Asymptotics of thermal spectral functions

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We use operator product expansion (OPE) techniques to study the spectral functions of currents and stress tensors at finite temperature, in the high-energy timelike region $\omega \gg T$. The leading corrections to these spectral functions are proportional to $\sim T^4$ expectation values in general, and the leading corrections $\sim g^2 T^4$ are calculated at weak coupling, up to an undetermined coefficient in the shear viscosity channel. Spectral functions are shown to be infrared safe, in the deeply virtual regime, up to order $g^8 T^4$. The convergence of (vacuum subtracted) sum rules in the shear and bulk viscosity channels is established in QCD to all orders in perturbation theory, though numerically significant tails $\sim T^4/(\log\omega)^3$ are shown to exist in the bulk viscosity channel. We argue that the spectral functions of currents and stress tensors in infinitely coupled $\mathcal{N} = 4$ super Yang-Mills theory do not receive any medium-dependent power correction.

I. INTRODUCTION

This paper is devoted to applying operator product expansion (OPE) [1] techniques to study asymptotics of real-time spectral functions $\rho(\omega, \vec{q})$ at finite temperature. We will obtain the leading corrections in an expansion in T/ω at large frequencies ω, with T a characteristic energy scale of the medium, to the spectral functions of currents (J^μ), scalar operators ($m\bar{\psi}\psi$) and stress tensors (in the shear and bulk viscosity channels).

The zero frequency, zero momentum limits of spectral functions are related in general to hydrodynamical transport coefficients, which, for the quark-gluon plasma as probed by the Relativistic Heavy Ion Collider [2], have been the focus of much recent work (for a more ample discussion we refer the reader to [3] and references therein). At finite timelike momenta, the vector channel spectral function is related to the production rate of lepton pairs by the plasma [4].

The OPE techniques employed in this paper, on the other hand, give information on the deeply virtual timelike region, $\omega \gg T, |\vec{q}|$. Spectral functions in this region probe the surrounding medium only on short time and length scales, which is why they are related, by the OPE, to the expectation values of local operators. The OPE analysis thus does not cover the region $\omega \sim T$ where the thermal corrections to the shape of ρ are the most important and where most of the, e.g., lepton pair emission, occurs.

This paper will cover the following applications. First of all, the compact expressions obtained in the OPE regime can be used as simple consistency checks on more complete calculations. For example, we will prove that the leading thermal corrections to spectral functions in QCD are proportional in general to T^4, as was observed in the early calculations [4], and we will give their coefficients in certain instances. Also, infrared divergences can be fully characterized in the OPE regime: we will show that, begin-

ning at order $g^8 T^4$, but not before, certain spectral functions cease to be computable perturbatively due to the so-called Linde problem.

An interesting sum rule was recently proposed by Kharzeev and Tuchin [5] and by Karsch, Kharzeev, and Tuchin [6], and used to estimate the QCD bulk viscosity near the deconfinement phase transition. We will show that their sum rule is sensitive to a numerically important ultraviolet tail, ignored in [5,6], which could affect their analysis away from very close to the phase transition.

In the shear viscosity channel we will demonstrate the convergence of sum rules in asymptotically free theories. Furthermore, we will argue that the left-hand side of such sum rules is saturated by a one-loop calculation, which however will not be performed here. We will also observe the possibility that such sum rules could possess, in certain theories, strong ultraviolet tails making them discontinuous in the free theory limit ($g^2 \rightarrow 0$).

Finally, we will briefly study spectral functions in the strong coupling limit of $\mathcal{N} = 4$ super Yang-Mills theory. We will argue that power corrections to spectral functions at large virtuality are restricted to polynomial terms in the momenta and forbidden for currents and stress tensors, generalizing observations of Teaney [7].

This paper is organized as follows. After describing our formalism in Sec. II, we apply it in the weak-coupling regime to the spectral functions enumerated above, in Sec. III. The physical implications of these results are then discussed in Sec. IV. Finally, in the Appendix, we reproduce a two-loop diagrammatic calculation which confirms the OPE prediction in the vector channel.

II. FORMALISM

The physical basis of the (Euclidean) OPE [1] is the separation of scales between that of a short-sized probe Δx and that of a typical wavelength $\sim T^{-1}$ in a medium,

leading to useful asymptotic expansions in $(T\Delta x)$. In Fourier space for the expectation value of a two-point function of operators $\mathcal{O}_{1,2}$ this gives rise to asymptotic expansions at high momenta, suppressing arguments other than the frequency:

$$G_{\mathrm{E}12}(\omega_{\mathrm{E}}) \sim \sum_n \langle \mathcal{O}_n \rangle \frac{c_{12}^n}{\omega_{\mathrm{E}}^{d_n}}. \qquad (2.1)$$

The locality of the operators $\mathcal{O}_n$ (which we will assume to be Hermitian, without loss of generality) follows from $T \ll \omega_E$. The powers d_n are determined by renormalization group equations (RGE), to be reviewed shortly.

Naively taking twice the imaginary part of the analytic continuation of Eq. (2.1) to Minkowski frequencies ω would yield asymptotics for the spectral function,

$$\rho_{12}(\omega) \sim \sum_n \langle \mathcal{O}_n \rangle 2 \, \mathrm{Im}\!\left(\frac{c_{12}^n}{(-i\omega)^{d_n}} \right), \qquad (2.2)$$

with $\rho(\omega)$ the commutator function $\langle [\mathcal{O}(\omega), \mathcal{O}(x=0)] \rangle$. Equation (2.2) is "naive" in that the asymptotics of an analytic function in one complex direction do not, in general, determine its asymptotics along other directions. It is nevertheless correct, because of dispersion relations, whenever $\rho(\omega)$ does admit an asymptotic expansion in inverse powers of ω (or logarithms), as will be proved shortly.

Equation (2.2) is the main result of this section. We read it as a Minkowski-space version of the OPE: heuristically, the locality of its operators is a consequence of the small time during which a pair of high-energy particles, created by a high-frequency operator, can travel before it is reabsorbed by its complex conjugate. Because of this picture, once its coefficients are determined we expect it to remain valid even in out-of-equilibrium situations, where an Euclidean formulation is not available.

A. Dispersion relations

Here we justify the passage from Eq. (2.1) to Eq. (2.2), *assuming* that $\rho(\omega)$ admits an asymptotic expansion in inverse powers of ω and logarithms. The argument is based on the dispersive representation of the Euclidean correlator,[1]

$$G_{\mathrm{E}}(\omega_{\mathrm{E}}) = P_n(\omega_{\mathrm{E}}) + \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi(\omega' - i\omega_{\mathrm{E}})} \rho(\omega'). \quad (2.3)$$

Note that ρ is always real (or a Hermitian matrix), $\rho = 2 \, \mathrm{Im}G$. $P_n(\omega_{\mathrm{E}})$ is a polynomial in ω_{E} that is not determined by the spectral density. In general, Eq. (2.3) is ultraviolet

divergent and a subtracted integral must be used, but this does not interfere with the present argument.

The basic point is that, if an upper cutoff $|\omega'| < \Lambda$ were imposed on the frequency integration in Eq. (2.3), the resulting $G_{\mathrm{E}}(\omega)$ would admit an expansion in purely integral powers of $1/\omega$ at large ω. Specifically, for $\rho(\omega)$ bounded by $|\omega|^{-k-\epsilon}$ ($\epsilon > 0$) at large ω, the kth derivative of Eq. (2.3) with respect to $1/\omega$ is shown to vanish at $\omega = \infty$. This shows that the asymptotic expansion of ρ directly translates into one for G_{E} modulo integral terms (e.g., terms that are killed by taking derivatives).

Thus it suffices to match the asymptotic expansions of Eq. (2.1) and (2.2) termwise. Two cases must be distinguished: nonintegral power terms (or powers times logarithms), and purely integral powers.

The former case is handled with the dispersive transform of a power law tail $\rho(\omega) \propto \omega^{-\alpha}$,

$$\int_{\omega_0}^{\infty} \frac{d\omega'}{2\pi(\omega' - i\omega_{\mathrm{E}})} \frac{1}{\omega^{\alpha}} \sim \frac{1}{2\sin\pi\alpha} \frac{1}{(-i\omega_{\mathrm{E}})^{\alpha}}$$
$$+ \sum_n \frac{D_n \mu^{n-\alpha}}{\omega^n}, \qquad (2.4)$$

where μ_0 is some infrared cutoff and the sum is an asymptotic series with n integers. The coefficients D_n in this sum depend on the details of infrared data (here on ω_0) but the nonanalytic term is a clean reflection of the large ω behavior of ρ, as expected. Its imaginary part at real ω produces the right asymptotics. In particular, the asymptotics at positive and negative ω can be reconstructed independently since their contributions are out of phase at real ω_{E}. Taking derivatives of Eq. (2.4) with respect to α gives identities for integrals with logarithms, for which the same argument applies.

The case of purely integral powers in ρ, d_n integer, is special because the Euclidean nonanalyticity is a single logarithm: it could cancel out between the contributions of positive and negative ρ. Such cancellations would occur when d_n is even and its contribution to $\rho(\omega)$ is even in ω, or when d_n is odd and its contribution is also odd. In these cases, the Euclidean function would be proportional to $1/(-i\omega_{\mathrm{E}})^n$ with an imaginary coefficient, e.g., it has a "wrong" phase. On the other hand, integral terms coming from small frequencies in Eq. (2.3), are proportional to $1/(-i\omega_{\mathrm{E}})^n$ with a real coefficient. Thus the two are cleanly separated by their phases, and we conclude that in all cases the asymptotics of ρ can be recovered from those of G_{E}.

In perturbation theory, we do not expect such "wrong phase" contributions to the Euclidean OPE, and in any case certainly none appears at the relatively low orders to which we will be working in this paper. Our corrections to spectral functions will come solely from nonanalytic terms in G_{E}.

We now comment on the assumption that $\rho(\omega)$ admits an expansion in inverse powers of ω, which we have assumed in proving Eq. (2.2). Possible violations of it at

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[1]At finite temperature, this gives a distinguished analytic continuation of G_{E} from the discrete set of Matsubara frequencies at which it is strictly defined [8]. Substituting $\omega_{\mathrm{E}} = -i\omega$ with ω in the upper-half plane, Eq. (2.3) always coincides with the retarded function $G_R(\omega)$.

the nonperturbative level (due, for instance, to oscillating terms) are discussed in [9]. However, in perturbation theory we find it hard to see how it could fail, for instance because there is no scale to provide an oscillation rate. Thus this assumption will be made throughout this paper.

B. Renormalization group equations

The OPE is based on a systematic separation of infrared and ultraviolet contributions (an interesting discussion may be found in [10]). A factorization scale μ (for us, $T \lesssim \mu \ll \omega$) is introduced, and all vacuum fluctuations from above this scale are integrated over, leaving more infrared and state-dependent fluctuations to be accounted for by the expectation values of operators. Since $\mu \ll \omega$ these operators can be taken to be local, in a systematic gradient expansion. The restriction to vacuum fluctuations ensures that this yields operator relations, that is, the OPE holds in any quantum state.

Omitting Lorentz and internal indices, this yields expansions of the form

$$\mathcal{O}_1^{(\mu)}(p)\mathcal{O}_2^{(\mu)}(x=0) \sim \sum_i C_{12}^i(p, \mu, \mu_{\overline{\text{MS}}}, g, m)\mathcal{O}_i^{(\mu)}(x=0),$$

$$(2.5)$$

with m and g standing for various intrinsic mass scales and couplings of the theory and $\mu_{\overline{\text{MS}}}$ its renormalization scale. The renormalized operators $\mathcal{O}_i^{(\mu)}$ obey the RGE, with γ a matrix of anomalous dimensions,

$$0 = [\mu \partial_\mu + \gamma]\mathcal{O}_i^{(\mu)}, \qquad (2.6)$$

from which we deduce, in the case that $\mathcal{O}_{1,2}^{(\mu)}$ are independent of μ (as for currents, which we will exclusively study in this paper), the RGE for the OPE coefficients:

$$[\mu \partial_\mu - \gamma^T]C_{12}^i = 0. \qquad (2.7)$$

Assuming the absence of microscopic scales between the factorization scale μ and p, the coefficient functions can depend only on three scales: the momentum p, the factorization scale μ and the renormalization scale $\mu_{\overline{\text{MS}}}$ of the theory. The $\mu_{\overline{\text{MS}}}$ dependence is determined by a RGE

$$\left[\mu_{\overline{\text{MS}}}\frac{\partial}{\partial \mu_{\overline{\text{MS}}}} + \beta(g)\frac{\partial}{\partial g} + \dots\right]\mathcal{O}_1(p)\mathcal{O}_2 = \sum_i D_{12}^i(p)\mathcal{O}_i.$$

$$(2.8)$$

The important point for us will be that the coefficients D_{12}^i can only be polynomials in the momenta p. This is because correlators at unequal positions are RGE-invariant and Fourier transforms can always be made insensitive to the coincidence limit by taking sufficiently many derivatives with respect to momenta.

Were there no right-hand side to Eq. (2.8) we would conclude, on dimensional grounds, that all logarithms in

the coefficient functions have to be of the form $\log(p^2/\omega^2)$. Equation (2.8) shows that terms which are polynomial in p can also contain logarithms of $\mu_{\overline{\text{MS}}}$.[2]

C. RGE in Minkowski signature

Since we interpret Eq. (2.2) as a Minkowski-space version of the OPE, it is worth specifying how we mean the RGE of operators directly in Minkowski space: we mean it to be exactly what it is in the Euclidean OPE, namely, vacuum fluctuations should be integrated over but not state-dependent ones. Provided equivalent regulators are used, this will reproduce the standard running of the operators in Euclidean space.[3]

Variations on this procedure can easily lead to difficulties due to transport phenomena (e.g., nonlocal phenomena), as may be illustrated by a concrete example: that of computing the contribution of a fluctuation at scale $\sim gT$ (with $g = \sqrt{4\pi\alpha_s T}$) to the expectation value of a local operator in a weakly coupled quark-gluon plasma. In this case, it would be natural to fully integrate out the scale T, which produces, at the one-loop level, the hard thermal loop effective theory [11]. It is nonlocal with support on the classical trajectories of plasma particles. The conclusion is that, employing any reasonable procedure, fully integrating out the scales $gT \lesssim \mu \lesssim T$ in real time must convert local operators at the scale T to *nonlocal* operators at the scale gT. This does not happen, however, when only vacuum fluctuations are integrated over,[4] as was assumed in the preceding subsection.

III. ASYMPTOTICS OF SPECTRAL FUNCTIONS

At weak coupling, the OPE coefficients for spectral functions, Eq. (2.2), are products of Euclidean OPE coefficients and anomalous dimensions, which we now compute in turn.

A. Conventions

For concreteness and simplicity we study the Euclidean Yang-Mills theory coupled to n_F Dirac fermions of the same mass m, $S_E = \frac{F_{\mu\nu}F_{\mu\nu}}{4g^2} + \sum_i \bar{\psi}_i(\slashed{p} - im)\psi_i$, with p^μ the covariant momentum. Some important operators (in Euclidean notation) will be

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[2]Examples of such terms are the logarithms in the Green's functions of free theories. Since they originate from ultraviolet divergences they do not depend on μ, and their nonanalytic behaviors $\sim \log(p^2/\mu_{\overline{\text{MS}}}^2)$ produce the free theory power tails in the spectral functions.
[3]Examples of equivalent regulators include a sharp momentum cutoff on $\vec{p}$ and dimensional regularization.
[4]A heuristic way to understand this fact is by analogy to the situation in the band theory of metals, in which completely filled bands do not conduct but only partially filled bands do: the field-theoretic vacuum is akin to a completely filled band.

$$T_g^{\mu\nu} = \frac{1}{g^2}\left[G^{\mu\alpha}G^\nu{}_\alpha - \frac{\delta^{\mu\nu}}{4}G^2\right], \tag{3.1a}$$

$$T_f^{\mu\nu} = \sum_i \bar\psi_i \frac{-i\overleftrightarrow{D}^\mu \gamma^\nu + -i\overleftrightarrow{D}^\nu \gamma^\mu}{4}\psi_i - [\text{trace part}], \tag{3.1b}$$

$$\mathcal{O}_m = -im\sum_i \bar\psi_i \psi_i. \tag{3.1c}$$

$T_g^{\mu\nu} + T_f^{\mu\nu}$ is the traceless part of the full stress tensor. We will also consider the trace of the stress tensor,

$$T^\mu{}_\mu = \frac{b_0 G^2}{32\pi^2} + [\text{fermion terms}], \tag{3.2}$$

where $b_0 = (\frac{11}{3}C_A - \frac{4}{3}n_F T_F)$ is the leading coefficient of the β function ($\beta(\alpha_s) \approx -b_0\alpha_s^2/2\pi$).

B. Euclidean OPE coefficients

At the leading order in perturbation theory the OPE of currents $J^\mu = \sum_i \bar\psi_i\gamma^\mu\psi_i$ is given by the first diagram of Fig. 1. A pedagogical introduction to this sort of calculation can be found in [12]. Up to dimension-four operators we find

$$J^\mu(q)J^\nu \sim \sum_i \bar\psi_i\left[\gamma^\mu \frac{1}{\slashed{q} - i\slashed{D} - im}\gamma^\nu\right.$$
$$\left. - \gamma^\nu \frac{1}{\slashed{q} + i\slashed{D} + im}\gamma^\mu\right]\psi_i$$
$$\sim \frac{2\epsilon^{\mu\nu\alpha\beta}q^\alpha}{q^2}\sum_i \bar\psi_i\gamma^\beta\gamma_5\psi_i - \frac{2(\delta^{\mu\nu} - \frac{q^\mu q^\nu}{q^2})}{q^2}\mathcal{O}_m$$
$$+ \frac{4}{q^4}T_f^{\alpha\beta}[\delta^{\mu\alpha}\delta^{\nu\beta}q^2 - \delta^{\nu\beta}q^\mu q^\alpha - \delta^{\mu\beta}q^\nu q^\alpha$$
$$+ \delta^{\mu\nu}q^\alpha q^\beta], \tag{3.3}$$

with $\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$ and the covariant derivative acting to its right on ψ. We will drop the first term (involving the axial current $\bar\psi_i\gamma^\beta\gamma_5\psi_i$) since it does not contribute to spectral functions, being scale invariant.

In isotropic media we decompose the current correlator into transverse and longitudinal components: we let $q = (q^4, 0, 0, |\vec{q}|)$, set $G^T = \langle J^1 J^1\rangle$, $G^L = \frac{q^2}{(q^0)^2}\langle J^3 J^3\rangle$ and employ $T_{f,g}^{ij} = -\frac{1}{3}\delta^{ij}T_{f,g}^{44}$ for the traceless operators. Equation (3.3) then gives

$$G^T(q) \sim \frac{8}{3q^4}(q_4^2 - \vec{q}^2)T_f^{44} - \frac{2}{q^2}\mathcal{O}_m, \tag{3.4a}$$

$$G^L(q) \sim \frac{8}{3q^2}T_f^{44} - \frac{2}{q^2}\mathcal{O}_m. \tag{3.4b}$$
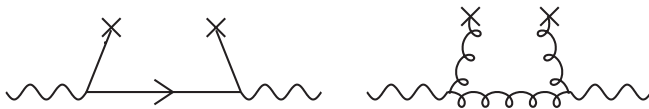

FIG. 1. Feynman diagrams giving the leading order OPE coefficients for currents and stress tensors.

Similarly, the OPE of $\mathcal{O}_m$ (scalar channel) is,

$$G^S(q) \sim \frac{3m^2}{q^2}\mathcal{O}_m + \frac{4m^2}{q^4}q_\alpha q_\beta T_f^{\alpha\beta} + \ldots; \tag{3.4c}$$

that of the trace anomaly T^μ_μ (bulk viscosity channel) is

$$G^\zeta(q) \sim b_0^2\alpha_s^2\left(4\frac{q_\mu q_\nu}{q^2}T_g^{\mu\nu} + \frac{G^2}{g^2}\right); \tag{3.4d}$$

and that of the shear mode T^{12} of the stress tensor, assuming isotropy, is

$$G^\eta(q) \sim \frac{2}{3q^2}(q_4^2 - \vec{q}^2)T_g^{44} + \frac{1}{6}\frac{G^2}{g^2}. \tag{3.4e}$$

The fermion contributions to Eqs. (3.4d) and (3.4e) begin at dimension-six and have been dropped. However, we must keep in mind that the OPE, like any two-point function, is really defined only modulo contact terms (terms purely polynomial in momenta).

C. Contact terms

The OPE coefficients of G^2 in Eqs. (3.4d) and (3.4e) are purely polynomial in q. According to the discussion at the end of Sec. II B, this means we have to decide whether the operators get evaluated at the scale $\mu_{\overline{\text{MS}}}$ or ω; alternatively, contact terms depending only on $\mu_{\overline{\text{MS}}}$ could be freely shifted in and out of the OPE as just mentioned.

In the shear channel Eq. (3.4e) it turns out that, had we computed the OPE for the general product $T^{\mu\nu}T^{\alpha\beta}$, we would have found that the coefficient of G^2 is purely polynomial and nontransverse (e.g., leading to $q_\mu T^{\mu\nu}T^{\alpha\beta} \neq 0$). Evaluating this operator at the scale ω would lead to a nontransverse spectral function, which is impossible. Therefore, the G^2 term in Eq. (3.4e) must be a pure contact term that runs with the scale $\mu_{\overline{\text{MS}}}$ and does not contribute to spectral functions. This issue is discussed in [13].

A similar ambiguity makes it possible to shift the coefficients of T_g^{44} and T_f^{44} in Eq. (3.4e) by p-independent constants. Although we believe that this could, in principle, be settled by studying the Ward identities obeyed by the full OPE of $T^{\mu\nu}T^{\alpha\beta}$, as in the above paragraph, this will not be done in this paper. This will translate in an indeterminacy for our shear channel spectral function.

The Ward identities are much harder to exploit in the bulk channel because the trace T^μ_μ is subleading at weak coupling. Thus it seems hard to determine, without an explicit calculation of running coupling effects, at which scale g^2G^2 is to be evaluated in this channel. Such a calculation will not be attempted in this work, so we will only be able to determine the asymptotics in this channel modulo g^2G^2. The term proportional to $q_\mu q_\nu T_g^{\mu\nu}/q^2$ in Eq. (3.4d) is unambiguous, however, since the Lorentz covariance of the OPE forbids the addition of spin-2 contact terms to spin-0 operator products.

D. Anomalous dimensions

The anomalous dimensions matrix of dimension-four, spin-two operators reads [14], acting on the basis $(T_g^{\mu\nu}, T_f^{\mu\nu})^T$,

$$\gamma = \frac{\alpha_s}{3\pi}\begin{pmatrix} 2n_F T_F & -4C_F \\ -2n_F T_F & 4C_F \end{pmatrix} + \mathcal{O}(\alpha_s^2), \qquad (3.5)$$

giving, e.g., the μ dependence of $T_f^{\mu\nu}$ in Eq. (3.4):

$$T_f^{\mu\nu}(\mu) \sim T_f^{\mu\nu}(\mu_0) + \frac{\alpha_s \log\frac{\mu_0^2}{\mu^2}}{3\pi}[2C_F T_f^{\mu\nu} - n_F T_F T_g^{\mu\nu}]$$
$$(3.6)$$

up to terms of order α_s^2.

The combination $\mathcal{O}_m$ does not run in perturbation theory, but the bare mass m has μ dependence given by $\gamma_m = 3\alpha_s C_F/2\pi + \mathcal{O}(\alpha_s^2)$, which must be included wherever it appears explicitly.

In theories with massless quarks, G^2 does not run at one-loop (at this order $G^2 \sim T^\mu{}_\mu$, which does not run to any order), though explicit powers of α_s do run according to $\gamma_{\alpha_s} = -\beta(\alpha_s)/\alpha_s = b_0\alpha_s/2\pi$.

E. Spectral functions

Frequency-dependent logarithms in G_E are determined by the RGE of the operators entering the right-hand side of the OPE, Eqs. (3.4), $\log(1/\mu^2) \to \log(1/\omega_E^2)$. To obtain ρ we take twice the imaginary part at real ω [or employ Eq. (2.2)], $\log(1/\omega_E^2) \to 2\pi$. To help clarify our conventions we recall the leading order, massless, zero-temperature results, with $q^2 = q_0^2 - \vec{q}^2$: $\rho^T(q)|^{vac} = \rho^L(q)|^{vac} = \frac{n_F d_F q^2}{6\pi}$, $\frac{\rho^S(q)}{m^2}|^{vac} = \frac{n_F d_F q^2}{4\pi}$, $\rho^\zeta(q)|^{vac} = \frac{b_0^2\alpha_s^2 d_A q^4}{128\pi^3}$, $\rho^\eta(q)|^{vac} = \frac{q^4}{80\pi}[d_A + \frac{1}{2}n_F d_F]$ [7]. This way we find the leading (dimension-four) thermal corrections:

$$\delta\rho^T(q) \sim \frac{16\alpha_s}{9q^2}\frac{q_0^2 + \vec{q}^2}{q^2}[2C_F T_f^{00} - n_F T_F T_g^{00}], \qquad (3.7a)$$

$$\delta\rho^L(q) \sim \frac{16\alpha_s}{9q^2}[2C_F T_f^{00} - n_F T_F T_g^{00}], \qquad (3.7b)$$

$$\delta\rho^S(q) \sim \frac{8\alpha_s m^2}{3q^2}\frac{q_\mu q_\nu}{q^2}\left[\frac{13}{2}C_F T_f^{\mu\nu} - n_F T_F T_g^{\mu\nu}\right]$$
$$- \frac{9\alpha_s m^2 C_F}{q^2}\mathcal{O}_m, \qquad (3.7c)$$

$$\delta\rho^\zeta(q) \sim \frac{b_0^2\alpha_s^3}{6\pi^2}\frac{q_\mu q_\nu}{q^2}\left[2C_F T_f^{\mu\nu} - \left(n_F T_F + \frac{3}{2}b_0\right)T_g^{\mu\nu}\right]$$
$$- C\frac{b_0^2\alpha_s^2}{4\pi}T^\mu{}_\mu, \qquad (3.7d)$$

$$\delta\rho^\eta(q) \sim \frac{4\alpha_s}{9}\left(D + \frac{2\vec{q}^2}{q^2}\right)[2C_F T_f^{00} - n_F T_F T_g^{00}]. \qquad (3.7e)$$

The presently undetermined coefficients C and D are due to the ambiguities discussed in Sec. III C, with $C = D = 1$

corresponding to choosing the renormalization scale ω in Eqs. (3.4d) and (3.4e). C and D are in principle both computable by more accurate calculations. For simplicity, Eq. (3.7d) accounts for the running of G^2 only for massless fermions, in general this operator can mix with $\mathcal{O}_m$. Equations (3.7c) and (3.7d) do not assume isotropy and the anisotropic case of Eqs. (3.7a) and (3.7b) may be obtained starting from Eq. (3.3). We note that $\delta\rho^\eta = 0$ when $n_F = 0$.

The operators in Eq. (3.7) are all Minkowski-signature operators, with the energy density being $\mathcal{E} = T^{00} = -T^{44}$. Please also note that Eqs. (3.7) are written in $(+ - --)$ metric convention, so both q^2 and $T^\mu{}_\mu$ have opposite sign relative to there Euclidean cousins [e.g., $T^\mu{}_\mu = \mathcal{E} - 3\mathcal{P}$ in Eq. (3.7)]. In Minkowski space with fermionic action $\sum_i \bar\psi_i(\not{p} - m)\psi_i$, $\mathcal{O}_m = m\sum_i \bar\psi_i\psi_i$.

At the Stefan-Boltzmann (free) level,

$$\langle T_g^{00}\rangle = \frac{\pi^2 T^4}{15}d_A, \qquad (3.8a)$$

$$\langle T_f^{00}\rangle = \frac{7\pi^2 T^4}{60}n_F d_F, \qquad (3.8b)$$

$$\langle \mathcal{O}_m\rangle = \frac{m^2 T^2}{3}n_F d_F + \mathcal{O}(m^4). \qquad (3.8c)$$

Equations (3.7), together with (3.8), constitute the main results of this paper.

IV. DISCUSSION

A. The photon spectral function

One motivation for doing the present calculations was to resolve, in a logically independent way, a discrepancy in the literature regarding the asymptotics of $\delta\rho^T(E)$. This (in fact the complete spectral function for $\omega \gg gT$ at zero spatial momentum) was computed a long time ago [4,15] to order α_s (two-loop order), and it was observed that the correction was proportional to g^2T^4/ω^2 at large energies ω. In contrast, a more recent calculation [16] instead observed a g^2T^2 behavior.

The OPE analysis presented in this paper makes it clear that the dominant thermal effects must scale like g^2T^4/ω^2, since the lowest dimension of a (gauge-invariant) local operator with a nontrivial anomalous dimensions in QCD is 4. In particular, a g^2T^2 asymptotic behavior is forbidden by the absence of local dimension-two operators. Thus we can confirm (at least qualitatively) the early findings [4,15]. It is difficult here, however, because of the different techniques employed, to comment explicitly on the careful calculation of [16].[5]

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[5]We do nevertheless agree with the main conclusion of [16], which was to rule out the infrared divergences claimed in [17] (see also the reply [18]).

In the Appendix, we reproduce the OPE result Eq. (4.1) by means of a more standard diagrammatic calculation in real-time perturbation theory.

For ρ^T at $\vec{q} = 0$ we find (upon reinstating the well-known $T = 0$ result, not computed here):

$$\rho^T(\omega) \sim \frac{n_{\mathrm{F}} d_{\mathrm{F}} \omega^2}{6\pi} \left[1 + \frac{3\alpha_{\mathrm{s}} C_{\mathrm{F}}}{4\pi} + \frac{16\pi^3 \alpha_{\mathrm{s}} C_{\mathrm{F}}}{9} \frac{T^4}{\omega^4} \right.$$
$$\left. + \mathcal{O}(\alpha_{\mathrm{s}}^2, T^6) \right]. \tag{4.1}$$

Interestingly, the correction, though parametrically small $\sim T^4/\omega^4$ at high energies, has a large numerical prefactor, suggesting that it could be useful down to not-so-large frequencies $\omega \gtrsim T$. Comparison with the complete two-loop calculation [4,15] should allow a precise determination of the regime of applicability of the OPE, which will not be pursued here.

B. Massive fermions ($m \gg T$)

Spectral functions of massive particles (with $m \gg T$) at $\vec{q} = 0$ have been considered in [19]. In this case the fermionic condensates $T_f^{\mu\nu}$ and $\mathcal{O}_m$ do not contribute, and our results Eqs. (3.7) for the thermal corrections [e.g., ignoring $\mathcal{O}(\alpha_{\mathrm{s}})$ vacuum corrections] become $\rho^T(q) \sim \frac{d_{\mathrm{F}} q^2}{6\pi} [1 - \frac{32\pi^3 \alpha_{\mathrm{s}} C_{\mathrm{F}}}{45} \frac{T^4}{q^4} + \ldots]$ and $\rho^S(q) \sim \frac{d_{\mathrm{F}} m^2 q^2}{4\pi} \times [1 - \frac{32\pi^3 \alpha_{\mathrm{s}} C_{\mathrm{F}}}{45} \frac{T^4}{q^4} + \ldots]$, in complete agreement with the findings[6] of [19].

C. Infrared divergences

The OPE analysis completely determines the cancellation pattern of infrared divergences at high energies: the OPE coefficients contain only, by definition, infrared safe zero-temperature physics. Infrared sensitivity can enter only through the expectation values of the local operators.

The leading order corrections to spectral functions ρ are proportional to g^2 anomalous dimensions times tree-level expectation values of dimension-four operators. The perturbative series for such expectation values should be similar to that for the thermodynamic pressure [20]. This implies that sensitivity of ρ to the gT scale (and the necessity for hard thermal loop resummation) will first enter at order $g^5 T^4/\omega^2$, and that nonperturbative physics associated with the $g^2 T$ scale (the so-called Linde problem [21]) will begin to contribute at order $g^8 T^4/\omega^2$.

This situation should be contrasted with that for the shape of the spectral function at $\omega \lesssim T$, for which infrared sensitivity shows up much earlier. At $\omega \sim T$, gT-scale physics appears at order g^3 [22], whereas at $\omega \sim gT$ it is

---

[6]To extract the complete asymptotics of [19] we observe that the mass subtraction (4.3) must be undone from their final results (4.7) and (C.11). In their notation, at order $1/\omega^2$ this means adding $\frac{g^2 d_{\mathrm{F}} C_{\mathrm{F}} M^2 T^2}{2\pi \omega^2}$ to (4.7) and $\frac{g^2 d_{\mathrm{F}} C_{\mathrm{F}} M^2}{8\pi}(-T^2 + 2M^2 T^2/\omega^2)$ to (C.11).

important already at the $\mathcal{O}(1)$ level [23]. Various integrals over the spectral functions, in the spirit of the famous $T = 0$ sum rules [24], however, are still governed by the Euclidean OPE, which becomes sensitive to the gT scale only at order g^3 and to the $g^2 T$ scale at order g^6. This seems to constrain, to a large extent, the corrections to the shape of ρ to only move things around in frequency space.

D. Bulk viscosity sum rules

A priori, the $\mathcal{O}(1)$ power tails in G^ζ, Eq. (3.7d), might appear sufficiently strong to make even the difference $\delta G_{\mathrm{E}}^\zeta = G_{\mathrm{E}}^\zeta - G_{\mathrm{E}}^{\zeta\mathrm{vac}}$, between the thermal Euclidean correlator of $T^\mu{}_\mu$ and its vacuum limit, require a subtracted dispersive representation. However, for asymptotically free theories, the RGE invariance of G^ζ forces us to evaluate the factors of g^2 in Eq. (3.7d) at the scale ω, in which case $\rho(\omega) \sim 1/\log^2 \omega$ at worse and an unsubtracted dispersion relation for $\delta G_{\mathrm{E}}(0)$ converges:

$$\delta G_{\mathrm{E}}^\zeta(0) = \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi\omega'} \delta\rho^\zeta(\omega'). \tag{4.2}$$

Higher order corrections to OPE coefficients will be suppressed by powers of $g^2(\omega') \sim 1/\log\omega'$ and will not affect convergence. We are not making any assumption here about the value of the coupling constant at the scale T, only the scale ω' is important to the OPE coefficients.

On the other hand, in [25] the Euclidean correlator $\delta \tilde{G}_{\mathrm{E}}^\zeta(0) = \lim_{q \to 0} \lim_{\omega \to 0} \delta G_{\mathrm{R}}(\omega, q)$ is evaluated, by means of broken scale invariance Ward identities ("low energy theorems") [25],

$$\delta \tilde{G}_{\mathrm{E}}^\zeta(0) = \left(T \frac{\partial}{\partial T} - 4 \right)(\mathcal{E} - 3\mathcal{P})$$
$$= (\mathcal{E} + \mathcal{P})\left(\frac{1}{c_{\mathrm{s}}^2} - 3 \right) - 4(\mathcal{E} - 3\mathcal{P}), \tag{4.3}$$

with $\mathcal{E} = T^{00}$ and $\mathcal{P} = T^{11}$.

Equation (4.3) together with the exact sum rule Eq. (4.2) were used recently in [5,6] to obtain information on the bulk viscosity $\zeta = \frac{1}{18} \lim_{\omega \to 0} \rho(\omega)/\omega$ near the QCD phase transition.

It is not our goal here to discuss the equality of the left-hand sides of Eqs. (4.2) and (4.3) nor possible contact terms to be added to the right-hand side of Eq. (4.2); this is discussed further by Romatschke and Son [26]. However, since the results of [5,6] were based on the assumption that Eq. (4.2) is saturated by low ω (together with an Ansatz for the shape of the spectral function) which is clearly in tension with the existence of the tail Eq. (3.7d), we would like to investigate the importance of this tail.

Let us thus try to estimate the contribution from the ultraviolet region $\omega \gtrsim \omega_{\mathrm{min}}$ in the pure glue theory. Setting $\alpha_{\mathrm{s}}(\omega) = 2\pi/b_0 \log(\omega/\Lambda_{\mathrm{QCD}})$ in Eq. (3.7d) yields

$$\delta G_E^\zeta(0)_{\rm UV} \approx \int_{\omega_{\rm min}}^\infty \frac{d\omega}{\pi\omega}\left[\frac{-C\pi T_\mu^\mu}{\log^2\frac{\omega}{\Lambda_{\rm QCD}}} - \frac{2\pi T^{00}}{\log^3\frac{\omega}{\Lambda_{\rm QCD}}}\right]$$

$$= C\frac{(3\mathcal{P}-\mathcal{E})}{\log\frac{\omega_{\rm min}}{\Lambda_{\rm QCD}}} - \frac{\mathcal{E}}{\log^2\frac{\omega_{\rm min}}{\Lambda_{\rm QCD}}}. \qquad (4.4)$$

The logarithms in Eq. (4.4) are never particularly large: setting $\omega_{\rm min} = 2\pi T$ we estimate $1/\log(\omega_{\rm min}/\Lambda_{\rm QCD}) \approx b_0\alpha_s(2\pi T)/2\pi \approx 0.4$ with $b_0 = 9$ in $n_F = 3$ QCD and $\alpha_s = 0.3$. Thus we conclude that, at least in the pure glue theory, whenever Eq. (4.3) is not parametrically large (e.g., $1/c_s^2$ large) compared to $0.4\mathcal{E}$, the sum rule Eq. (4.2) is very much sensitive to the ultraviolet tail and is not a clean probe of the $\omega \lesssim \omega_{\rm min}$ region. It seems that this could affect the analysis of [5,6], at least away from very close to the phase transition.

The weak-coupling limit of Eq. (4.2) is particularly interesting: its left-hand side is of order g^6T^4 (g being evaluated at the scale T from now on) whereas its right-hand side receives a contribution of order g^4T^4 from the $\omega \sim T$ region [27],

$$\int_{\omega\sim T}\frac{d\omega}{\pi\omega}\delta\rho(\omega) \approx \int_0^\infty \frac{d\omega}{\pi\omega}\frac{d_A b_0^2\alpha_s^2\omega^4}{64\pi^3}n_B\left(\frac{\omega}{2}\right)$$

$$= \frac{d_A b_0^2\alpha_s^2 T^4}{60}. \qquad (4.5)$$

At order g^4T^4 there is another contribution: from the $\sim g^6T^4$ ultraviolet tail integrated over a $\sim 1/g^2$ logarithmic range. In the weak-coupling limit this contribution is well separated and is given by Eq. (4.4) evaluated at $\omega_{\rm min} \sim T$. The dominant term is the second term, $-\mathcal{E}b_0^2\alpha_s^2/4\pi^2$, which, remarkably, using Eq. (3.8a) for $\mathcal{E}$, is seen to exactly cancel Eq. (4.5). The presently undetermined coefficient C would only become important at order g^6T^4. Thus, the sum rule Eq. (4.2) is obeyed at order g^4T^4 but only when the ultraviolet tail is included. This resolves a puzzle raised in [27].

E. Shear viscosity channel and discontinuity at $g^2 \to 0$

From Eq. (3.7e), the asymptotics of the thermal correction $\delta\rho^\eta$ are proportional to g^2 times an operator of strictly positive anomalous dimension γ. Upon resummation of logarithms its behavior will be proportional to $(\log\omega)^{-1-a}$ (the 1 coming from the running of g^2 in an asymptotically free theory and $a > 0$ coming from the anomalous dimension): thus the unsubtracted dispersive integral Eq. (2.3) converges. As in the preceding subsection, higher order corrections will not modify this result because the theory is asymptotically free. The integrals should also converge in conformal theories such as the $\mathcal{N} = 4$ super Yang-Mills theory, because in this case all tails are associated with operators of strictly positive anomalous dimensions γ, and decay like $\omega^{-\gamma}$.

Since the dispersive integral vanishes at $\omega_E \to \infty$ we can write, in general,

$$\delta G_E^\eta(\omega_E) = \delta G_E^\eta(\infty) + \int_{-\infty}^\infty \frac{d\omega'}{2\pi(\omega' - i\omega_E)}\delta\rho^\eta(\omega').$$

$$(4.6)$$

Note that convergence of the integral implies that δG_E^η approaches a constant at infinity, so that $\delta G_E^\eta(\infty)$ is well defined. According to [26], the left-hand side at $\omega_E = 0$ is determined by hydrodynamical considerations. In asymptotically free theories we believe that the OPE coefficients of $\delta G_E^\eta(\infty)$ are saturated by a one-loop computation.[7] Knowledge of both of these ingredients should yield interesting exact sum rules, involving, at most, the pressure, energy density and chiral condensates of QCD. We hope to return to this question in the future.

Here we wish only to discuss a possible discontinuity in the free theory limit $g^2 \to 0$ of individual terms on the right-hand side of Eq. (4.6), if the undetermined coefficient D in Eq. (3.7e) turns out to be nonzero. Consider the term T_g^{00} on the right-hand side of the Euclidean OPE Eq. (3.4e). At $g = 0$ and $\omega = \infty$ it contributes a finite amount T_g^{00} to $\delta G_E(\infty)$, which discontinuously changes to $(T_g^{00} + T_f^{00})/(1 + n_F T_F/2C_F)$ at any small but finite coupling due to running, Eq. (3.5). On the other hand, at any finite but small coupling one has the $\mathcal{O}(g^2)$ tail Eq. (3.7e) in the spectral function, which is to be integrated over a $\mathcal{O}(1/g^2)$ logarithmic range similarly to the preceding subsection. Its contribution is thus $\mathcal{O}(1)$. It is easy to convince oneself that it exactly compensates for the discontinuity in $G_E(\infty)$.

Thus it might happen that equations such as Eq. (4.6) are only continuous at $g^2 = 0$ when both terms on the right-hand side are included. We hope to return in future work to shear channel sum rules in QCD and in other theories, and to the question of whether they actually contain strong ultraviolet tails. In pure Yang-Mills theory, the coefficient D is irrelevant and at the leading order such tails are absent.

F. Strongly coupled $\mathcal{N} = 4$ super Yang-Mills theory

The operator spectrum of strongly coupled multicolor $\mathcal{N} = 4$ super Yang-Mills (SYM) theory has the very peculiar property, that the only operators which do not develop large anomalous dimensions $\sim\lambda^{1/4}$ are protected by supersymmetry and have strictly vanishing anomalous dimensions [28], where $\lambda = g^2 N_c$ is the 't Hooft coupling.

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[7]Because other twist-two operators acquire positive anomalous dimensions, only three operators can appear in $\delta G_E^\eta(\infty)$: the traceless and trace part of the full stress tensor $T^{\mu\nu}$, and $\mathcal{O}_m$. The coefficient of T^μ_μ/g^2 vanishes at tree level [13] but a one-loop computation is needed to find that of T^μ_μ. The coefficients of twist-two operators and of $\mathcal{O}_m$ are determined at the tree level, but we believe a one-loop anomalous dimension matrix is necessary to carefully separate the total $T^{\mu\nu}$ from other twist-two operators, and $\mathcal{O}_m$ from the trace T^μ_μ.

There are no small nontrivial anomalous dimensions in this theory, even at finite λ.

One way to find power corrections in spectral functions is, as discussed in Sec. II A, if the Euclidean OPE coefficients have wrong phases. This would certainly seem peculiar but we have no general argument against this possibility. However, the results of [29] suggest that the OPE coefficients of protected operators in $\mathcal{N} = 4$ SYM theory are identical to those of the free theory, for which this definitely does not happen.

Thus we will assume that power corrections to spectral functions at high frequencies are associated with nonanalytic terms in Euclidean correlators. Without anomalous dimensions the only sort of nonanalyticity allowed by the RGE (see Sec. II B) are polynomial terms in momenta, times single logarithms $\log(p^2/\mu_{\mathrm{MS}}^2)$. They lead to strictly polynomial terms in spectral functions.

By dimensional analysis and transversality these are strictly forbidden in the spectral functions of currents and stress tensors (except if they multiply the unit operator). Thus these spectral functions are strictly protected against medium-dependent power corrections. It would be interesting to see whether polynomial corrections actually occur in other spectral functions. At finite but large λ, we expect power tails $\sim \omega^{-n}$ with $n \sim \lambda^{1/4}$.

Thermal corrections to the spectral functions of R currents and stress tensors at strong coupling have been studied by Teaney [7] and observed, remarkably, to decay exponentially fast at high energies. This section generalizes his observation.

ACKNOWLEDGMENTS

APPENDIX: DIAGRAMMATIC EVALUATION OF ρ_μ^μ

This Appendix reproduces a calculation of the trace $\rho_\mu^\mu(p)$ of the current spectral function, using real-time Feynman diagrams. We work in the high-energy limit where $\rho = \Pi^>$ and drop all terms suppressed by Boltzmann factors $\propto \exp(-p^0/2T)$, but keep all power corrections. The aim is to confirm the OPE result, Eq. (3.7), for $-\Pi^{>\mu}{}_\mu = 2\rho^T + \rho^L$. For notational simplicity in this section we assume $n_{\mathrm{F}} = 1$.

1. Outline of calculation

The two-loop diagrams contributing to the Wightman self-energy $\Pi^>(q)$ are shown in Figs. 2 and 3. The use the cutting rule of Weldon [30], which expresses the

Wightman self-energy as a product of two *retarded* amplitudes separated by Wightman (on shell) propagators (depicted as the main cut in the figures). Its physical interpretation is as follows: the main cut sums over intermediate states, as is expected for a Wightman function, and the amplitudes are retarded because intermediate states are expanded in a basis of "in" states (e.g., free theory states defined at $t \to -\infty$).

To evaluate the retarded amplitudes on each side of the cut we use the so-called Schwinger-Keldysh (ra) formalism, as described in [31].[8] The resulting expressions are summarized graphically in the figures: the arrowed propagators are retarded propagators and the propagators with the double cut are the fluctuation functions of this formalism (G_{rr} propagators); vertices are as in ordinary zero-temperature perturbation theory (e.g., no complex conjugation appears).

Following the OPE philosophy we look for propagators which can become soft, $q \sim T \ll p$. Visual inspection reveals that no more than one propagator can ever become soft simultaneously: at least two hard particles must traverse the main cut, and to channel their hard momenta to the external legs in all cases requires at least two other hard propagators. Thus we will organize the calculation around the propagator which becomes soft.

At order α_s there is no need for HTL resummation and the retarded propagators are temperature-independent. The temperature dependence is due to the statistical factors entering the Wightman and rr propagators,

$$G_R^{\mu\nu}(p) = \frac{-i\delta^{\mu\nu}}{p^2 + i\epsilon p^0}, \qquad S_R(p) = \frac{i\not{p}}{p^2 + i\epsilon p^0}, \tag{A1a}$$

$$\delta G_{>,<,rr}^{\mu\nu}(p) = -\delta^{\mu\nu}\tilde{G}_{\mathrm{B}}(p), \qquad \delta S_{<,>,rr}(p) = -\not{p}\tilde{G}_{\mathrm{F}}(p), \tag{A1b}$$

with the vacuum cuts obeying $G_>(p) = 2\,\mathrm{Re}\,G_R(p)\theta(p^0)$. We will not use the explicit forms $\tilde{G}_{\mathrm{B,F}}(p) = 2\pi\delta(p^2)n_{\mathrm{B,F}}(|p^0|)$ until the end of the calculation; up to then the sole purpose of the Ansatzes Eqs. (A1b) is to simplify polarization sums. Our metric signature is $(+ - - -)$.

2. Gluon condensate

First we allow the gluon propagator in diagrams (a) and (b) of Fig. 2 to become soft. Upon evaluating the Dirac trace this contribution may be written
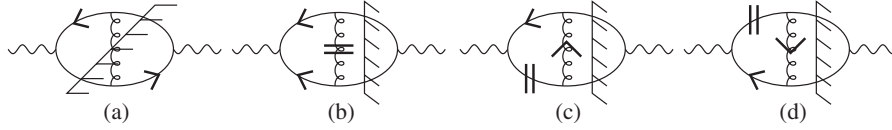

FIG. 2.   Real-time diagrams of first topology contributing to $\Pi^>$ (with the complex conjugate diagrams omitted). The arrows show the time flow along retarded propagators, not the charge flow; the doubly-dashed propagator is the fluctuation function G_{rr}.
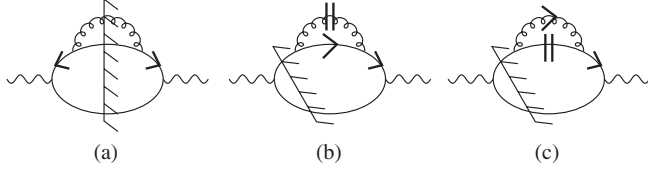

FIG. 3.   Real-time diagrams of second topology contributing to $\Pi^>$, in the notation of Fig. 2. Not shown, the complex conjugates to (b) and (c).

$$\frac{-\Pi^{>\mu}{}_\mu(p)}{g^2 C_F d_F} \supset 32 \int_k [\tilde{G}_{\rm B}(k) + \tilde{G}_{\rm B}(-k)]$$
$$\times \int_l 4\pi^2 \left(\frac{\delta(l^2)\delta(l_p^2)}{l_k^2 l_{pk}^2} + \frac{\delta(l^2)\delta(l_{pk}^2)}{l_k^2 l_p^2}\right) l$$
$$\cdot\, l_{pk} l_p \cdot l_k, \tag{A2}$$

where we have introduced the abbreviations $\int_l = \int \frac{d^4 l}{(2\pi)^4}$, $l_k = l - k$, $l_p = l - p$ and $l_{pk} = l - p - k$, to be used in all what follows, and $k \ll l \sim p$ is the soft momentum. It is kinematically impossible for two denominators in Eq. (A2) to vanish simultaneously and only the real part (e.g., principal value) of the propagators contributes. The l integration is Lorentz-covariant and becomes elementary in the rest frame that is singled out by the δ functions. Thus Eq. (A2) yields

$$\frac{1}{\pi} \int_k \tilde{G}_{\rm B}(k)\left[2 + \frac{(p^2 + k^2)^2}{2p \cdot k\Delta} \ln\left(\frac{1 - \frac{(p\cdot k+\Delta)^2}{p^2}}{1 - \frac{(p\cdot k-\Delta)^2}{p^2}}\right)\right.$$
$$\left. + \frac{p^2 + k^2}{\Delta} \ln\left(\frac{1 + \frac{p\cdot k+\Delta}{p^2}}{1 + \frac{p\cdot k-\Delta}{p^2}} \frac{1 + \frac{-p\cdot k+\Delta}{p^2}}{1 + \frac{-p\cdot k-\Delta}{p^2}}\right)\right], \tag{A3}$$

with $\Delta = \sqrt{(p \cdot k)^2 - p^2 k^2}$.

To evaluate the diagrams of the second topology, Fig. 3, without having to deal with ill-defined expressions such as $\delta(l^2)/l^2$ [which would appear in too literal an interpretation of diagrams (b)-(c)], we write their sum as a discontinuity,

$$\frac{-\Pi^{>\mu}{}_\mu(p)}{g^2 C_F d_F} \supset 16 \int_k [\tilde{G}_{\rm B}(k) + \tilde{G}_{\rm B}(-k)] \int_l 2\pi\delta(l_p^2)$$
$$\times 2\,{\rm Im}\,\frac{2l \cdot l_p l \cdot l_k - l^2 l_p \cdot l_k}{(l^2 + i\epsilon l^0)^2(l_k^2 + i\epsilon l_k^0)}, \tag{A4}$$

where we have also included the contribution with the self-

energy inserted on the lower propagator. In our kinematic regime the poles of the denominators are disjoint and occur at positive energies, $l^0 > 0$, $l_k^0 > 0$. The discontinuity across the squared propagator $1/(l^2)^2$ may be conveniently evaluated by integration by parts along $l_p^\mu \partial_{p^\mu}$, yielding the formula[9]:

$$2\,{\rm Im}\int_l \frac{\delta(l_p^2)F(l)}{(l^2 + i\epsilon l^0)^2} = \frac{1}{p^2} \int_l \delta(l_p^2) 2\pi\delta(l^2)\left(1 + l_p^\mu \frac{\partial}{\partial l^\mu}\right) F(l),$$
$$\tag{A5}$$

for $F(l)$ any function of l regular at $l^2 = 0$.

The total imaginary part in Eq. (A4) is the sum of that from Eq. (A5) and from that across the $1/l_k^2$ propagator, $2\,{\rm Im}\,1/(l_k^2 + i\epsilon l_k^0) = -2\pi\delta(l_k^2)$. Upon performing the l integration we obtain

$$\frac{1}{\pi} \int_k \tilde{G}_{\rm B}(k)\left[-4 + \frac{p \cdot k}{\Delta} \ln\left(\frac{1 - \frac{(p\cdot k+\Delta)^2}{p^2}}{1 - \frac{(p\cdot k-\Delta)^2}{p^2}}\right)\right]. \tag{A6}$$

Our final result for the sensitivity to the gluon distribution in the medium is the sum of Eqs. (A3) and (A6).

3. Fermion condensate

Letting the lower fermion propagator become soft in Fig. 2(a) and in its left-right flip, or the rr propagators with similar positions in 2(c) and its conjugate, gives a contribution:

$$\frac{-\Pi^{>\mu}{}_\mu(p)}{g^2 C_F d_F} \supset -64 \int_k \tilde{G}_{\rm F}(k) \int_l 4\pi^2\left[\frac{\delta(l^2)\delta(l_{pk}^2)}{l_p^2}\right.$$
$$\left. + \frac{\delta(l^2)\delta(l_p^2)}{l_{pk}^2}\right] \times \frac{l_p \cdot (p + k) l \cdot k}{(p + k)^2}$$
$$= \frac{1}{\pi} \int_k \tilde{G}_{\rm F}(k)\left[\frac{2p \cdot k}{(p + k)^2}\right.$$
$$\left. - \frac{(p^2 - k^2)(2k \cdot p + k^2)}{(p + k)^2 \Delta} \ln\frac{1 + \frac{p\cdot k+\Delta}{p^2}}{1 + \frac{p\cdot k-\Delta}{p^2}}\right], \tag{A7}$$

where our notation and techniques are as in the previous

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[9]An alternative way of deriving this result is to treat the self-energy insertion as a correction to the external states, in which case at this order one gets thermal mass shifts and wave function renormalization factors. See, for instance, [15].

subsection. Contributions in which the upper fermion propagators are allowed to become soft are similar, but with k replaced by $-k$; these will have to be included at the end.

The diagram of Fig. 3 receives a contribution from when the upper fermion propagator becomes soft,

$$
\frac{-\Pi^{>\mu}{}_{\mu}(p)}{g^2 C_F d_F} \supset -32 \operatorname{Im} \int_k \tilde{G}_F(k)
$$
$$
\times \int_l \frac{2\pi \delta(l_p^2)(l^2 l_p \cdot k - 2l \cdot k l \cdot l_p)}{(l^2 + i\epsilon l^0)^2 (l_k^2 + i\epsilon l_k^0)}
$$
$$
= \frac{1}{\pi} \int_k \tilde{G}_F(k) \left[-1 + \frac{-\frac{1}{2}p^2 + p \cdot k}{\Delta} \right.
$$
$$
\left. \times \ln\left(\frac{1 - \frac{p \cdot k + \Delta}{p^2}}{1 - \frac{p \cdot k - \Delta}{p^2}} \right) \right],
\tag{A8}
$$

and from when the lower fermion propagator becomes soft,

$$
\frac{-\Pi^{>\mu}{}_{\mu}(p)}{g^2 C_F d_F} \supset 16 \int_k \tilde{G}_F(k) \int_l \delta(l^2)\delta(l_{pk}^2)
$$
$$
\times \frac{k \cdot (p+k) - l \cdot k}{(p+k)^2}
$$
$$
= \frac{1}{\pi} \int_k \tilde{G}_F(k) \left[\frac{k \cdot (p+k)}{(p+k)^2} \right].
\tag{A9}
$$

The total sensitivity of $\Pi^{>\mu}{}_{\mu}$ to fermions moving in the medium is the sum of Eqs. (A7)–(A9) and the same objects with $(k \to -k)$.

4. Expansion in $1/p$

Each of the contributions, Eqs. (A3) and (A6)–(A9), is free of infrared divergences and is moreover *local* in k. That is, each admits a Taylor expansion in positive powers of $p \cdot k/p^2$ and Δ^2, as is readily verified from their parity under $\Delta \to -\Delta$.

This is the main point of this analysis: divergences and nonlocalities have canceled out in the sum over cuts, for each individual diagram. A Minkowski-space OPE thus works for each diagram. Furthermore, upon summing the diagrams, we find that the term of order $(p^2)^0$ cancels in the $1/p^2$ expansion of the sum of Eqs. (A3) and (A6) as required by gauge invariance, since this would correspond to a noninvariant $A_\mu A^\mu$ condensate: gauge invariance is restored upon summing diagrams.

The leading nontrivial term in the expansion arise at order $1/p^2$,

$$
\frac{-\Pi^{>\mu}{}_{\mu}(p)}{g^2 C_F d_F} \approx \frac{1}{\pi p^2} \int_k \left[\tilde{G}_F(k) \frac{8(p \cdot k)^2 + \frac{8}{3}\Delta^2}{p^2} \right.
$$
$$
\left. - \tilde{G}_B(k) \frac{2(p \cdot k)^2 + \frac{2}{3}\Delta^2}{p^2} + \mathcal{O}\left(\frac{k^4}{p^2}\right) \right]
$$
$$
\simeq \frac{4 p_\mu p_\nu}{3\pi p^4} \left[\frac{2}{d_F} \langle T_f^{\mu\nu} \rangle - \frac{1}{d_A} \langle T_g^{\mu\nu} \rangle \right],
\tag{A10}
$$

in complete agreement with the OPE result for $(2\rho^T + \rho^L)(p)$ when $n_F = 1$, Eqs. (3.7).

[1] K. G. Wilson, Phys. Rev. **179**, 1499 (1969); K. G. Wilson and W. Zimmermann, Commun. Math. Phys. **24**, 87 (1972).

[2] K. Adcox *et al.* (PHENIX Collaboration), Nucl. Phys. **A757**, 184 (2005); B. B. Back *et al.*, Nucl. Phys. **A757**, 28 (2005); I. Arsene *et al.* (BRAHMS Collaboration), Nucl. Phys. **A757**, 1 (2005); J. Adams *et al.* (STAR Collaboration), Nucl. Phys. **A757**, 102 (2005).

[3] U. W. Heinz, arXiv:0901.4355; See also, D. Teaney, Phys. Rev. C **68**, 034913 (2003); P. Romatschke and U. Romatschke, Phys. Rev. Lett. **99**, 172301 (2007).

[4] T. Altherr and P. Aurenche, Z. Phys. C **45**, 99 (1989); R. Baier, B. Pire, and D. Schiff, Phys. Rev. D **38**, 2814 (1988).

[5] D. Kharzeev and K. Tuchin, J. High Energy Phys. 09 (2008) 093.

[6] F. Karsch, D. Kharzeev, and K. Tuchin, Phys. Lett. B **663**, 217 (2008).

[7] D. Teaney, Phys. Rev. D **74**, 045025 (2006).

[8] J. I. Kapusta and C. Gale, *Finite-temperature field theory: Principles and applications* (Univ. Pr., Cambridge, UK, 2006), p. 428.

[9] M. A. Shifman, arXiv:hep-ph/0009131.

[10] V. A. Novikov, M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, Yad. Fiz. **41**, 1063 (1985) [Nucl. Phys. **B249**, 445 (1985)].

[11] E. Braaten and R. D. Pisarski, Nucl. Phys. **B337**, 569 (1990); J. C. Taylor and S. M. H. Wong, Nucl. Phys. **B346**, 115 (1990); E. Braaten and R. D. Pisarski, Phys. Rev. D **45**, R1827 (1992).

[12] V. A. Novikov, M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, Fortschr. Phys. **32**, 585 (1984); reprinted in *Vacuum Structure and QCD Sum Rules*, edited by M. A. Shifman (North-Holland, Amsterdam, 1992).

[13] V. A. Novikov, M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, Nucl. Phys. **B174**, 378 (1980).

[14] M. E. Peskin and D. V. Schroeder, *An Introduction To Quantum Field Theory* (Addison-Wesley, Reading, MA, 1995).

[15] Y. Gabellini, T. Grandou, and D. Poizat, Ann. Phys. (N.Y.) **202**, 436 (1990).

[16] A. Majumder and C. Gale, Phys. Rev. C **65**, 055203 (2002).

[17] J. I. Kapusta and S. M. H. Wong, Phys. Rev. D **62**, 037301

(2000).

[18] P. Aurenche *et al.*, Phys. Rev. D **65**, 038501 (2002).

[19] Y. Burnier, M. Laine, and M. Vepsalainen, J. High Energy Phys. 02 (2009) 008.

[20] E. Braaten and A. Nieto, Phys. Rev. D **53**, 3421 (1996); K. Kajantie, M. Laine, K. Rummukainen, and Y. Schroder, Phys. Rev. D **67**, 105008 (2003).

[21] A. D. Linde, Phys. Lett. **96B**, 289 (1980).

[22] P. Aurenche, M. E. Carrington, and N. Marchal, Phys. Rev. D **68**, 056001 (2003).

[23] E. Braaten, R. D. Pisarski, and T. C. Yuan, Phys. Rev. Lett. **64**, 2242 (1990).

[24] M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, Nucl. Phys. **B147**, 385 (1979); **B147**, 448 (1979).

[25] P. J. Ellis, J. I. Kapusta, and H. B. Tang, Phys. Lett. B **443**, 63 (1998); I. A. Shushpanov, J. I. Kapusta, and P. J. Ellis, Phys. Rev. C **59**, 2931 (1999).

[26] P. Romatschke and D. T. Son, arXiv:0903.3946.

[27] G. D. Moore and O. Saremi, J. High Energy Phys. 09 (2008) 015.

[28] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri, and Y. Oz, Phys. Rep. **323**, 183 (2000).

[29] S. Lee, S. Minwalla, M. Rangamani, and N. Seiberg, Adv. Theor. Math. Phys. **2**, 697 (1998).

[30] H. A. Weldon, Phys. Rev. D **28**, 2007 (1983).

[31] K. C. Chou, Z. B. Su, B. L. Hao, and L. Yu, Phys. Rep. **118**, 1 (1985). See also, M. v. Eijck, R. Kobes, and Ch. v. Weert, Phys. Rev. D **50**, 4097 (1994).

[32] T. S. Evans, Nucl. Phys. **B374**, 340 (1992); F. Guerin, Nucl. Phys. **B432**, 281 (1994).