Uniqueness theorem for charged rotating black holes in five-dimensional minimal supergravity

Shinya Tomizawa,[1] Yukinori Yasui,[2] and Akihiro Ishibashi[1]

[1]*Cosmophysics Group, Institute of Particle and Nuclear Studies, KEK, Tsukuba, Ibaraki, 305-0801, Japan*

[2]*Department of Mathematics and Physics, Graduate School of Science, Osaka City University, 3-3-138 Sugimoto, Sumiyoshi, Osaka 558-8585, Japan*

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We show a uniqueness theorem for charged rotating black holes in the bosonic sector of five-dimensional minimal supergravity. More precisely, under the assumptions of the existence of two commuting axial isometries and spherical topology of horizon cross sections, we prove that an asymptotically flat, stationary charged rotating black hole with finite temperature in the five-dimensional Einstein-Maxwell-Chern-Simons theory is uniquely characterized by the mass, charge, and two independent angular momenta and therefore is described by the Chong-Cvetič-Lü-Pope solution. We also discuss a generalization of our uniqueness theorem for spherical black holes to the case of black rings.

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I. INTRODUCTION

In string theory and various related contexts, higher-dimensional black holes and other extended black objects have played an important role. In particular, physics of black holes in the five-dimensional Einstein-Maxwell-Chern-Simons (EMCS) theory has recently been the subject of increased attention, as the five-dimensional EMCS theory describes the bosonic sector of five-dimensional minimal supergravity, a low-energy limit of string theory. Various types of black hole solutions in the EMCS theory [1–22] have so far been found, with the help of, in part, recent development of solution generating techniques [20,23–40]. However, the classification of those black hole solutions has not been achieved yet. The purpose of this paper is to show a uniqueness theorem for charged rotating black holes in the five-dimensional EMCS theory, as a partial solution to the black hole classification problem in the string theory.

It is now evident that even within the framework of vacuum Einstein gravity, there is a much richer variety of black hole solutions in higher dimensions [37,41–47], the classification of which still remains a major open issue. As shown by Emparan and Reall [42], five-dimensional vacuum Einstein gravity admits the coexistence of a rotating spherical hole and two rotating rings with the same conserved charges, illustrating explicitly the nonuniqueness property in higher dimensions. However, it is possible to show type of uniqueness theorems for some restricted cases in which certain additional conditions are imposed on some parameters/properties, other than the global conserved charges. For example, restricting attention to static solutions, Gibbons *et al.* [48] showed that the only asymptotically flat, static vacuum black hole is the Schwarzschild-Tangherlini solution [49]. For the rotating case, by assuming the existence of two axial Killing symmetries and spherical topology of the event horizon, Morisawa and Ida [50] succeeded in proving that five-dimensional asymptotically flat, stationary vacuum rotating black holes must be in the Myers-Perry family. Their theorem was recently generalized to a class of asymptotically flat solutions with nonspherical horizon topology [51,52]. For other cases (such as cases including Maxwell field), see [53–61].

In this paper, we generalize the boundary value analysis of Morisawa and Ida [50] performed in vacuum Einstein gravity to the case of the bosonic sector of five-dimensional minimal supergravity. We are concerned with stationary black hole spacetimes that are asymptically flat in the standard sense: Namely, we demand that the exterior region of the black hole is globally hyperbolic, having a spherical spatial infinity, and that the metric and other physical fields, such as Maxwell field, falloff in a certain manner at large distances. (The asymptotic falloff conditions are given later.) Furthermore, for simplicity, we focus on the single black hole case, that is, the event horizon is connected. Then, we note that in five-dimensional EMCS theory the Chong-Cvetič-Lü-Pope black hole solution [4,5] (with vanishing cosmological constant) appears to be the most general such solutions that describe an asymptotically flat, stationary charged rotating black hole with spherical horizon topology, characterized by four conserved charges, i.e., the mass, two independent angular momenta, and electric charge, and that encompass the known asymptotically flat, spherical black hole solutions in a subclass of EMCS theory, such as the Myers-Perry solution [41], in a certain limit. Thus, we wish to show the following theorem.

Theorem. Consider, in five-dimensional Einstein-Maxwell-Chern-Simons theory [*given by Eq.*. (1) *below*], *a stationary charged rotating black hole with finite temperature that is regular on and outside the event horizon and asymptotically flat in the standard sense with spherical spatial infinity. If (1) the black hole spacetime admits, besides the stationary Killing vector field, two mutually commuting axial Killing vector fields so that the isometry group is* $\mathbb{R} \times U(1) \times U(1)$ *and (2) the topology of the horizon cross sections is spherical*, S^3, *and the topology of the black hole exterior region is* $\mathbb{R} \times \{\mathbb{R}^4 \setminus \mathbb{B}^4\}$, *then the* *black hole spacetime is uniquely characterized by its mass, electric charge, and two independent angular momenta, and hence must be isometric to the Chong-Cvetič-Lü-Pope solution*.

Before presenting our proof, we would like to make a few comments concerning the assumptions made in our theorem. In order to obtain global results, we need the symmetry-condition (1), which, in particular, makes it possible to reduce five-dimensional minimal supergravity to a nonlinear sigma model with certain symmetries as shown in [62,63]. Since all known exact black hole solutions in higher dimensions admit multiple axial isometries, our additional symmetry-condition (1) does not appear to be too restrictive. However, we should note that the rigidity theorem [64] (see also [65]) in higher dimensions—which is recently shown to be applicable also to EMCS theory [66]—only guarantees the existence of a single rotational isometry (provided the spacetime metric and other fields are real, analytic), and therefore at present, the condition (1) is not yet fully justified. In this respect, note also that the possibility for higher-dimensional black holes with fewer isometries than $\mathbb{R} \times U(1) \times U(1)$ has been suggested [67]. Since the rigidity theorem yields that the event horizon is a Killing horizon, the notion of surface gravity is well-defined. Then, by *finite temperature* we mean that the event horizon is of nondegenerate type, having nonvanishing surface gravity and a bifurcate surface [68,69]. For extremal (zero-temperature) black holes with vanishing surface gravity, the event horizon is of degenerated type and does not possess a bifurcate surface. Then, our boundary conditions to be imposed on target space fields at the event horizon would not appear to straightforwardly apply to such a case that the horizon has no bifurcate surface. It would be of great interest to consider the classification problem of such extremal (zero-temperature) black holes. In this respect, there have recently appeared some attempts to classify near-horizon geometries of extremal black objects, rather than extremal black objects themselves (see e.g., [70–75] and references therein).

We also need to additionally impose the topology-condition (2), in order to explicitly specify boundary conditions on target space variables at the event horizon, in terms of certain coordinates, globally defined over the black hole exterior region. The topological censorship, together with our assumption of asymptotic flatness described above, immediately implies that the exterior region is topologically $\mathbb{R} \times V^{(4)}$ with $V^{(4)}$ being some four-dimensional simply connected Riemannian manifold. However, the simple connectedness by itself does not completely determine the topology of $V^{(4)}$. Therefore, in the present theorem, we simply demand that $V^{(4)} \approx \{\mathbb{R}^4 \setminus \mathbb{B}^4\}$, which is in accordance with the topology of the Chong-Cvetič-Lü-Pope solution. Our boundary conditions—in particular, the *rod structure*, which was first introduced by Harmark [76] based on earlier work for static solutions [77]—are accordingly specified in the manner discussed in Sec. IV. The topology theorem [78–80] yields that in five dimensions, cross sections of the event horizon must be topologically either a sphere, a ring, or a lens-space. The requirement (2) excludes some interesting class of solutions to be dealt with. It would be interesting to consider generalization of our uniqueness theorem to include solutions with nonspherical horizon topology.

We would like to emphasize that even under these restrictive assumptions (1) and (2), still it is not at all obvious whether black holes in EMCS theory are uniquely specified by their global charges. In fact, it has been shown by numerical studies [81] that when the value of the Chern-Simons coupling is larger than some critical value, spherical black holes in such a general EMCS theory no longer enjoy the uniqueness property. In the present paper, motivated from sting theory, we restrict attention to a special class of EMCS theory, that is, five-dimensional minimal supergravity and then are able to show the above uniqueness theorem. It would be interesting to find the precise onset of this nonuniqueness property in general EMCS theory, using the formulas developed in this paper.

The rest of the paper is devoted to prove the above uniqueness theorem. In the next section, we present the metric and the gauge potential in Einstein-Maxwell-Chern-Simons theory with three Killing symmetries, introduce the Weyl-Papapetrou coordinates, and reduce the system to a nonlinear sigma model with certain symmetries. In Sec. III, using the matrix representation of the sigma model, we derive a divergence identity/Mazur identity associated with our nonlinear sigma model. A good part of the material in Sec. II and the first part of Sec. III concerning the matrix representation is discussed in [36]. Then, in Sec. IV, presenting our boundary conditions for our sigma model fields and using the Mazur identity, we show that if two asymptotically flat black hole solutions have the same conserved charges, i.e., the mass, electric charge, and two angular momenta, then they must coincide with each other, and complete our proof of the uniqueness theorem. In Sec. V, we summarize our results and discuss possible generalization of our theorem to include nonspherical black objects. We discuss that in order to have a uniqueness theorem for black ring solutions in EMCS theory, we need to specify rod-data, besides global charges and horizon topology. In Appendix A, we explicitly compute relevant components of the Maxwell-field. In Appendix B, we provide the black hole solution of Chong-Cvetič-Lü-Pope, and study, in terms of the Weyl-Papapetrou coordinates, the limiting behavior of the solution near relevant boundaries.

II. EINSTEIN-MAXWELL-CHERN-SIMONS SYSTEM WITH SYMMETRIES

We consider the bosonic sector of five-dimensional minimal supergravity theory, which can be obtained by a suitable truncation of 11-dimensional supergravity. The five-dimensional action is given by

$$S = \frac{1}{16\pi}\left[\int dx^5\sqrt{-g}\left(R - \frac{1}{4}F^2\right) - \frac{1}{3\sqrt{3}}\int F\wedge F\wedge A\right], \tag{1}$$

where we set a Newton constant to be unity and $F = dA$. Varying this action (1), we derive the Einstein equation

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{1}{2}\left(F_{\mu\lambda}F_{\nu}{}^{\lambda} - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right), \tag{2}$$

and the Maxwell equation

$$d*F + \frac{1}{\sqrt{3}}F\wedge F = 0. \tag{3}$$

The purpose of this section is to reduce the above five-dimensional Einstein-Maxwell-Chern-Simons system to a nonlinear sigma model with certain target space symmetries. We first consider consequences of the existence of commuting Killing vector fields in our spacetime and identify the target space variables in Sec.. II A. Then, having another (stationary) Killing vector field, we introduce the Weyl-Papapetrou coordinates and write down explicitly the desired nonlinear sigma model action in Sec. II B.

A. Two Killing system

Let $\xi_a (a = 1, 2)$ be two mutually commuting Killing vector fields, so that $[\xi_a, \xi_b] = 0$, $\mathcal{L}_{\xi_a}g = 0$, and $\mathcal{L}_{\xi_a}F = 0$. Then, introducing the coordinates x^a as Killing parameters of ξ_a (so that $\xi_a = \partial/\partial x^a$), one can express the metric g and the gauge potential one-form A, respectively, as

$$ds^2 = \lambda_{ab}(dx^a + a^a{}_i dx^i)(dx^b + a^b{}_j dx^j) + |\tau|^{-1}h_{ij}dx^i dx^j, \tag{4}$$

$$A = A_a dx^a + A_i dx^i, \tag{5}$$

where the functions $\tau := -\det(\lambda_{ab})$, $a^a{}_i$, h_{ij}, A_a, and A_i $(i = 3,4,5)$ are independent of the coordinates x^a.

Let us define the electric one-form E_a with respect to ξ_a by

$$E_a = -i_{\xi_a}F. \tag{6}$$

Then the exterior derivatives of the electric one-forms yield

$$dE_a = i_{\xi_a}dF - \mathcal{L}_{\xi_a}F = 0, \tag{7}$$

where $F = dA$ is used. Hence there exist locally the potentials ψ_a such that

$$d\psi_a = -\frac{1}{\sqrt{3}}i_{\xi_a}F. \tag{8}$$

Hence, the gauge potential can be written as

$$A = \sqrt{3}\psi_a dx^a + A_i dx^i, \tag{9}$$

where ψ_a is also independent of the coordinates x^a. Next, define the magnetic one-form B by

$$B = *(\xi_1\wedge\xi_2\wedge F). \tag{10}$$

Noting that B can be rewritten as $B = *(\xi_1\wedge\xi_2\wedge F) = -i_{\xi_2}*(\xi_1\wedge F) = i_{\xi_2}i_{\xi_1}*F$ and using the identity $di_{\xi_2}i_{\xi_1} = i_{\xi_2}i_{\xi_1}d + i_{\xi_1}\mathcal{L}_{\xi_2} - i_{\xi_2}\mathcal{L}_{\xi_1}$, we can write the exterior derivative of B as

$$dB = i_{\xi_2}i_{\xi_1}d*F. \tag{11}$$

Then, using the Maxwell Eq. (3), we find that

$$dB = -\frac{1}{\sqrt{3}}i_{\xi_2}i_{\xi_1}F\wedge F = \frac{2}{\sqrt{3}}E_1\wedge E_2 = 2\sqrt{3}d\psi_1\wedge d\psi_2 = \sqrt{3}d(\psi_1 d\psi_2 - \psi_2 d\psi_1). \tag{12}$$

This immediately implies that there exists the magnetic potential μ such that

$$d\mu = \frac{1}{\sqrt{3}}B - \epsilon^{ab}\psi_a d\psi_b, \tag{13}$$

where $\epsilon^{12} = -\epsilon^{21} = 1$. We also introduce the twist one-form by

$$V_a = *(\xi_1\wedge\xi_2\wedge d\xi_a). \tag{14}$$

Using the Einstein-equation, we can write the exterior derivative of V_a as

$$\begin{aligned} dV_a &= 2*(\xi_1\wedge\xi_2\wedge R(\xi_a)) \\ &= -\tau^{-1}i_{\xi_2}i_{\xi_1}*^2(\xi_1\wedge\xi_2\wedge E_a\wedge B) \\ &= -E_a\wedge B \\ &= -3d\psi_a\wedge(d\mu + \epsilon^{bc}\psi_b d\psi_c) \\ &= -3d[\psi_a d\mu] - d[\psi_a\epsilon^{bc}\psi_b d\psi_c], \end{aligned} \tag{15}$$

where $R(\xi_a)$ in the first line is the Ricci one-form. Therefore, there exists the twist potentials ω_a that satisfy

$$d\omega_a = V_a + \psi_a(3d\mu + \epsilon^{bc}\psi_b d\psi_c). \tag{16}$$

Thus, as a consequence of the existence of isometries ξ_a, we have eight scalar fields λ_{ab}, ω_a, ψ_a, μ, $(a = 1, 2)$, which we denote collectively by coordinates $\Phi^A = (\lambda_{ab}, \omega_a, \psi_a, \mu)$. As we will see soon, other components, such as $a^a{}_i$, A_i are determined by Φ^A. Then, we can find that the equations of motion, Eqs. (2) and (3), are cast into a set of equations derived from the following action for sigma-model Φ^A coupled with three-dimensional gravity with respect to the metric h_{ij},

$$S = \int_\Sigma\left(\mathcal{R}^h - G_{AB}\frac{\partial\Phi^A}{\partial x^i}\frac{\partial\Phi^B}{\partial x^j}h^{ij}\right)\sqrt{|h|}dx^3, \tag{17}$$

where the target space metric, G_{AB}, is given by

$$G_{AB}d\Phi^A d\Phi^B = \frac{1}{4}\mathrm{Tr}(\lambda^{-1}d\lambda\lambda^{-1}d\lambda) + \frac{1}{4}\tau^{-2}d\tau^2 + \frac{3}{2}d\psi^T\lambda^{-1}d\psi - \frac{1}{2}\tau^{-1}V^T\lambda^{-1}V - \frac{3}{2}\tau^{-1}(d\mu + \epsilon^{ab}\psi_a d\psi_b)^2, \tag{18}$$

where $\lambda = (\lambda_{ab})$, $\psi = (\psi_1, \psi_2)^T$, $\omega = (\omega_1, \omega_2)^T$ and $V = d\omega - \psi(3d\mu + \epsilon^{bc}\psi_b d\psi_c)$. Varying the action by h_{ij}, we obtain the equations

$$R^h_{ij} = G_{AB}\frac{\partial\Phi^A}{\partial x^i}\frac{\partial\Phi^B}{\partial x^j}, \tag{19}$$

where R^h_{ij} denotes the Ricci tensor with respect to h_{ij}. Next varying the action by Φ^A, we derive the equation

$$\Delta_h\Phi^A + h^{ij}\Gamma^A_{BC}\frac{\partial\Phi^B}{\partial x^i}\frac{\partial\Phi^C}{\partial x^j} = 0, \tag{20}$$

where Δ_h is the Laplacian with respect to the three-dimensional metric h_{ij} and Γ^A_{BC} is the Christoffel symbol with respect to the target space metric G_{AB}.

B. Weyl-Papapetrou form

Now we consider another Killing vector field ξ_3 which is assumed to commute with the other Killing vectors ξ_a and will be identified below as the asymptotic time-translation Killing vector field. Let us consider the condition that the two-dimensional distribution orthogonal to three Killing vector fields $\xi_I (I = 1, 2, 3)$ becomes integrable. The commutativity of Killing vector fields, $[\xi_I, \xi_J] = 0$, enables us to find coordinate system x_I $(I = 1, 2, 3)$, so that $\xi_I = \partial/\partial x^I$ and the coordinate components of the metric become independent of x^I. We now recall the following theorem about the integrability of two-planes orthogonal to Killing vector fields [77,76]:

Proposition. If three mutually commuting Killing vector fields ξ_I $(I = 1, 2, 3)$, *in a five-dimensional spacetime satisfy the following two conditions*

(1) $\xi_1^{[\mu_1}\xi_2^{\mu_2}\xi_3^{\mu_2]}D^\nu\xi_I^\rho = 0$ *holds at least one point of the spacetime for a given* $I = 1, 2, 3$,

(2) $\xi_I^\nu R_\nu^{[\rho}\xi_1^{\mu_1}\xi_2^{\mu_2}\xi_3^{\mu_2]} = 0$ *holds for all* $I = 1, 2, 3$, *then the two-planes orthogonal to the Killing vector fields* ξ_I $(I = 1, 2, 3)$ *are integrable*.

Note here that one can replace a pair of Killing vector fields (ξ_1, ξ_2) above by another pair (ξ_2, ξ_3). We denote the corresponding quantities in the choice (ξ_2, ξ_3) with *tilde*˜. For example, we denote the twist one-forms with respect to (ξ_2, ξ_3) by

$$\tilde V_{\tilde a} = *(\xi_2 \wedge \xi_3 \wedge d\xi_{\tilde a}), \tag{21}$$

where $\tilde a = 2, 3$. Then, using $i_{\xi_I}d\psi_a = i_{\xi_I}d\mu = 0$, $i_{\xi_I}d\tilde\psi_a = i_{\xi_I}d\tilde\mu = 0$, and Eq. (16), we show $i_{\xi_I}dV_a = i_{\xi_I}d\tilde V_{\tilde a} = 0$, and hence have

$$*(\xi_1 \wedge \xi_2 \wedge \xi_3 \wedge R(\xi_a)) = -i_{\xi_3} * (\xi_1 \wedge \xi_2 \wedge R(\xi_a)) = -\frac{1}{2}i_{\xi_3}dV_a = 0, \tag{22}$$

and

$$*(\xi_1 \wedge \xi_2 \wedge \xi_3 \wedge R(\xi_3)) = -i_{\xi_1} * (\xi_2 \wedge \xi_3 \wedge R(\xi_3)) = -\frac{1}{2}i_{\xi_1}d\tilde V_3 = 0. \tag{23}$$

This implies that the condition 2 holds in our present system (33) with three commuting Killing vector fields. Furthermore, the axial symmetry of at least one of ξ_I $(I = 1, 2, 3)$ implies that the condition 1 also holds on the axis of rotation. Therefore, the two-dimensional surface orthogonal to three ξ_I is integrable.

Now, without loss of generality, we choose our three coordinates (x^1, x^2, x^3) as the three Killing parameters, so that $\xi_3 = \partial/\partial t$ denotes the stationary (asymptotic time-translation) Killing vector field in our spacetimes and $\xi_1 = \partial/\partial\phi$ and $\xi_2 = \partial/\partial\psi$ are two independent axial Killing symmetries. Then, from the above observation, we can express the three-dimensional metric h_{ij} by $h = h_{pq}dx^p dx^q - \rho^2 dt^2$ $(p\ q = 4, 5)$, where $\rho^2 = -\det(g_{IJ})$. Note that the function ρ is globally well-defined [82]. That ρ is a harmonic function can be seen by looking at the (tt)-component of Eq. (19), which is written

$$R_{tt} = \rho\hat D^2\rho = 0, \tag{24}$$

where $\hat D_p$ is the covariant derivative associated with the two-dimensional metric h_{pq}. Let z be harmonic function conjugate to ρ which satisfies $\hat D^2 z = 0$, $\hat D_p\rho\hat D^p z = 0$, $\hat D_p\rho\hat D^p\rho = \hat D_p z\hat D^p z$. Choose the coordinates (x^4, x^5) as $x^4 = \rho$ and $x^5 = z$. Then, the metric can be written in the Weyl-Papapetrou type form as

$$ds^2 = \lambda_{\phi\phi}(d\phi + a^\phi{}_t dt)^2 + \lambda_{\psi\psi}(d\psi + a^\psi{}_t dt)^2 + 2\lambda_{\phi\psi}(d\phi + a^\phi{}_t dt)(d\psi + a^\psi{}_t dt) + |\tau|^{-1}[e^{2\sigma}(d\rho^2 + dz^2) - \rho^2 dt^2], \tag{25}$$

where all the metric components depend only on ρ and z.

In this coordinate system, Φ^A are determined by the equations of motion

$$\Delta_\gamma\Phi^A + \Gamma^A_{BC}[\Phi^B_{,\rho}\Phi^C_{,\rho} + \Phi^C_{,z}\Phi^C_{,z}] = 0, \tag{26}$$

where Δ_γ is the Laplacian with respect to the abstract three-dimensional metric $\gamma = d\rho^2 + dz^2 + \rho^2 d\varphi^2$. On the other hand, once Φ^A are given, one can completely determine σ, $a^\phi{}_t$, $a^\psi{}_t$, A_i. In fact, the function σ is determined by

$$\frac{2}{\rho}\sigma_{,\rho} = R^h_{\rho\rho} - R^h_{zz} = G_{AB}[\Phi^A_{,\rho}\Phi^B_{,\rho} - \Phi^A_{,z}\Phi^B_{,z}], \tag{27}$$

$$\frac{1}{\rho}\sigma_{,z} = R^h_{\rho z} = G_{AB}\Phi^A_{,\rho}\Phi^B_{,z}. \tag{28}$$

The integrability $\sigma_{,\rho z} = \sigma_{,z\rho}$ is assured by Eq. (26). From Eq. (16), the metric functions $a^a{}_t$ are determined by

$$a^a{}_{t,\rho} = \rho\tau^{-1}\lambda^{ab}(\omega_{b,z} - 3\psi_b\mu_{,z} - \psi_b\epsilon^{cd}\psi_c\psi_{d,z}) \quad (29)$$

$$a^a{}_{t,z} = -\rho\tau^{-1}\lambda^{ab}(\omega_{b,\rho} - 3\psi_b\mu_{,\rho} - \psi_b\epsilon^{cd}\psi_c\psi_{d,\rho}). \quad (30)$$

As shown in Appendix A, we can set $A_\rho = A_z = 0$. Therefore it follows from Eq. (13) that the t-component of the gauge potential A is determined by

$$A_{t,\rho} = \sqrt{3}[a^a{}_t\psi_{a,\rho} - \rho\tau^{-1}(\mu_{,z} + \epsilon^{bc}\psi_b\psi_{c,z})], \quad (31)$$

$$A_{t,z} = \sqrt{3}[a^a{}_t\psi_{a,z} + \rho\tau^{-1}(\mu_{,\rho} + \epsilon^{bc}\psi_b\psi_{c,\rho})]. \quad (32)$$

Thus, once we determine $\Phi^A = (\lambda_{ab}, \omega_a, \psi_a, \mu)$, we can specify the solutions of the system given originally by the action, Eq. (1), with our Killing symmetry assumption. It turns out that the above equations of motion, Eq. (26), for Φ^A are derived from the following action

$$\begin{aligned} S &= \int d\rho dz\rho[G_{AB}(\partial\Phi^A)(\partial\Phi^B)] \\ &= \int d\rho dz\rho\Big[\frac{1}{4}\,\mathrm{Tr}(\lambda^{-1}\partial\lambda\lambda^{-1}\partial\lambda) + \frac{1}{4}\tau^{-2}\partial\tau^2 \\ &\quad + \frac{3}{2}\partial\psi^T\lambda^{-1}\partial\psi - \frac{1}{2}\tau^{-1}\upsilon^T\lambda^{-1}\upsilon - \frac{3}{2}\tau^{-1}(\partial\mu \\ &\quad + \epsilon^{ab}\psi_a\partial\psi_b)^2\Big], \end{aligned} \quad (33)$$

where $\upsilon = \partial\omega - \psi\ (3\partial\mu + \epsilon^{bc}\psi_b\partial\psi_c)$. This action is invariant under the global $G_{2(2)}$ transformation.

III. MAZUR IDENTITY

In the proof of uniqueness theorems for four-dimensional charged rotating black holes, a key role was played by a certain global identity—called the Mazur identity. This is also the case for five-dimensional charged rotating black holes. In this section, we present the Mazur type identity for our nonlinear sigma models derived in the previous section. The derivation parallels that for the vacuum Einstein case given in other literature, e.g., Morisawa and Ida [50], and therefore we present here only some key formulas.

Following [36], we introduce the $G_{2(2)}/SO(4)$ coset matrix, M, defined by

$$M = \begin{pmatrix} \hat{A} & \hat{B} & \sqrt{2}\hat{U} \\ \hat{B}^T & \hat{C} & \sqrt{2}\hat{V} \\ \sqrt{2}\hat{U}^T & \sqrt{2}\hat{V}^T & \hat{S} \end{pmatrix}, \quad (34)$$

where $\hat{A}$ and $\hat{C}$ are symmetric 3×3 matrices, $\hat{B}$ is a 3×3 matrix, $\hat{U}$ and $\hat{V}$ are 3-component column matrices, and $\hat{S}$ is a scalar, defined, respectively, by

$$\hat{A} = \begin{pmatrix} [(1-y)\lambda + (2+x)\psi\psi^T - \tau^{-1}\tilde{\omega}\tilde{\omega}^T + \mu(\psi\psi^T\lambda^{-1}\hat{J} - \hat{J}\lambda^{-1}\psi\psi^T)] & \tau^{-1}\tilde{\omega} \\ \tau^{-1}\tilde{\omega}^T & -\tau^{-1} \end{pmatrix},$$

$$\hat{B} = \begin{pmatrix} (\psi\psi^T - \mu\hat{J})\lambda^{-1} - \tau^{-1}\tilde{\omega}\psi^T\hat{J} & [(-(1+y)\lambda\hat{J} - (2+x)\mu + \psi^T\lambda^{-1}\tilde{\omega})\psi + (z - \mu\hat{J}\lambda^{-1}\tilde{)}\omega] \\ \tau^{-1}\psi^T\hat{J} & -z \end{pmatrix},$$

$$\hat{C} = \begin{pmatrix} (1+x)\lambda^{-1} - \lambda^{-1}\psi\psi^T\lambda^{-1} & \lambda^{-1}\tilde{\omega} - \hat{J}(z - \mu\hat{J}\lambda^{-1})\psi \\ \tilde{\omega}^T\lambda^{-1} + \psi^T(z + \mu\lambda^{-1}\hat{J})\hat{J} & [\tilde{\omega}^T\lambda^{-1}\tilde{\omega} - 2\mu\psi^T\lambda^{-1}\tilde{\omega} - \tau(1 + x - 2y - xy + z^2)] \end{pmatrix},$$

$$\hat{U} = \begin{pmatrix} (1 + x - \mu\hat{J}\lambda^{-1})\psi - \mu\tau^{-1}\tilde{\omega} \\ \mu\tau^{-1} \end{pmatrix}, \qquad \hat{V} = \begin{pmatrix} (\lambda^{-1} + \mu\tau^{-1}\hat{J})\psi \\ \psi^T\lambda^{-1}\tilde{\omega} - \mu(1 + x - z) \end{pmatrix}, \qquad \hat{S} = 1 + 2(x - y),$$

with

$$\tilde{\omega} = \omega - \mu\psi, \quad (35)$$

$$x = \psi^T\lambda^{-1}\psi, \qquad y = \tau^{-1}\mu^2, \qquad z = y - \tau^{-1}\psi^T\hat{J}\,\tilde{\omega}, \quad (36)$$

and the 2×2 matrix,

$$\hat{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (37)$$

We note that this 7×7 matrix M is symmetric, $M^T = M$, and unimodular, $\det(M) = 1$. Since we choose the Killing vector fields ξ_ϕ and ξ_ψ to be spacelike, all the eigenvalues of M are real and positive. Therefore, there exists an $G_{2(2)}$ matrix $\hat{g}$ such that

$$M = \hat{g}\hat{g}^T. \quad (38)$$

We define a current matrix as

$$J_i = M^{-1}\partial_i M, \quad (39)$$

which is conserved if the scalar fields are the solutions of the equation of motion derived by the action (33). Then, the action (33) can be written in terms of J and M as follows

$$\begin{aligned} S &= \frac{1}{4}\int d\rho dz\rho\,\mathrm{tr}(J_iJ^i) \\ &= \frac{1}{4}\int d\rho dz\rho\,\mathrm{tr}(M^{-1}\partial_i MM^{-1}\partial^i M). \end{aligned} \quad (40)$$

Thus, the matrix M completely specify the solutions to our system.

Let us now consider two sets of field configurations, $M_{[0]}$ and $M_{[1]}$, that satisfy the equations of motion derived from the action, Eq. (33). We denote the difference between the value of the functional obtained from the field configuration $M_{[1]}$ and the value obtained from $M_{[0]}$ as a bull's eye $^{\circ}$, e.g.,

$$\overset{\circ}{J}{}^{i} = J^{i}_{[1]} - J^{i}_{[0]}, \tag{41}$$

where the subscripts $_{[0]}$ and $_{[1]}$ denote, respectively, the quantities associated with the field configurations $M_{[0]}$ and $M_{[1]}$. The deviation matrix, Ψ, is then defined by

$$\Psi = \overset{\circ}{M} M^{-1}_{[0]} = M_{[1]} M^{-1}_{[0]} - \mathbf{1}, \tag{42}$$

where $\mathbf{1}$ is the unit matrix. Taking the derivative of this, we have the relation between the derivative of the deviation matrix and $\overset{\circ}{J}{}^{i}$,

$$D^{i}\Psi = M_{[1]} \overset{\circ}{J}{}^{i} M^{-1}_{[0]}, \tag{43}$$

where D_i is a covariant derivative associated with the abstract three-metric γ. Taking, further, the divergence of the above formula and also the trace of the matrix elements, we have the following divergence identity

$$D_i D^i \mathrm{tr}\Psi = \mathrm{tr}(M_{[1]} \overset{\circ}{J} M^{-1}_{[0]}), \tag{44}$$

where we have also used the conservation equation $D_i J^i = 0$. Then, integrating this divergence identity over the region $\Sigma = \{(\rho, z) | \rho \geq 0, -\infty < z < \infty\}$, we obtain the Mazur identity,

$$\int_{\partial\Sigma} \rho \partial_p \mathrm{tr}\Psi dS^p = \int_{\Sigma} \rho \hat{h}_{pq} \mathrm{tr}(\mathcal{M}^{Tp} \mathcal{M}^{q}) d\rho dz, \tag{45}$$

where $\hat{h}_{pq}$ is the two-dimensional flat metric

$$\hat{h} = d\rho^2 + dz^2, \tag{46}$$

and the matrix $\mathcal{M}$ is defined by

$$\mathcal{M}^{p} = \hat{g}^{-1}_{[0]} \overset{\circ}{J}{}^{Tp} \hat{g}_{[1]}. \tag{47}$$

Now we note that the right-hand side of the identity, (45), is non-negative. Therefore, if we impose the boundary conditions at $\partial\Sigma$, under which the left-hand side of Eq. (45) vanishes, then we must have $\overset{\circ}{J}{}^{i} = 0$. In that case, it follows from Eq. (43) that Ψ must be a constant matrix over the region Σ. Therefore, in particular, if Ψ is shown to be zero on some part of the boundary $\partial\Sigma$, it immediately follows that Ψ must be identically zero over the base space Σ, implying that the two solutions $M_{[0]}$ and $M_{[1]}$ must coincide with each other. This is indeed the case under our boundary conditions discussed in the next section.

IV. BOUNDARY VALUE PROBLEMS

In this section, we derive necessary boundary conditions for determining the scalar fields $\Phi^A = (\lambda_{ab}, \omega_a, \psi_a, \mu)$, requiring asymptotic flatness at infinity, regularity on the two rotation axes (i.e., the ϕ-invariant plane and the ψ-invariant plane), and on the event horizon (of which cross sections are assumed to be topologically spherical). Note that by *asymptotically flat*, we mean that the spacetime metric has the following falloff behavior at large distances,

$$\begin{aligned} ds^2 \simeq & \left(-1 + \frac{8M_{\mathrm{ADM}}}{3\pi r^2} + \mathcal{O}(r^{-3})\right) dt^2 - \left(\frac{8J_{\phi}\sin^2\theta}{\pi r^2} \right. \\ & \left. + \mathcal{O}(r^{-3})\right) dt d\phi - \left(\frac{8J_{\psi}\cos^2\theta}{\pi r^2} + \mathcal{O}(r^{-3})\right) dt d\psi \\ & + (1 + \mathcal{O}(r^{-1}))(dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2 \\ & + \cos^2\theta d\psi^2)), \end{aligned} \tag{48}$$

having the spherical spatial infinity, S^3_{∞}. Here the constants M_{ADM} and J_a are the asymptotic conserved mass and angular momenta. Since we are concerned with stationary, axisymmetric spacetimes with Killing symmetries ξ_I, the conserved charges M_{ADM} and J_a are defined, respectively, by

$$M_{\mathrm{ADM}} = -\frac{3}{32\pi} \int_{S^3_{\infty}} dS^{\mu\nu} \nabla_{\mu} (\xi_3)_{\nu}, \tag{49}$$

$$J_a = \frac{1}{16\pi} \int_{S^3_{\infty}} dS^{\mu\nu} \nabla_{\mu} (\xi_a)_{\nu}. \tag{50}$$

We write below our boundary conditions for Φ^A in terms of the Weyl-Papapetrou coordinates. Therefore, in particular, relevant conditions at infinity—see below Eqs. (87)–(93)—are derived from the above falloff behavior, Eq. (48), by the coordinate transformation

$$\rho = \frac{1}{2} r^2 \sin 2\theta, \qquad z = \frac{1}{2} r^2 \cos 2\theta. \tag{51}$$

Then, we can find that the boundary conditions given in this section are, in fact, the same as the limiting behavior of Φ^A for the exact solution of Chong-Cvetič-Lü-Pope black hole [4,5] at the corresponding boundaries, which we discuss in Appendix B.

In terms of the Weyl-Papapetrou coordinate system introduced in Sec. II B, and the rod structure [76], the boundary $\partial\Sigma$ of the base space $\Sigma = \{(\rho, z) | \rho > 0, -\infty < z < \infty\}$ is described as a set of three rods and the infinity: Namely,

(i) the ϕ-invariant plane: $\partial\Sigma_{\phi} = \{(\rho, z) | \rho = 0, k^2 < z < \infty\}$ with the rod vector $v = (0, 1, 0)$,

(ii) the horizon: $\partial\Sigma_{\mathcal{H}} = \{(\rho, z) | \rho = 0, -k^2 < z < k^2\}$,

(iii) the ψ-invariant plane: $\partial\Sigma_{\psi} = \{(\rho, z) | \rho = 0, -\infty < z < -k^2\}$ with the rod vector $v = (0, 0, 1)$,

(iv) the infinity: $\partial\Sigma_{\infty} = \{(\rho, z) | \sqrt{\rho^2 + z^2} \to \infty$ with $z/\sqrt{\rho^2 + z^2}$ finite$\}$, where here and hereafter $\mathcal{H}$ denotes a spatial cross section of the event horizon. Accordingly, the boundary integral in the left-hand side of the Mazur identity, Eq. (45), is decomposed into the integrals over the three rods (i)–(iii), and the integral at infinity (iv), as

$$\int_{\partial\Sigma}\rho\partial_p \operatorname{tr}\Psi dS^p = \int_{-\infty}^{-k^2}\rho\frac{\partial \operatorname{tr}\Psi}{\partial z}dz + \int_{-k^2}^{k^2}\rho\frac{\partial \operatorname{tr}\Psi}{\partial z}dz + \int_{k^2}^{\infty}\rho\frac{\partial \operatorname{tr}\Psi}{\partial z}dz + \int_{\partial\Sigma_\infty}\rho\partial_a \operatorname{tr}\Psi dS^a. \tag{52}$$

In order to evaluate this boundary integral, let us first consider the integrals of the twist one-forms $d\omega_a$ along the z-axis. By definition, the partial derivatives with respect to z of the twist potentials ω_a vanish on both rotation axes. This means that the twist potentials ω_a are constant over the ϕ-invariant plane and the ψ-invariant plane. Therefore, the integral can be written as

$$\int_{-\infty}^{\infty}\omega_{a,z}dz = \int_{-k^2}^{k^2}\omega_{a,z}dz \tag{53}$$

$$= [\omega_a]_{z=-k^2}^{z=k^2}. \tag{54}$$

On the other hand, by Stokes's theorem, the integral of $d\omega_a$ on the horizon is evaluated as

$$\int_{\partial\Sigma_{\mathcal{H}}}d\omega_a = \int_{\partial\Sigma_\infty}d\omega_a = \int_{\partial\Sigma_\infty}V_a + \int_{\partial\Sigma_\infty}\psi_a(3d\mu + \epsilon^{bc}\psi_b d\psi_c). \tag{55}$$

We find that the first integral in the right-hand side of Eq. (55) is proportional to the angular momenta J_a, defined by Eq. (50) above. As will be seen later, the second integral vanishes at infinity. Hence, using the degrees of freedom in adding a constant to ω_a, we can always set the value of ω_a on the two rotation axes to be

$$\omega_a(z) = -\frac{2J_a}{\pi}, \tag{56}$$

for $z \in [k^2, \infty]$, and

$$\omega_a(z) = \frac{2J_a}{\pi}, \tag{57}$$

for $z \in [-\infty, -k^2]$.

Next, consider the integral of $\mu_{,z}$ on the horizon $\partial\Sigma_{\mathcal{H}}$. The derivative of the potential, $d\mu$, vanishes on the two rotation axes by definition. Hence the integral along the z-axis becomes

$$\int_{-\infty}^{\infty}\mu_{,z}dz = \int_{-k^2}^{k^2}\mu_{,z}dz = [\mu]_{z=-k^2}^{z=k^2}. \tag{58}$$

We find that this integral is proportional to the electric charge Q defined by

$$Q = \frac{1}{16\pi}\int_{\mathcal{H}}\left(*F + \frac{1}{\sqrt{3}}A\wedge F\right). \tag{59}$$

In fact, straightforward calculation shows

$$\begin{aligned}\frac{1}{16\pi}\int_{\mathcal{H}}\left(*F + \frac{1}{\sqrt{3}}A\wedge F\right) &= \frac{\pi}{4}\int_{-k^2}^{k^2}\Big[\frac{\tau}{\rho}(A_{t,\rho} - a^\phi{}_t A_{\phi,\rho} - a^\psi{}_t A_{\psi,\rho}) \\ &\quad - \frac{1}{\sqrt{3}}(A_\phi A_{\psi,z} - A_\psi A_{\phi,z})\Big]dz \\ &= \frac{\pi\sqrt{3}}{4}\int_{-k^2}^{k^2}\mu_{,z}dz.\end{aligned} \tag{60}$$

Hence, without loss of generality, μ can be set to be

$$\mu = -\frac{2Q}{\sqrt{3}\pi}, \tag{61}$$

for $\rho = 0$, $z \in [-\infty, -k^2]$, and

$$\mu = \frac{2Q}{\sqrt{3}\pi}, \tag{62}$$

for $\rho = 0$, $z \in [k^2, \infty]$.

Now we would like to show that the boundary integral, Eq. (52), indeed vanishes under our preferable boundary conditions that require the regularity on the three rods and asymptotic flatness at infinity. For this purpose, in the following we evaluate the limiting behavior of the integrand, $\rho\partial_z \operatorname{tr}\Psi$, of Eq. (52), separately on each boundary (i)–(iv)

(i) ϕ-invariant plane: $\partial\Sigma_\phi = \{(\rho, z)|\rho = 0, k^2 < z < \infty\}$. The regularity on the ϕ-invariant plane requires that for $\rho \longrightarrow 0$, the scalar fields behave as

$$\lambda_{\phi\phi} \simeq \mathcal{O}(\rho^2), \tag{63}$$

$$\lambda_{\psi\psi} \simeq \mathcal{O}(1), \tag{64}$$

$$\lambda_{\phi\psi} \simeq \mathcal{O}(\rho^2), \tag{65}$$

$$\omega_\phi \simeq -\frac{2J_\phi}{\pi} + \mathcal{O}(\rho^2), \tag{66}$$

$$\omega_\psi \simeq -\frac{2J_\psi}{\pi} + \mathcal{O}(\rho^2), \tag{67}$$

and

$$\psi_\phi \simeq \mathcal{O}(\rho^2), \tag{68}$$

$$\psi_\psi \simeq \mathcal{O}(1), \tag{69}$$

$$\mu \simeq \frac{2Q}{\sqrt{3}\pi} + \mathcal{O}(\rho^2), \tag{70}$$

where the boundary conditions, Eqs. (63)–(65) and Eqs. (68) and (69), come from the requirement that $\partial\Sigma_\phi$ is the ϕ-invariant plane, i.e., the plane invariant under the rotation with respect to the axial Killing vector $\partial/\partial\phi$. The conditions, Eqs. (66) and (67), are derived from Eq. (56). In the derivation of the condition (70), Eq. (62) is used. Then for two solutions, $M_{[0]}$ and $M_{[1]}$, with the same mass, the same angular momenta, and the same electric charge, $\rho\,\mathrm{tr}\Psi$ behaves as

$$\rho\partial_z \mathrm{tr}\Psi \simeq O(\rho). \tag{71}$$

(ii) Horizon: $\partial\Sigma_{\mathcal{H}} = \{(\rho, z)|\rho = 0, -k^2 < z < k^2\}$. The regularity on the horizon requires that for $\rho \to 0$,

$$\lambda_{ab} \simeq \mathcal{O}(1), \qquad \omega_a \simeq \mathcal{O}(1), \tag{72}$$

$$\psi_a \simeq \mathcal{O}(1), \qquad \mu \simeq \mathcal{O}(1). \tag{73}$$

Therefore, for $\rho \to 0$, $\rho\,\mathrm{tr}\Psi$ behaves as

$$\rho\partial_z \mathrm{tr}\Psi \simeq O(\rho). \tag{74}$$

(iii) ψ-invariant plane: $\partial\Sigma_\psi = \{(\rho, z)|\rho = 0, -\infty < z < -k^2\}$. Similarly to the case (i), the regularity on the ϕ-invariant plane requires

$$\lambda_{\psi\psi} \simeq \mathcal{O}(\rho^2), \tag{75}$$

$$\lambda_{\phi\phi} \simeq \mathcal{O}(1), \tag{76}$$

$$\lambda_{\phi\psi} \simeq \mathcal{O}(\rho^2), \tag{77}$$

$$\omega_\phi \simeq \frac{2J_\phi}{\pi} + \mathcal{O}(\rho^2), \tag{78}$$

$$\omega_\psi \simeq \frac{2J_\psi}{\pi} + \mathcal{O}(\rho^2), \tag{79}$$

and

$$\psi_\phi \simeq \mathcal{O}(1), \tag{80}$$

$$\psi_\psi \simeq \mathcal{O}(\rho^2), \tag{81}$$

$$\mu \simeq -\frac{2Q}{\sqrt{3}\pi} + \mathcal{O}(\rho^2). \tag{82}$$

Therefore, for $\rho \to 0$, $\rho\,\mathrm{tr}\Psi$ behaves as

$$\rho\partial_z \mathrm{tr}\Psi \simeq O(\rho). \tag{83}$$

(iv) Infinity: $\partial\Sigma_\infty = \{(\rho, z)|\sqrt{\rho^2 + z^2} \to \infty$ with $z/\sqrt{\rho^2 + z^2}$ finite$\}$. Recall that the three-dimensional metric $g = (g_{IJ})$ $(I, J = t, \phi, \psi)$ is subject to the constraint

$$\det(g) = -\rho^2. \tag{84}$$

Therefore, using the constraint and the formula,

$$\begin{aligned}\det(g + \delta g) &= \det[g(1 + g^{-1}\delta g)] \\ &= -\rho^2(1 + \mathrm{tr}(g^{-1}\delta g) + \det(g^{-1}\delta g)) \\ &\simeq -\rho^2(1 + \mathrm{tr}(g^{-1}\delta g)),\end{aligned} \tag{85}$$

we can see in the next order that the metric has to satisfy the constraint

$$\sum_{I=t,\phi,\psi} \frac{\delta g_{II}}{g_{II}} = 0, \tag{86}$$

which is the same constraint as in the vacuum case [76]. Then, the asymptotic flatness, Eq. (48), requires that the limiting behavior of the metric be

$$g_{tt} \simeq -1 + \frac{4M_{\mathrm{ADM}}}{3\pi}\frac{1}{\sqrt{\rho^2 + z^2}} + \mathcal{O}\left(\frac{1}{\rho^2 + z^2}\right), \tag{87}$$

$$g_{t\phi} \simeq -\frac{J_\phi}{\pi}\frac{\sqrt{\rho^2 + z^2} - z}{\rho^2 + z^2} + \mathcal{O}\left(\frac{1}{\rho^2 + z^2}\right), \tag{88}$$

$$g_{t\psi} \simeq -\frac{J_\psi}{\pi}\frac{\sqrt{\rho^2 + z^2} + z}{\rho^2 + z^2} + \mathcal{O}\left(\frac{1}{\rho^2 + z^2}\right), \tag{89}$$

$$\lambda_{\phi\phi} \simeq (\sqrt{\rho^2 + z^2} - z)\left(1 + \frac{2(M_{\mathrm{ADM}} + \eta)}{3\pi\sqrt{\rho^2 + z^2}} + \mathcal{O}\left(\frac{1}{\rho^2 + z^2}\right)\right), \tag{90}$$

$$\lambda_{\psi\psi} \simeq (\sqrt{\rho^2 + z^2} + z)\left(1 + \frac{2(M_{\mathrm{ADM}} - \eta)}{3\pi\sqrt{\rho^2 + z^2}} + \mathcal{O}\left(\frac{1}{\rho^2 + z^2}\right)\right), \tag{91}$$

$$\lambda_{\phi\psi} \simeq \zeta\frac{\rho^2}{(\rho^2 + z^2)^{3/2}} + \mathcal{O}\left(\frac{1}{\rho^2 + z^2}\right), \tag{92}$$

$$g_{\rho\rho} = g_{zz} \simeq \frac{1}{2\sqrt{\rho^2 + z^2}} + \mathcal{O}\left(\frac{1}{\rho^2 + z^2}\right), \tag{93}$$

where the constant M_{ADM} denotes the conserved mass defined by Eq. (49) and J_ϕ and J_ψ the angular momenta, defined by Eq. (50). Here η is a constant that comes from gauge degrees of freedom in the choice of the coordinate z, i.e., degrees of freedom with respect to shift translation $z \to z + \alpha$. (This gauge freedom exists even after the gauge freedom of the conjugate coordinate, ρ, is fixed at infinity.) Since in our proof we choose the coordinate z such that the horizons are located at the interval $[-k^2, k^2]$ for two configurations $M_{[0]}$ and $M_{[1]}$, we choose the same values of η for the two solutions.

The left-hand side of the Einstein-Maxwell equation behaves as $\mathcal{O}((\rho^2+z^2)^{-1})$ in a neighborhood of the infinity. The energy-momentum tensor of the Maxwell field must also behave as $\mathcal{O}((\rho^2+z^2)^{-1})$. Hence from the asymptotic flatness, the gauge potential must behave as

$$A_t \simeq \frac{2Q}{\pi\sqrt{\rho^2+z^2}} + \mathcal{O}\left(\frac{1}{\rho^2+z^2}\right), \tag{94}$$

$$\psi_\phi \simeq \mathcal{O}\left(\frac{1}{\sqrt{\rho^2+z^2}}\right), \tag{95}$$

$$\psi_\psi \simeq \mathcal{O}\left(\frac{1}{\sqrt{\rho^2+z^2}}\right). \tag{96}$$

Next, we derive the behavior of μ and ω_a near infinity. The magnetic potential, μ, is determined by Eq. (13). From Eqs. (95) and (96), the second term in the right-hand side of Eq. (13) behaves as $\mathcal{O}((\rho^2+z^2)^{-1})$. The leading term $\mu^{(0)}$, where $\mu \simeq \mu^{(0)} + \mathcal{O}((\rho^2+z^2)^{-1/2})$, is derived from the equations

$$\mu^{(0)}_{,z} \simeq -\frac{\rho}{\sqrt{3}}A_{t,\rho}, \qquad \mu^{(0)}_{,\rho} \simeq \frac{\rho}{\sqrt{3}}A_{t,z}. \tag{97}$$

Using the asymptotic behavior (94) of the gauge field A_t, we obtain

$$\mu^{(0)} = \frac{2Qz}{\pi\sqrt{3}\sqrt{\rho^2+z^2}}. \tag{98}$$

The twist potential, ω_a, is determined by Eq. (16). The second term behaves as $\mathcal{O}((\rho^2+z^2)^{-1})$. Hence, the leading term $\omega^{(0)}_a$, where $\omega_a \simeq \omega^{(0)}_a + \mathcal{O}((\rho^2+z^2)^{-1/2})$, is derived from the equations

$$\omega^{(0)}_{a,z} \simeq \frac{\tau}{\rho}\lambda_{ab}a^b{}_{t,\rho}, \tag{99}$$

$$\omega^{(0)}_{a,\rho} \simeq -\frac{\tau}{\rho}\lambda_{ab}a^b{}_{t,z}. \tag{100}$$

The functions $a^a{}_t$ behaves as

$$a^\phi{}_t = \frac{\lambda_{\phi\psi}g_{t\psi} - \lambda_{\psi\psi}g_{t\phi}}{\tau} \simeq -\frac{J_\phi}{\pi}\frac{1}{\rho^2+z^2}, \tag{101}$$

$$a^\psi{}_t = \frac{\lambda_{\phi\psi}g_{t\phi} - \lambda_{\phi\phi}g_{t\psi}}{\tau} \simeq -\frac{J_\psi}{\pi}\frac{1}{\rho^2+z^2}. \tag{102}$$

Therefore, solving Eqs. (99) and (100), we obtain

$$\omega^{(0)}_\phi = \frac{J_\phi}{\pi}\left(\frac{\rho^2}{\rho^2+z^2} - \frac{2z}{\sqrt{\rho^2+z^2}}\right), \tag{103}$$

$$\omega^{(0)}_\psi = \frac{J_\psi}{\pi}\left(\frac{\rho^2}{\rho^2+z^2} - \frac{2z}{\sqrt{\rho^2+z^2}}\right). \tag{104}$$

Then, for $\sqrt{\rho^2+z^2} \to \infty$, $\rho\,\mathrm{tr}\Psi$ behaves as

$$\rho\,\mathrm{tr}\Psi \simeq \mathcal{O}\left(\frac{1}{\rho^2+z^2}\right). \tag{105}$$

Therefore,

$$\rho\partial_p\,\mathrm{tr}\Psi dS^p \simeq \mathcal{O}\left(\frac{1}{\sqrt{\rho^2+z^2}}\right). \tag{106}$$

Thus, we find from (i)–(iv) that the boundary integral, Eq. (52), vanishes on each rod and the infinity. The deviation matrix, Ψ, is constant and has the asymptotic behavior, $\Psi \to 0$. Therefore, Ψ vanishes over Σ, and the two configurations, $M_{[0]}$ and $M_{[1]}$, coincide with each other. Furthermore, as shown in Appendix B, the boundary conditions derived above are the same as the limiting behavior of the Chong-Cvetič-Lü-Pope solution at each corresponding boundary. Therefore, the data $M_{[0]}$ (and now equivalently $M_{[1]}$) must also be the same as the corresponding matrix to the Chong-Cvetič-Lü-Pope solution. This completes our proof for the uniqueness theorem.

V. SUMMARY

We have shown the uniqueness theorem which states that in five-dimensional Einstein-Maxwell-Chern-Simons theory, an asymptotically flat, stationary charged rotating black hole with finite temperature is uniquely specified by its asymptotic conserved charges and therefore is described by the Chong-Cvetič-Lü-Pope solution, if (1) it admits two independent axial Killing symmetries and (2) the topology of the event horizon cross section is spherical. Our theorem generalizes the uniqueness theorem for spherical black holes in five-dimensional vacuum Einstein gravity [50] to the case of EMCS theory. In our proof, in addition to the symmetry-assumption (1), the Chern-Simons term in the theory, Eq. (33), plays an important role to reduce the system into a nonlinear sigma model with desired symmetry property, $G_{2(2)}/SO(4)$, as discussed in [62,63]. Then, having this symmetry property on the target space, we have obtained the matrix representation of [36], in which our system is completely determined by $G_{2(2)}/SO(4)$ coset matrix M. We then derived the Mazur identity, and used the identity to show that if two solutions, i.e., two matrices, $M_{[0]}$ and $M_{[1]}$, satisfy the same boundary conditions (imposed at infinity, on two rotational axis, and on the horizon), then the solutions $M_{[0]}$ and $M_{[1]}$ must coincide with each other. We have shown that our boundary conditions (the asymptotic flatness and the regularity) are the same as the limiting behavior of the Chong-Cvetič-Lü-Pope solution.

In the present theorem, we restrict attention to topologically spherical black holes by the assumption (2). Our theorem can be generalized to the case of charged rotating black ring solutions by imposing certain additional conditions. We first note that under the same symmetry condition (1), the analysis in Sec. II, III, and IV, apply also for black ring solutions (if they exist) in EMCS theory. See [7] for such a ring solution.) The only difference from the spherical black hole case arises in the boundary value analysis. Now we also note that asymptotically flat, five-dimensional black ring solutions that satisfy the symmetry assumption (1) have the following rod structure: (i) $[c, \infty]$, $v = (0, 1, 0)$, (ii) $[ck^2, c]$, $v = (0, 0, 1)$, (iii) $[-ck^2, ck^2]$, (iv) $[-\infty, -ck^2]$, $v = (0, 0, 1)$, where $c > 0$, $k^2 < 1$ and v's are eigenvectors with respect to a zero eigenvalue of the three-dimensional matrix g_{IJ} for each segment. It should be noted that we are not concerned with a lens space throughout discussion here, and therefore the only nontrivial rod-data are given by rod intervals. Then, after fixing the scale c, one can completely specify the rod data in terms of k^2. The finite spacelike rod (ii) is the main difference from the rod structure for topologically spherical black holes considered in Sec. IV. We believe that by appropriately specifying rod structure, one can determine the topology of the horizon, as well as the topology of black hole exterior region. In this respect, it has recently been shown [61] that the topology and symmetry structure of the black hole spacetime can be completely determined in terms of *rod intervals*, which is similar to but somewhat different from the rod structure of Harmark [76]. In the charged black ring case, some additional parameters other than the global conserved charges may also come to play a role. These issues deserve further study.

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APPENDIX A: MAXWELL FIELD WITH SYMMETRIES

Let F denote the stationary and axisymmetric Maxwell field, i.e., that satisfies

$$\mathcal{L}_{\xi_I} F = 0, \tag{A1}$$

with ξ_I ($I = \phi,\ \psi,\ t$) being commuting Killing vector fields for the axial-symmetries and the stationary symmetry, discussed in sec. II B. From the Maxwell equation, $dF = 0$, and the identity

$$d i_{\xi_I} i_{\xi_J} = -i_{\xi_I} \mathcal{L}_{\xi_J} + i_{\xi_J} \mathcal{L}_{\xi_I} + d i_{\xi_I} i_{\xi_J} d, \tag{A2}$$

we have

$$d i_{\xi_I} i_{\xi_J} F = -i_{\xi_I} \mathcal{L}_{\xi_J} F + i_{\xi_J} \mathcal{L}_{\xi_I} F + d i_{\xi_I} i_{\xi_J} dF = 0. \tag{A3}$$

Similarly, using the identity

$$\begin{aligned} d i_{\xi_I} i_{\xi_J} i_{\xi_K} = {}& i_{\xi_I} i_{\xi_J} \mathcal{L}_{\xi_K} - i_{\xi_I} i_{\xi_K} \mathcal{L}_{\xi_J} + i_{\xi_J} i_{\xi_K} \mathcal{L}_{\xi_I} \\ & - i_{\xi_I} i_{\xi_J} i_{\xi_K} d, \end{aligned} \tag{A4}$$

we have

$$\begin{aligned} d i_{\xi_I} i_{\xi_J} i_{\xi_K} * F &= i_{\xi_I} i_{\xi_J} \mathcal{L}_{\xi_K} * F - i_{\xi_I} i_{\xi_K} \mathcal{L}_{\xi_J} * F \\ &\quad + i_{\xi_J} i_{\xi_K} \mathcal{L}_{\xi_I} * F - i_{\xi_I} i_{\xi_J} i_{\xi_K} d * F \\ &= i_{\xi_I} i_{\xi_J} * \mathcal{L}_{\xi_K} F - i_{\xi_I} i_{\xi_K} * \mathcal{L}_{\xi_J} F \\ &\quad + i_{\xi_J} i_{\xi_K} * \mathcal{L}_{\xi_I} F - i_{\xi_I} i_{\xi_J} i_{\xi_K} d * F \\ &= \frac{1}{\sqrt{3}} i_{\xi_I} i_{\xi_J} i_{\xi_K} F \wedge F = 0. \end{aligned} \tag{A5}$$

Therefore, $F(\xi_I, \xi_J)$ and $(*F)(\xi_I, \xi_J, \xi_K)$, are constant. Since they vanish, at least, on rotation axes, these imply

$$F(\xi_I, \xi_J) = 0, \tag{A6}$$

$$(*F)(\xi_I, \xi_J, \xi_K) = 0. \tag{A7}$$

In terms of the coordinates (t, ϕ, ψ, ρ, z), these can be written as

$$F_{t\phi} = F_{t\psi} = F_{\phi\psi} = 0, \tag{A8}$$

$$F_{\rho z} = 0. \tag{A9}$$

Then, from (A9), using the gauge degrees of freedom, $A \to A - d\chi$, with the function χ satisfying $A_\rho = \chi_{,\rho}, A_\theta = \chi_{,\theta}$ we can show

$$A_\rho = A_z = 0. \tag{A10}$$

APPENDIX B: CHONG-CVETICĆ-LÜ-POPE SOLUTION

Here we present the asymptotically flat stationary charged rotating black hole solution in five-dimensional Einstein-Maxwell-Chern-Simons theory, found by Cvetič *et al.* [4,5]. The solution has three mutually commuting Killing vectors that generate isometries $\mathbb{R} \times U(1) \times U(1)$, and spherical topology of the horizon cross sections. We observe that the limiting behavior of relevant scalar functions of the solution, which correspond to Φ^A, are in perfect accordance with our general boundary conditions discussed in Sec. IV.

The metric and the gauge potential in [4,5] for $g = 0$ are given, respectively, by

$$
\begin{aligned}
ds^2 = &-dt^2 - \frac{2q}{\tilde{\rho}^2}\nu(dt-\omega) + \frac{f}{\tilde{\rho}^4}(dt-\omega)^2 + \frac{\tilde{\rho}^2 r^2}{\Delta}dr^2 \\
&+ \tilde{\rho}^2 d\theta^2 + (r^2+a^2)\sin^2\theta d\phi^2 \\
&+ (r^2+b^2)\cos^2\theta d\psi^2,
\end{aligned} \tag{B1}
$$

and

$$A = \frac{\sqrt{3}q}{\tilde{\rho}^2}(dt-\omega), \tag{B2}$$

where

$$\nu = b\sin^2\theta d\phi + a\cos^2\theta d\psi, \tag{B3}$$

$$\omega = a\sin^2 d\phi + b\cos^2\theta d\psi, \tag{B4}$$

$$f = 2m\tilde{\rho}^2 - q^2, \tag{B5}$$

$$\Delta = (r^2+a^2)(r^2+b^2) + q^2 + 2abq - 2mr^2, \tag{B6}$$

$$\tilde{\rho}^2 = r^2 + a^2\cos^2\theta + b^2\sin^2\theta. \tag{B7}$$

The scalar fields $\Phi^A = (\lambda_{ab}, \omega_a, \psi_a, \mu)$ for the solution (B1) and (B2) are computed as

$$\lambda_{\phi\phi} = \frac{2q}{\tilde{\rho}^2}ab\sin^4\theta + \frac{f}{\tilde{\rho}^4}a^2\sin^4\theta + (r^2+a^2)\sin^2\theta, \tag{B8}$$

$$\lambda_{\psi\psi} = \frac{2q}{\tilde{\rho}^2}ab\cos^4\theta + \frac{f}{\tilde{\rho}^4}b^2\cos^4\theta + (r^2+b^2)\cos^2\theta, \tag{B9}$$

$$\lambda_{\phi\psi} = \frac{q}{\tilde{\rho}^2}(a^2+b^2)\cos^2\theta\sin^2\theta + \frac{f}{\tilde{\rho}^4}ab\cos^2\theta\sin^2\theta, \tag{B10}$$

$$
\begin{aligned}
\omega_\phi = &\frac{(2am+bq)(-4\cos 2\theta + \cos 4\theta)}{8} \\
&- \frac{2(a^2-b^2)(2aq^2 + (2am+bq)F)\cos^2\theta\sin^4\theta}{F^2},
\end{aligned} \tag{B11}
$$

$$
\begin{aligned}
\omega_\psi = &-\frac{(2bm+aq)(4\cos 2\theta + \cos 4\theta)}{8} \\
&- \frac{2(a^2-b^2)(2bq^2 + (2bm+aq)F)\cos^4\theta\sin^2\theta}{F^2},
\end{aligned} \tag{B12}
$$

$$\psi_\phi = -\frac{qa\sin^2\theta}{\tilde{\rho}^2}, \tag{B13}$$

$$\psi_\psi = -\frac{qb\cos^2\theta}{\tilde{\rho}^2}, \tag{B14}$$

$$\mu = \frac{1}{2}q\cos 2\theta - \frac{2(b^2-a^2)q\cos^2\theta\sin^2\theta}{F}, \tag{B15}$$

where the function F is defined by

$$F = a^2 + b^2 + 2r^2 + (a^2-b^2)\cos 2\theta. \tag{B16}$$

Let us introduce the coordinates (ρ, z) defined by

$$\rho = \frac{1}{2}\sqrt{\Delta}\sin 2\theta, \qquad z = \frac{2r^2 + a^2 + b^2 - 2m}{4}\cos 2\theta. \tag{B17}$$

Then, the base space $\Sigma = \{(\rho, z) | \rho > 0, -\infty < z < \infty\}$ has four boundaries, which exactly correspond to the four boundaries discussed in Sec. IV: Namely, (i) ϕ-invariant plane, i.e., the plane which is invariant under the rotation with respect to the Killing vector field $\partial/\partial\phi$: $\partial\Sigma_\phi = \{(\rho, z) | \rho = 0, k^2 < z < \infty\}$, (ii) Horizon: $\partial\Sigma_{\mathcal{H}} = \{(\rho, z) | \rho = 0, -k^2 < z < k^2\}$, (iii) ψ-invariant plane, i.e., the plane which is invariant under the rotation with respect to the Killing vector field $\partial/\partial\psi$: $\partial\Sigma_\psi = \{(\rho, z) | \rho = 0, -\infty < z < -k^2\}$, and (iv) Infinity: $\partial\Sigma_\infty = \{(\rho, z) | \sqrt{\rho^2 + z^2} \to \infty$ with $z/\sqrt{\rho^2+z^2}$ finite$\}$, where the constant k^2 is given by

$$k^2 = \frac{\sqrt{(2m - a^2 - b^2)^2 - 4(ab+q)^2}}{4}. \tag{B18}$$

Let us examine the behavior of the scalar fields on each boundary.

(i) Near the ϕ-invariant plane $\partial\Sigma_\phi$, each scalar field behaves as

$$\lambda_{\phi\phi} \simeq \mathcal{O}(\rho^2), \qquad \lambda_{\psi\psi} \simeq \mathcal{O}(1), \qquad \lambda_{\phi\psi} \simeq \mathcal{O}(\rho^2), \tag{B19}$$

$$
\begin{aligned}
\omega_\phi &\simeq -\frac{3}{8}(2am+bq) + \mathcal{O}(\rho^2), \\
\omega_\psi &\simeq -\frac{5}{8}(2bm+aq) + \mathcal{O}(\rho^2),
\end{aligned} \tag{B20}
$$

$$
\begin{aligned}
\psi_\phi \simeq \mathcal{O}(\rho^2), \qquad \psi_\psi \simeq \mathcal{O}(1), \\
\mu \simeq \frac{1}{2}q + \mathcal{O}(\rho^2).
\end{aligned} \tag{B21}
$$

(ii) Near the horizon $\partial\Sigma_{\mathcal{H}}$, the scalar fields behave as

$$\lambda_{ab} \simeq \mathcal{O}(1), \qquad \omega_a \simeq \mathcal{O}(1), \tag{B22}$$

$$\psi_a \simeq \mathcal{O}(1), \qquad \mu \simeq \mathcal{O}(1). \tag{B23}$$

(iii) Near the ψ-invariant plane $\partial\Sigma_\psi$, each potential behaves as

$$\lambda_{\phi\phi} \simeq \mathcal{O}(1), \qquad \lambda_{\psi\psi} \simeq \mathcal{O}(\rho^2), \qquad \lambda_{\phi\psi} \simeq \mathcal{O}(\rho^2), \tag{B24}$$

$$\omega_\phi \simeq \frac{5}{8}(2am + bq) + \mathcal{O}(\rho^2), \quad \omega_\psi \simeq \frac{3}{8}(2bm + aq) + \mathcal{O}(\rho^2), \tag{B25}$$

$$\psi_\phi \simeq \mathcal{O}(1), \qquad \psi_\psi \simeq \mathcal{O}(\rho^2), \quad \mu \simeq -\frac{q}{2} + \mathcal{O}(\rho^2). \tag{B26}$$

In the neighborhood of infinity $\partial\Sigma_\infty$, the behavior of the potentials becomes

$$\lambda_{\phi\phi} \simeq (\sqrt{\rho^2 + z^2} - z)\left(1 + \frac{a^2}{2\sqrt{\rho^2 + z^2}}\right) + \mathcal{O}\left(\frac{1}{\rho^2 + z^2}\right), \tag{B27}$$

$$\lambda_{\psi\psi} \simeq (\sqrt{\rho^2 + z^2} + z)\left(1 + \frac{2m - a^2}{2\sqrt{\rho^2 + z^2}}\right) + \mathcal{O}\left(\frac{1}{\rho^2 + z^2}\right), \tag{B28}$$

$$\lambda_{\psi\phi} \simeq \frac{(a^2q + b^2q + 2abm)\rho^2}{8(\rho^2 + z^2)^{3/2}} + \mathcal{O}\left(\frac{1}{\rho^2 + z^2}\right), \tag{B29}$$

$$\omega_\phi \simeq \frac{1}{8}(2am + bq)(-4\cos 2\theta + \cos 4\theta) + \mathcal{O}\left(\frac{1}{\sqrt{\rho^2 + z^2}}\right), \tag{B30}$$

$$\omega_\psi \simeq -\frac{1}{8}(2bm + aq)(4\cos 2\theta + \cos 4\theta) + \mathcal{O}\left(\frac{1}{\sqrt{\rho^2 + z^2}}\right), \tag{B31}$$

$$\psi_\phi \simeq -\frac{qa(\sqrt{\rho^2 + z^2} - z)}{4(\rho^2 + z^2)} + \mathcal{O}\left(\frac{1}{\rho^2 + z^2}\right), \tag{B32}$$

$$\psi_\psi \simeq -\frac{qb(\sqrt{\rho^2 + z^2} + z)}{4(\rho^2 + z^2)} + \mathcal{O}\left(\frac{1}{\rho^2 + z^2}\right), \tag{B33}$$

$$\mu \simeq \frac{qz}{2\sqrt{\rho^2 + z^2}} + \mathcal{O}\left(\frac{1}{\sqrt{\rho^2 + z^2}}\right). \tag{B34}$$

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