Classifying and avoiding singularities in the alternative gravity dark energy models

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(Received 14 March 2009; published 8 June 2009)

The future finite-time singularities emerging in alternative gravity dark energy models are classified and studied in Jordan and Einstein frames. It is shown that such singularity may occur even in flat spacetime for the specific choice of the effective potential. The conditions for the avoidance of finite-time singularities are presented and discussed. The problem is reduced to the study of a scalar field evolving on an effective potential by using the conformal transformations. Some viable modified gravity models are analyzed in detail and the way to cure singularity is considered by introducing the higher order curvature corrections. These results may be relevant for the resolution of the conjectured problem in the relativistic star formation in such modified gravity where finite-time singularity is also manifested.

DOI: 10.1103/PhysRevD.79.124007 PACS numbers: 04.50.Kd, 11.25.−w, 95.36.+x, 98.80.−k

I. INTRODUCTION

Several assumptions of the cosmological standard model have been ruled out by the advent of the so-called *precision cosmology* capable of probing physics at very large redshifts. The old picture, based upon radiation and baryonic matter, has to be revised. Besides the introduction of dark matter, needed to fit the astrophysical dynamics at galactic and galaxy cluster scales (i.e. to explain clustered structures), a new ingredient is requested in order to explain the observed accelerated behavior of the Hubble flow: the so-called dark energy. Essentially, data coming from the luminosity distance of type Ia supernovae [1], the deep and wide galaxy surveys [2], and the anisotropy of cosmic microwave background [3] suggest that the so-called cosmological concordance model (ΛCDM) is spatially flat, dominated by cold dark matter (CDM $\sim 25\%$) and dark energy ($\Lambda \sim 70\%$). The first ingredient should be able to explain the dynamics of clustered structures while the latter, in the form of an "effective" cosmological constant, should give rise to the late-time accelerated expansion.

The cosmological constant is the most relevant candidate to interpret the cosmic acceleration, but, in order to overcome its intrinsic shortcomings associated with the energy scale, several alternative proposals have been suggested (see recent reviews [4–6]): quintessence models, where the cosmic acceleration is generated by means of a scalar field, in a way similar to the early-time inflation, acting at large scales and recent epochs [7]; models based on exotic fluids like the Chaplygin gas [8], or nonperfect fluids [9]; phantom fields, based on scalar fields with anomalous signature in the kinetic term [10], higher dimensional theories (braneworld) [11]. All of these models have the common feature to introduce new sources into the cosmological dynamics, but, from an "economic" point of view, it would be preferable to develop scenarios consistent with observations without invoking extra parameters or components nontestable (up to now) at a fundamental level.

Alternative theories of gravity, which extend in some way general relativity (GR), allow one to pursue this different approach (no further unknown sources), giving rise to suitable cosmological models where a late-time accelerated expansion is naturally realized.

The idea that Einstein gravity should be extended or corrected at large scales (infrared limit) or at high energies (ultraviolet limit) is suggested by several theoretical and observational aspects. Quantum field theories in curved spacetime, as well as the low-energy limit of string theory, both imply semiclassical effective Lagrangians containing higher order curvature invariants or scalar-tensor terms. In addition, GR has been definitely tested only at Solar System scales while it may show several shortcomings, if checked at higher energies or larger scales. Besides, in the opinion of several authors, the Solar System experiments are not so conclusive as to state that the *only reliable theory at these scales is GR*.

Of course modifying the gravitational action asks for several fundamental challenges. These models can exhibit instabilities [12] or ghostlike behaviors [13], while, on the other hand, they should be matched with observations and experiments in the low-energy limit [in other words, Solar System tests and parametrized post-Newtonian limits should reproduce the results of GR appropriately]. Despite all of these issues, in the past years, several interesting results have been achieved in the framework of the so-called modified gravity at cosmological, galactic and solar system scales (see Refs. [14,15] for a review).

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For example, cosmological solutions exist that lead to the accelerated expansion of the Universe at late times in specific models of $f(R)$ gravity as is discovered in Refs. [16–18]. In some of the realistic theories of this sort the problems indicated in [19] may be overcome [14].

Viable $f(R)$ models exist that can satisfy both background cosmological constraints and stability conditions [20–26] as well as local tests. Recently many works have been devoted to place constraints on $f(R)$ models using the observations of cosmic microwave background anisotropies and galaxy power spectrum [27,28].

Besides, considering $f(R)$ gravity in the low-energy limit, it is possible to obtain corrected gravitational potentials capable of explaining the flat rotation curves of spiral galaxies and galaxy cluster haloes without considering huge amounts of dark matter [25,29–34] and, furthermore, this seems the only self-consistent way to reproduce the universal rotation curve of spiral galaxies [35]. On the other hand, several anomalies in Solar System experiments could be framed and addressed in this picture [36].

However, a fundamental task which has to be faced for any alternative gravity model is to classify singularities which could emerge at a finite time and propose the way to avoid it. From a physical point of view, this point is crucial in order to achieve viable and self-consistent models, especially in the possible applications.

In this paper, we discuss the future singularities which can, in principle, appear in dark energy models coming from alternative gravity theories (higher order or nonminimally coupled gravity).

In fact, when dark energy models with the effective equation of state parameter close to -1 were added to the list of admissible cosmological theories to explain the observed accelerated behavior, due to the violation of all (or part) of the energy conditions, strange features emerged. For instance, it is well known that phantom dark energy brings the universe to a finite-time *big rip* singularity [37,38]. Moreover, the effective quintessence dark energy cosmologies may end up in (softer) finite-time singularity [39,40]. For such effective quintessence dark energy models only part of the energy conditions does not work in the standard way. Nevertheless, they show the rip singularity behaviors which have been classified in Ref. [39].

It is clear that, qualitatively, the same situation should also occur in modified gravity cosmologies [14]. Indeed, it is quite well known that some versions of modified gravity [like $f(R)$] have an effective ideal fluid description [41]. Hence, precisely the same singular behavior should be typical for the (effective phantom/quintessence) modified gravities in the future too. Indeed, it was found some time ago [42] that modified gravity becomes invalid (complex theory) at the point where mathematically equivalent scalar-tensor dark energy theory enters the big rip singularity. Moreover, the effective phantom behavior may enter a transient phase and future singularity does not occur if some higher order terms (like R^2) are added to initially phantomlike models [43]. The same approach has recently been considered in Ref. [44] to remove the singularity in order to avoid the conjectured problems with neutron stars formation in modified gravity.

In this paper we want to discuss, in general, the problem of finite-time singularities and discuss some ways to avoid them in viable models which well fit data at local and cosmological scales.

The layout of the paper is the following. In Sec. II, we describe, in general, the problem of finite-time singularities in dark energy models coming from $f(R)$ to scalar-tensor modified gravities. Section III is devoted to the conditions for singularity avoidance in $f(R)$ gravity and in its scalar-tensor counterpart. In Sec. IV, we discuss the singularity problem in some physically viable $f(R)$ models adopting a conformal transformation approach. This method allows, in principle, to discriminate singularities by studying the behavior of effective scalar field potential after dynamics has been conformally reduced to the Einstein frame. In particular, we study the effect of adding a correction term, proportional to R^n with $n \geq 2$, to modify the structure of the potential at large values of R and cure the singularity. Discussion and conclusions are drawn in Sec. V.

II. FINITE-TIME SINGULARITIES IN DARK ENERGY MODELS: FROM $F(R)$ GRAVITY TO SCALAR-TENSOR THEORY

Let us start with a generic action of $F(R)$ gravity which is a straightforward extension of general relativity:

$$S_{F(R)} = \int d^4x\sqrt{-g}\left\{\frac{F(R)}{2\kappa^2} + \mathcal{L}_m\right\}. \tag{1}$$

Here $F(R) = R + f(R)$ is an appropriate function of the scalar curvature R and $\mathcal{L}_m$ is the Lagrangian density of matter. By the variation over the metric tensor $g_{\mu\nu}$, we obtain the fourth-order field equations:

$$\frac{1}{2}g_{\mu\nu}F(R) - R_{\mu\nu}F'(R) - g_{\mu\nu}\Box F'(R) + \nabla_\mu\nabla_\nu F'(R) = -\frac{\kappa^2}{2}T_{(m)\mu\nu}. \tag{2}$$

In Eq. (2), $T_{(m)\mu\nu}$ is the matter energy-momentum tensor. Contracting Eq. (2) with respect to μ and ν, we obtain the trace equation:

$$2F(R) - RF'(R) - 3\Box F'(R) = -\frac{\kappa^2}{2}T. \tag{3}$$

To recover, formally, general relativity, Eq. (3) can be rewritten as

$$R + 2f(R) - Rf'(R) - 3\Box f'(R) = -\frac{\kappa^2}{2}T. \tag{4}$$

In order to study how finite-time singularities emerge and can be discussed, let us consider, for the moment, classes of models which are paradigmatic for our purposes. For example, in Ref. [22], a model which easily passes local tests and several cosmological bounds, has been proposed:

$$f_{\rm HS}(R) = -\frac{m^2 c_1 (R/m^2)^n}{c_2 (R/m^2)^n + 1}, \tag{5}$$

or otherwise written as

$$f_{\rm HS}(R) = -\lambda R_c \frac{(\frac{R}{R_c})^n}{(\frac{R}{R_c})^n + 1}. \tag{6}$$

Here m is a proper scale and c_1, c_2, n, and λ are dimensionless positive constants (n is not restricted to be an integer) and R_c is a positive constant. When the curvature is sufficiently large at dark energy epoch, this model can be approximated as follows:

$$f(R) \sim -2\Lambda + \frac{\alpha}{R^n}. \tag{7}$$

In case of (5), one may identify

$$2\Lambda = \frac{m^2 c_1}{c_2}, \qquad \alpha = \frac{m^{2m+2} c_1}{c_2^2}. \tag{8}$$

Then Eq. (4) reduces to

$$R + 3\alpha \Box (R^{-n-1}) \sim 0. \tag{9}$$

Since the large curvature regime is considered, the cosmological constant term appears as a next-to-leading order correction, compared with the first term in (9), and we neglect it. If we define χ as

$$\chi \equiv R^{-n-1}, \tag{10}$$

and the Friedmann-Robertson-Walker (FRW) metric with a flat spatial part is chosen,

$$ds^2 = -dt^2 + a(t)^2 \sum_{i=1,2,3} (dx^i)^2. \tag{11}$$

Equation (9) has the following form:

$$\ddot{\chi} + 3H\dot{\chi} = \frac{1}{3\alpha}\chi^{-1/(n+1)}. \tag{12}$$

Note $\chi = 0$ corresponds to the curvature singularity $R \to \infty$. Note that as other dark energy models with an equation of state (EoS) parameter around -1, the above gravitational alternative for dark energy also has the singularity as will be explained below. The fact that such $F(R)$ gravity may show the phantomlike behavior has been established in Ref. [43]. It was demonstrated there that account of R^2 (or similar nature term) makes the phantom phase transient and removes the singularity. In principle, the phantom phase in $F(R)$ gravity may end up as a big rip-like type singularity [37] as was demonstrated in Ref. [42].

First we consider the classical equation of motion:

$$\ddot{x} = \frac{1}{3\alpha} x^{-1/(n+1)}. \tag{13}$$

The difference between the cosmological equation (12) and the classical equation (13) is the second term depending on H. This second term gives the only subleading contribution, which will be shown in the analysis from (15), where the H dependence will be explicitly included and it will be shown that the result from the classical analysis here will be reproduced. For Eq. (13), one gets an exact solution:

$$x = C(t_0 - t)^{2(n+1)/(n+2)}. \tag{14}$$

Here C and t_0 are constants. Note $2 > 2(n+1)/(n+2) > 1$. Then x vanishes in a finite time $t = t_0$, which corresponds to the curvature singularity in (13).

We now investigate the asymptotic solution when the curvature is large, that is, χ is small. As there is a curvature singularity, one may assume

$$H \sim h_0 (t_0 - t)^{-\beta}. \tag{15}$$

Here h_0 and β are constants, h_0 is assumed to be positive, and $t < t_0$ as it should be for the expanding universe. Even for noninteger $\beta < 0$, some derivative of H and therefore the curvature becomes singular. The case $\beta = 1$ corresponds to the big rip singularity, Fig. 1. Furthermore, since $\beta = 0$ corresponds to the de Sitter space, which has no singularity, it is assumed $\beta \neq 0$.

When $\beta > 1$, the scalar curvature R behaves as

$$R \sim 12H^2 \sim 12h_0^2 (t_0 - t)^{-2\beta}. \tag{16}$$

On the other hand, when $\beta < 1$, the scalar curvature R behaves as

$$R \sim 6\dot{H} \sim -6h_0\beta (t_0 - t)^{-\beta-1}. \tag{17}$$

Then R diverges when $\beta > -1$ but $\beta \neq 0$.

We now consider four cases (compare with [45]): (1) $\beta = 1$, (2) $\beta > 1$, (3) $1 > \beta > 0$, and (4) $0 > \beta > -1$; see Fig. 2.

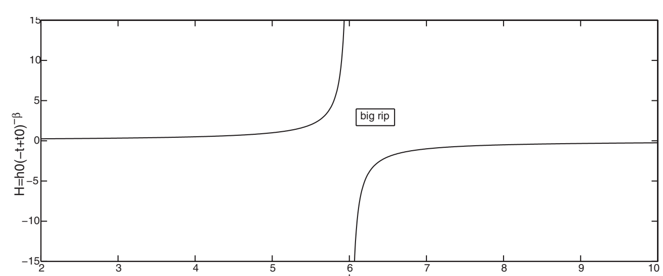

FIG. 1. Plot of $H \sim h_0(t_0 - t)^{-\beta}$. The big rip singularity occurs for $\beta = 1$ and $t = t_0 = 6$.

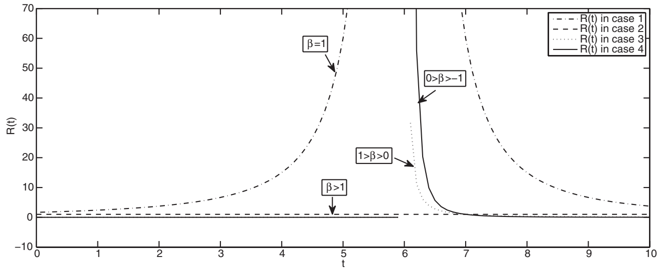

FIG. 2. Behavior of $R(t)$ for the four cases in the text. The dash-dotted line represents R for $\beta = 1$ in Eq. (18); the dashed line is for $\beta > 1$ in Eq. (22); the dotted line is $R(t)$ for $1 > \beta > 0$; and finally, the solid line $R(t)$ is for $0 > \beta > -1$ in Eq. (27), respectively.

(i) In case (1) $\beta = 1$, since

$$R \sim \frac{12h_0^2 + 6h_0}{(t_0 - t)^2}, \tag{18}$$

and therefore, from (10), we find

$$\chi \sim (t_0 - t)^{2(n+1)}, \tag{19}$$

and the left-hand side (lhs) of (12) behaves as

$$\ddot{\chi} + 3H\dot{\chi} \sim (t_0 - t)^{2n}, \tag{20}$$

but the right-hand side (rhs) behaves as

$$\frac{1}{3\alpha}\chi^{-1/(n+1)} \sim (t_0 - t)^{-2}, \tag{21}$$

which is inconsistent since the powers of both sides do not coincide with each other. Therefore, $\beta \neq 1$.

(ii) In case (2) $\beta > 1$, we find

$$R = 12H^2 + 6\dot{H} \sim 12H^2 \sim (t_0 - t)^{-2\beta}, \tag{22}$$

and therefore

$$\chi \sim (t_0 - t)^{2\beta(n+1)}. \tag{23}$$

In the lhs of (12), the second term dominates and the lhs behaves as

$$\ddot{\chi} + 3H\dot{\chi} \sim 3H\dot{\chi} \sim (t_0 - t)^{\beta(2n+1)-1}. \tag{24}$$

On the other hand, the rhs behaves as

$$\frac{1}{3\alpha}\chi^{-1/(n+1)} \sim (t_0 - t)^{-2\beta}. \tag{25}$$

Then by comparing the powers of both sides, one gets

$$\beta(2n + 1) - 1 = -2\beta, \tag{26}$$

which gives $\beta = 1/(2n + 3)$, but this conflicts with the assumption $\beta > 1$.

(iii) In case (3) $1 > \beta > 0$ or case (4) $0 > \beta > -1$, we find

$$R = 12H^2 + 6\dot{H} \sim 6\dot{H} \sim (t_0 - t)^{-\beta-1}, \tag{27}$$

and therefore

$$\chi \sim (t_0 - t)^{(\beta+1)(n+1)}. \tag{28}$$

Then on the lhs of (12), the first term dominates and the lhs behaves as

$$\ddot{\chi} + 3H\dot{\chi} \sim \ddot{\chi} \sim (t_0 - t)^{\beta(n+1)+n-1}. \tag{29}$$

On the other hand, the rhs behaves as

$$\frac{1}{3\alpha}\chi^{-1/(n+1)} \sim (t_0 - t)^{-\beta-1}. \tag{30}$$

Then by comparing the powers of the left-hand side and the right-hand side, the consistency gives

$$\beta(n + 1) + n - 1 = -\beta - 1 \quad \text{or} \quad \beta = -n/(n + 2). \tag{31}$$

This conflicts with case (3) $0 < \beta < 1$ but is consistent with case (4) $0 > \beta > -1$. In fact, by substituting (31) into (28), we get

$$\chi \sim (t_0 - t)^{2(n+1)/(n+2)}, \tag{32}$$

which corresponds to (14). Since $0 > \beta > -1$, this singularity corresponds to type II in [39].

Thus, the sudden finite-time curvature singularity really appears in the Hu-Sawicki (HS) model.

In [39], the classification of the finite-time singularities was suggested in the following way:

(i) Type I (big rip): For $t \to t_s$, $a \to \infty$, $\rho \to \infty$, and $|p| \to \infty$. This also includes the case of ρ, p being finite at t_s.

(ii) Type II ("sudden"): For $t \to t_s$, $a \to a_s$, $\rho \to \rho_s$, and $|p| \to \infty$

(iii) Type III: For $t \to t_s$, $a \to a_s$, $\rho \to \infty$, and $|p| \to \infty$

(iv) Type IV: For $t \to t_s$, $a \to a_s$, $\rho \to 0$, $|p| \to 0$, and higher derivatives of H diverge. This also includes the case when p (ρ) or both of them tend to some finite values while higher derivatives of H diverge.

Here t_s, a_s, and ρ_s are constants with $a_s \neq 0$. We now identify t_s with t_0. Type I corresponds to the $\beta > 1$ or the $\beta = 1$ case and we have a big rip singularity [38], whereas type II to the $-1 < \beta < 0$ case and corresponds to the sudden future singularity, type III to the $0 < \beta < 1$ case and is different from the sudden future singularity in the sense that ρ diverges, and type IV to $\beta < -1$ but β is not any integer case.

Let us now remember that the type II singularity has been already discussed in several dark energy models [40] besides $F(R)$ gravity. Here we consider several theories in the FRW spacetime with the flat spatial part (11). For type II singularity the Hubble rate $H \equiv \dot{a}/a$ has the following form:

$$H = H_0 + H_1(t_0 - t)^\gamma. \tag{33}$$

Here H_0, H_1, t_0, and γ are constants. We now choose $0 < \gamma < 1$. Then H is finite $H \to H_0$ in the limit of $t \to t_0$ but $\dot{H}$ diverges as

$$\dot{H} = H_1 \gamma (t_0 - t)^{\gamma - 1}, \tag{34}$$

which generates the singularity in the scalar curvature R since (see Fig. 3)

$$R = 12H^2 + 6\dot{H} \sim 6H_1\gamma(t_0 - t)^{\gamma-1}. \tag{35}$$

We should note that the energy density ρ is finite since the first FRW equation gives

$$\rho = \frac{3}{\kappa^2} H^2, \tag{36}$$

and therefore $\rho \to (3/\kappa^2)H_0^2 < \infty$ in the limit $t \to t_0$. Hence, the curvature singularity could occur even if the energy density is finite as in some other quintessence models.

We now give an explicit example of the ideal fluid which gives the singularity in (33). First we should note that the second FRW equation has the following form:

$$p = -\frac{1}{\kappa^2}(2\dot{H} + 3H^2). \tag{37}$$

For the Hubble rate H in (33), Eqs. (36) and (37) give

$$\rho = \frac{3}{\kappa^2}(H_0 + H_1(t_0 - t)^\gamma)^2,$$
$$\rho + p = -\frac{2H_1\gamma}{\kappa^2}(t_0 - t)^{\gamma-1}. \tag{38}$$

Then by deleting t in the two equations of (38), we find

$$\rho = \frac{3}{\kappa^2}\left(H_0 + H_1\left(-\frac{\kappa^2(\rho + p)}{2H_1\gamma}\right)^{\gamma/(\gamma-1)}\right)^2. \tag{39}$$

Equation (39) can be regarded as an EoS. Conversely if we consider the perfect fluid satisfying the EoS (39), the singularity (33) occurs.

We now consider the occurrence of singularity (33), in terms of the scalar-tensor theory, whose action is given by

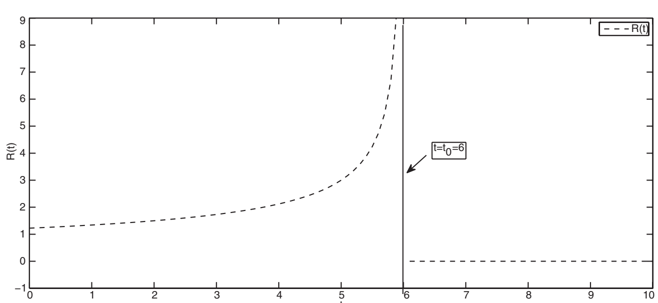

FIG. 3. Behavior of R as given in Eq. (35). We have assumed $\gamma = \frac{1}{2}$.

$$S = \int dx^4 \sqrt{-g}\left[\frac{1}{2\kappa^2}R - \frac{1}{2}\omega(\phi)\partial_\mu\phi\partial^\mu\phi - V(\phi)\right], \tag{40}$$

where $V(\phi)$ is the scalar potential and $\omega(\phi)$ is the kinetic function, respectively. Note that for convenience the kinetic factor is introduced. We should note that the scalar field may be always redefined so that the kinetic function is absorbed. In the spatially flat FRW spacetime (11), the energy density and the pressure of the scalar field are given by

$$\rho_\phi = \tfrac{1}{2}\omega(\phi)\dot{\phi}^2 + V(\phi), \qquad p_\phi = \tfrac{1}{2}\omega(\phi)\dot{\phi}^2 - V(\phi). \tag{41}$$

Combining the FRW equations (36) and (37) with (41), one obtains

$$\omega(\phi)\dot{\phi}^2 = -\frac{2}{\kappa^2}\dot{H}, \qquad V(\phi) = \frac{1}{\kappa^2}(3H^2 + \dot{H}). \tag{42}$$

We now consider the theory in which $V(\phi)$ and $\omega(\phi)$ are given by

$$\omega(\phi) = -\frac{2}{\kappa^2}f'(\phi), \qquad V(\phi) = \frac{1}{\kappa^2}[3f(\phi)^2 + f'(\phi)], \tag{43}$$

where $f(\phi)$ is a proper function of ϕ. Then the following solution is found:

$$\phi = t, \qquad H(t) = f(t). \tag{44}$$

In case of (33), we find

$$\omega(\phi) = -\frac{2H_1\gamma}{\kappa^2}(t_0 - \phi)^{\gamma-1},$$
$$V(\phi) = \frac{1}{\kappa^2}[(3H_0 + 3H_1(t_0 - \phi)^\gamma)^2 + H_1\gamma(t_0 - \phi)^{\gamma-1}]. \tag{45}$$

Thus, with such potential choice, the singularity (33) could be easily realized and the energy density is, of course, finite. It is easy to construct the models showing the finite future singularity in other models, say, the scalar Gauss-Bonnet theory, nonminimal theories [46], etc.

We now consider what kind of scalar potential can generate the singularity in the scalar field in the flat spacetime background. As a form of the singularity, we now assume

$$\phi = \phi_0 + \phi_1(t_0 - t)^\gamma. \tag{46}$$

Here ϕ_0, ϕ_1, and γ are constants. One may take γ to be positive but not an integer. Then some derivative of the scalar field has singularity.

In the next section, we will rewrite the $F(R)$ gravity in the scalar-tensor form, where the metric is rescaled as given in (53). Because of the scale transformation, the curvature singularity in the original frame of the $F(R)$ gravity does not appear in the rescaled metric in the

the scalar-tensor frame. The singularity occurs in the scalar field as follows from (46). Hence, such scalar field singularity (46) is called finite-time singularity even in flat Minkowski space.

In the flat spacetime background, the equation of the scalar field is given by

$$\ddot{\phi} + V'(\phi) = 0. \tag{47}$$

Here $V(\phi)$ is a potential of the scalar field. By substituting (46) into (47), we find

$$\phi_1\gamma(\gamma - 1)(t_0 - t)^{\gamma-2} + V'(\phi) = 0. \tag{48}$$

Since (46) can be rewritten as

$$t_0 - t = \left(\frac{\phi - \phi_0}{\phi_1}\right)^{1/\gamma}, \tag{49}$$

by substituting the expression (49) into (48), we find

$$\phi_1\gamma(\gamma - 1)\left(\frac{\phi - \phi_0}{\phi_1}\right)^{1-2/\gamma} + V'(\phi) = 0, \tag{50}$$

which gives the form of the potential as

$$V(\phi) = V_0 - \frac{\phi_1\gamma^2}{2}\left(\frac{\phi - \phi_0}{\phi_1}\right)^{2-2/\gamma}. \tag{51}$$

Hence, it is found that finite-time singularity (46) can be realized by the potential (51) even in flat spacetime. Thus, we demonstrated that for variety of dark energy models including modified gravity the finite-time singularity easily occurs even in the situation when the effective EoS parameter is bigger than -1 (the effective quintessence).

III. THE AVOIDANCE OF FINITE-TIME SINGULARITY IN MODIFIED $F(R)$ GRAVITY

Let us consider the action (1) again. By introducing the auxiliary field A, we rewrite the action (1) of the $F(R)$ gravity in the following form:

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\{F'(A)(R - A) + F(A)\}. \tag{52}$$

Here we neglect the contribution from the matter. By the variation over A, one obtains $A = R$. Substituting $A = R$ into the action (52), one can reproduce the action in (1). Furthermore, we rescale the metric in the following way (conformal transformation) [18]:

$$g_{\mu\nu} \rightarrow e^{\sigma}g_{\mu\nu}, \qquad \sigma = -\ln F'(A). \tag{53}$$

Hence, the Einstein frame action is obtained:

$$S_E = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left(R - \frac{3}{2}g^{\rho\sigma}\partial_\rho\sigma\partial_\sigma\sigma - V(\sigma)\right),$$
$$V(\sigma) = e^{\sigma}g(e^{-\sigma}) - e^{2\sigma}f(g(e^{-\sigma})) = \frac{A}{F'(A)} - \frac{F(A)}{F'(A)^2} \tag{54}$$

Here $g(e^{-\sigma})$ is given by solving the equation $\sigma = -\ln(1 + f'(A)) = \ln F'(A)$ as $A = g(e^{-\sigma})$. In terms of $f(R)$ the potential $V(\sigma)$ could be rewritten as

$$V(\sigma) = \frac{Af'(A) - f(A)}{(1 + f'(A))^2}. \tag{55}$$

For the class of models with $f(R)$ behaving as in (7), we find

$$V(\sigma) \sim -2\Lambda - \frac{(n + 1)\alpha}{R^n}, \tag{56}$$

for large scalar curvature. When the curvature is large in the model (7), one gets

$$\sigma \sim \frac{\alpha n}{R^{n+1}}. \tag{57}$$

By combining (56) and (57), the potential $V(\sigma)$ is given in terms of σ as

$$V(\sigma) \sim -2\Lambda - (n + 1)\alpha\left(\frac{\sigma}{\alpha n}\right)^{n/(n+1)}. \tag{58}$$

By comparing (56) with (51) and identifying σ with $(\phi - \phi_0)/\phi_0$, we find

$$\frac{n}{n + 1} = 2 - \frac{2}{\gamma}, \tag{59}$$

or

$$\gamma = \frac{2(n + 1)}{n + 2}. \tag{60}$$

Since γ is fractional in general, the scalar field σ generates the singularity in (46), which now corresponds to the curvature singularity in the Jordan frame. We should note that the curvature in the Einstein frame is in general not singular.

Let us consider the realistic models which unify the early-time inflation and late-time acceleration and which were introduced in Refs. [25,26].

In order to construct such models, the following conditions are used:

(i) Condition that inflation occurs:

$$\lim_{R\to\infty} f(R) = -\Lambda_i. \tag{61}$$

Here Λ_i is an effective early-time cosmological constant.

(ii) The condition that there is flat spacetime solution is given as

$$f(0) = 0. \tag{62}$$

(iii) The condition that late-time acceleration occurs should be

$$f(R_0) = -2\tilde{R}_0, \qquad f'(R_0) \sim 0. \tag{63}$$

Here R_0 is the current curvature of the Universe and we assume $R_0 > \tilde{R}_0$. Because of the condition (63), $f(R)$ becomes almost constant in the present universe and plays the role of the effective small cosmological constant: $\Lambda_{\rm eff} \sim -f(R_0) = 2\tilde{R}_0$.

An example which satisfies the conditions (61)–(63) is given by the following action [25]:

$$f(R) = -\frac{(R-R_0)^{2n+1} + R_0^{2n+1}}{f_0 + f_1\{(R-R_0)^{2n+1} + R_0^{2n+1}\}}. \tag{64}$$

Here n is a positive integer.

The conditions (61) and (63) require

$$\frac{R_0^{2n+1}}{f_0 + f_1 R_0^{2n+1}} = 2\tilde{R}_0, \qquad \frac{1}{f_1} = \Lambda_i. \tag{65}$$

One can show that for the above class of models, the singularity does not occur or it is difficult to see that the singularity occurs in the visible future. We will now work in the scalar-tensor form or the Einstein frame (54).

In the case of the model in [22], in the limit of $A = R \to +\infty$, $f(A)$ becomes a small constant $-\Lambda_{\rm eff}$ corresponding to the small effective cosmological constant in the present universe. In the limit, we find $f'(A) \to 0$ and the potential (55) behaves as in (56) and therefore

$$V(\sigma) \to \Lambda_{\rm eff}. \tag{66}$$

Then the value of V is very small and the curvature singularity could be easily generated. In the models exhibiting this kind of singularity due to the fact that singularity is at a finite distance with the current energy scale, it was conjectured [44] that neutron stars cannot be formed. [This is really a conjecture because the derivation of the stars formation follows the same approximation as in the usual Einstein gravity, actually neglecting the higher derivative terms typical in $F(R)$ gravity. This derivation should be reconsidered within the real $F(R)$ gravity with account of higher derivative terms as well as nonlinearity that is technically not an easy task. For instance, the nonlinear structure of modified gravity leads to oscillations [47] already in the very simple approximation.] The conjecture is that if the density of the matter becomes finite but large and reaches a critical value, the curvature singularity occurs and the star could collapse. Therefore the star with density larger than the critical value could not be formed. As is mentioned above these simple considerations may be not valid with account of higher derivative terms where the star formation process should be reconsidered from the very beginning.

On the other hand, for the class of models satisfying the conditions (61)–(63) like the model (65), the singularity does not occur easily. For this class of the models, $f'(R)$ vanishes at $R = R_0$ and in the limit of $R \to \infty$. When $R = R_0$, the condition (61) gives

$$V(\sigma) \to 2\tilde{R}_0, \tag{67}$$

which could be quite small. Since the value of R is finite, (67) does not correspond to any singularity. On the other hand, in the limit of $R \to +\infty$, by using the condition (63), we find

$$V(\sigma) \to \Lambda_i. \tag{68}$$

Since Λ_i corresponds to the effective cosmological constant during the inflation, the energy scale is not small, typically it is the grand unification scale. Therefore the value of $V(\sigma)$ could be very large. Then the singularity could be generated only at the energy density larger then the energy density corresponding to the inflation of the early universe but it does not occur around the energy density which is typical for neutron star.

Even for the class of dark energy models where singularity occurs at smaller energies, there is a scenario to avoid the singularity proposed in Ref. [43]. Indeed, let us consider the model where $f(R)$ behaves as

$$f(R) \sim f_0 R^\alpha, \tag{69}$$

with constants f_0 and $\alpha > 1$. If the ideal fluid, which could be the matter with the constant EoS parameter w: $p = w\rho$, couples with the gravity, when the $f(R)$ term dominates compared with the Einstein-Hilbert term, an exact solution is [43]

$$\begin{aligned} a &= a_0 t^{h_0}, \qquad h_0 \equiv \frac{2\alpha}{3(1+w)}, \\ a_0 &\equiv \Big[-\frac{6f_0 h_0}{\rho_0}(-6h_0 + 12h_0^2)^{\alpha-1}\{(1-2\alpha)(1-\alpha) \\ &\quad - (2-\alpha)h_0\}\Big]^{-1/3(1+w)}. \end{aligned} \tag{70}$$

When $\alpha = 1$, the result $h_0 = \frac{2}{3(1+w)}$ in the Einstein gravity is reproduced. The effective EoS parameter $w_{\rm eff}$ may be defined by

$$h_0 = \frac{2}{3(1+w_{\rm eff})}. \tag{71}$$

By using (70), one finds

$$w_{\rm eff} = -1 + \frac{1+w}{\alpha}. \tag{72}$$

Hence, if w is greater than -1 (effective quintessence or even usual ideal fluid with positive w), when α is negative, we obtain the effective phantom phase where $w_{\rm eff}$ is less than -1. This is different from the case of pure modified gravity. On the other hand, when $\alpha > w + 1$ (it can be even positive), $w_{\rm eff}$ could be negative for negative w. Hence, it follows that modified gravity minimally coupled with usual (or quintessence) matter may reproduce a quintessence (or phantom) evolution phase for the dark energy universe in an easier way than without such a coupling.

If we choose α to be negative in (69), when the curvature is small, the $f(R)$ term becomes dominant compared with the Einstein-Hilbert term. Then from (72), we have an effective phantom even if $w > -1$. Usually the phantom generates the big rip singularity. However, near the big rip singularity, the curvature becomes large and the Einstein-Hilbert term becomes dominant. In this case, we have $w_{\rm eff} \sim w > -1$, which prevents the big rip singularity.

One may add an extra term to $f(R)$ (69) as [43,46]

$$f(R) = f_0 R^\alpha + f_1 R^\beta. \tag{73}$$

Here we choose $f_1 > 0$ and $\beta > 1$. Then for the large curvature, the second term could dominateand the potential (55) behaves as

$$V = \frac{\beta - 2}{f_1 R^{\beta-2}}. \tag{74}$$

Then if $1 < \beta < 2$, the potential is positive and diverges near the curvature singularity $R \to \infty$, which could prevent the curvature singularity even if the singularity is the big rip type (type I) or type II [46] or other softer singularity [46].

Thus, we demonstrated that for a large class of viable $F(R)$ gravities the finite-time singularity occurs in such a distant future that it cannot influence the current universe processes. From another side, there exists the trick introduced in Refs. [43,46] of adding an extra term to modified gravity to remove the singularity.

As is clear from (52), for $F'(R) = 1 + f'(R) > 0$, the square of the effective gravitational coupling is positive. However, due to $\kappa^2_{\rm eff} \equiv \kappa^2/F'(A)$, the theory could enter an antigravity regime. In order to avoid this antigravity problem from the very beginning, we may consider a model given by

$$f(R) = -f_0 \int_0^R dR e^{-[\alpha R_1^{2n}/(R-R_1)^{2n}]-(R/\beta\Lambda_i)}. \tag{75}$$

Here α, β, f_0, and R_1 are constants and we assume $0 < R_1 \ll \Lambda_i$. In this model, the correction to the Newton law is very small. Then by construction, as long as $0 < f_0 < 1$, we find $f'(R) > -1$ or $F'(R) > 0$ and therefore there is no antigravity problem. Since

$$f(R_1) \sim -f_0 \int_0^{R_1} dR e^{-[\alpha R_1^{2n}/(R-R_1)^{2n}]} = -f_0 A_n(\alpha) R_1,$$
$$A_n(\alpha) \equiv \int_0^1 dx e^{-\alpha/x^{2n}}, \tag{76}$$

and $-f(R_1)$ could be identified with the effective cosmological constant $2\tilde{R}_0$, we find

$$f_0 A_n(\alpha) R_1 = \tilde{R}_0. \tag{77}$$

Note that $A_n(0) = 1$, $A_n(+\infty) = 0$, and $A'(x) < 0$. On the other hand, since

$$f(+\infty) \sim \int_0^\infty dR e^{-R/\beta\Lambda_i} = -f_0 \beta \Lambda_i, \tag{78}$$

and $-f(+\infty)$ could be identified with the effective cosmological constant at the inflationary epoch, Λ_i, we find

$$f_0 \beta = 1. \tag{79}$$

Then the conditions (61) and (63) are satisfied. The condition (61) is, of course, satisfied by construction (75). As discussed around Eq. (68), as long as Λ_i is large enough, the potential becomes large when the scalar curvature R is large and there could not occur or could be difficult to realize the singularity. We also note that if we add the second term $f_1 R^\beta$ with $1 < \beta < 2$ in (73), the singularity is completely removed since the potential diverges in the limit of $R \to \infty$. Thus, for a large class of viable modified gravity models the singularity may be easily removed by adding anextra term which is actually relevant at the early universe. Moreover, the typical energy scales of the neutron star and the singularity formation process (above the inflation scale) are at large distances and they are not relevant to each other.

IV. THE SINGULARITY PROBLEM IN *f*(*R*)-VIABLE MODELS

In this section, we are going to discuss the curvature singularity problem that affects several infrared-modified $f(R)$ models [22]. As we have said previously, considering $f(R) \neq R$ gravity means that a new scalar degree of freedom has to be taken into account. Conformal transformations of the metric can be used to make it explicit in the action [48,49].

For our goals, we consider a class of $f(R)$ models which do not contain a cosmological constant and are explicitly designed to satisfy cosmological and Solar System constraints [50]. In practice, we choose a class of functional forms of $f(R)$ capable of matching, in principle, observational data [51]. First of all, any viable cosmological model has to reproduce the cosmic microwave background radiation constraints in the high-redshift regime. Secondly, it should give rise to an accelerated expansion, at low redshift, according to the ΛCDM model. Thirdly, these models should give rise to a large mass for the scalaron [52] in the high-density region where local gravity experiments are carried out. In such a regime, the perturbation in R can be larger than the background value R_0, which means that the linear approximation to derive the Newtonian effective gravitational constant in a spherically symmetric spacetime ceases to be valid [53,54]. In the nonlinear regime with a heavy scalaron mass, however, it is known that a spherically symmetric body has a thin shell [22,28,53,55] through the so-called *chameleon mechanism* [56,57] (see also Refs. [58,59]). When a thin shell is formed, an effective coupling that mediates the fifth force gets smaller. This allows the possibility that the $f(R)$ models which have a large scalaron mass in the high-density region can be compatible with local gravity experiments. Then there should be sufficient degrees of freedom in the parametrization to encompass low redshift phenomena (e.g. the large scale structure) according to the observations [60]. Finally, small deviations from GR should be consistent with Solar System tests. All these requirements suggest that we have to assume the limits

$$\lim_{R\to\infty} f(R) = \text{const}, \tag{80}$$

$$\lim_{R\to 0} f(R) = 0, \tag{81}$$

which are satisfied by a general class of broken power law models, as those proposed in [22]

$$f_{\rm HS}(R) = -\lambda R_c \frac{(\frac{R}{R_c})^{2n}}{(\frac{R}{R_c})^{2n} + 1}. \tag{82}$$

Since $f(R = 0) = 0$, the cosmological constant has to disappear in a flat spacetime. The parameters $\{n, \lambda, R_c\}$ are constants which should be determined by experimental bounds.

Other interesting models with similar features have been studied in [23–26]. In all these models, a de Sitter stability point, responsible for the late-time acceleration, exists for $R = R_1 (>0)$, where R_1 is derived by solving the equation $R_1 f_{,R}(R_1) = 2f(R_1)$ [61].

In the region $R \gg R_c$, model (82) behaves as

$$f_{\rm hybrid\ 1}(R) \simeq -\lambda R_c[1 - (R_c/R)^{2s}], \tag{83}$$

where s is a positive constant. The model approaches ΛCDM in the limit $R/R_c \to \infty$.

Finally, let also consider the class of models [20,21,28]

$$f_{\rm hybrid\ 2}(R) = -\lambda R_c \left(\frac{R}{R_c}\right)^q. \tag{84}$$

Also in this case λ, q, and R_c are positive constants (note that n, s, and q have to converge toward the same values to match the observations). We do not consider the models with negative q, because they suffer for instability problems associated with negative $f_{,RR}$ [27].

A. The role of conformal transformations

Conformal transformations can play a key role in classifying singularities in modified gravity (apart from the general classification presented in the second section). As we have shown in the previous section, at scale R_c and beyond, the expansion rate of the Universe is set primarily by the matter density, just as in standard cosmology, with small corrections. Once the local curvature drops below R_c, according to the chameleon mechanism, the expansion rate feels the effect of modified gravity. The spacetime curvature, on the other hand, is controlled by the further scalar degree of freedom σ which gravity acquires. It obeys the usual scalar field equation with potential $V(\sigma)$, the shape of which is directly determined by function $f(R)$, and a driving term from the trace of matter stress-energy tensor. But a problem surfaces at this point: it turns out that precisely those functions $f(R)$ that lead to Einstein-like gravity action in the large curvature regime yield a potential V with an *unprotected* curvature singularity. (Note that just the same occurs for a number of realistic quintessence dark energies.) Let us consider, for example, the model (82) which has been constructed to avoid linear instabilities. The conformal transformation gives $\sigma = -\ln(1 + f'(R)) = -\ln F'(R)$ with the potential defined in Eq. (55). For the model (82), the scalar field σ is given by

$$\sigma(x) = -\ln\left(2 - \frac{2nx^{2n-1}\lambda}{(x^{2n}+1)^2}\right), \tag{85}$$

where the crossover curvature scale R_c can be reabsorbed into a rescaling of coordinate (which can be assumed dimensionless and can be measured in length units corresponding to R_c). Suitable coordinates are $x = \frac{R}{R_c} > 0$ and $R_c \sim \rho_g \sim 10^{-24}$ g/cm^3 for the galactic density in the solar vicinity and $R_c \sim \rho_g \sim 10^{-29}$ g/cm^3 for the present cosmological density. For the large curvature limit $R \to \pm\infty$, we have that $\sigma = -0.70$ while, for $R \to 0$, it corresponds to $\sigma = -0.70$, which gives us a hint that the potential is going to be a multivalued function. For $R \to 1$, we have $\sigma = 0$. We have $V(\sigma) \to \infty$ for $R \to 1$, then it is singular for this curvature value, while for $R \to \infty$, we have $V(\sigma) = 2$ and flat spacetime in the limit $R \to 0$. The analytic expression is

$$V(\sigma(x)) = \frac{x^2(x^{2n} - 2n + 1)(x^n + x^{3n})^2\lambda}{(2(x - n\lambda)x^{2n} + x^{4n+1} + x)^2}. \tag{86}$$

For our goals, it is important to study the trend of the potential and of the field σ through a parametric plot of two functions with respect to $x = \frac{R}{R_c}$. In this way, singularities can be easily identified (see Fig. 4).

For the hybrid-1 model, σ assumes the form

$$\sigma = -\ln\left(2 - 2s\left(\frac{1}{x}\right)^{2s+1}\lambda\right). \tag{87}$$

From Fig. 5, for $\lambda = 2$ and $s = 1$, we can see that we have $\sigma = 0$ in the limit $R \to -\infty$ and $\sigma \to \infty$ for $R = 0$.

The effective potential is

$$V(\sigma) = \frac{x^{2s+2}(x^{2s} - 2s - 1)\lambda}{(x^{2s+1} - 2s\lambda)^2}, \tag{88}$$

and, from a rapid inspection of Fig. 6 with $\lambda = 2$ and $s = 1$, we have $V = 2$ for $R \to \pm\infty$.

Finally, let us analyze the model (84). The scalar field assumes the form

$$\sigma = -\ln\left(q\lambda\left(\frac{1}{x}\right)^{q+1} + 2\right) \tag{89}$$

and then $\sigma \to -\infty$ for $R \to 0$; see Fig. 7.

The potential is

$$V(\sigma) = \frac{(q+1)x^{q+2}\lambda}{(x^{q+1} + q\lambda)^2} \tag{90}$$

which gives $V = 0$ for $R \to \pm\infty$ and for $R = 0$; see Figs. 8 and 9.

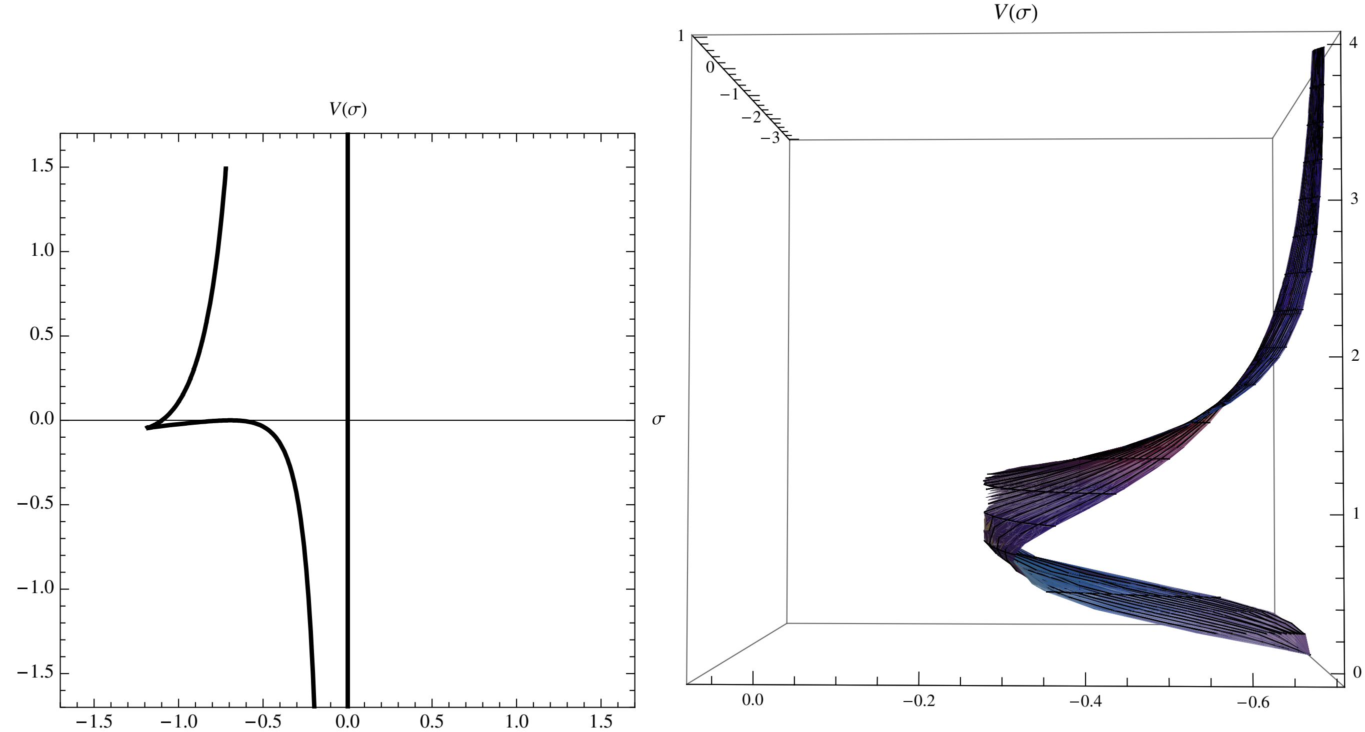

FIG. 4 (color online). Qualitative evolution of the scalar degree of freedom for the model (82) in two dimensions $\{V(\sigma), \sigma\}$ and in three dimensions $\{V(\sigma), \sigma, x\}$ with $\lambda = 2$ and $n = 1$.

As a concluding remark, we can say that the conformal transformations allow one to recast the singularity problem in terms of the scalar field and its potential. This fact suggests an easier interpretation of the singularities, at least, in some specific cases.

B. Singularities vs chameleon mechanism

In order to compare these results, we can take into account another approach where the matter coupling is considered as has also been studied by Dev *et al.* [62]. This allows one to compare singularities with density scale and could be particularly useful to construct models where finite singularities are avoided at infrared scales.

Starting again with action (1) which leads to the equations of motion (2), the evolution of the scalar degree of freedom is given by the trace:

$$2F(R) - RF'(R) - 3\Box F'(R) = -\frac{\kappa^2}{2}T. \qquad (91)$$

Another convenient way to define the scalar function σ is

$$\sigma \equiv F'(R) - 1, \qquad (92)$$

which is expressed through the Ricci scalar once $F(R)$ is

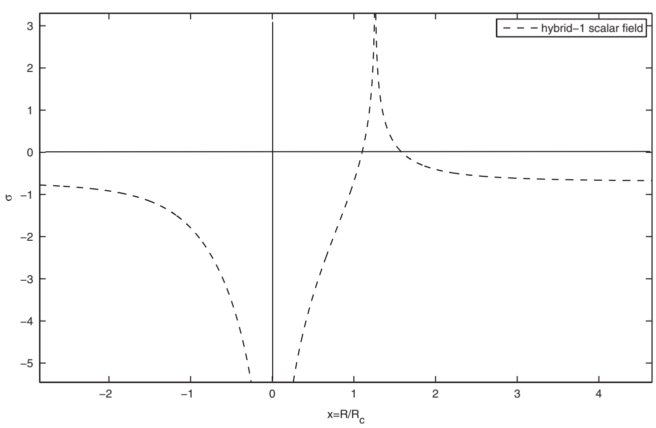

FIG. 5. Plot of Eq. (87) for $\lambda = 2$ and $s = 1$.

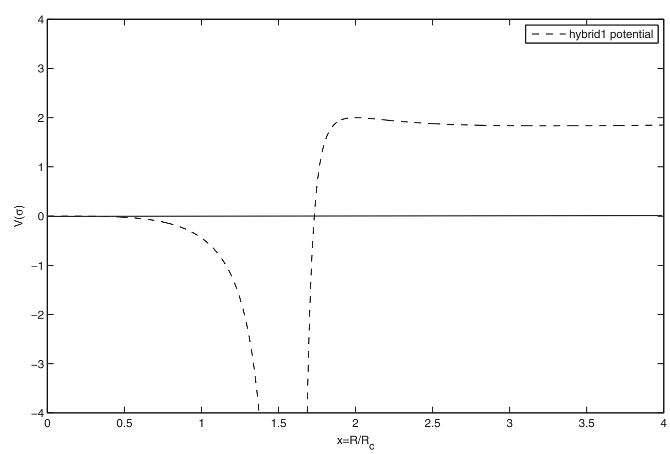

FIG. 6. Plot of Eq. (88) for $\lambda = 2$ and $s = 1$.

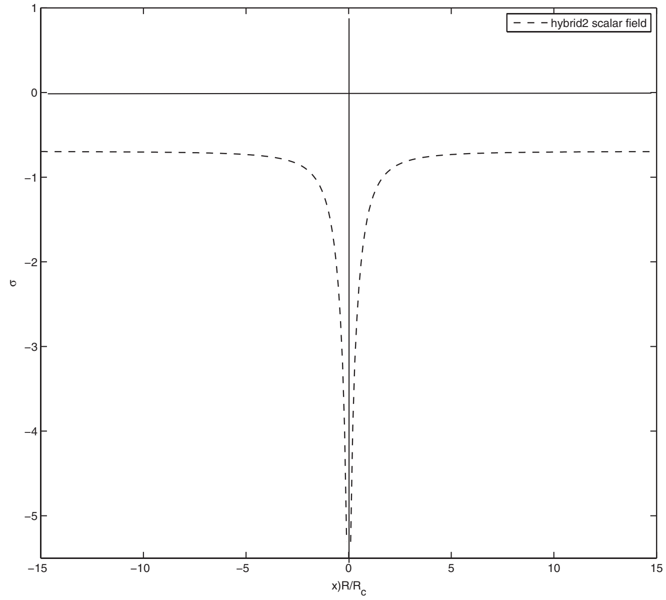

FIG. 7. Plot of Eq. (89) for $\lambda = 2$ and $q = 1$.

specified. We can write the trace equation [Eq. (3)] in terms of V and T as

$$\Box\sigma = \frac{dV}{d\sigma} + \frac{\kappa^2}{6}T. \tag{93}$$

The potential can be evaluated using the following relation:

$$\frac{dV}{dR} = \frac{dV}{d\sigma}\frac{d\sigma}{dR} = \frac{1}{3}(2F(R) - F'(R)R)F''(R). \tag{94}$$

Then $V(\sigma)$ is given by the pair of functions $\{\sigma(R), V(R)\}$. Let us consider now the model (82) as is also shown in [62]. The scalar field σ is given by

$$\sigma(R) = -\frac{2nx^{2n-1}\lambda}{(x^{2n}+1)^2}. \tag{95}$$

We can compute $V(R)$ for a given value of n. In the case of $n = 1$ and $\lambda = 1$, we have

$$V = \frac{1}{24}\left(\frac{-3x^7 - 24x^6 + 21x^5 - 56x^4 + 11x^3 - 40x^2 + 3x - 8}{(x^2+1)^4} - 3\tan^{-1}(x)\right), \tag{96}$$

where $x = R/R_c$. In the FRW background, the trace equation can be rewritten in the convenient form

$$\ddot{\sigma} + 3H\dot{\sigma} + \frac{dV}{d\sigma} = -\frac{\kappa^2}{6}\rho. \tag{97}$$

The effective scalar potential is plotted in Fig. 10 for $\lambda = 1$, and it is multivalued indeed. It has a minimum depending on the values of n and λ. For generic values of the parameters, the minimum of the potential is close to $\sigma = 0$, corresponding to infinitely large curvature $R = \pm\infty$. Thus, while the field is evolving toward the minimum, it evolves oscillating toward a singular point. We have a stable de Sitter minimum and an unstable de Sitter maximum. The point $R \to 0$ corresponds to a flat spacetime, which, although a solution for the model, is unstable. Cusps that occur at $R = \pm 1/\sqrt{3}$ are critical points with $f'' = 0$. However, depending upon the values of the parameters, we can choose a finite range of initial conditions for which scalar field σ evolves to the minimum of the potential without hitting the singularity. Note again that as

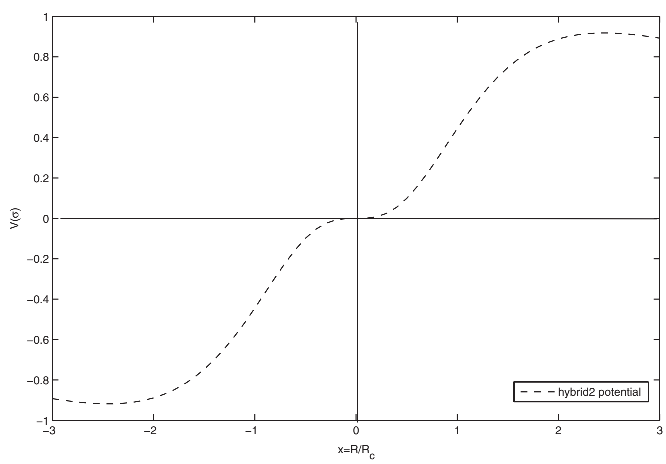

FIG. 8. Plot of Eq. (90) for $\lambda = 2$ and $q = 1$. It is worth noting the double sigmoid trend.

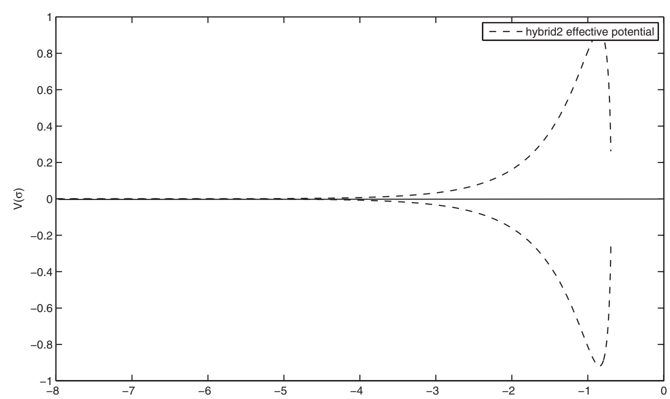

FIG. 9. Parametric plot of Eqs. (89) and (90) for $\lambda = 2$ and $q = 1$.

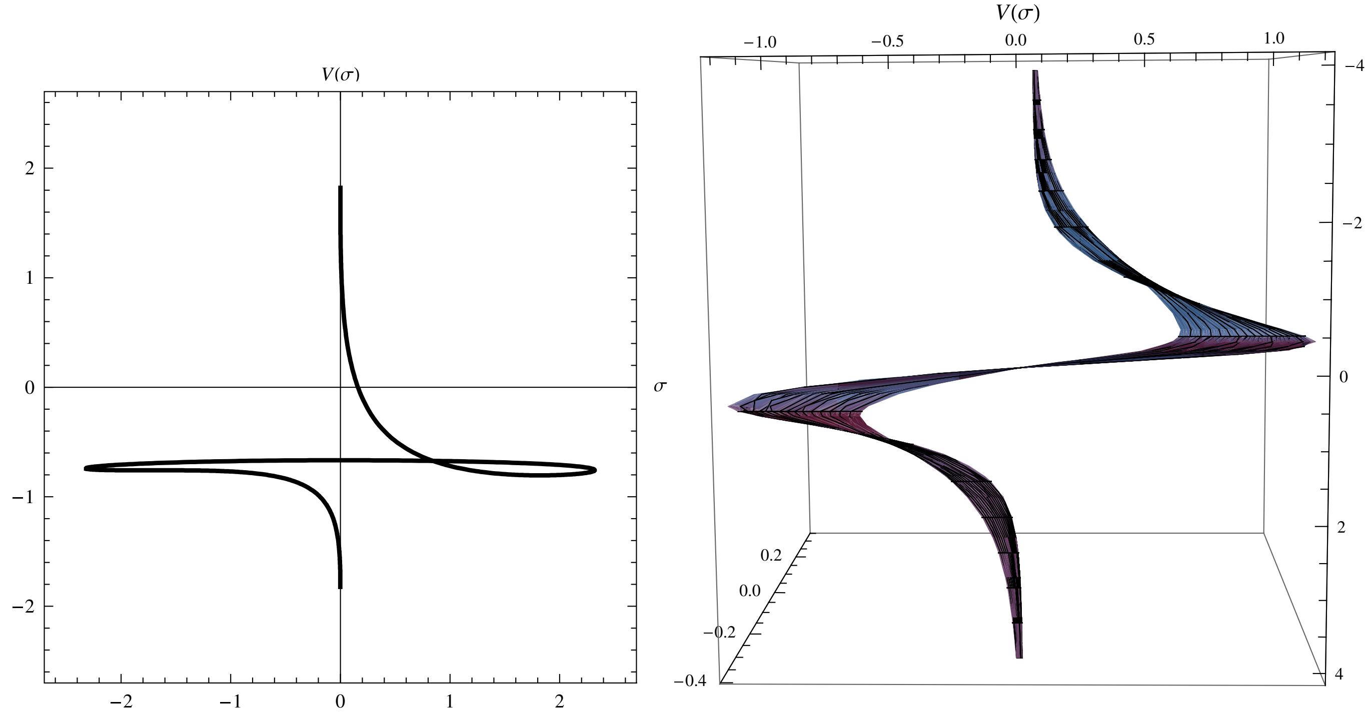

FIG. 10 (color online). Qualitative parametric plot of the evolution of the scalaron potential V vs σ and x for $n = 1$ and $\lambda = 1$ in two and three dimensions for the model (82).

it was demonstrated in the second section, it is just a manifestation of type II finite-time singularity. In Fig. 11, the cosmological behavior of σ is plotted. It is clear the contribution of matter in changing the local behavior.

The time-time component of the equation of motion (2) gives the Hubble equation

$$H^2 + \frac{d(\ln F'(R))}{dt} H + \frac{1}{6} \frac{F(R) - F(R)R}{f'(R)} = -\frac{\kappa^2}{6}\rho. \quad (98)$$

The Einstein gravity is recovered in the limit $f' = 1$. The picture of dynamics which appears here is the following: above the infrared scale (R_c), the expansion rate is set by the matter density and once the local curvature falls below R_c the expansion rate gets the effects of modified gravity.

For pressureless dust, the effective potential presents an extremum at

$$2F(R) - RF'(R) = -\frac{\kappa^2}{2}\rho. \quad (99)$$

For viable late-time cosmologies, the field evolves near the minimum of the effective potential. The finite-time singu-

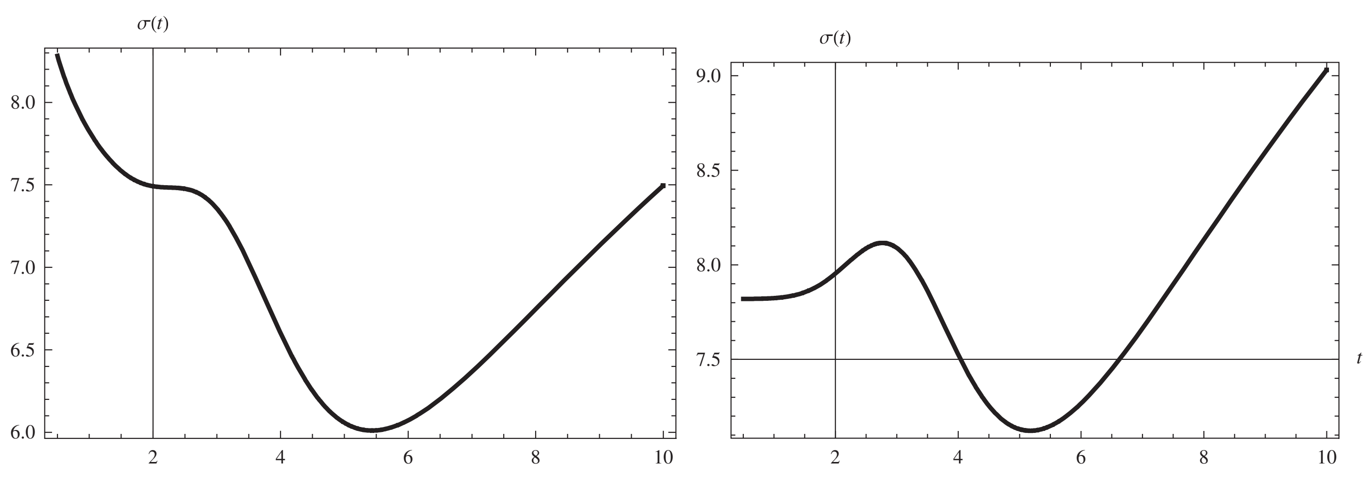

FIG. 11. Cosmological behavior of the scalar field σ as given in Eq. (97) plotted vs the cosmological time in the presence of matter (left panel) and without matter (right panel). It is possible to note that the matter contribution changes the local behavior toward $t = 0$.

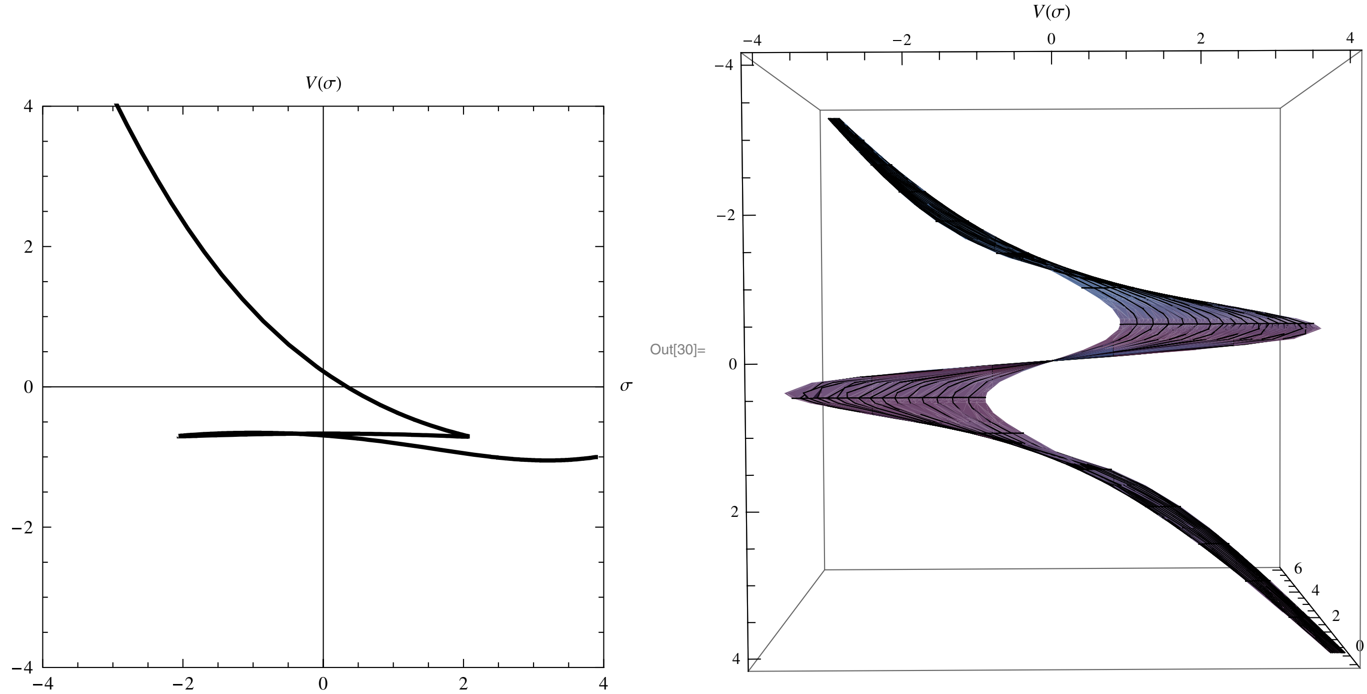

FIG. 12 (color online). Parametric plot of the effective potential for $n = 2$, $\lambda = 2$, and $\alpha = 1/2$ in the presence of the R^2 correction in two and three dimensions for the model (82).

larity, which occurs for the class of models under consideration, severely constrains the field dynamics.

C. Adding higher curvature corrections to cure singularities

It is well known that for large curvature regimes, the quantum effects become important leading to higher curvature corrections. Then the program of this paper can be enhanced by considering if higher curvature corrections (as is already stressed in the third section) added to the original models can solve the singularity problem. In general, higher curvature corrections change the structure of the effective potential around the singularity [62]. Keeping this in mind, let us consider the modification of the model (82). Although we focus on a specific model, similar results hold for the models of the classes considered here. In cosmology, higher curvature corrections appear as Lagrangian contributions like $\mathcal{L} = \alpha_2 R^2 + \alpha_3 R^3 + \cdots$, and so the most natural choice for the leading order term is $\alpha R^2/R_c^2$.[1] It is well known that the R^2 term may be responsible for inflation in the *early* Universe if R_c is set to be at an inflationary scale [62]. In the case we are considering, we have

$$F(R) = -\frac{\lambda(\frac{R}{R_c})^{2n}}{(\frac{R}{R_c})^{2n} + 1} + \alpha\frac{R^2}{R_c^2} + \frac{R}{R_c}, \tag{100}$$

then the field σ becomes

$$\sigma(R) = 2\frac{R}{R_c}\alpha - \frac{2n\lambda}{(\frac{R}{R_c})^{2n+1} + \frac{R}{R_c}} + \frac{2n\lambda}{\frac{R}{R_c}((\frac{R}{R_c})^{2n} + 1)^2}. \tag{101}$$

When $|R|$ is large in modulus, the first term which comes from αR^2 dominates. In this case, the curvature singularity $R = \pm\infty$ corresponds to $\sigma = \pm\infty$. Hence, by this modification, the minimum of the effective potential is separated from the curvature singularity by the infinite distance in the $\{\sigma, V(\sigma)\}$ plane.

For $n = 2$, $\lambda = 2$, and $\alpha = 0.5$, we have a large range of the initial condition for which the scalar field evolves to the minimum of the potential as shown in Fig. 12. In conclusion, the introduction of the R^2 term formally allows one to avoid the singularity as it was suggested earlier in [43,45,46].

V. CONCLUSIONS

In this paper we have discussed the future finite-time singularities which can, in principle, appear in the dark energy universe coming from modified gravity as well as in other dark energy theories. Considering $f(R)$ gravity models that satisfy cosmological viability conditions (chameleon mechanism), it is possible to show that finite-time singularities emerge in several cases. Such singularities can be classified according to the values of the scale factor

[1] Higher order corrections naturally include terms like $R_{\mu\nu}R^{\mu\nu}$, but in this paper we focus on the $f(R)$-type modified gravity and hence simply assume that the corrections are also given by a function of the Ricci scalar.

$a(t)$, the density ρ, and the pressure p as done in Sec. II. To avoid the singularities, suitable boundary conditions have to be imposed which depend, in general, on the parameters of the model as seen in Sec. III. It is interesting that in static spherically symmetric spacetime the finite-time singularity manifests itself as singularity at some specific value of curvature.

Besides, the problem can be analyzed by considering the mass of an auxiliary scalar field coming from the further degrees of freedom of $f(R)$ gravity. Such a scalar field is heavy in the high-curvature regime whose density is much larger than the present cosmological density. Such a field allows one to study the singularity problem using the conformal transformations. In this case, we have to consider singularities of the scalar field and the related effective potential and try to see if they can be avoided in the conformal picture. For example, the most striking feature of the potential in Fig. 4, and the core of the problem for infrared-modified $f(R)$ models, is that curvature singularity is at a finite distance both in field and energy values. The scalar field σ directly feels the matter distribution; for suitable values of the parameters, the force is directed to the right, and drives the field σ up the wall toward infinite curvature. The characteristic scale of the potential $V(\sigma)$ is the curvature R_c which is of the same order of magnitude of the today observed cosmological constant. Such a value is extremely low compared to the matter densities we encounter at local Solar System and Galactic scales. Given the scales involved, it is easy to drive the scalar field in order to jump the potential well considering the dynamics of standard matter which could cause catastrophic curvature singularity. If this were to happen, this is in contrast with any viable model. Similarly, matter with a sufficiently stiff equation of state can destabilize the model by driving the field to unstable points. It is also remarkable that even in flat spacetime, some classes of the classical potential may bring the theory to finite-time singularity.

Finally we have discussed the possibility to address and solve the singularity problem adopting higher curvature corrections. In such a way, the future cosmological era has no singularity as was demonstrated earlier in [43,45,46]. Then, it also does not manifest itself in spherically symmetric spacetime. The interesting result of this analysis is the fact that finite-time singularities, which are present in some modified gravity, may not influence the conjectured problem with relativistic star formation indicated in Ref. [44]. Of course, this fact depends on the specific characteristics of the adopted modified gravity model which can be even totally free of future, finite-time singularities.

ACKNOWLEDGMENTS

This research was supported by the INFN-CSIC bilateral project, by the grant *Azione Integrata Italia-Spagna 2007* (MIUR Project No. 464, 13/3/2006), and by MICINN (Spain) Projects No. FIS2006-02842 and No. PIE2007-50I023. The work by S. N. was supported by the Ministry of Education, Science, Sports and Culture of Japan under Grant No. 18549001.

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