k.a sigma-model kink) is a holomorphic function on the parameter space and, therefore, cannot depend [5,10] on the nonholomorphic parameter ξ. Thus, we can reduce ξ all the way to $\xi = 0$, and the mass of the confined monopole stays intact. At $\xi = 0$, on the Coulomb branch, the monopoles are no longer confined, and their masses are given by the exact Seiberg-Witten solution of the bulk theory. This leads us to the conclusion that the kink masses in the two-dimensional sigma model (2.24) should coincide with those of monopoles/dyons in the four-dimensional bulk theory on the Coulomb branch in the $r = N$ vacuum. As was mentioned above, this fact was earlier observed"experimentally" in [38,41].

Now the same logic leads us to one another conclusion. Since the confined monopole masses in the bulk theory do not depend on ξ, we can reduce ξ and safely pass from domain I to III, keeping the BPS spectrum unchanged. In domain I the spectrum of confined monopoles is determined by the BPS spectrum of the sigma model (2.24), while in domain III it is determined by the BPS spectrum of the dual sigma model (7.1). Thus, we arrive at the conclusion, that BPS spectra of two dual world-sheet models (2.24) and (7.1) should coincide.

It is instructive to explicitly check this assertion. Let us start from (2.24) at positive β and use the description of the supersymmetric model (2.24) in terms of exact superpotentials [41,42]. Following [29] and integrating out fields n^P and ρ^K we can describe the model by an exact twisted superpotential

$$\mathcal{W}_{\rm eff} = \frac{1}{4\pi}\sum_{P=1}^{N}(\sqrt{2}\Sigma + m_P)\ln\frac{\sqrt{2}\Sigma + m_P}{\Lambda} - \frac{1}{4\pi}\sum_{K=N+1}^{N_F}(\sqrt{2}\Sigma + m_K)\ln\frac{\sqrt{2}\Sigma + m_K}{\Lambda} - \frac{N-\tilde{N}}{4\pi}\sqrt{2}\Sigma, \tag{7.5}$$

where Σ is a twisted superfield [29] (with σ being its lowest scalar component). Minimizing this superpotential with respect to σ we find

$$\prod_{P=1}^{N}(\sqrt{2}\sigma + m_P) = \Lambda_\sigma^{(N-\tilde{N})}\prod_{K=N+1}^{N_f}(\sqrt{2}\sigma + m_K). \tag{7.6}$$

Note that the roots of this equation coincide with the double roots of the Seiberg-Witten curve (3.2) of the bulk theory [38,41].

The BPS kink masses are given by differences of the superpotential (7.5) calculated at distinct roots,

$$m_{ij}^{\rm BPS} = 2|\mathcal{W}_{\rm eff}(\sigma_i) - \mathcal{W}_{\rm eff}(\sigma_j)|. \tag{7.7}$$

It is easy to show that the above masses coincide with those of monopoles and dyons in the bulk theory given by the period integrals of the Seiberg-Witten curve presented in (5.21) [this equation is written down for certain particular roots]. As was mentioned above, this coincidence of the BPS spectra of the world-sheet and bulk theories was expected.

Now let us consider the effective superpotential of the dual world-sheet theory (7.1). It has the form

$$\mathcal{W}_{\rm eff}^{\rm dual} = \frac{1}{4\pi}\sum_{K=N+1}^{N_f}(\sqrt{2}\Sigma + m_K)\ln\frac{\sqrt{2}\Sigma + m_K}{\Lambda} - \frac{1}{4\pi}\sum_{P=1}^{N}(\sqrt{2}\Sigma + m_P)\ln\frac{\sqrt{2}\Sigma + m_K}{\Lambda} - \frac{\tilde{N}-N}{4\pi}\sqrt{2}\Sigma. \tag{7.8}$$

We see that it coincides with the superpotential (7.5) up to the sign. Clearly, both the root equations and the BPS spectra are the same for both dual sigma models, as expected. They are given by Eqs. (7.6) and (7.7), respectively.

VIII. CONCLUSIONS

In this paper we continued our explorations of the transition from weak to strong coupling in $\mathcal{N} = 2$ supersymmetric QCD in the course of variation of the parameter ξ. These explorations began in [1], where we analyzed the case $N_f = N$ and discovered a crossover transition from the original weakly coupled (at large ξ) non-Abelian theory to a strong-coupling regime (at small ξ) described by a dual weakly coupled Abelian theory. Now we expanded this study to cover the $N_f > N$ case.

We found that at strong coupling (i.e. small ξ) a dual weakly coupled $\mathcal{N} = 2$ theory exists but it is non-Abelian, based on the gauge group U($\tilde{N}$). This non-Abelian dual describes low-energy physics at small ξ. The dual theory has N_f flavors of light dyons, to be compared with N_f quarks in the original U(N) theory. Both, the original and dual theories are Higgsed and share the same global symmetry SU(N) $\times$ SU($\tilde{N}$) $\times$ U(1), albeit the physical meaning of the SU(N) and SU($\tilde{N}$) factors is different in the large- and small-ξ regimes. Both regimes support non-Abelian semilocal strings.

In each of these two regimes particles that are in the adjoint representations with respect to one of the factor groups exist in two varieties: elementary fields and composite states bound by strings. These varieties interchange upon transition from one regime to the other. We conjecture that the composite stringy states can be related to Seiberg's M fields.

We demonstrated that non-Abelian confinement in our theory is a combined effect of the Higgs screening, decay processes on CMS and confining string formation. Strings always confine monopoles or dyons (whose charges can be represented as a sum of a monopole and W-boson charges) in both original and dual theories. As we pass from domain I to III, the screened quarks and gauge bosons decay into (anti)monopoles and dyons, which are still bound together in pairs by strings and form mesons. These mesons form the adjoint representation of the SU(N) factor of the global group. And *vice versa*, when we go from the domain III to I, the screened dyons and dual gauge fields of the dual theory decay into pairs of confined (anti)monopoles and dyons and form the corresponding stringy mesons, which fall into the adjoint representation of the SU($\tilde{N}$) factor of the global group. A level crossing takes place on the way.

The bulk duality that we observed translates into a two-dimensional duality on the world sheet of the non-Abelian strings. At large ξ the internal dynamics of the semilocal non-Abelian strings is described by the sigma model of N orientational and $(N_f - N)$ size moduli, while at small ξ the roles of orientational and size moduli interchange. The BPS spectra of two dual sigma models (describing confined monopoles/dyons of the bulk theory) coincide.

It would be extremely interesting to trace parallels between the non-Abelian duality we detected and string theory constructions. We conjecture that such parallels must exist.

ACKNOWLEDGMENTS

This work is supported in part by DOE Grant No. DE-FG02-94ER408. The work of A. Y. was supported by FTPI, University of Minnesota, by RFBR Grant No. 09-02-00457a and by Russian State Grant for Scientific Schools RSGSS-11242003.2.

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