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Topological regularization and self-duality in four-dimensional anti–de Sitter gravity

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It is shown that the addition of a topological invariant (Gauss-Bonnet term) to the anti–de Sitter gravity action in four dimensions recovers the standard regularization given by the holographic renormalization procedure. This crucial step makes possible the inclusion of an odd parity invariant (Pontryagin term) whose coupling is fixed by demanding an asymptotic (anti) self-dual condition on the Weyl tensor. This argument allows one to find the dual point of the theory where the holographic stress tensor is related to the boundary Cotton tensor as $T^i_j = \pm(\ell^2/8\pi G)C^i_j$, which has been observed in recent literature in solitonic solutions and hydrodynamic models. A general procedure to generate the counterterm series for anti–de Sitter gravity in any even dimension from the corresponding Euler term is also briefly discussed.

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I. INTRODUCTION

On the gravity side of the AdS/CFT correspondence, the relevant information that realizes this duality is encoded in the finite part of the boundary stress tensor [1]. That identification requires the cancellation of the infrared divergences in the bulk theory made by the holographic renormalization procedure [2], which is based on an asymptotic analysis of the metric in the Fefferman-Graham (FG) coordinate system [3]

$$ds^2 = \frac{\ell^2}{4\rho^2}d\rho^2 + \frac{1}{\rho}g_{ij}dx^i dx^j, \tag{1}$$

where $h_{ij} = g_{ij}/\rho$ corresponds to the boundary metric. For asymptotically anti–de Sitter (AAdS) spaces, $g_{ij}(x, \rho)$ accepts a regular expansion near the boundary $\rho = 0$, i.e., $g_{ij}(x,\rho) = g_{(0)ij} + \rho g_{(1)ij} + \cdots$. Solving the Einstein equations in this frame leads to the holographic reconstruction of the spacetime from a given boundary data $g_{(0)ij}$ that is essential in determining the series of intrinsic counterterms $\mathcal{L}_{\mathrm{ct}}$ which renders finite the boundary stress tensor [4].

However, the algorithm which produces $\mathcal{L}_{\mathrm{ct}}$ becomes extremely complex as the spacetime dimension increases, such that there is not a closed formula for counterterms for an arbitrary dimension. This argument motivates the search for alternative approaches.

On the other hand, any other regularization scheme, even if properly removes the asymptotic divergences, might spoil the holographic interpretation of the theory within the AdS/CFT framework because of different boundary conditions.

In particular, a regularization mechanism for AdS gravity in any dimension which consists of the addition of counterterms that depend on the extrinsic curvature K_{ij} (Kounterterms method) has been recently proposed [5,6]. In this case, the on-shell variation of the regularized action I_{reg} contains terms of the type δK_{ij}that make a definition of the quasilocal stress tensor more elusive. But one knows that in AAdS spacetimes the leading order of the asymptotic expansion in K_{ij} coincides with the leading order of the induced metric h_{ij}, i.e.,

$$K_{ij} = \frac{1}{\ell}\frac{g_{(0)ij}}{\rho} + \mathcal{O}(\rho). \tag{2}$$

The above relation inspires a reformulation of holographic renormalization in terms of an expansion of the extrinsic curvature [7]. This suggests it might be still possible to obtain a regularized stress tensor $\langle T_{ij}\rangle$ associated to $g_{(0)ij}$ even though Kounterterms regularization does not lend itself to a Brown-York stress tensor definition $T_{ij} = \frac{2}{\sqrt{-h}} \times \frac{\delta I_{\mathrm{reg}}}{\delta h^{ij}}$. It also motivates a direct comparison with the standard procedure that, until now, has been performed in Einstein gravity only in three dimensions [8]. For four and higher even dimensions, this is carried out below.

II. GAUSS-BONNET INVARIANT IN 4-DIMENSIONAL ADS GRAVITY

Let us consider the Einstein-Hilbert action with negative cosmological constant in four dimensions supplemented by the Gauss-Bonnet (GB) term $\mathcal{E}_4$ with an arbitrary coupling constant α

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$$I = \int_M d^4x\sqrt{-\mathcal{G}}\Big[\frac{1}{16\pi G}(R-2\Lambda) + \alpha(R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - 4R_{\mu\nu}R^{\mu\nu} + R^2)\Big], \qquad (3)$$

where $\Lambda = -3/\ell^2$ is the cosmological constant in terms of the AdS radius ℓ. It was shown in Ref. [9] that a well-posed action principle for gravity with AdS asymptotics removes the arbitrariness in the GB coupling. Since $\mathcal{E}_4$ is a topological invariant, it does not modify the field equations. However, it still contributes to the surface term when the total action is varied

$$\delta I = \int_M EOM + \int_{\partial M} d^3x\sqrt{-h}n_\sigma \delta^{[\sigma\lambda\mu\nu]}_{[\gamma\delta\alpha\beta]}\mathcal{G}^{\delta\varepsilon}\delta\Gamma^\gamma_{\lambda\varepsilon} \times \left(\frac{1}{64\pi G}\delta^{[\alpha\beta]}_{[\mu\nu]} + \alpha R^{\alpha\beta}_{\mu\nu}\right), \qquad (4)$$

where n_μ is the normal vector to the boundary [10]. The total action is rendered stationary demanding $\delta I = 0$ on shell for a given boundary condition. For asymptotically locally AdS spacetimes, i.e.,

$$R^{\alpha\beta}_{\mu\nu} + \frac{1}{\ell^2}\delta^{[\alpha\beta]}_{[\mu\nu]} = 0$$

at ∂M, the variational principle fixes the coupling constant as $\alpha = \ell^2/(64\pi G)$, which produces finite Noether charges [9]. Surprisingly enough, the same value of α regularizes the Euclidean action in a background-independent way [5] and casts Eq. (3) into a MacDowell-Mansouri form [11]

$$I_4 = \frac{\ell^2}{256\pi G}\int_M d^4x\sqrt{-\mathcal{G}}\delta^{[\sigma\lambda\mu\nu]}_{[\gamma\delta\alpha\beta]}\left(R^{\gamma\delta}_{\sigma\lambda} + \frac{1}{\ell^2}\delta^{[\gamma\delta]}_{[\sigma\lambda]}\right) \times \left(R^{\alpha\beta}_{\mu\nu} + \frac{1}{\ell^2}\delta^{[\alpha\beta]}_{[\mu\nu]}\right). \qquad (5)$$

Using the field equations, one proves that the Weyl tensor is

$$W^{\alpha\beta}_{\mu\nu} = R^{\alpha\beta}_{\mu\nu} + \frac{1}{\ell^2}\delta^{[\alpha\beta]}_{[\mu\nu]}, \qquad (6)$$

where the right-hand side is the curvature of the AdS group (the rest corresponds to the torsion, which vanishes in Riemann gravity). This fact implies that the action (5) is on shell equivalent to conformal gravity

$$I_4 = \frac{\ell^2}{64\pi G}\int_M d^4x\sqrt{-\mathcal{G}}W_{\mu\nu\alpha\beta}W^{\mu\nu\alpha\beta}, \qquad (7)$$

because any trace of $W_{\mu\nu\alpha\beta}$ is identically zero [12].

In what follows, we show that the addition of a topological invariant of the Euler class recovers the standard counterterm regularization and holographic stress tensor, by considering its equivalent boundary formulation.

III. BOUNDARY FORMULATION

In a four-dimensional manifold without boundaries, the integration of the GB term is proportional to the Euler characteristic $\chi(M)$. When a boundary is introduced, a correction to $\chi(M)$ is required, such that the Euler theorem reads

$$\int_M d^4x\mathcal{E}_4 = 32\pi^2\chi(M) + \int_{\partial M} d^3x B_3, \qquad (8)$$

where B_3 is a boundary term known as the second Chern form. If the spacetime is foliated using Gaussian (radial) coordinates $ds^2 = N^2(\rho)d\rho^2 + h_{ij}(\rho,x)dx^i dx^j$, the term B_3 is given as a polynomial in the extrinsic curvature $K_{ij} = -\frac{1}{2N}\partial_\rho h_{ij}$ and the intrinsic curvature $\mathcal{R}^{ij}_{kl}(h)$ as [5]

$$B_3 = 4\sqrt{-h}\delta^{[i_1i_2i_3]}_{[j_1j_2j_3]}K^{j_1}_{i_1}\left(\frac{1}{2}\mathcal{R}^{j_2j_3}_{i_2i_3}(h) - \frac{1}{3}K^{j_2}_{i_2}K^{j_3}_{i_3}\right). \qquad (9)$$

There is a reason to consider the boundary formulation of topological invariants beyond the purpose of comparison with the counterterm regularization. The boundary dynamics does not tell between the Euler and the boundary term B_3, as they are locally equivalent. However, computations of the Euclidean action show that the Euler term shifts the black hole entropy S by a constant proportional to $\chi(M)$ [5] that can also be obtained using Wald's entropy formula. Thus, S may take negative values for topological black holes with a hyperbolic spatial section that can only be avoided by supplementing the action with the Kounterterm B_3 instead.

In order to compare to the standard regularization procedure, one can simply add and subtract the Gibbons-Hawking term from the Einstein-Hilbert action plus the boundary term B_3,

$$I_4 = I_{\rm EH} - \frac{1}{8\pi G}\int_{\partial M} d^3x\sqrt{-h}K + \int_{\partial M} d^3x\mathcal{L}_{\rm ct}. \qquad (10)$$

The first two terms define the Dirichlet problem in gravity, while the quantity $\mathcal{L}_{\rm ct}$ is given by

$$\mathcal{L}_{\rm ct} = \frac{\ell^2}{16\pi G}\sqrt{-h}\delta^{[i_1i_2i_3]}_{[j_1j_2j_3]}K^{j_1}_{i_1}\left(\frac{1}{2}\mathcal{R}^{j_2j_3}_{i_2i_3}(h) - \frac{1}{3}K^{j_2}_{i_2}K^{j_3}_{i_3} + \frac{1}{\ell^2}\delta^{j_2}_{i_2}\delta^{j_3}_{i_3}\right). \qquad (11)$$

For the boundary metric $h_{ij} = g_{ij}/\rho$, the intrinsic curvature and the determinant rescale as $\mathcal{R}^{ij}_{kl}(h) = \rho\mathcal{R}^{ij}_{kl}(g)$ and $\sqrt{-h} = \sqrt{-g}/\rho^{3/2}$, respectively. This also implies

$$K^j_i = K_{ik}h^{kj} = \frac{1}{\ell}(\delta^j_i - \rho k^j_i) \qquad (12)$$

for the extrinsic curvature, with the definition $k^j_i = g^{jk}\partial_\rho g_{ki}$. Expanding Eq. (11) in FG form, one notices that k^i_j is absent from the divergent terms,

$$\mathcal{L}_{\rm ct} = \frac{1}{8\pi G}\frac{\sqrt{-g}}{\rho^{3/2}}\left(\frac{2}{\ell} + \frac{\ell}{2}\rho\mathcal{R}(g)\right) + \mathcal{O}(\rho^{1/2}) \qquad (13)$$

such that one recovers the Balasubramanian-Kraus local counterterms

$$\mathcal{L}_{\rm ct} = \frac{1}{8\pi G}\sqrt{-h}\left(\frac{2}{\ell} + \frac{\ell}{2}\mathcal{R}(h)\right). \qquad (14)$$

The agreement with the standard holographic renormalization can also be seen from the on-shell variation of the action (4), which for the radial foliation and the value $\alpha = \ell^2/(64\pi G)$ adopts the form

$$\delta I_4 = \frac{\ell^2}{32\pi G}\int_{\partial M} d^3x\sqrt{-h}\,\delta^{[jkl]}_{[mnp]}\left(\delta K^m_j + \frac{1}{2}K^m_i(h^{-1}\delta h)^i_j\right) \times\left(R^{j_2j_3}_{i_2i_3} + \frac{1}{\ell^2}\delta^{[j_2j_3]}_{[i_2i_3]}\right). \qquad (15)$$

Expanding the fields in the FG frame, the first term vanishes at the boundary, whereas the second gives a stress tensor

$$\tau^j_i = \frac{\ell^2}{32\pi G}\delta^{[jkl]}_{[mnp]}K^m_i\left(R^{np}_{kl} + \frac{1}{\ell^2}\delta^{[np]}_{[kl]}\right). \qquad (16)$$

Note that any conserved quantity constructed with this stress tensor will vanish for spacetimes which are globally of constant curvature, as the AdS vacuum. Using Gauss-Codazzi relations, one might also notice that τ^j_i contains higher powers in the extrinsic curvature. However, it is straightforward to prove that τ^j_i coincides up to the relevant order in ρ with the Balasubramanian-Kraus stress tensor T^j_i, when it is appropriately rewritten

$$T^j_i = \frac{1}{8\pi G}\left(K^j_i - \delta^j_i K + \frac{2}{\ell}\delta^j_i - \left(\mathcal{R}^j_i(h) - \frac{1}{2}\delta^j_i\mathcal{R}(h)\right)\right) = \frac{\rho\ell}{32\pi G}\delta^{[jkl]}_{[inp]}\left(\mathcal{R}^{np}_{kl}(g) + \frac{4}{\ell^2}\delta^n_k k^p_l\right). \qquad (17)$$

The above derivation shows that the divergence cancellation provided by the counterterm series can be regarded as a *topological* regularization, since it comes from the addition of the GB term with a coupling such that the regularized action takes the MacDowell-Mansouri form (5).

In the holographic renormalization framework, the information on the holographic stress tensor in four dimensions is carried by the coefficient $g_{(3)}$ in FG expansion

$$g_{ij}(x,\rho) = g_{(0)ij} + \rho g_{(1)ij} + \rho^{3/2}g_{(3)ij} + \cdots. \qquad (18)$$

It is just after solving the Einstein equations order by order in ρ that the vanishing of the Weyl anomaly comes from the zero trace of $g_{(3)}$ [2]. On the other hand, the anomaly $\mathcal{A}$ can also be read off from a Weyl transformation with infinitesimal parameter σ on the regularized action, that is, $\delta_\sigma I_{\rm reg} = \int_{\partial M} d^{D-1}x\sqrt{g_{(0)}}\sigma\mathcal{A}$. This means that one might have concluded the same by simple inspection of Eq. (7), since it is manifestly invariant under conformal transformations.

Up to the relevant order, the stress tensor (16) can be rewritten as

$$\tau^j_i = \frac{\ell}{8\pi G}W^j_{\mu i\nu}n^\mu n^\nu, \qquad (19)$$

using the traceless property and index symmetries of the Weyl tensor.

The conformal completion technique [13] defines an AAdS spacetime in such a way that the metric $\mathcal{G}_{\mu\nu}$, which obeys the Einstein equations, can be conformally mapped into an *unphysical* one $\tilde{\mathcal{G}}_{\mu\nu} = \Omega^{-2}\mathcal{G}_{\mu\nu}$ by a smooth conformal factor Ω which satisfies precise falloff conditions. The procedure gives rise to a background-independent conserved charge for every asymptotic symmetry ξ^i as the integral on the spatial section $\tilde{\Sigma}$ of the boundary

$$\mathcal{H}_\xi = \frac{\ell}{8\pi G}\int_{\tilde{\Sigma}}\tilde{E}^j_i\xi^i\tilde{u}_j d\tilde{\Sigma}, \qquad (20)$$

where $\tilde{E}^j_i = \Omega^{3-D}\tilde{W}^j_{\mu i\nu}\tilde{n}^\mu\tilde{n}^\nu/(D-3)$ is the *electric* part of the unphysical Weyl tensor, $d\tilde{\Sigma}$ is the integration element on $\tilde{\Sigma}$, and $\tilde{u}_j$ is the unit timelike normal to $\tilde{\Sigma}$. Rescaling all the quantities into the ones of the spacetime metric, it is easy to prove that the conserved quantities $Q_\xi \equiv \int_\Sigma \tau^j_i\xi^i u_j d\Sigma$ coming from the Kounterterms regularization in $D = 4$ are the same as the Ashtekar-Magnon-Das formula (20).

IV. PONTRYAGIN TERM AND SELF-DUAL SOLUTIONS

In four dimensions there exists an additional (odd parity) topological invariant known as the Pontryagin term $\mathcal{P}_4$, which is locally equivalent to the derivative of the gravitational Chern-Simons term

$$\mathcal{P}_4 = -\tfrac{1}{4}\epsilon^{\mu\nu\alpha\beta}R^{\sigma\lambda}_{\mu\nu}R_{\sigma\lambda\alpha\beta} = \epsilon^{\mu\nu\alpha\beta}\partial_\mu\left(\Gamma^\sigma_{\nu\lambda}\partial_\alpha\Gamma^\lambda_{\beta\sigma} + \tfrac{2}{3}\Gamma^\sigma_{\nu\lambda}\Gamma^\lambda_{\alpha\varepsilon}\Gamma^\varepsilon_{\beta\sigma}\right), \qquad (21)$$

where $\epsilon^{\mu\nu\alpha\beta}$ is the constant Levi-Cività tensor density.

The Pontryagin term $F\wedge F$ in four-dimensional Maxwell electromagnetism modifies the dynamics such that the Lorentz boost and the parity invariance are lost when it is coupled through an external, fixed quantity.

We will consider here the addition of the Pontryagin term with a constant coupling β to the regularized action, i.e.,

$$\tilde{I} = I_{\rm EH} + \frac{\ell^2}{64\pi G}\int_M d^4x\mathcal{E}_4 + \beta\int_M d^4x\mathcal{P}_4, \qquad (22)$$

with Euclidean signature. Therefore, in a fashion similar to the case of the addition of the Euler term, the bulk dynamics cannot fix the Pontryagin coupling. However, one may expect that again the variational principle would provide a criterion to remove the arbitrariness in β.

The on-shell variation of the total action produces

$$\delta\tilde{I} = \int_{\partial M} d^3x \sqrt{\mathcal{G}} \frac{n_\sigma}{N} \delta\Gamma^\gamma_{\varepsilon\lambda} \Big(\frac{\ell^2}{64\pi G} \delta^{[\sigma\lambda\mu\nu]}_{[\gamma\delta\alpha\beta]} \mathcal{G}^{\delta\varepsilon} W^{\alpha\beta}_{\mu\nu} + \beta \frac{\epsilon^{\sigma\lambda\mu\nu}}{\sqrt{\mathcal{G}}} \mathcal{G}_{\gamma\tau} W^{\varepsilon\tau}_{\mu\nu} \Big), \tag{23}$$

where in the last term the part along $\delta^{[\varepsilon\tau]}_{[\mu\nu]}$ is identically zero. The total surface term must vanish identically for certain boundary conditions. The argument here is different from the one used to fix the GB coupling in Eq. (3). In that case, α is also determined from the cancellation of the leading-order divergences in the Euclidean action that can be seen, e.g., from evaluating it for the Schwarzschild-AdS black hole

$$-TI_{\mathrm{SAdS}} = TS - \frac{\pi r^3}{4G\ell^2}\Big(1 - \frac{64\pi G}{\ell^2}\alpha\Big) - \frac{M}{2}\Big(1 + \frac{64\pi G}{\ell^2}\alpha\Big), \tag{24}$$

where S and M are the black hole entropy and mass, respectively, and T is the Hawking temperature. It is clear that the correct black hole thermodynamics is reproduced only by the same value of α as before. Moreover, for a given cosmological constant, it is not possible to express the variation Eq. (23) only in terms of the Weyl tensor unless α takes the value fixed in the previous sections.

The result (24) remains unchanged when $\mathcal{P}_4$ is added to the action, as it vanishes for static AdS_4 black holes. In general, it can be shown that the contribution of the Pontryagin term to the action is at most finite.

This means that we should look for asymptotic conditions in the next-to-leading order in the curvature of the AdS group (6). Considering (anti) self-duality in the Weyl tensor

$$W_{\mu\nu\alpha\beta} = \pm\tfrac{1}{2}\sqrt{\mathcal{G}}\epsilon_{\mu\nu\lambda\sigma} W^{\lambda\sigma}_{\alpha\beta} \tag{25}$$

in the asymptotic region, we can fix the coupling constant of $\mathcal{P}_4$ as

$$\beta = \pm\frac{\ell^2}{32\pi G}, \tag{26}$$

demanding a well-posed action principle.

For arbitrary β, the variation of the action (22) projected to the boundary indices defines a total stress tensor $\mathcal{T}^i_j$

$$\delta\tilde{I} = \frac{1}{2}\int_{\partial M} d^3x \sqrt{h}\mathcal{T}^i_j (h^{-1}\delta h)^j_i = \frac{1}{2}\int_{\partial M} d^3x\sqrt{h}(T^i_j + \beta C^i_j)(h^{-1}\delta h)^j_i, \tag{27}$$

where T^i_j is the stress tensor (17) and

$$C^i_j = \frac{1}{\sqrt{h}}\epsilon^{ikl}\nabla_k\Big(\mathcal{R}_{lj} - \frac{1}{4}h_{lj}\mathcal{R}\Big)$$

is the Cotton tensor, obtained from the functional variation of the gravitational Chern-Simons term with respect to the induced metric h_{ij}. The Cotton tensor is symmetric, traceless, and covariantly conserved, and contributes as above to the total stress tensor of the theory when h_{ij} is held fixed on the boundary (Dirichlet problem).

The term $\mathcal{P}_4$ does not modify the AdS asymptotics, such that we can use FG expansion and find the finite part of Eq. (27), which is given by

$$\delta\tilde{I} = \frac{1}{2}\int_{\partial M} d^3x\sqrt{g_{(0)}}\Big(-\frac{3}{16\pi G\ell} g^j_{(3)i} + \beta C^i_j(g_{(0)})\Big) \times (g^{-1}_{(0)}\delta g_{(0)})^j_i. \tag{28}$$

In a similar fashion, (anti) self-duality reads

$$\rho\mathcal{W}^{np}_{kl} + \frac{3\rho^{3/2}}{2\ell^2}(g^n_{(3)k}\delta^p_l - g^n_{(3)l}\delta^p_k + \delta^n_k g^p_{(3)l} - \delta^n_l g^p_{(3)k}) + \mathcal{O}(\rho^2) = \mp\frac{\rho^{3/2}}{\ell\sqrt{g_{(0)}}}\epsilon^{npm}(\nabla_{(0)k}g_{(1)ml} - \nabla_{(0)l}g_{(1)mk}) + \mathcal{O}(\rho^2), \tag{29}$$

where $\mathcal{W}$ is the Weyl tensor of $g_{(0)}$.

As a consequence, when the condition (25) holds, the value $\beta = \pm\ell^2/(32\pi G)$ corresponds to the self-dual point where the total stress tensor vanishes identically, i.e., $\mathcal{T}^i_j = 0$.

This reproduces the relation between the holographic stress tensor T^i_j and the Cotton tensor

$$T^i_j = \pm\frac{\ell^2}{8\pi G}C^i_j, \tag{30}$$

which has been observed in recent literature for solitonic solutions [14], electric-magnetic transformations in the fields in first-order gravity [12], and axial-polar perturbations in hydrodynamic models in AdS_4 [15].

The full duality between the renormalized stress tensor and the Cotton tensor has been obtained in [16] by relating two dual boundary conformal field theories (CFTs) which correspond to Dirichlet and Neumann boundary conditions (for related work on boundary conditions, see [17]). The two descriptions are mapped one into another by a Legendre transformation generated by a gravitational Chern-Simons term.

The total action for the particular value of β which realizes the relation (30) can be written in tetrad formalism as

$$\tilde{I} = \frac{\ell^2}{64\pi G}\int_M(\epsilon_{ABCD}W^{AB}W^{CD} \mp 2W^{AB}W_{AB}), \tag{31}$$

with the Weyl 2-form $W^{AB} = W^{\alpha\beta}_{\mu\nu} e^A_\alpha e^B_\beta dx^\mu dx^\nu$ in terms of the local orthonormal basis $e^A = e^A_\mu dx^\mu$. In this notation the (anti) self-duality condition (25) reads $W_{AB} = \pm * W_{AB} = \pm \frac{1}{2}\epsilon_{ABCD} W^{CD}$, with $** = +1$ for Euclidean signature.

Using the identity

$$\epsilon_{ABCD} W^{AB} W^{CD} = \tfrac{1}{2}\epsilon_{ABCD}(W^{AB}W^{CD} + *W^{AB} * W^{CD})$$

and also that

$$\epsilon_{ABCD} W^{AB} * W^{CD} = -2\mathcal{P}_4 d^4 x$$

(in an analogous way as in Yang-Mills theory), the total action (31) can be cast into the form

$$\tilde{I} = \frac{\ell^2}{128\pi G}\int_M \epsilon_{ABCD}(W^{AB} \mp *W^{AB})(W^{CD} \mp *W^{CD}) = \frac{\ell^2}{16\pi G}\int_M \sqrt{\det(W^{AB} \mp *W^{AB})}. \tag{32}$$

It is evident from the form of Eq. (5) that the value of the action reaches an absolute minimum for spacetimes which are globally of constant curvature (vacuum states of AdS gravity). The action (32) naturally generalizes this property to states which are globally (anti) self-dual in AdS gravity.

V. CONCLUSIONS

We have shown that the standard regularization of AdS gravity with counterterms is indeed topological, as it can be obtained from the addition of the Gauss-Bonnet invariant or the corresponding boundary term.

We have also considered the odd parity Pontryagin invariant, which accounts for viscosity in hydrodynamic models and for *magnetic* properties of solitonic solutions in AdS_4 [by analogy to the charge formula (20) which involves the electric part of the Weyl tensor due to the addition of the Euler term]. It is shown that the inclusion of this term is consistent assuming an asymptotic (anti) self-dual condition on the Weyl tensor. This reasoning explains the holographic stress tensor/Cotton tensor relation (30) recently found in different setups in the literature, and interprets it as coming from a duality between topological invariants.

The addition of topological invariants of the Euler class to the Einstein-Hilbert gravity action in $D = 2n$ dimensions was studied in Ref. [18], with the purpose of rendering finite the Noether charges for AAdS spacetimes. The variational principle singles out the value of the Euler coupling which produces a regularizing effect. One can instead consider the action supplemented by a boundary term $I_{2n} = I_{\rm EH} + c_{2n-1}\int_{\partial M} d^{2n-1}x B_{2n-1}$, where c_{2n-1} is a constant. In Ref. [5], it is claimed that the term B_{2n-1} which solves the regularization problem in even-dimensional AdS gravity is always prescribed by the Euler theorem and written using a parametric integration as a polynomial in the intrinsic and extrinsic curvatures

$$B_{2n-1} = 2n\sqrt{-h}\int_0^1 dt\,\delta^{[i_1\cdots i_{2n-1}]}_{[j_1\cdots j_{2n-1}]} K^{j_1}_{i_1}\left(\frac{1}{2}\mathcal{R}^{j_2 j_3}_{i_2 i_3} - t^2 K^{j_2}_{i_2}K^{j_3}_{i_3}\right) \times\cdots\cdots\times\left(\frac{1}{2}\mathcal{R}^{j_{2n-2}j_{2n-1}}_{i_{2n-2}i_{2n-1}} - t^2 K^{j_{2n-2}}_{i_{2n-2}}K^{j_{2n-1}}_{i_{2n-1}}\right) \tag{33}$$

with a coupling constant $c_{2n-1} = (-\ell^2)^{n-1}/(16\pi G n(2n-2)!)$. On purpose, we have not absorbed the constant in the boundary term, in order to stress the geometrical origin of the Kounterterm B_{2n-1}, as it is linked to topological invariants.

Furthermore, it has been proved that the term (33) regulates the Euclidean action and conserved quantities in any gravity theory of a Lovelock type with AdS asymptotics—including Einstein-Gauss-Bonnet AdS—, and where the information on a particular theory is contained only in its coupling constant [19].

As in the four-dimensional case, we add and subtract the Gibbons-Hawking term, i.e.,

$$I_{2n} = I_{\rm EH} - \frac{1}{8\pi G}\int_{\partial M} d^{2n-1}x\sqrt{-h}K + \int_{\partial M} d^{2n-1}x\mathcal{L}_{\rm ct},$$

where $\mathcal{L}_{\rm ct}$ is given by

$$\mathcal{L}_{\rm ct} = \frac{(-\ell^2)^n}{8\pi G(2n-2)!}\sqrt{-h}\,\delta^{[i_1\cdots i_{2n-1}]}_{[j_1\cdots j_{2n-1}]}K^{j_1}_{i_1} \times\int_0^1 dt\left[\left(\frac{1}{2}\mathcal{R}^{j_2 j_3}_{i_2 i_3} - t^2 K^{j_2}_{i_2}K^{j_3}_{i_3}\right)\cdots\cdots \times\left(\frac{1}{2}\mathcal{R}^{j_{2n-2}j_{2n-1}}_{i_{2n-2}i_{2n-1}} - t^2 K^{j_{2n-2}}_{i_{2n-2}}K^{j_{2n-1}}_{i_{2n-1}}\right) + \frac{(-1)^n}{\ell^{2n-2}}\delta^{j_2}_{i_2}\cdots\delta^{j_{2n-1}}_{i_{2n-1}}\right]. \tag{34}$$

When all the fields are expanded in the FG frame, one can collect terms as a power series in ρ and perform explicitly the parametric integration. It is useful to express the extrinsic curvature expansion as

$$K^i_j = \frac{1}{\ell}\delta^i_j - \rho\ell S^i_j(g) + \mathcal{O}(\rho^2),$$

where

$$S^i_j(g) = \frac{1}{D-3}\left(\mathcal{R}^i_j(g) - \frac{1}{2(D-2)}\delta^i_j\mathcal{R}(g)\right)$$

is the Schouten tensor of the metric $g_{ij}(x,\rho)$. Owing to the rescaling properties of the boundary Riemann tensor, the result can be written as a series of intrinsic counterterms

$$\mathcal{L}_{\rm ct} = \frac{\sqrt{-h}}{8\pi G}\Bigg[\frac{(2n-2)}{\ell} + \frac{\ell}{2(2n-3)}\mathcal{R} + \frac{\ell^3}{2(2n-3)^2(2n-5)} \times\left(2\mathcal{R}^{ij}\mathcal{R}_{ij} - \frac{(2n+1)}{4(2n-2)}\mathcal{R}^2 - \frac{(2n-3)}{4}\mathcal{R}^{ijkl}\mathcal{R}_{ijkl}\right) + \cdots\Bigg], \tag{35}$$

which includes a rather unusual (Riemann)^2 contribution. However, the falloff conditions for AAdS solutions imply that the Weyl tensor is such that

$$\sqrt{-h}W^{ijkl}W_{ijkl} \sim \frac{1}{r^{D-1}}$$

in Schwarzschild-like coordinates (see also [13]). Using this property for $D \geq 6$, we trade off the Riemman-squared term for the other curvature-squared terms, i.e.,

$$\mathcal{R}^{ijkl}\mathcal{R}_{ijkl} = \frac{4}{(2n-3)}\left(\mathcal{R}^{ij}\mathcal{R}_{ij} - \frac{1}{2(2n-2)}\mathcal{R}^2\right).$$

Remarkably enough, the series $\mathcal{L}_{\rm ct}$ adopts the form of standard counterterms obtained by holographic renormalization

$$\mathcal{L}_{\rm ct} = \frac{\sqrt{-h}}{8\pi G}\Bigg[\frac{(2n-2)}{\ell} + \frac{\ell}{2(2n-3)}\mathcal{R} + \frac{\ell^3}{2(2n-3)^2(2n-5)}\left(\mathcal{R}^{ij}\mathcal{R}_{ij} - \frac{(2n-1)}{4(2n-2)}\mathcal{R}^2\right) + \cdots\Bigg], \tag{36}$$

where cubic terms in the curvature are required by the regularization problem only for $D \geq 8$ dimensions.

We will provide the details of this derivation in a forthcoming publication.

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