Modified clock inequalities and modified black hole lifetime

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Based on a generalized uncertainty principle, Salecker-Wigner inequalities are modified. When applied to black holes, they give a modified black hole lifetime: $T_{\rm MB} \sim \frac{M^3}{m_p^3}(1 - m_p^2/M^2)t_p$, and the number of bits required to specify the information content of the black hole as the event horizon area in Planck units $N \sim \frac{M^2}{m_p^2}(1 - m_p^2/M^2)$.

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I. INTRODUCTION

The conventional derivation of the Hawking lifetime uses the Heisenberg's uncertainty principle on the event horizon scale R_g to determine a temperature for the black hole which, under the assumption that the black hole is a black body, then allows one to use the Stefan-Boltzmann law to calculate the lifetime of the black hole for complete evaporation (see, e.g., [1,2]).

By applying Salecker-Wigner's clock inequalities to black holes, Barrow obtained the same result [3]. The heuristic way is as follows: According to Heisenberg's uncertainty principle: $\Delta p \sim \hbar/\Delta x$, if a clock of mass M has quantum position uncertainty Δx, then its momentum uncertainty is $\hbar \Delta x^{-1}$. The clock to be considered should have an accuracy τ (the minimum time interval that the clock is capable of resolving) and be able to measure time intervals up to a maximum T. After a time t, the uncertainty in position of the clock will grow to $\Delta x' = \Delta x + \hbar t M^{-1} \Delta x^{-1}$. If the effects on mass are neglected, then this will be a minimum when $\Delta x = \sqrt{\hbar t/M}$. Hence, to keep the clock accurate over the total running time T, its linear spread λ must be limited:

$$\lambda \geq 2\sqrt{\hbar T/M}, \tag{1}$$

the same order of magnitude of the position uncertainty, meaning that the size of the clock must be larger than the uncertainty in its position. This is Salecker-Wigner's first clock inequality [4]. To give time to within an accuracy τ, the quantum position uncertainty must not be larger than the minimum wavelength of the quanta striking it (in order to read the time); that is, $\Delta x' \leq c\tau$. The use of a signal with nonzero rest mass would give a more rigorous limit. This condition gives a bound on the minimum mass of the clock:

$$M \geq \frac{4\hbar}{c^2 \tau}\left(\frac{T}{\tau}\right). \tag{2}$$

This is Salecker-Wigner's second clock inequality [4]. This inequality is more restrictive than that imposed by Heisenberg's energy-time uncertainty principle because it requires that a clock still show proper time after being read: the quantum uncertainty in its position must not introduce significant inaccuracies in its measurement of time over the total running time. To derive Salecker-Wigner's clock inequalities (1) and (2), it assumes unsqueezed, unentangled, and Gaussian wave packets without any detailed phase information; they are valid only for single analog clocks (black holes can be seen as analog clocks [5]), not for digital quantum clocks.

Barrow applied Salecker-Wigner's size limit (1) to a black hole, assuming that the minimum clock size is the Schwarzschild radius $R_g = 2GM/c^2$ and found the maximum running time of the black hole is [3]

$$T \sim \frac{G^2 M^3}{\hbar c^4} = \frac{M^3}{m_p^3} t_p, \tag{3}$$

where $t_p \equiv \sqrt{G\hbar/c^5}$ and $m_p \equiv \sqrt{c\hbar/G}$ are the Planck time and mass. The maximum running time of a black hole is the Hawking lifetime [6]. If we had not known of the existence of black hole evaporation, Eq. (3) would have implied that there is a maximum lifetime for a black hole state. Compared with the conventional method, the application of the Salecker-Wigner inequality (1) to the event horizon scale predicts the Hawking lifetime (3) without the assumption that the black hole is a black body radiator.

But, one may suggest, when considering black holes, the effect of gravity may be taken into account. In this work, we obtain modified clock inequalities based on a generalized uncertainty principle that takes into account some properties of black holes, and find a modified black hole lifetime, which may throw light on quantum gravity at the Planck scale.

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II. MODIFIED CLOCK INEQUALITIES

Salecker-Wigner's clock inequalities are based on the Heisenberg's position-momentum uncertainty principle: $p \sim \hbar/\Delta x$. But, if we combine quantum theory and some basic concepts of gravity, Heisenberg's position-momentum uncertainty principle may be modified [7–24], and so do Salecker-Wigner's clock inequalities. Using Heisenberg's uncertainty principle and some properties of black holes, Scardigli had shown how a generalized uncertainty principle (GUP) can be derived from a measure gedanken experiment [25]

$$\Delta x \geq \frac{\hbar}{\Delta p} + l_p^2 \frac{\Delta p}{\hbar}, \tag{4}$$

where $l_p^2 = \sqrt{G\hbar/c^3}$ is the Planck distance. As Scardigli argued, this GUP is independent from particular versions of quantum gravity. This GUP also arises from quantum fluctuation of the background space-time metric [26]. Note, however, this GUP is firstly derived in Ref. [13]. The GUP (4) can be written in a general form $\Delta x \geq \hbar(1/\Delta p + \beta \Delta p)$, where β is a constant [27]. Introduction of the GUP has drawn considerable attention, and many authors considered various problems in the framework of GUP, such as Refs. [28–60]. Note, however, it should be kept in mind that this GUP is derived based upon only heuristic arguments, and is thus far from proven.

Basing on the GUP (4), Adler *et al.* obtained a modified black hole lifetime with the conventional method [2].

$$T_{\rm ACS} = \frac{1}{16}\left\{\frac{8}{3}\left(\frac{M}{m_p}\right)^3 - 8\frac{M}{m_p} - \frac{m_p}{M} + \frac{8}{3}\left[\left(\frac{M}{m_p}\right)^2 - 1\right]^{3/2} - 4\sqrt{\left(\frac{M}{m_p}\right)^2 - 1} + 4\arccos\left(\frac{m_p}{M}\right) + \frac{19}{3}\right\} t_{\rm ch}, \tag{5}$$

where $t_{\rm ch} = 16^2 \times 60\pi t_p$. To derive this black hole lifetime, Adler *et al.* also assume that the black hole is a black body radiator, and the dispersion relation $E = pc$ holds. But if the uncertainty principle is modified, the dispersion relation may also be modified (see, e.g., [61]).

Because the space-time fluctuation will be significant when the measured length scale approaches to the Planck distance, it is reasonable to expect that the linear spread of a clock must not be less than the Planck distance. In fact, the GUP (4) implies a minimum length $2l_p$, which can be considered as a limit on the linear spread of a clock. This limit can be improved, as we see below. From Eq. (4), if a clock of mass M has quantum position uncertainty Δx, then its momentum uncertainty will be $\Delta p \sim \frac{\Delta x \hbar}{2l_p^2} \times [1 - \sqrt{1 - 4l_p^2/\Delta x^2}]$ [2]. Following the steps to derive the Salecker-Wigner's clock inequalities, Eq. (1) is modified as (see Appendix)

$$\lambda \geq 2l_p \sqrt{1 + \frac{\hbar T}{M l_p^2}}, \tag{6}$$

stronger than limit (1) and come back to limit (1) for $\hbar T \gg M l_p^2$. Here, we also require that the position uncertainty created by the measurement of time must not be larger than the minimum wavelength of the quanta used to read the clock. Then Salecker-Wigner's second clock inequality (2) is modified as

$$M \geq \frac{4\hbar T}{c^2 \tau^2} \frac{1}{1 - 4t_p^2/\tau^2}. \tag{7}$$

This inequality links the mass, total running time, accuracy of the clock, and the Planck time together, and may link together our concepts of gravity and quantum uncertainty. Obviously, it firstly gives a limit on the accuracy of the clock $\tau > 2t_p$. Like Salecker-Wigner inequalities, (1), (2), (6), and (7) are valid for single analog clocks, not for digital quantum clocks.

III. MODIFIED BLACK HOLE LIFETIME

Now applying modified clock inequality (6) to black holes and assuming that the minimum clock size is the Schwarzschild radius $R_g = 2GM/c^2$, one may find the maximum running time of the black hole is modified as

$$T_{\rm MB} \sim \frac{M R_g^2}{4\hbar}(1 - 4l_p^2/R_g^2) = \frac{M^3}{m_p^3}(1 - m_p^2/M^2) t_p, \tag{8}$$

which has a term $M t_p/m_p$ different from the Hawking lifetime (3) and holds for $M \geq m_p$. This difference may throw light on quantum gravity in some sense at Planck scale. Using the GUP (4), Adler *et al.* found that the thermal radiation of the black hole will stop at the Planck distance, and the black hole becomes an inert remnant, possessing only gravitational interaction [2], consistent the results obtained in modified clock inequalities background. Aside from about a factor of $16^2 \times 60\pi$, the first two terms of the Adler-Chen-Santiago lifetime $T_{\rm ACS}$ is consistent with the modified black hole lifetime $T_{\rm MB}$. The comparison among the Hawking lifetime $T_{\rm H}$, the modified black hole lifetime $T_{\rm MB}$, and Adler-Chen-Santiago lifetime $T_{\rm ACS}$ are shown in Fig. 1.

The minimum interval that the black hole can be used to measure is given by the light travel time across the black hole [3,5]: $\tau \sim 2GM/c^3 = R_g/c$. Thus, we are led to view the black hole as an information-processing system in which the number of computational steps is

$$N \equiv \frac{T_{\rm MB}}{\tau} \sim \frac{M^2}{m_p^2}(1 - m_p^2/M^2). \tag{9}$$

As expected from the identification of a black hole entropy [62] or holographic principle [63,64], this gives the number

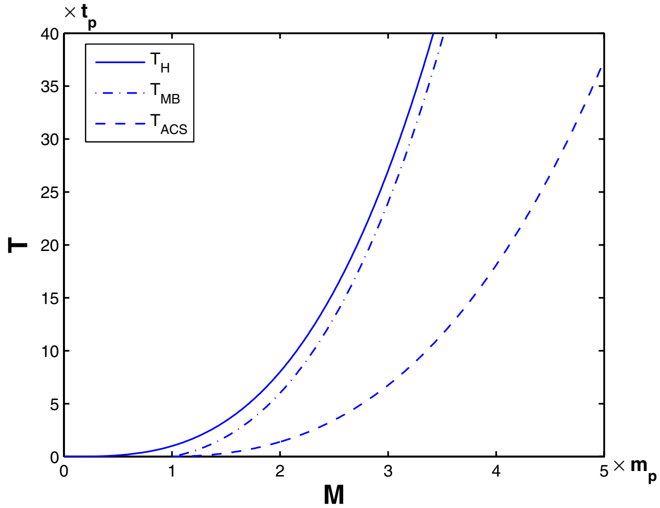

FIG. 1 (color online). Comparison among the Hawking lifetime $T_{\rm H}$, modified clock inequality lifetime $T_{\rm MB}$, and Adler-Chen-Santiago lifetime $T_{\rm ACS}$, aside from a numerical factor $16^2 \times 60\pi$.

of bits required to specify the information content of the black hole as the event horizon area in Planck units.

IV. SUMMARY

To summarize, based on a generalized uncertainty principle, we obtain modified clock inequalities, which give bounds on the size and the accuracy of the analog clock that must be larger than 2 times the Planck distance l_p and time t_p, respectively. As an application, we discussed the case of black holes, and obtained a modified black hole lifetime $T_{\rm MB} \sim \frac{M^3}{m_p^3} t_p (1 - m_p^2/M^2)$, which is different from Hawking lifetime and give a limit on the mass of black holes naturally. Viewing a black hole as an information-processing system, we also find the number of bits required to specify the information content of the black hole as the event horizon area in Planck units $N \sim \frac{M^2}{m_p^2}(1 - m_p^2/M^2)$. These results reinforce the central importance of black holes as the simplest and most fundamental constructs of space-time, linking together our concept of gravity, information, and quantum uncertainty. Note, however, applying clock inequalities to obtain the lifetimes of other type black holes is still an open interesting problem, work is in progress in this direction.

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APPENDIX

According to the generalized uncertainty principle

$$\Delta x \geq \frac{\hbar}{\Delta p} + l_p^2 \frac{\Delta p}{\hbar}, \tag{A1}$$

where $l_p^2 = \sqrt{G\hbar/c^3}$ is the Planck distance, if a clock with mass M has quantum position uncertainty Δx, then its momentum uncertainty will be

$$\Delta p \sim \frac{\Delta x \hbar}{2l_p^2}\left[1 - \sqrt{1 - 4l_p^2/\Delta x^2}\right]. \tag{A2}$$

After a time t the uncertainty in position of the clock becomes

$$\Delta x' = \Delta x + \frac{\Delta x \hbar t}{2Ml_p^2}\left[1 - \sqrt{1 - 4l_p^2/\Delta x^2}\right]. \tag{A3}$$

To obtain the minimal value of $\Delta x'$ in this case, using the condition

$$0 = \frac{d\Delta x'}{d\Delta x} = 1 + \frac{\hbar t\left[1 - \sqrt{1 - 4l_p^2/\Delta x^2}\right]}{2Ml_p^2} - \frac{2t\hbar}{M\Delta x^2\sqrt{1 - 4l_p^2/\Delta x^2}}, \tag{A4}$$

we get $\Delta x = [2Ml_p^2 + t\hbar]/\sqrt{M(Ml_p^2 + t\hbar)}$. By inserting this value into Eq. (A3), we obtain the minimal value of $\Delta x'$

$$\Delta x'_{\rm min} = 2l_p\sqrt{1 + \frac{t\hbar}{Ml_p^2}}. \tag{A5}$$

By taking t as the total running time T during which the clock can remain accurate, and considering the condition that the linear spread of clock λ must not be less than the uncertainty in position $\Delta x'$, that's $\lambda \geq \Delta x' \geq \Delta x'_{\rm min}$, we obtain Eq. (6).

[1] P.K. Townsend, arXiv:gr-qc/9707012v1.
[2] R.J. Adler, P. Chen, and D.I. Santiago, Gen. Relativ. Gravit. **33**, 2101 (2001); P. Chen and R.J. Adler, Nucl. Phys. B, Proc. Suppl. **124**, 103 (2003).
[3] J.D. Barrow, Phys. Rev. D **54**, 6563 (1996).
[4] H. Salecker and E.P. Wigner, Phys. Rev. **109**, 571 (1958).
[5] Y.J. Ng, Phys. Rev. Lett. **86**, 2946 (2001).
[6] S.W. Hawking, Nature (London) **248**, 30 (1974).

[7] G. Veneziano, Europhys. Lett. **2**, 199 (1986).
[8] D. J. Gross and P. F. Mende, Nucl. Phys. **B303**, 407 (1988).
[9] D. Amati, M. Ciafaloni, and G. Veneziano, Phys. Lett. B **216**, 41 (1989).
[10] K. Konishi, G. Paffuti, and P. Provero, Phys. Lett. B **234**, 276 (1990).
[11] R. Guida, K. Konishi, and P. Provero, Mod. Phys. Lett. A **6**, 1487 (1991).
[12] M. Kato, Phys. Lett. B **245**, 43 (1990).
[13] M. Maggiore, Phys. Lett. B **304**, 65 (1993).
[14] E. Witten, Phys. Today **49**, 24 (1996).
[15] L. J. Garay, Int. J. Mod. Phys. A **10**, 145 (1995).
[16] C. Bambi, Classical Quantum Gravity **25**, 105003 (2008).
[17] M. Park, Phys. Lett. B **659**, 698 (2008).
[18] W. Kim, J. S. Edwin, and M. Yoon, J. High Energy Phys. 01 (2008) 035.
[19] R. Machluf, arXiv:0807.2190v1.
[20] B. Nath Tiwari, arXiv:0801.3402v1.
[21] S. Capozziello, G. Lambiase, and G. Scarpetta, Int. J. Theor. Phys. **39**, 15 (2000).
[22] A. Camacho, Gen. Relativ. Gravit. **34**, 1839 (2002).
[23] A. Bina, K. Atazadeh, and S. Jalalzadeh, Int. J. Theor. Phys. **47**, 1354 (2008).
[24] C. Castro, Found. Phys. Lett. **10**, 273 (1997).
[25] F. Scardigli, Phys. Lett. B **452**, 39 (1999).
[26] R. J. Adler and D. I. Santiago, Mod. Phys. Lett. A **14**, 1371 (1999).
[27] L. N. Chang, D. Minic, N. Okamura, and T. Takeuchi, Phys. Rev. D **65**, 125028 (2002).
[28] A. J. M. Medved and E. C. Vagenas, Phys. Rev. D **70**, 124021 (2004).
[29] A. Ashoorioon, A. Kempf, and R. B. Mann, Phys. Rev. D **71**, 023503 (2005).
[30] B. Vakili and H. R. Sepangi, Phys. Lett. B **651**, 79 (2007).
[31] B. Bolen and M. Cavaglia, Gen. Relativ. Gravit. **37**, 1255 (2005).
[32] K. Nozari and T. Azizi, Gen. Relativ. Gravit. **38**, 735 (2006).
[33] W. Kim, Y. W. Kim, and Y. J. Park, Phys. Rev. D **74**, 104001 (2006).
[34] D. V. Ahluwalia, Phys. Lett. B **339**, 301 (1994).
[35] F. Nasseri, Phys. Lett. B **632**, 151 (2006).
[36] M. Maziashvili, Phys. Lett. B **635**, 232 (2006).
[37] Y. S. Myung, Y.-W. Kim, and Y.-J. Park, Phys. Lett. B **645**, 393 (2007).
[38] R. Akhoury and Y. P. Yao, Phys. Lett. B **572**, 37 (2003).
[39] S. Doplicher *et al.*, Phys. Lett. B **331**, 39 (1994).
[40] S. Kalyana Rama, Phys. Lett. B **519**, 103 (2001).
[41] S. Hossenfelder, M. Bleicher, S. Hofmann, J. Ruppert, S. Scherer, and H. Stoecker, Phys. Lett. B **575**, 85 (2003).
[42] F. Scardigli and R. Casadio, Classical Quantum Gravity **20**, 3915 (2003).
[43] Y.-W. Kim, H. W. Lee, and Y. S. Myung, Phys. Lett. B **673**, 293 (2009).
[44] A. Larranaga and H. J. Hortua, arXiv:0901.3727v1.
[45] M. Liu, Y. Gui, and H. Liu, Phys. Rev. D **78**, 124003 (2008).
[46] M. Sakhawat Hossain and S. B. Faruque, Phys. Scr. **78**, 035006 (2008).
[47] M. V. Battisti and G. Montani, Phys. Lett. B **656**, 96 (2007); Phys. Rev. D **77**, 023518 (2008).
[48] J. Y. Bang and M. S. Berger, Phys. Rev. D **74**, 125012 (2006).
[49] Y. Shibusa, Int. J. Mod. Phys. A **22**, 5279 (2007).
[50] T. Matsuo and Y. Shibusa, Mod. Phys. Lett. A **21**, 1285 (2006).
[51] P. S. Custódio and J. E. Horvath, Classical Quantum Gravity **20**, L197 (2003).
[52] M. R. Setare, Phys. Rev. D **70**, 087501 (2004).
[53] S. Das and E. C. Vagenas, arXiv:0901.1768v1.
[54] Y. Ko, S. Lee, and S. Nam, arXiv:hep-th/0608016v2.
[55] C. M. Sarris and A. N. Proto, Physica A (Amsterdam) **377**, 33 (2007).
[56] X. Li and X. Q. Wen, arXiv:0901.0603v2.
[57] K. Nouicer, Phys. Lett. B **646**, 63 (2007).
[58] I. Arraut, D. Batic, and M. Nowakowski, arXiv:0810.5156v1.
[59] P. Galan and G. A. Mena Marugan, Phys. Rev. D **74**, 044035 (2006).
[60] A. E. Shalyt-Margolin, arXiv:0807.3485v1.
[61] K. Nozari and B. Fazlpour, Gen. Relativ. Gravit. **38**, 1661 (2006); G. Amelino-Camelia, M. Arzano, Y. Ling, and G. Mandanici, Classical Quantum Gravity **23**, 2585 (2006).
[62] J. D. Bekenstein, Phys. Rev. D **7**, 2333 (1973).
[63] G.'t Hooft, in *Salamfestschrift*, edited by A. Ali, J. Ellis, and S. Randjbar-Daemi (World Scientific, Singapore, 1993), p. 284.
[64] L. Susskind, J. Math. Phys. (N.Y.) **36**, 6377 (1995).