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Geometrical properties of the trans-spherical solutions in higher dimensions

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We investigate the geometrical properties of static vacuum p-brane solutions of Einstein gravity in $D = n + p + 3$ dimensions, which have spherical symmetry of S^{n+1} orthogonal to the p directions and which are invariant under the translation along them. The solutions are characterized by the mass density and p number of tension densities. The causal structure of the higher-dimensional solutions is essentially the same as that of the five-dimensional ones. Namely, a naked singularity appears for most solutions except for the Schwarzschild black p-brane and the Kaluza-Klein bubble. We show that some important geometric properties such as the area of S^{n+1} and the total spatial volume are characterized only by the three parameters (the mass density, the sum of tension densities, and the sum of tension density squares), rather than individual tension densities. These geometric properties are analyzed in detail in this parameter space and are compared with those of the five-dimensional case.

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I. INTRODUCTION

Recently, the physical meaning of the two parameters in static vacuum hypercylindrical solutions in five dimensions [1] was correctly interpreted in Ref. [2]. In the analogy of weak field solutions for a cylindrical matter source distributed uniformly along the fifth direction, the author identified the two parameters as ‘‘mass’’ and ‘‘tension’’ densities.[1] It is also pointed out that the well-known Schwarzschild black string solution corresponds to the case in which the tension-to-mass ratio is exactly one-half. This means that the Schwarzschild black string, which was believed to be a family of solutions characterized by the mass density only, is indeed a special case of a wider class of solutions characterized by tension density as well. Note also that in five-dimensional spacetime, there is another class of stationary solutions which is characterized by the mass and ‘‘momentum’’ densities along the extra directions [4,5]. The Schwarzschild black string background is known to be unstable under small gravitational perturbations along the fifth direction—the so-called Gregory-Laflamme (GL) instability [6,7]. Therefore, studying the physical role of tension might give a better understanding about what really causes the GL instability.

In Ref. [8] the geometric properties of this class of spacetimes with arbitrary tension in five dimensions were investigated in detail. Some of the main properties are as follows.

(i) The solutions are classified by the tension-to-mass ratio a. The event horizon exists only when the tension density is half of the mass density, i.e. $a = 1/2$. Only in this case can the spacetime be called a black string. All spacetimes having values of the tension-to-mass ratio other than $1/2$ and 2 have a naked singularity at the ‘‘center.’’

(ii) Even though there is a naked singularity instead of an event horizon, light radiated from it is infinitely redshifted, provided that $a < 2$.

(iii) The geometry of some subspaces behaves interestingly. For $a < 1/2$ or $a > 2$ the area of an S^2 sphere monotonically decreases down to zero as one approaches the naked singularity from infinity, as usual. For $1/2 < a < 2$, however, it bounces up and increases again to infinity at the naked singularity, as in the geometry of a wormhole spacetime.

(iv) On the other hand, the proper length of a segment along the fifth direction shrinks down to zero for $a > 1/2$, but expands to infinity for $a < 1/2$ as one approaches the naked singularity.

(v) Although the S^2 area and the segment length compete with each other, the total area of the segment $S^2 \times L$ turns out to decrease monotonically down to zero at the singularity, except for $a = 1/2$. For the case of $a = 1/2$ the area of S^2 becomes finite at the

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[1]They are indeed gravitational mass and tension densities rigorously defined at asymptotic infinity [3].

horizon and the scale factor of the fifth direction is constant.

The geodesic motions in this spacetime were also studied in Ref. [9].

In this paper we investigate how much these features change if the spacetime dimensionality becomes higher than five. For the case of $a < 1/2$, for instance, since the area of an S^2 sphere shrinks to zero whereas the proper length along the fifth direction diverges, one may expect that the total area might not shrink down to zero if one increases the number of extra dimensions enough. In addition, since there will be more tension parameters as the number of extra dimensions increases, one may wonder if there are more black brane solutions in addition to higher-dimensional Schwarzschild black branes.

Higher-dimensional spacetime solutions having translational symmetry along extra dimensions and spherical symmetry on slices perpendicular to them have been discovered by many authors in the literature in different contexts [10–15]. For instance, much more general solutions, even in the presence of dilatonic scalar fields and antisymmetric forms, were found in Refs. [11,12,14]. (See also references therein.) However, a full, detailed analysis of the geometry and a correct interpretation of the "tension" parameter have not yet been done, as far as we know.

In Sec. II, we study the geometric properties of the trans-spherical vacuum solutions in $D = n + p + 3$ dimensions in detail. In Sec. III, we analyze the causal structure of the solution. Finally, in Sec. IV, we summarize our results and discuss their physical implications.

II. GEOMETRIC PROPERTIES OF SOLUTIONS

We consider static solutions for the vacuum Einstein equations in $D = n + p + 3$ dimensions, which are invariant under translations along the extra p dimensions and are spherically symmetric on the $(n + 2)$ dimensions transverse to the p dimensions. The most general form of the metric with the symmetries may be written as

$$\begin{aligned} ds^2 &= g_{\mu\nu}dx^\mu dx^\nu \\ &= -H_0(\rho)dt^2 + G(\rho)(d\rho^2 + \rho^2 d\Omega^2_{(n+1)}) \\ &\quad + \sum_{i=1}^{p} H_i(\rho)dz_i^2. \end{aligned} \tag{1}$$

Here H_0, G, and H_i are functions of the isotropic coordinate ρ only. This class of spacetime solutions will be characterized by the p number of tension densities τ_i along z_i and the ADM mass density M associated with the spatial and time translation symmetries, respectively [3]. Interestingly, it turns out that the exponents in the metric component functions depend only on the dimensionless tension-to-mass ratio a_i defined as

$$\tau_i = a_i M. \tag{2}$$

For convenience, we define the sums of a_i and a_i^2 by

$$a = \sum_{i=1}^{p} a_i, \qquad \bar{a}^2 = \sum_{i=1}^{p} a_i^2, \tag{3}$$

where i runs over spatial extra dimensions. Note that these definitions for a and $\bar{a}$ restrict values of $\bar{a}$ such that $\bar{a} \geq |a/\sqrt{p}|$, as shown by the shaded region in Fig. 1 below. The solutions in these general higher dimensions were found by several authors [11,12,14] and can be expressed as

$$\begin{aligned} H_0(\rho) &= \left|\frac{1 - m/\rho^n}{1 + m/\rho^n}\right|^{\sqrt{(n+1)/n}[2(p+n-a)/(p+n+1)/\sqrt{\bar{a}^2+1-(a+1)^2/(p+n+1)}]}, \\ G(\rho) &= \left(1 + \frac{m}{\rho^n}\right)^{4/n}\left|\frac{1 - m/\rho^n}{1 + m/\rho^n}\right|^{2/n[1-(a+1)/(p+n+1)(\sqrt{n(n+1)}/\sqrt{\bar{a}^2+1-(a+1)^2/(p+n+1)})]}, \\ H_i(\rho) &= \left|\frac{1 - m/\rho^n}{1 + m/\rho^n}\right|^{\sqrt{(n+1)/n}[2(a_i-(a+1)/(p+n+1))/\sqrt{\bar{a}^2+1-(a+1)^2/(p+n+1)}]} \end{aligned} \tag{4}$$

by using the ADM mass and ADM tensions. Here the integration constant m is related to the ADM mass M by

$$m = \left[\frac{n}{n+1}\left(\bar{a}^2 + 1 - \frac{(a+1)^2}{p+n+1}\right)\right]^{1/2}\frac{4\pi G_D M}{\Omega_{(n+2)}}, \tag{5}$$

where $\Omega_q = q\pi^{q/2}/\Gamma(q/2 + 1)$, and G_D is the D-dimensional Newton constant. Note that the quantity inside the square root in Eq. (5) is positive definite for all real values of a_i. The solutions (4) are described by $p + 1$-independent parameters, m and a_i. It is interesting to see that the functions H_0, G, and $\prod_{i\geq 1} H_i$ depend only on the sums of a_i and a_i^2 (i.e., a and $\bar{a}^2$), not on the individual values of a_i. The double-Wick rotations $t \to iz_i$ and $z_i \to it$ of the metric (1) also satisfy the vacuum Einstein equation.

Let us analyze geometrical properties of the spacetime solutions described above and see how they depend on the spacetime dimensionality. First of all, one sees that these spacetimes become flat as either $\rho \to \infty$ or $\rho \to 0$. The same happens in 4D Schwarzschild and Reissner-Nordstrom solutions written in isotropic coordinates. From the form of solutions, we find that the component

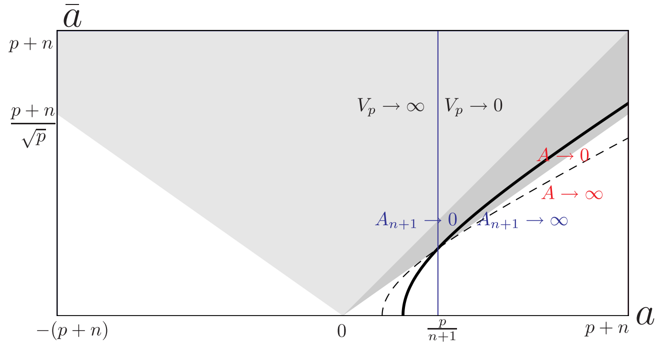

FIG. 1 (color online). Schematic phase diagram of higher-dimensional p-brane solutions in $(a, \bar{a})$. The horizontal and vertical axes stand for a and $\bar{a}$, respectively. The definitions of a and $\bar{a}$ allow their values on the shaded region satisfying $\bar{a} \geq |a/\sqrt{p}|$. The dark shaded region is for physical parameters based on the strong energy condition. As ρ goes to K, the area of the $(n+2)$-dimensional sphere A_{n+1}, the volume V_p for extra p dimensions, and the total volume A become nonvanishing finite values on the thick curve, on the vertical black line, and on the dashed curve, respectively.

functions of the metric, H_0, G, and H_i, show different behaviors as $\rho \to K$, depending on the sign of their exponents. Here

$$K = |m|^{1/n}. \tag{6}$$

Note that the region with $0 < \rho < K$ is another copy of the spacetime region covered by $K < \rho < \infty$, as in the case of the five-dimensional one [8].[2] Although arbitrary values of m (e.g., M) are allowed, we assume non-negative values of ADM mass density for the discussions below. Thus, $m = K^n$.

Note also that the case of uniform tensions with $a_i = 1/(n+1)$ for all i corresponds to the well-known Schwarzschild black p-brane in higher dimensions,

$$ds^2 = -\left(\frac{1-K^n/\rho^n}{1+K^n/\rho^n}\right)^2 dt^2 + G(d\rho^2 + \rho^2 d\Omega_{(n+1)}) + \sum_{j=1}^{p} dz_j^2, \tag{7}$$

where the event horizon is located at $\rho = K$. The other well-known special cases are given by ($a_i = n+1$, $a_{j\neq i} = 1$), the Kaluza-Klein bubble solution,

$$ds^2 = -dt^2 + G(d\rho^2 + \rho^2 d\Omega_{(n+1)}) + \left(\frac{1-K^n/\rho^n}{1+K^n/\rho^n}\right)^2 dz_i^2 + \sum_{j\neq i}^{p} dz_j^2, \tag{8}$$

which are related to the Schwarzschild black p-brane solution mentioned above through the double-Wick rotation. Notice also that the metric becomes singular at $\rho = K$. It turns out that this is a coordinate singularity for the case of the Schwarzschild black p-brane and the Kaluza-Klein bubble.[3] Otherwise, it is a curvature singularity, as will be shown below in detail.

By investigating the spatial geometries of several spacelike hypersurfaces, we classify the solutions in the parameter space of a_i. Note first that, in 4 + 1 dimensions with $n = 1 = p$, the area of S^2 diverges if $1/2 < a < 2$ and is finite if $a = 1/2$ or 2, whereas it vanishes if $a > 2$ or $a < 1/2$ [8]. In general, the area of the S^{n+1} sphere at the $z_1, \cdots, z_p =$ constant surface is given by

$$A_{n+1}(\rho) = \Omega_{(n+2)}(\sqrt{G}\rho)^{n+1} = \Omega_{(n+2)}\rho^{n+1}\left(1-\frac{K^n}{\rho^n}\right)^{[(n+1)/n](1-\alpha)} \times \left(1+\frac{K^n}{\rho^n}\right)^{[(n+1)/n](1+\alpha)}. \tag{9}$$

Here α is defined as

$$\alpha = \frac{a+1}{p+n+1}\frac{\sqrt{n(n+1)}}{\sqrt{\bar{a}^2+1-\frac{(a+1)^2}{p+n+1}}}. \tag{10}$$

As $\rho \to \infty$, this area increases as usual. However, as $\rho \to K$, one can see that the property is crucially dependent on the signature of the exponent $(1-\alpha)$. The area of the $\rho = K$ surface has a nonvanishing finite value if $1 - \alpha = 0$. As shown by the thick curve in Fig. 1, this equation gives the hyperbola defined by the following equation on a and $\bar{a}$ space:

$$\frac{(n+1)^2+p}{(n+p+1)^2}(a+1)^2 - \bar{a}^2 = 1, \tag{11}$$

with a restriction of $a > -1$. Note that this hyperbola intersects with the line $\bar{a} = a/\sqrt{p}$ one time if $n \geq (p+1)/(p-1)$ and two times if $1 \leq n < (p+1)/(p-1)$. On the hyperbola, the area of the $\rho = K$ surface becomes

$$A_{n+1} = 4^{1+(1/n)}\Omega_{(n+2)}K^{n+1} = \Omega_{(n+2)}^{-1/n}\left(\frac{16n\pi G_D M}{n+p+1}\right)^{(n+1)/n}(a+1)^{(n+1)/n}. \tag{12}$$

Therefore we see that, while only the cases of $a = 1/2$ and 2 in five dimensions give a nonvanishing finite area A_2, all solutions whose gravitational tensions are properly balanced in accordance with Eq. (11) have a nonvanishing finite area A_{n+1}. Namely, in the parameter space of p-dimensional a_i, this region is given as the intersection of the $(p-1)$-dimensional sphere $\sum_{i=1}^{p} a_i^2 = \bar{a}^2$ and a

[2] This can be easily seen by the coordinate transformation $\rho \to K^2/\rho$.

[3] The conical singularity in the case of the Kaluza-Klein bubble is removed by compactifying the z_i coordinate, e.g., $z_i \sim z_i + (2^{2+2/n}\pi K)/\sqrt{n}$.

$(p-1)$-dimensional plane $\sum_{i=1}^{p} a_i = a$ for each $(a, \bar{a})$ on the hyperbola (11) in the shaded region in Fig. 1.

If $1-\alpha < 0$, the behavior of the area is basically the same as the case of $1/2 < a < 2$ in five dimensions. Namely, as ρ decreases to K from infinity, this area decreases only up to its minimum point $\rho = \rho_m$ given by

$$\rho_m = \left(\frac{4\pi G_D M}{\Omega_{(n+2)}}\right)^{1/n}\left(n(a+1) + \sqrt{\frac{(n+1)^2+p}{(p+n+1)^2}(a+1)^2 - (\bar{a}^2+1)}\right)^{1/n}, \quad (13)$$

and then starts to increase up to infinity instead of decreasing down monotonically. Therefore, this spatial geometry looks like a wormhole. However, we point out that the $\rho = K$ surface can be reached from the throat (i.e., $\rho = \rho_m$) in a finite affine time. For $n \geq (p+1)/(p-1)$, the value of ρ_m does not have a maximum. On the other hand, for $n < (p+1)/(p-1)$, it does have maximum. In Fig. 1, the corresponding parameter space is designated by the lower right end of the thick curve in the shaded region.

On the other hand, if $1-\alpha > 0$ this area monotonically decreases and shrinks to zero at $\rho = K$, as in the cases of $a > 2$ or $a < 1/2$ in five dimensions. In Fig. 1, the corresponding parameter space is designated by the upper left end of the thick curve in the shaded region.

In the special case of string configurations (i.e., $p = 1$), a and $\bar{a}$ become the same, and A_{n+1} diverges as ρ goes to K in the region given by

$$\frac{1}{n+1} < a < n+1. \quad (14)$$

At $\rho = K$, A_{n+1} takes a nonvanishing finite value for the cases of $a = \frac{1}{n+1}$ or $a = n+1$. For the cases of $a < 1/(n+1)$ or $a > n+1$, it vanishes. For $n = 1$ we recover the five-dimensional results.

Now let us consider the spatial geometry in extra p dimensions. Note first that, if $a_i = \tau_i/M = 1/(n+1)$ for all i, all scale factors of the extra space are constant since all $H_i = 1$. This case corresponds to the Schwarzschild black p-brane. For given values of tension-to-mass ratios $\{a_1, a_2, \cdots, a_{i-1}, a_{i+1}, \cdots, a_p\}$, the z_i direction becomes flat if

$$a_i = \frac{1 + a_1 + a_2 + \cdots + a_{i-1} + a_{i+1} + \cdots + a_p}{p+n}, \quad (15)$$

or equivalently, $a_i = \frac{a+1}{p+n+1}$. For other values of tension ratios, we see that H_i in Eq. (4) becomes singular as ρ goes to K. Namely, as ρ decreases to K, the proper length of a unit segment with $\Delta z_i = 1$ at the $\rho =$ constant surface shrinks to zero if $a_i > \frac{a+1}{p+n+1}$, but it infinitely expands if $a_i < \frac{a+1}{p+n+1}$. However, we point out that all these cases do not necessarily correspond to singular spatial geometry in extra p dimensions. Consider a Kaluza-Klein bubble solution along the z_1 direction, in which $H_1(\rho)$ vanishes at $\rho = K$ with $H_{i\neq 1}(\rho) = 1$. We can easily check that there exists no curvature singularity in this case. Indeed, the $\rho = K$ surface turns out to be a fixed point whose conical singularity can be removed by compactifying the z_i coordinate suitably, as mentioned before. Hence, the z_1 direction becomes a circle. In general, if $p \geq 2$, one can find many solutions where some extra directions become circles in this way. However, all those solutions contain curvature singularities at $\rho = K$, except for the case of the Kaluza-Klein bubble solutions. To summarize, we find that the regular spatial geometry in extra p dimensions happens only for the Schwarzschild black p-brane and Kaluza-Klein bubble solutions.

It is interesting to see how the total spatial volume A for a given ρ behaves as $\rho \to K$. The total volume for the $\rho =$ constant surface per unit length of z_i with respect to an asymptotic observer is given by

$$A = A_{n+1} V_p, \quad (16)$$

where V_p is the spatial volume of extra p dimensions,

$$V_p = \prod_{i=1}^{p} \int_{z_i}^{z_i+1} \sqrt{H_i}\, dz_i = \left(\frac{1 - K^n/\rho^n}{1 + K^n/\rho^n}\right)^{[\sqrt{(n+1)/n}][(n+1)a - p/(p+n+1)]/\sqrt{\bar{a}^2+1-[(a+1)^2/(p+n+1)]}}. \quad (17)$$

We see that the spatial volume V_p becomes unit when $a = p/(n+1)$, which is designated by the vertical line in Fig. 1. It is interesting to see that the total volume V_p could be finite if the divergence in a certain direction is precisely canceled out by the shrinking of some other directions. As ρ decreases to K, V_p monotonically decreases to zero for $a > p/(n+1)$ and monotonically increases to infinity for $a < p/(n+1)$.

The total spatial volume of the $\rho =$ constant section is given by

$$A = \Omega_{(n+2)}\rho^{n+1}\left(1+\frac{K^n}{\rho^n}\right)^{2(n+1)/n}\left|\frac{1-K^n/\rho^n}{1+K^n/\rho^n}\right|^{(n+1)/n[1-\sqrt{n/(n+1)}/\sqrt{\bar{a}^2+1-(a+1)^2/(p+n+1)}]}. \tag{18}$$

Thus the behavior of the total volume is determined by the exponent of $[1 - K^n/\rho^n]$,

$$\frac{(p+n+1)[(n+1)\bar{a}^2+1]-(n+1)(a+1)^2}{n(p+n+1)\sqrt{\bar{a}^2+1-\frac{(a+1)^2}{p+n+1}}\left(\sqrt{\bar{a}^2+1-\frac{(a+1)^2}{p+n+1}}+\sqrt{\frac{n}{n+1}}\right)}, \tag{19}$$

which vanishes on a hyperbola (the dashed curve given in Fig. 1). From the definitions of a and $\bar{a}$, the numerator of Eq. (19) can be reexpressed in the sum of squared terms, given by

$$(n+1)\sum_{i=1}^{p}\sum_{j=i+1}^{p}(a_i-a_j)^2+\sum_{i=1}^{p}[(n+1)a_i-1]^2. \tag{20}$$

Equation (20) is non-negative for all real values of a_i. It vanishes only when

$$a_1 = a_2 = \ldots = a_p = \frac{1}{n+1}. \tag{21}$$

The total volume with this set of a_i becomes

$$A = 2^{2(n+1)/n}\Omega_{(n+2)}K^{n+1}. \tag{22}$$

This case of uniform tensions is exactly the same as the case in which all extra p directions are regular, as mentioned above. In addition, we point out that the geometry of the sphere is regular since this case satisfies Eq. (11) as well. In fact, the solution with parameters in Eq. (21) corresponds to the Schwarzschild black p-brane (7) in higher dimensions.

Since either A_{n+1} or V_p diverges, depending on the values of the tensions, one may expect that the total volume A also diverges. However, we claim that this never happens for any value of the tension parameters. This is because the exponent of $[1 - K^n/\rho^n]$ takes a positive value for all cases except for the case of uniform tensions, so that the total volume A vanishes as $\rho \to K$. This happens because, even if the volume of the sphere diverges for certain values of parameters a_i, V_p shrinks more strongly for those values. Similarly, for the case in which V_p diverges, A_{n+1} shrinks more strongly. The idea of making the total volume A divergent by arbitrarily adding expanding extra dimensions does not work. This is because, as p increases, both the expanding rate in V_p and the shrinking rate in A_{n+1} change as well so that the net effect always gives vanishing total volume.

So far, we have not restricted the values of a_i. However, their physical ranges may be given by some energy conditions. Note that the strong energy condition in five-dimensional spacetime restricts the value of the tension [8]. In d-dimensional spacetimes, this extends as follows:

$$T_{00}+\frac{1}{d-2}\left(-T_{00}+\sum^{p}T_{z_iz_i}\right)\geq 0 \Rightarrow (d-3)M \geq \sum \tau_i. \tag{23}$$

Although we do not know whether the gravitational tensions satisfy the same condition, we assume that the same restriction holds as in the case of matter fields. Therefore, the physical range of a may be given as

$$0 \leq a \leq d-3 = p+n, \tag{24}$$

where the first inequality comes from the positivity theorem for gravitational tension, $a_i \geq 0$ [3].

Note that there exists a coordinate singularity at $\rho = K$. Let us see whether this singularity is genuine or not. The Kretschmann invariant for the metric (4) is

$$\begin{aligned} R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} &= 16nK^{2n}\rho^{4+2n}(\rho^n-K^n)^{-(4/n)(1+n-\beta_1)}(K^n+\rho^n)^{-(4/n)(1+n+\beta_1)}\{2[1+4\beta_1^2+\beta_1^4+n(3+(5\beta_1^2+2\beta_2)\\ &\quad+\beta_1^2(\beta_1^2+4\beta_2))+n^2(2+(\beta_1^2+3\beta_2)+2\beta_1(\beta_1\beta_2+2\beta_3))+n^3(\beta_2+\beta_2^2+\beta_4)]K^{2n}\rho^{2n}-4(1+n)\\ &\quad\times(\beta_1(1+\beta_1^2)+n\beta_1(1+2\beta_2)+n^2\beta_3)(K^{3n}\rho^n+K^n\rho^{3n})+(1+n)(2+n)(\beta_1^2+n\beta_2)(K^{4n}+\rho^{4n})\}, \end{aligned} \tag{25}$$

where $\beta_q = \sum_{i=0}^{p}(\alpha_i)^q$ with $q = 1, 2, 3, 4$. Here α_i with $i = 0, 1, \ldots, p$ is half of the exponent of $\frac{1-K^n/\rho^n}{1+K^n/\rho^n}$ in H_i. We see that the right-hand side of the first line becomes singular as $\rho \to K$ since $1 + n - \beta_1 = 1 + n - \alpha > 0$. Thus, there occurs a curvature singularity at $\rho = K$, unless the term inside the curly brackets cancels this divergence. We find that such a cancellation happens only for the cases satisfying all the following conditions:

$$\beta_1 = \beta_2 = \beta_3 = \beta_4 = 1. \tag{26}$$

By solving these algebraic equations above, we obtain either

$$\alpha_0 = 1, \qquad \alpha_1 = \alpha_2 = \cdots = \alpha_p = 0, \tag{27}$$

or $(i = 1, 2, \ldots, p)$

$$\alpha_i = 1,$$
$$\alpha_0 = \alpha_1 = \alpha_2 = \cdots = \alpha_{i-1} = \alpha_{i+1} = \cdots = \alpha_p = 0. \tag{28}$$

The first case (27) corresponds to the Schwarzschild black p-brane. The second case corresponds to the p number of different Kaluza-Klein bubble solutions, which are related to the Schwarzschild black p-brane by the double-Wick rotations in Eq. (1). For the nonsingular cases, the value of the Kretschmann invariant becomes

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \frac{32n(n+1)^2K^{2n}}{\rho^{2(n+2)}}\left(1+\frac{K^n}{\rho^n}\right)^{-[4(n+2)/n]}. \tag{29}$$

III. CAUSAL STRUCTURE

Let us consider the causal structure of these spacetime solutions. It is enough to consider the z_i = constant surface with fixed angles. Then, the metric becomes

$$ds^2 = -H_0dt^2 + Gd\rho^2 = -H_0(dt + d\rho^*)(dt - d\rho^*). \tag{30}$$

Here a tortoise coordinate ρ^* is defined as

$$\rho^* = \int^\rho \sqrt{\frac{G}{H_0}}d\rho = \int^\rho d\rho\left(1+\frac{K^n}{\rho^n}\right)^{2/n}\left|\frac{1-K^n/\rho^n}{1+K^n/\rho^n}\right|^{1/n[1-(\sqrt{n(n+1)}/\sqrt{\bar{a}^2+1-(a+1)^2/(p+n+1)})]}. \tag{31}$$

The ingoing and outgoing null coordinates are defined by

$$v = t + \rho^*, \qquad u = t - \rho^*, \tag{32}$$

respectively.

In order to see the causal properties of the $\rho = K$ surface, let us consider whether the null rays can escape from the surface or not. The geodesic motion of an outgoing light in (v, ρ) coordinates becomes

$$\frac{d\rho}{dv} = \frac{1}{2}\sqrt{\frac{H_0}{G}} \sim |\rho - K|^{-q} \tag{33}$$

in the vicinity of the $\rho = K$ surface. Here q is given by

$$q = \frac{1}{n}\left(1 - \sqrt{\frac{n(n+1)}{\bar{a}^2 + 1 - \frac{(a+1)^2}{p+n+1}}}\right) \geq -1, \tag{34}$$

where the equality holds only for $a_i = 1/(n+1)$ for all positive i. The elapsed value v for a light traveling from $\rho = K + \epsilon$ to $\rho = \rho'$ outside becomes

$$\Delta v = 2\int_{K+\epsilon}^{\rho'}\sqrt{\frac{G}{H_0}}d\rho \sim \begin{cases} \ln|\frac{\rho'-K}{\epsilon}| & \text{for } a_i = \frac{1}{n+1}, \quad \forall i, \\ \ln|\rho' - K|^{q+1} - \epsilon^{q+1} & \text{for } a_i \neq \frac{1}{n+1}, \quad \exists i. \end{cases} \tag{35}$$

For the case of $a_i = 1/(n+1)$, which corresponds to the Schwarzschild black brane, $\Delta v \to \infty$ as $\epsilon \to 0$. This implies that the light cannot escape from the $\rho = K$ surface, and consequently, the $\rho = K$ surface is indeed an event horizon. On the other hand, Δv takes a finite value for other cases. Therefore, the light can actually escape from the $\rho = K$ surface in a finite time. In this sense, the surface is not an event horizon and, in fact, is interpreted as a naked singularity because of the curvature singularity there. The Penrose diagram for the spacetime at z_i = constant is given in Fig. 2. Figure 2(a) is for the case in which all tension-to-mass ratios are $1/(n+1)$, which corresponds to the Schwarzschild black brane. The surface of $\rho = K$ is indeed an event horizon. The region of $0 < \rho < K$ is not the inside of the $\rho = K$ surface, but it is another copy of the spacetime, as is well known in the isotropic coordinate

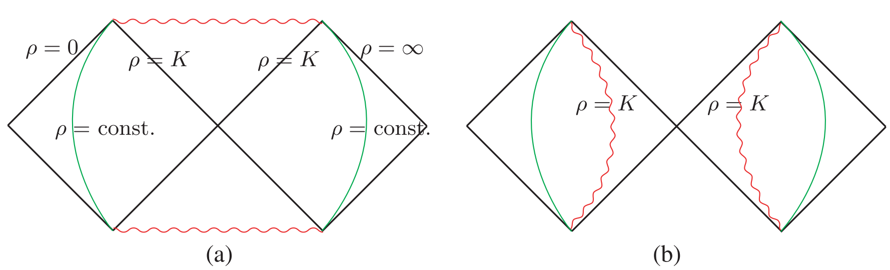

FIG. 2 (color online). Penrose diagram at a z_i = constant slice. The angular coordinates are suppressed here. Figure (a) is for the case in which all tension-to-mass ratios are $1/(n+1)$, which corresponds to the Schwarzschild black brane. Figure (b) is for the case having a naked singularity at the $\rho = K$ surface.

system. The inside region can be obtained by analytic continuation of the spacetime covered by $0 < \rho < \infty$, revealing a spacelike singularity at the center. The case of the Kaluza-Klein bubble is not shown here. Figure 2(b) is for the other case. Here the $\rho = K$ surface becomes a timelike singularity (i.e., a naked singularity), and the spacetime cannot be analytically continued beyond this surface like the spacetime around the spacelike singularity at the center in Fig. 2(a).

IV. CONCLUSIONS

We have investigated the geometrical properties of static vacuum p-brane solutions of Einstein gravity in $n + p + 3$ dimensions, which have a spherical symmetry of S^{n+1} orthogonal to the p directions and which are invariant under the translation along them.

The solutions are characterized by $(p + 1)$ parameters that are mass density M and p-number of tension densities. Interestingly, some important geometric properties such as the area of $S^{(n+1)}$ and the total spatial volume (18) are characterized only by the three parameters $(M, a, \bar{a})$, not by the individual values of tension ratios a_i, where a and $\bar{a}^2$ are the summations of tension-to-mass ratios and of their squares, respectively, as defined in Eq. (3). The surface area of the S^{n+1} sphere shows interesting behavior, depending on the values of a and $\bar{a}$. There are three classes. Namely, (i) $\frac{(n+1)^2+p}{(p+n+1)^2}(a + 1)^2 - \bar{a}^2 > 1$: As ρ decreases, this area does not decrease monotonically, but increases again as ρ approaches K. In other words, the spatial geometry of the $z_i =$ constant surface behaves like a wormhole geometry. (ii) $\frac{(n+1)^2+p}{(p+n+1)^2}(a + 1)^2 - \bar{a}^2 = 1$: This case is designated by the thick curve (A_{n+1}) in Fig. 1. The Schwarzschild black p-brane and the Kaluza-Klein bubble solutions belong to this class. The area monotonically decreases to a nonvanishing finite value at $\rho = K$. (iii) $\frac{(n+1)^2+p}{(p+n+1)^2}(a + 1)^2 - \bar{a}^2 < 1$: This area monotonically decreases to zero at $\rho = K$.

The spatial volume V_p of unit extra p directions takes a nonvanishing finite value only when $a = p/(n + 1)$. As ρ decreases to K, it monotonically increases to infinity for $a < p/(n + 1)$ and monotonically decreases to zero for $a > p/(n + 1)$. Although each A_{n+1} and V_p behaves variously as ρ goes to K, the total volume $A = A_{n+1}V_p$ always shrinks to zero except for the case of $a_i = 1/(n + 1)$ for all i. It is interesting to see that this value is independent of the number of extra dimensions p. For such an exceptional case, A takes a finite value and it corresponds to the Schwarzschild p-brane solution. The curvature square diverges at $\rho = K$ except for the cases of the Schwarzschild black p-brane (7) and Kaluza-Klein bubble (8). This curvature singularity at the surface $\rho = K$ turns out to be naked.

Some geometrical properties explicitly depend on each value of the tension. In particular, the spatial geometry for each z_i direction is dependent on a_i, in addition to a and $\bar{a}$. The proper length along z_i becomes neither shrinking nor expanding as ρ approaches K if the ith tension takes a specific value which is determined by the values of other tensions, as in Eq. (15). The regular spatial geometry in all extra p dimensions happens only for the Schwarzschild black p-brane and Kaluza-Klein bubble solutions.

We have seen that the causal structure of the higher-dimensional solutions is essentially the same as those of the five-dimensional solutions [8]. Namely, only the solutions where the values of the tensions are equal to the mass divided by the number of angular coordinates, i.e. $M/(n + 1)$, have an event horizon located at $\rho = K$. As is well known, for these Schwarzschild black p-brane solutions, the spacetime can be continued beyond the $\rho = K$ surface and a spacelike curvature singularity appears inside the event horizon, as shown in Fig. 2. The Kaluza-Klein bubble solution, where one of the tensions is given by the mass times the number of angular coordinates and all other tensions are equal to the mass, do not possess an event horizon and the signature of this spacetime changes across the $\rho = K$ surface.[4] Other than these two cases, the curvature singularity appears at the $\rho = K$ surface and light can escape from this surface toward spatial infinity. Consequently, this singularity is naked in nature.

Investigation of such singular spacetimes might be physically meaningless. However, studying the geometrical structure of such singular spacetimes in detail may be very important for the following reason. It is interesting to see that the Schwarzschild black brane is so special for geometrical properties in the whole solution space. This fact was already pointed out in Refs. [12,13]. In particular, when one considers spherically symmetric perturbations with translational symmetries along extra directions untouched, the Schwarzschild black brane solution is singled out because it is stable under such perturbations, whereas all other solutions possessing naked singularities are catastrophically unstable [13,16].[5]

On the other hand, the spacetimes we considered may be formed from the higher-dimensional trans-spherical distribution of normal matter. Our study shows that a slight deviation of tension and mass densities from those evolving to the Schwarzschild black brane will end up with a spacetime configuration possessing a naked singularity. Assuming that the cosmic censorship conjecture is valid in higher dimensions as well, we do not expect the development of a naked singularity. This observation may indicate

[4]In order to see this, we need to analytically continue the spacetime region covered by $K < \rho < \infty$ beyond the $\rho = K$ surface. Note that, in the isotropic coordinate system, $\rho < K$ does not describe the inside of the $\rho = K$ surface.

[5]Note, however, that even the Schwarzschild black brane solution becomes unstable as some perturbations that depend on the extra-dimensional coordinates are turned on, the so-called Gregory-Laflamme instability [7].

that some quantum effect becomes important right before forming a naked singularity during the gravitational collapse. In fact, Kobayashi and Tanaka [17] showed numerically that the naked singularity existing at some static spacetimes near the Schwarzschild one in parameter space becomes null if the Gauss-Bonnet gravity term is added in five dimensions. It is interesting to see how much various quantum effects change the formation of a naked singularity in trans-spherical gravitational collapse. Finally, it is also interesting to study how much the geometrical properties are modified if the angular momentum or charge is taken into consideration. These interesting issues deserve future work.

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