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Minimal uncertainty in momentum: The effects of IR gravity on quantum mechanics

M. Zarei[1] and B. Mirza[1,2]

[1]*Department of Physics, Isfahan University of Technology, Isfahan 84156-83111, Iran*
[2]*Institute of Theoretical Science, University of Oregon, Eugene, Oregon 97403-5203, USA*
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The effects of the IR aspects of gravity on quantum mechanics is investigated. At large distances where, due to gravity, the space-time is curved, there appears nonzero minimal uncertainty Δp_0 in the momentum of a quantum mechanical particle. We apply the minimal uncertainty momentum to some quantum mechanical interferometry examples and show that the phase shift depends on the area surrounded by the path of the test particle. We also put some limits on the related parameters. This prediction may be tested through future experiments. The assumption of minimal uncertainty in momentum can also explain the anomalous excess of the mass of the Cooper pair in a thin, rotating, superconductor ring.

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I. INTRODUCTION

The quantum mechanics (QM) on a Riemannian curved space-time background is the simplest part of the fundamental problem associated with general relativity and the quantum world. On the other hand, it is not possible to detect the classical gravitational effects on quantum mechanical experiments except during the interferometry experiments with neutrons, as discussed in [1]. In this paper we investigate another possibility for studying the effects of gravity on quantum mechanical systems using minimal uncertainty in momentum assumption. It is known that for large distances, where the curvature of space-time becomes important, there is no notion of a plane wave on a general curved space-time [2]. This means that there is a limit to the precision with which the corresponding momentum can be described. One can express this as a nonzero minimal uncertainty in momentum (MUM) measurement. The minimal uncertainty in momentum appears as an IR effect of gravity on a quantum mechanical system.

On the other hand, it has been discussed in the literature that a minimal uncertainty in position will be inevitable when a test particle tries to resolve short distances [2]. In order to probe the short distances of the order of Planck length, ℓ_P, test particles require very high energies. According to Einstein's equation, the gravity effects of high energy test particles must significantly disturb the space-time structure when probed. Because of this phenomenon, one expects a finite limit to the possible resolution of distances. Therefore, due to the UV effects of gravity, quantum mechanics experiences a minimal uncertainty in position. It has been shown that the minimal uncertainty in position can also be obtained by assuming a noncommutative space-time [3]. In one dimension, the minimal uncertainty in position and momentum can be generalized in quantum mechanics as follows [2]:

$$\Delta x \Delta p \geq \frac{\hbar}{2}(1 + \alpha(\Delta x)^2 + \beta(\Delta p)^2 + \gamma), \qquad (1)$$

where α, β, and γ are positive and independent of Δx and Δp. While in ordinary QM the uncertainty Δp can be made arbitrarily small by letting Δx grow correspondingly, this is no longer the case in the modified Heisenberg uncertainty relation (1). Equation (1) can be deduced from the commutation relation of the form

$$[\hat{x}_i, \hat{p}_j] = \frac{\hbar}{i}\delta_{ij}(1 + \alpha r^2 + \beta p^2), \qquad (2)$$

with $\gamma = \alpha\langle \mathbf{x}\rangle^2 + \beta\langle \mathbf{p}\rangle^2$ and $r^2 = \sum x_i^2$. In order to have a closure form for the commutation relations, Eq. (2) is extended to the following form:

$$[\hat{x}_i, \hat{x}_j] \neq 0, \qquad [\hat{p}_i, \hat{p}_j] \neq 0, \qquad (3)$$

where the commutator $[\hat{x}_i, \hat{x}_j] \neq 0$ implies a noncommutative structure for space-time.

The UV aspects of these UV/IR effects of gravity on quantum mechanics, i.e., minimal uncertainty in position, have been investigated widely in previous years [4]. Also, it has been shown recently that this may be testable in a STM device [5].

At first, we concentrate on the IR aspects of gravity and its effects on quantum mechanics. So, we assume that in (1) $\beta = 0$, and we study only the quantum mechanics with minimal uncertainty in momentum. The Heisenberg uncertainty principle has been verified experimentally for the fullerence molecules in [6]. We used the data generated in this work, fit the generalized relation (1) for $\beta = 0$ into these data, and found the following bound on α:

$$\sqrt{\alpha} \leq 3 \times 10^6\ \mathrm{m}^{-1}. \qquad (4)$$

Also, it is possible to apply the MUM assumption to other quantum mechanical examples for constraining the parameter α.

In order to construct an effective quantum theory and compare it with experimental data, similar to the minimal length theories [7], we redefine the $\hat{x}$ and $\hat{p}$ in terms of the coordinate and momentum which satisfy the usual Heisenberg uncertainty relation. Then the whole effect of the MUM is transferred to a new effective Schrödinger equation which has terms dependent on the parameter α. Therefore, we find an effective quantum mechanical theory with corrections coming from the MUM assumption.

The generalized Heisenberg algebra with nonzero α and β can be represented on a position space wave function $\psi(x) = \langle x|\psi\rangle$ by letting x and p act as operators.

$$\hat{p}_i \cdot \psi(x) = \frac{\hbar}{i}(1 + \alpha r^2)\partial_i \psi(x), \tag{5}$$

$$\hat{x}_i \cdot \psi(x) = x_i \psi(x). \tag{6}$$

Here, it must be noticed that the x's are noncommutative coordinates. Thus, in order to study each quantum mechanical problem with the minimal uncertainty in momentum, it is sufficient to modify the Schrödinger equation to a noncommutative one and replace the momentum operator with the modified one given in (5).

Here, we study the effects of minimal uncertainty momentum on well-understood quantum mechanic phenomena, such as the Aharonov-Bohm (AB) effect [8], the Aharonov-Casher (AC) effect [9], the COW effect [10], flux quantization in superconductors [11], a rotating superconducting quantum interference device (SQUID) [12], the Sagnac effect [13], the gravitational AB effect [14], and the hydrogen atom. It will be shown that MUM makes a universal area-dependent correction in all these phenomena. This correction disturbs the topological properties of these effects and may be observed by suitably constructed experiments.

II. THE AHARONOV-BOHM EFFECT

In 1959, Aharonov and Bohm proposed an experiment to explore the effects of the electromagnetic potential in the quantum domain [8]. The standard configuration considered was the interferometer pattern of the two slit diffraction experiments involving a magnetic flux enclosed by two charged particle beams to detect the phase shift. Excellent agreement was found between the measured phase shift and the theoretical prediction,

$$\delta\varphi_0 = \frac{q}{\hbar}\oint \mathbf{A}\cdot d\ell, \tag{7}$$

where q is the charge of the particle, $\mathbf{A}$ is the vector potential, and the contour encloses the magnetic flux of a solenoid. In this section, we calculate the modification in the phase shift due to MUM using quantum mechanics and semiclassical approaches.

A. The quantum mechanics approach

In noncommutative space the Schrödinger equation can be written as [15]

$$H \star |\psi\rangle = E|\psi\rangle, \tag{8}$$

where the $\star$ product is defined as

$$(\mathbf{A}\star\mathbf{B})(x) = \mathbf{A}(x_1)e^{(i/2)\theta^{ij}\partial_i^{(1)}\partial_j^{(2)}}\mathbf{B}(x_2)|_{x_1=x_2=x}. \tag{9}$$

The Schrödinger equation (8) in the presence of a vector potential is

$$H\star\psi = \frac{1}{2m}\mathcal{D}_j\star\mathcal{D}_j\star\psi = k_jk_j\psi, \tag{10}$$

where the variables k_j are the eigenvalues of the operators $\mathcal{D}_j = \hat{p}_j - qA_j$,

$$\mathcal{D}_j\star\psi = k_j\psi. \tag{11}$$

Here, A_j is the vector potential associated with magnetic field $\mathbf{B}$, q is the charge of the test particle, and the operator $\hat{p}$ is the α-dependent momentum operator defined in (5). One can solve this equation by choosing ψ as

$$\psi = e^{\Lambda}. \tag{12}$$

Now we assume that noncommutativity is small; then Eq. (11) can be solved by perturbative expansion of A_j and Λ as follows:

$$A_j = A_j^{(0)} + \theta A_j^{(1)} + \cdots, \tag{13}$$

$$\Lambda = \Lambda^{(0)} + \theta\Lambda^{(1)} + \cdots. \tag{14}$$

Then, in the two slit experiments the wave function ψ of the charged particle satisfies the following equation:

$$\begin{aligned}\mathcal{D}_j\star\psi &= -i\hbar(1+\alpha r^2)\partial_j e^{\Lambda} - qA_j\star e^{\Lambda}\\ &= e^{\Lambda}\Big[-i\hbar(1+\alpha r^2)\partial_j\Lambda - qA_j\\ &\quad + \frac{iq}{2}\theta^{lm}\partial_l A_j\partial_m\Lambda + \mathcal{O}(\alpha\theta,\theta^2)\Big]\\ &= k_j\psi.\end{aligned} \tag{15}$$

In order for the Schrödinger equation to be gauge invariant, Λ must satisfy the following equation:

$$-i\hbar(1+\alpha r^2)\partial_j\Lambda - qA_j + \frac{iq}{2}\theta^{lm}\partial_l A_j\partial_m\Lambda = k_j. \tag{16}$$

Now, for the first order of A_j and Λ, one finds

$$-i\hbar(1+\alpha r^2)\partial_j\Lambda^{(0)} - qA_j^{(0)} = k_j, \tag{17}$$

which can be solved by a simple computation,

$$\begin{aligned}\partial_j\Lambda^{(0)} &= \frac{iq}{\hbar}\frac{1}{1+\alpha r^2}(A_j^{(0)} + k_j)\\ &\approx \Big(\frac{iq}{\hbar} - \frac{iq\alpha}{\hbar}r^2\Big)(k_j + A_j^{(0)}).\end{aligned} \tag{18}$$

Therefore,

$$\Lambda^{(0)} = \frac{iq}{\hbar}\int_{\ell_1}^{\ell_2} \mathbf{k}\cdot d\ell - \frac{iq\alpha}{\hbar}\int_{\ell_1}^{\ell_2} r^2\mathbf{k}\cdot d\ell + \frac{iq}{\hbar}\int_{\ell_1}^{\ell_2} \mathbf{A}\cdot d\ell - \frac{iq\alpha}{\hbar}\int_{\ell_1}^{\ell_2} r^2\mathbf{A}\cdot d\ell, \tag{19}$$

where the terms containing k_j are the free particle solutions in the absence of a magnetic field and will be discussed in Sec. IV. The terms containing A_j give the Aharonov-Bohm phase shift φ_{AB}. For a closed path the phase φ_{AB} is corrected as

$$\delta\varphi_{AB} = \delta\varphi_0 + \delta\varphi_1 = \frac{q}{\hbar}\int \mathbf{B}\cdot d\mathbf{s} - \frac{q\alpha}{\hbar}\oint r^2\mathbf{A}\cdot d\ell = \frac{q}{\hbar}\Phi - \frac{q\alpha}{\hbar}\int \nabla\times(r^2\mathbf{A})\cdot\hat{\mathbf{n}}ds. \tag{20}$$

For a finite-radius solenoid, the vector potential for the regions inside and outside of the solenoid are given by

$$\mathbf{A}_{\text{in}} = \frac{B}{2}(-y, x, 0), \tag{21}$$

$$\mathbf{A}_{\text{out}} = \frac{B}{2}\frac{a^2}{x^2+y^2}(-y, x, 0), \tag{22}$$

where a is the radius of the solenoid. Using this, the surface integral in (20) for both inside and outside regions of the solenoid is written as

$$\int \nabla\times(r^2\mathbf{A})\cdot\hat{n}ds = 2\int_{\text{in}}(\mathbf{r}\times\mathbf{A})\cdot\hat{n}ds + \int_{\text{in}} r^2\nabla\times\mathbf{A}\cdot\hat{n}ds + 2\int_{\text{out}}(\vec{r}\times\mathbf{A})\cdot\hat{n}ds + \int_{\text{out}} r^2\nabla\times\mathbf{A}\cdot\hat{\mathbf{n}}ds. \tag{23}$$

Inserting (21) and (22) into (23), one finds

$$\delta\varphi_1 = -\frac{q}{\hbar}\Phi\left(\alpha\frac{S_{\text{out}}}{\pi} - \alpha\frac{2}{\pi a^2}\int_{\text{in}} r^2 ds\right), \tag{24}$$

where, provided that the radius of the solenoid, a, is small, the second term becomes negligible. Consequently, S_{out} can be approximated by S, the total area of the surface bounded by a closed path. Therefore, φ_{AB} can be written as

$$\delta\varphi_{AB} = \frac{q}{\hbar}\Phi\left(1 - \alpha\frac{S}{\pi}\right). \tag{25}$$

The second term in the phase shift causes the AB effect to lose its topological properties. Before discussing the phenomenological importance of (25), we will derive this result using a semiclassical method in the next subsection. The parameter α on the right-hand side of (25) may be dependent on the cosmological constant. The example that we have studied here is $2+1$-dimensional systems living on a $\text{Re}^{(2)} - \{0\}$ manifold. The use of three-dimensional gravity has been suggested as a test bed for the quantization of gravity [16]. Because of the smaller number of dimensions, this theory has tremendous mathematical simplicity. The Einstein theory of gravity in $2+1$ space-time dimensions has a well-know result; namely, the only nontrivial solution for the Einstein equation is the de Sitter (or anti–de Sitter) one and, therefore, the α coefficient should correspond to the cosmological constant. Then the α-dependent part of Eq. (10) is actually the Schrödinger one in a de Sitter background.

If one retains the terms of the order of the noncommutative parameter θ, the following equation is found:

$$-i\hbar(1+\alpha r^2)\partial_j\Lambda^{(1)} - qA_j^{(1)} + \frac{iq}{2}\theta^{lm}\partial_l A_j^{(0)}\partial_m\Lambda = 0, \tag{26}$$

which, by integrating, gives the $\Lambda^{(1)}$ term contributing the noncommutative effects in the AB phase shift. Here, we do not deal with this equation anymore, and the solution can be found in [15,17]. In the following we ignore the θ-dependent terms and only consider the α-dependent corrections to the ordinary phase shifts.

B. The semiclassical approach

In an interferometry experiment, the wave function of two spatially separated beams, $\psi_1(\vec{r}, t) = \phi_1(r)e^{i\omega t}$ and $\psi_2(\mathbf{r}, t) = \phi_2(\mathbf{r})e^{i\omega t}$, can be described in terms of particle trajectories. In this semiclassical approximation, $\phi_i(\mathbf{r})$ is written in the following form [1],

$$\phi_i(\mathbf{r}) = \sqrt{\rho_i(\mathbf{r})}\exp\left(\frac{i\mathcal{S}_i(\mathbf{r})}{\hbar}\right), \tag{27}$$

where $\mathcal{S}$ can be identified with the classical action. Substitution of (27) in the minimal momentum Schrödinger equation will show that $\mathcal{S}$ obeys the following minimal momentum eikonal equation,

$$((1+\alpha r^2)\nabla\mathcal{S}_i)^2 = \mathbf{p}_c^2, \tag{28}$$

where $\mathbf{p}_c$ is the canonical momentum. Then, Eq. (28) can easily be solved, and the wave function will be

$$\phi_i(\mathbf{r}) = \sqrt{\rho_i(\mathbf{r})}\exp\left(\frac{i\mathcal{S}_i(\mathbf{r})}{\hbar}\right) \equiv \phi_0\exp\left(\frac{i}{\hbar}\oint\frac{\mathbf{p}_c\cdot d\ell}{1+\alpha r^2}\right), \tag{29}$$

where ϕ_0 is the unperturbed part of the wave function. In the AB experiment, the integration encloses the interferometry surface, and $\mathbf{p}_c$ is defined by the Lagrangian of a charged particle in a magnetic field,

$$\mathcal{L} = \frac{p^2}{2m} + q\mathbf{v}\cdot\mathbf{A}. \tag{30}$$

From this, the canonical momentum will be given by $\mathbf{p}_c = m\dot{\mathbf{r}} + q\mathbf{A}$. The phase of the perturbed part of the wave function will then be given by

$$\mathcal{S} = q\oint \frac{\mathbf{A}\cdot d\ell}{1+\alpha r^2} \approx q\Big[\oint \mathbf{A}\cdot d\ell - \alpha\oint r^2\mathbf{A}\cdot d\ell\Big]. \tag{31}$$

Therefore, the correction to the Aharonov-Bohm phase is calculated as

$$\delta\varphi_{\rm AB} = \mathcal{S}/\hbar = \delta\varphi_0\Big(1-\alpha\frac{S}{\pi}\Big), \tag{32}$$

where $\delta\varphi_0 = \frac{q}{\hbar}\Phi$. An estimation for the upper bound on the parameter of α can be made using the available experimental data on the Aharonov-Bohm effect. The ratio of the modified phase shift due to MUM and the usual phase shift is

$$\Big|\frac{\delta\varphi_1}{\delta\varphi_0}\Big| = \frac{\alpha S}{\pi}. \tag{33}$$

Fitting the ratio (33) into the accuracy bound of the experiments to verify the AB effect, one obtains a bound on the parameter α. The experiment reported in [18] with an error of 11% gives a constraint on α as follows,

$$\sqrt{\alpha} \le 6\times 10^2\ \mathrm{m}^{-1}, \tag{34}$$

where we have estimated the area surrounded by two electron beams as $S \approx 1\ \mu\mathrm{m}^2$.

III. THE AHARONOV-CASHER EFFECT

Following the semiclassical approach of the previous section, we study the AC effect [9] by assuming a minimal uncertainty in momentum and find variation in the AC phase shift $\delta\varphi_{\rm AC}$. The AC effect is the dual of the AB effect. The AC phase will accumulate when a test particle carrying a magnetic moment μ travels around a charged wire. It is simple to verify that the canonical momentum for the MUM condition is given by

$$\mathbf{p}_{\mathbf{c}} = m\dot{\mathbf{r}} + \frac{1}{1+\alpha r^2}\mu\times\mathbf{E}, \tag{35}$$

where the electric field $\mathbf{E}$ is

$$\mathbf{E} = \frac{\lambda}{2\pi r}\hat{r}, \tag{36}$$

where λ is the charge per unit length. The phase shift for a test particle diffracting around the line charge is calculated by

$$\begin{aligned}\delta\varphi_{\rm AC} &= \frac{1}{\hbar}\oint\frac{\mathbf{p}_{\mathbf{c}}\cdot d\ell}{1+\alpha r^2} = \frac{\lambda}{2\pi\varepsilon_o\hbar}\oint\frac{\mu\times\mathbf{E}\cdot d\ell}{r^2(1+\alpha r^2)}\\ &\approx \frac{\lambda}{2\pi\varepsilon_o\hbar}\oint\frac{\mu\times\mathbf{r}\cdot d\ell}{r^2} - \alpha\frac{\lambda}{2\pi\varepsilon_o\hbar}\oint\mu\times\mathbf{r}\cdot d\ell\\ &= \delta\varphi_0 - \alpha\frac{\lambda}{2\pi\varepsilon_o\hbar}\int\nabla\times(\mu\times\mathbf{r})\cdot\hat{\mathbf{n}}ds.\end{aligned} \tag{37}$$

Therefore,

$$\delta\varphi_{\rm AC} = \delta\varphi_0\Big(1-\frac{\alpha S}{\pi}\Big), \tag{38}$$

where $\delta\varphi_0 = \lambda\mu$. Similar to the previous section, the experimental observations on the AC phase shift can be used to put a limit on the α parameter. In the experiment described in [19], the area can be approximated as $S \approx 4\ \mathrm{cm}^2$. Then, fitting (38) into the accuracy bound of this experiment, which is 24%, one obtains

$$\sqrt{\alpha} \le 0.5\times 10^2\ \mathrm{m}^{-1}, \tag{39}$$

which is close to the AB case. The effect of noncommutativity on the AC phase shift has been studied in [15].

IV. THE COW EFFECT

The effect of the MUM assumption on the results of an experiment was studied by Colella, Overhauser, and Werner [10] in 1975 with neutron interferometry in the gravitational field of the Earth. In this experiment the beam of a neutron is split into two parts, such that they can travel at different heights in the gravitational field of the Earth with different velocities. Using the first term of the canonical momentum, $m\dot{\mathbf{r}}$, the phase shift in the interferometer experiment in a situation where the neutron goes through the ABCD loop is given as

$$\delta\varphi_0 = \frac{m}{\hbar}\oint\mathbf{v}\cdot d\ell \approx \frac{m(v_0-v_1)}{\hbar}\overline{\mathrm{AB}}, \tag{40}$$

where v_0 and v_1 denote the velocities along the paths $\overline{\mathrm{AB}}$ and $\overline{\mathrm{CD}}$. Then, following the discussion of the previous sections, the MUM modification phase shift is given by

$$\delta\varphi = \delta\varphi_0\Big(1-\alpha\frac{S}{\pi}\Big), \tag{41}$$

and using the 1% accuracy confirmed by the COW experiment, the following bound on α is obtained:

$$\sqrt{\alpha} \le 0.5\times 10\ \mathrm{m}^{-1}. \tag{42}$$

V. THE FLUX QUANTIZATION

It is well known that the magnetic flux passing through any area bounded by a superconducting ring is quantized [11]. The quantization of the magnetic flux is closely related to the Aharonov-Bohm effect. The quantum of this magnetic flux is universal, independent of the ring properties, and is equal to

$$\Phi_0 = \frac{\pi\hbar}{e} \approx 2\times 10^{-7}\ \text{gauss-cm}. \tag{43}$$

But this value changes when we consider the MUM assumption. Under such conditions, the flux quantization is modified as

$$\Phi = \frac{q}{\hbar}\oint\frac{\mathbf{A}\cdot d\ell}{1+\alpha r^2}. \tag{44}$$

The only physical requirement is that there can be only one value of the wave function,

$$\psi = \sqrt{\rho} e^{i(q/\hbar)\oint \mathbf{A}\cdot d\ell/(1+\alpha r^2)}, \tag{45}$$

on a closed path. Then the flux becomes

$$\Phi \approx \frac{\pi n\hbar}{e}\left(1 - \frac{\alpha S}{\pi}\right), \tag{46}$$

where S is the horizontal area bounded by a supercurrent (superconducting electrical current). Hence, the quantum of the magnetic flux is changed as

$$\Phi_0 \approx \frac{\pi\hbar}{e}\left(1 - \frac{\alpha S}{\pi}\right). \tag{47}$$

Among the experiments for measuring Φ_0 which can be used to constrain α, Ref. [20] has found that the quantum of the magnetic flux trapped in a hollow superconductor is $\pi\hbar/e \pm 4\%$. In this experiment, the area is $S \approx 3 \times 10^{-8}$ cm^2; therefore, we will have

$$\sqrt{\alpha} \leq 4 \times 10^5 \text{ m}^{-1} \tag{48}$$

which is near the bound obtained in Sec. I.

VI. THE SAGNAC EFFECT

The Sagnac effect [13] for light waves is also valid for matter waves, and it has been verified experimentally [1]. In the well-known Sagnac effect, an extra shift $\delta\varphi_S$ arises when observing the interference between two beams in flat space-time along a closed path due to the rotation of the interferometer. In this section, we study the effect of the rotating frame on an electron-wave interferometry experiment. Similar to the previous section, the eikonal approximation is used in combination with the assumption that the interferometer is small in comparison to the radius of the Earth. Using the Lagrangian of a charged particle on the rotating Earth, one can find the canonical momentum $\mathbf{p_c}$ as

$$\mathbf{p}_{\mathbf{c}} = m\dot{\mathbf{r}} + m\omega \times \mathbf{r}, \tag{49}$$

where m is the mass of the particle and ω is the angular velocity. The phase shift induced by the rotation of the frame is given by

$$\delta\varphi_S = \frac{m}{\hbar}\oint \omega \times \mathbf{r}\cdot d\ell. \tag{50}$$

If one assumes that the MUM condition holds, then the phase shift is modified as follows:

$$\delta\varphi_S = \frac{m}{\hbar}\oint \frac{\omega \times \mathbf{r}\cdot d\ell}{1+\alpha r^2} = \delta\varphi_0 - \frac{m\alpha}{\hbar}\oint r^2\omega \times \mathbf{r}\cdot d\ell. \tag{51}$$

The calculation of the second term gives the following solution:

$$\begin{aligned}\delta\varphi_1 &= -\frac{m\alpha}{\hbar}\oint r^2\omega \times \mathbf{r}\cdot d\ell \\ &= -\frac{m\gamma}{\hbar}\int \nabla \times [r^2\omega \times \mathbf{r}]\cdot \hat{\mathbf{n}} ds\end{aligned} \tag{52}$$

For horizontally incident beams, $\delta\varphi^{(1)}$ becomes

$$\delta\varphi_1 = -\frac{4m}{\hbar}\alpha \int r^2 ds \approx -\delta\varphi_0\alpha\left(\frac{2S}{\pi}\right), \tag{53}$$

where $\delta\varphi_0 = (2m/\hbar)\omega S$. The dependence on the surface indicates that the Sagnac effect is not as topological as the Aharonov-Bohm effect. The total Sagnac phase shift is given by

$$\delta\varphi_S = \delta\varphi_0\left(1 - \alpha\frac{2S}{\pi}\right). \tag{54}$$

In Ref. [21], the phase shift caused by rotation of an electron biprism interferometer placed on a turntable has been measured for different areas. In the experimental setup with $\omega/2\pi = 0.5$ s^{-1}, area $S \approx 2.8$ mm^2, and an error of about 30%, the parameter α is bounded as

$$\sqrt{\alpha} \leq 4 \times 10^2 \text{ m}^{-1}. \tag{55}$$

For different areas, this bound will have the same order of magnitude.

VII. ROTATING SUPERCONDUCTOR

An interesting investigation of the Sagnac effect can be made in a rotating superconducting system by the use of a SQUID [12]. We consider a SQUID involving an interference between currents that flow through a pair of Josephson junctions, which rotate with a constant angular speed ω about the z axis perpendicular to the SQUID. Now if x_μ denotes the coordinates of the initial framework and x'_μ denotes the coordinates of the rotating framework, then one can verify that

$$x = x'\cos\omega t' - y'\sin\omega t', \tag{56}$$

$$y = x'\sin\omega t' + y'\cos\omega t', \tag{57}$$

and the Lagrangian for motion in the rotating frame becomes

$$\mathcal{L} = -m(g_{\mu\nu}\dot{x}'^{\mu}\dot{x}'^{\nu})^{1/2}, \tag{58}$$

where $g_{\mu\nu}$ is defined as

$$g_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\alpha}{\partial x'^\nu}\eta_{\alpha\beta}, \tag{59}$$

with $\eta_{\alpha\beta}$ the Minkowski metric. The $g_{\mu\nu}$ component is given by

$$\begin{aligned} &g_{00} = 1 - (\omega \times \mathbf{x}')\cdot(\omega \times \mathbf{x}'), \\ &g_{0i} = (\mathbf{x}' \times \omega)_i, \qquad g_{ij} = \delta_{ij}, \end{aligned} \tag{60}$$

and therefore

$$\mathcal{L} = -m + \tfrac{1}{2}m\dot{\mathbf{x}}'^2 + e\tilde{\phi} - e\dot{\mathbf{x}}' \cdot \tilde{\mathbf{A}}, \tag{61}$$

where the effective scalar and vector potentials, $\tilde{\phi}$ and $\tilde{\mathbf{A}}$, due to rotation are defined as

$$\tilde{\phi} = \frac{m}{2e}(\omega \times \mathbf{x}') \cdot (\omega \times \mathbf{x}'), \qquad \tilde{\mathbf{A}} = -\frac{m}{e}(\mathbf{x}' \times \omega), \tag{62}$$

and the angular velocity ω can be interpreted as a magnetic B field which becomes zero within the rotating superconductor. This leads to a phase shift $\delta\varphi_0$ in the interference pattern between two supercurrents flowing through different paths in the SQUID. For the MUM condition, the phase shift is calculated as follows,

$$\delta\varphi = \frac{2e}{\hbar}\oint \frac{\tilde{\mathbf{A}} \cdot d\ell}{1 + \alpha r^2} \approx \delta\varphi_0\left(1 - \alpha\frac{S}{\pi}\right), \tag{63}$$

where $\delta\varphi_0$ is given by

$$\delta\varphi_0 = \frac{4m\omega S}{\hbar}, \tag{64}$$

and S is the area of the nonsuperconducting region enclosed by two supercurrents. An experiment on the rotating superconductor has been reported in [22], in which the phase shift (64) is measured indirectly with an error of about 30%. In this experiment, $S \approx 0.074\ \mathrm{cm}^2$ and $\omega = 10\ \mathrm{rad\,s}^{-1}$. With this information, the bound on α is given as

$$\sqrt{\alpha} \leq 3.6 \times 10^2\ \mathrm{m}^{-1}. \tag{65}$$

It can also be possible to use the MUM assumption in order to interpret the Cabrera and Tate experiment [23], which reported an anomalous excess of mass for the Cooper pair in a thin, rotating, superconductor ring. In this experiment the difference between the experimental Cooper mass m^* and its theoretical prediction m is reported as

$$\begin{aligned} \Delta m &= m^* - m \\ &= 1.000\,084(21) \times 2m_e - 0.999\,992 \times 2m_e \\ &= 94.147\,240(21)\ \mathrm{eV}. \end{aligned} \tag{66}$$

So far, this experimental result has never received an explanation in the context of superconductor physics. Hence, some work has been done in the context of quantum gravity and dark energy to interpret it [24]. For the case in which MUM is assumed, one finds that

$$\frac{\hbar}{m^*} = 2S\Delta\nu\left(1 - \alpha^*\frac{S}{\pi}\right), \tag{67}$$

where S is the area bounded by the closed path, $\Delta\nu$ is the flux null spacing, and α^* is the constant of MUM defined in the rotating frame. As shown in [25], the coefficient α can be dependent on the metric of space-time. Therefore, one expects that α in a rotating frame will be different from α in a nonrotating frame. Hence, one can write

$$\frac{\Delta m}{m} \approx S\Delta\alpha, \tag{68}$$

where Δm was defined in (66) and $\Delta\alpha = \alpha^* - \alpha$. So MUM can give an alternative interpretation for the anomalous Cooper pair excess. From the numerical values of the Cabrera and Tate experiment, we find $\Delta\alpha$ as

$$\Delta\alpha \approx 5 \times 10^{-2}\ \mathrm{m}^{-2}. \tag{69}$$

VIII. THE GRAVITATIONAL AHARONOV-BOHM EFFECT

In this section, we will consider the minimal uncertainty in momentum in the gravitational analog of the Aharonov-Bohm effect. This teleparallel approach to gravitation consists in a shift similar to the Aharonov-Bohm effect, but produced by the presence of a gravitational gauge potential. This analog phase shift is zero in Newtonian gravity. In general relativity, both mass and angular momentum act as the source of gravity, but in Newtonian gravity only the mass, not the angular momentum, gravitates. These 2 degrees of freedom are analog to the charge and magnetic moments in an electromagnetic field.

Therefore, the analog of the AB effect for gravity is the phase shift $\delta\varphi$ acquired by a mass going around a string or rod that has angular momentum.

We now consider the linearized limit of general relativity and write the metric as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $h_{\mu\nu} \ll 1$ and $\vec{h}_0 = -(h_{01}, h_{02}, h_{03})$ are the Coriolis vector potentials. The phase shift due to gravitational fields in a situation where a particle with mass m and spin S interferes around a cylinder is given by

$$\delta\varphi = -\frac{1}{2\hbar}\oint h_{\mu a,b}S^{ab}dx^{\mu}, \tag{70}$$

where S^{ab} $(a, b = 0, 1, 2, 3)$ is a generator for Lorentz transformation in the spin space related to the spin vector S^a and the 4-velocity v^b as

$$S^{ab} = \epsilon^{abcd}v_c S_d. \tag{71}$$

The above phase shift is the gravitational analog of the AC phase shift [26] and can be simplified further to

$$\delta\varphi = -\frac{2}{\hbar}\oint \mathbf{g} \times \mathbf{S} \cdot d\ell, \tag{72}$$

where S is the spin of the test particle and $\mathbf{g}$ is the acceleration due to gravity, defined as $\mathbf{g} = -\frac{1}{2}\nabla h_{00}$. The dual of this situation is the gravitational analog of the AB effect [27], which is given by

$$\delta\varphi = \frac{m}{\hbar}\oint h_{0,\mu}dx^{\mu}. \tag{73}$$

This phase shift is given by interfering a mass m around the cylinder with an angular momentum $\mathbf{J}$ given by

$$\vec{h}_0 = -\frac{4G}{\rho_0^2}\mathbf{J}\times\mathbf{r}, \tag{74}$$

where ρ_0 is the radius of the cylinder and G is Newton's constant. The two phase shifts obtained above can also be derived through a Lagrangian of the form

$$\mathcal{L} = \frac{1}{2m}(\mathbf{p} - m\vec{h}_0 - 2\mathbf{S}\times\mathbf{g})^2 + \frac{1}{2}mh_{00}. \tag{75}$$

Using this Lagrangian, one can extend the phase shifts for the case of the minimal uncertainty in momentum condition. In a semiclassical situation, the modified phase shifts are given by

$$\begin{aligned}\delta\varphi_m &= \delta\varphi^{(1)} + \delta\varphi^{(2)}\\ &= -\frac{2}{\hbar}\oint\frac{\mathbf{g}\times\mathbf{S}\cdot d\ell}{1+\alpha r^2} + \frac{m}{\hbar}\oint\frac{\vec{h}_0\cdot d\ell}{1+\alpha r^2}\end{aligned} \tag{76}$$

where calculations are straightforward and lead to results similar to those in the previous sections.

The AB phase shift is given by a solenoid producing a vector potential, which satisfies the Maxwell equation. Corresponding to this solenoid, for the Einstein equation there is a spinning cosmic string solution which has angular momentum and mass. A gravitational AB phase shift can be produced by such a cosmic string; i.e. the solenoid may be replaced by a cosmic string. Outside the cosmic string, the curvature and torsion vanish, which is analogous to the vanishing of the electromagnetic field strength outside the solenoid. Then a quantum mechanical particle with the state $|\psi\rangle$ enclosing the cosmic string develops an AB phase shift.

Another interesting issue is to consider the AB scattering of a cosmic string with a quantum-charge particle. The AB effect is a mechanism for detecting cosmic strings [28]. This effect is similar to the case of a charge particle scattering off an infinitesimally thin solenoid. For the interaction of a cosmic string with a relativistic quantum particle, the differential cross section is given by writing the Dirac equation and then calculating the scattering amplitude. The MUM assumption will modify the AB cross section into a cross section with α corrections.

IX. THE HYDROGEN ATOM

The effects of MUM can also be studied at atomic scales. We have calculated the effects of MUM on a hydrogen atom, where energy levels are modified as

$$E_{n,l} = E_{n,l}^{(0)} + \frac{(\alpha\hbar)^2}{2m_e}\left((n+l)^2 + l(l+1) - \frac{l(l+1)}{n}\right), \tag{77}$$

where $E_{n,l}^{(0)}$ is the energy related to the usual hydrogen system. In order to constrain α, we use measurements on the hydrogen energy levels similar to what has been done in Ref. [29]. In that paper, the hydrogen 1S-2S transition frequency ω has been measured with an accuracy of 1.8 parts in 10^{14}. Using this measurement, one has

$$\frac{\delta\omega}{\omega} \leq 1.8\times 10^{-14}. \tag{78}$$

Then, the parameter α is constrained as

$$\sqrt{\alpha} \leq 1.6\times 10^2\ \mathrm{m}^{-1}. \tag{79}$$

X. CONCLUSION

We argued that QM on a curved space-time experiences a minimal uncertainty in momentum. This feature can be considered as the effect of IR aspects of gravity on quantum mechanics. Based on this assumption, we resolved some quantum mechanics examples such as the Aharonov-Bohm effect, the Aharonov-Casher effect, the COW effect, flux quantization, the Sagnac effect, rotating superconductors, the gravitational Aharonov-Bohm effect, and hydrogen atom energy levels. We found that the phase shift due to these effects is corrected by an area-dependent term. By fitting this new phase shift with the experiments performed to verify the above effects and also the data of the experiment performed for exploring the Heisenberg uncertainty relation, we found that the parameter α is bounded as $\sqrt{\alpha} \leq 10 - 10^6\ \mathrm{m}^{-1}$. In order to explore the area-dependent phase shift predicted in this paper, one can, for instance, prepare and perform the interferometry experiment with different area conditions similar to the setup used in [21]. It was also shown that the assumption of minimal uncertainty in momentum can explain the anomalous excess of the mass of the Cooper pair in a thin, rotating, superconductor ring in the Cabrera and Tate experiment.

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[1] G. van der Zouw, http://www.quantum.univie.ac.at/publications/thesis /gvdzdiss; H. Rauch and S. Werner, *Neutron Interferometry: Lessons in Experimental Quantum Mechanics* (Oxford University Press, New

York, 2000); D. Greenberger, Rev. Mod. Phys. **51**, 43 (1979).

[2] A. Kempf, arXiv:hep-th/9405067; H. Hinrichsen and A. Kempf, J. Math. Phys. (N.Y.) **37**, 2121 (1996); A. Kempf, J. Math. Phys. (N.Y.) **38**, 1347 (1997).

[3] M. Maggiore, Phys. Lett. B **304**, 65 (1993).

[4] L. N. Chang, D. Minic, N. Okamura, and T. Takeuchi, Phys. Rev. D **65**, 125027 (2002); J. C. Niemeyer, Phys. Rev. D **65**, 083505 (2002); L. N. Chang, D. Minic, N. Okamura, and T. Takeuchi, Phys. Rev. D **65**, 125028 (2002); S. Benczik, L. N. Chang, D. Minic, N. Okamura, S. Rayyan, and T. Takeuchi, Phys. Rev. D **66**, 026003 (2002); A. Kempf and J. C. Niemeyer, Phys. Rev. D **64**, 103501 (2001).

[5] S. Das and E. C. Vagenas, Phys. Rev. Lett. **101**, 221301 (2008).

[6] O. Nairz, M. Arndt, and A. Zeilinger, Phys. Rev. A **65**, 032109 (2002).

[7] S. Benczik, L. N. Chang, D. Minic, and T. Takeuchi, Phys. Rev. A **72**, 012104 (2005).

[8] Y. Aharonov and D. Bohm, Phys. Rev. **115**, 485 (1959).

[9] Y. Aharonov and A. Casher, Phys. Rev. Lett. **53**, 319 (1984).

[10] R. Colella, A. W. Overhauser, and S. A. Werner, Phys. Rev. Lett. **34**, 1472 (1975).

[11] R. Feynman, R. Leighton, and M. Sands, *The Feynman Lectures on Physics* (Addison-Wesley, Reading, MA, 1965), Vol. III.

[12] B. R. Holstein, Am. J. Phys. **59**, 1080 (1991).

[13] L. A. Page, Phys. Rev. Lett. **35**, 543 (1975).

[14] J. Anandan, arXiv:gr-qc/9505011; R. Aldrovandi, J. G. Pereira, and K. H. Vu, Classical Quantum Gravity **21**, 51 (2004).

[15] H. Falomir, J. Gamboa, M. Loewe, F. Mendez, and J. C. Rojas, Phys. Rev. D **66**, 045018 (2002); B. Mirza and M. Zarei, Eur. Phys. J. C **32**, 583 (2004); B. Mirza, R. Narimani, and M. Zarei, Eur. Phys. J. C **48**, 641 (2006).

[16] E. Witten, Nucl. Phys. **B311**, 46 (1988).

[17] M. Chaichian, A. Demichev, P. Presnajder, M. M. Sheikh-Jabbari, and A. Tureanu, Phys. Lett. B **527**, 149 (2002).

[18] A. Tonomura, N. Osakabe, T. Matsuda, T. Kawasaki, and J. Endo, Phys. Rev. Lett. **56**, 792 (1986); N. Osakabe, T. Matsuda, T. Kawasaki, J. Endo, and A. Tonomura, Phys. Rev. A **34**, 815 (1986).

[19] A. Cimmino, G. I. Opat, A. G. Klein, H. Kaiser, S. A. Werner, M. Arif, and R. Clothier, Phys. Rev. Lett. **63**, 380 (1989).

[20] B. Lischke, Phys. Rev. Lett. **22**, 1366 (1969).

[21] F. Hasselbach and M. Nicklaus, Phys. Rev. A **48**, 143 (1993).

[22] J. E. Zimmerman and J. E. Mercereau, Phys. Rev. Lett. **14**, 887 (1965).

[23] J. Tate, B. Cabrera, S. B. Felch, and J. T. Anderson, Phys. Rev. Lett. **62**, 845 (1989).

[24] C. J. de Matos, arXiv:0812.4993v1.

[25] P. Kuusk and J. Ord, Phys. Lett. B **421**, 99 (1998).

[26] B. Reznik, Phys. Rev. D **51**, 3108 (1995).

[27] B. S. DeWitt, Phys. Rev. Lett. **16**, 1092 (1966).

[28] A. Shapere and F. Wilczek, *Geometric Phases in Physics* (World Scientific, Singapore, 1988); A. E. Everett, Phys. Rev. D **24**, 858 (1981); M. G. Alford and F. Wilczek, Phys. Rev. Lett. **62**, 1071 (1989).

[29] M. Niering, R. Holzwarth, J. Reichert, P. Pokasov, Th. Udem, M. Weitz, and T. W. Hänsch, Phys. Rev. Lett. **84**, 5496 (2000).