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Wormhole dynamics in spherical symmetry

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A dynamical theory of traversable wormholes is detailed in spherical symmetry. Generically a wormhole consists of a tunnel of trapped surfaces between two mouths, defined as temporal outer trapping horizons with opposite senses, in mutual causal contact. In static cases, the mouths coincide as the throat of a Morris-Thorne wormhole, with surface gravity providing an invariant measure of the radial curvature or "flaring-out". The null energy condition must be violated at a wormhole mouth. Zeroth, first, and second laws are derived for the mouths, as for black holes. Dynamic processes involving wormholes are reviewed, including enlargement or reduction, and interconversion with black holes. A new area of wormhole thermodynamics is suggested.

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I. INTRODUCTION

Space-time wormholes, short cuts between otherwise distant or unconnected regions of the universe, have become a popular research topic since the influential paper of Morris and Thorne [1]. Early work was reviewed in the book of Visser [2] and there is an extensive recent review by Lobo [3]. The Morris-Thorne study was restricted to static, spherically symmetric space-times, and initially there were various *ad hoc* attempts to generalize the definition of wormhole by inserting time-dependent factors into the metric, despite the problem that such metrics become singular precisely at the throat, so do not necessarily describe a traversable wormhole in any sense.

A more geometrically founded generalization was proposed by the author [4] in terms of trapping horizons, also associated with black holes. This allows a substantial body of theory developed for black holes to be applied to wormholes. That reference deferred details to a longer article, of which this is a version, ten years late. It seems timely if only due to a recent resurgence of *ad hoc* approaches.

A Morris-Thorne wormhole throat, at a given static time, is a minimal surface in the static hypersurface, i.e. locally minimizing area among surfaces in the hypersurface. Their "flaring-out" condition expresses strict minimality [2]. A natural generalization is a spatial surface which is minimal in some spatial hypersurface. It is easy to show that, except in the doubly marginal case, this is a (future or past) trapped surface, more usually associated with black or white holes. Since this is a generic condition, a generic wormhole must consist of a space-time region. Then it is natural to look for the boundaries of this region, which one would expect to be trapping horizons, i.e. composed of marginal surfaces, which are extremal in null hypersurfaces. For a two-way traversable wormhole, there should be two temporal boundaries in mutual causal contact. Prosaically, the wormhole consists of a *tunnel between two mouths*. In static cases, the tunnel shrinks away and the two mouths coincide as the throat. In nonstatic cases, "throat" evidently means different things to different people, so the terminology will be avoided here.

This viewpoint has various consequences. First, a Morris-Thorne wormhole throat is a double trapping horizon, which will generally bifurcate under a dynamic perturbation, such as someone crossing it. This raises the issues of stability and, if stable, maintenance, i.e. returning a perturbed wormhole to a static state. Also, since black holes may also be defined locally in terms of trapping horizons, it is possible for a wormhole to collapse to a black hole, or for a black hole to be converted to a traversable wormhole. Concrete examples of such processes, both analytical and numerical, in toy models and full Einstein gravity, were given in a series of papers [5–13].

For pedagogical reasons, this article will be restricted to spherical symmetry, though everything can be generalized as outlined in the original reference [4]. Einstein gravity will be assumed, though the key ideas can be generalized to other metric-based theories and other dimensions. Section II reviews the necessary geometrical ideas, Sec. III defines wormhole mouths, Sec. IV checks the static limit, and Sec. V derives some basic laws of wormhole dynamics and cites examples.

II. GEOMETRY

In spherical symmetry, the area A of the spheres of symmetry is a geometrical invariant. It is convenient to use the area radius $r = \sqrt{A/4\pi}$, so that

$$A = 4\pi r^2. \tag{1}$$

A sphere is said to be *untrapped, marginal*, or *trapped*, respectively, if $g^{-1}(dr)$ is spatial, null, or temporal, where g is the metric and g^{-1} its inverse. If the space-time is time orientable and $g^{-1}(dr)$ is future (respectively, past) causal, then the sphere is said to be *future* (respectively, *past*) trapped or marginal. A hypersurface foliated by marginal spheres is called a *trapping horizon* [14,15].

The Kodama vector [16] is

$$k = g^{-1}(*dr), \tag{2}$$

where $*$ is the Hodge operator in the space normal to the spheres of symmetry, i.e.

$$k \cdot dr = 0, \qquad g(k, k) = -g^{-1}(dr, dr). \tag{3}$$

This vector gives a preferred flow of time, coinciding with the static Killing vector of standard black holes such as Schwarzschild and Reissner-Nordström. Note that k is temporal, null or spatial, respectively, on untrapped, marginal, or trapped spheres.

Both k and the corresponding energy-momentum density

$$j = -g^{-1}(T \cdot k), \tag{4}$$

where T denotes the energy-momentum-stress tensor, are covariantly conserved [17,18]:

$$\nabla \cdot k = 0, \tag{5}$$

$$\nabla \cdot j = 0, \tag{6}$$

where ∇ denotes the covariant derivative operator and the second property uses the Einstein equation. These Noether currents therefore admit Noether charges

$$V = -\int_\Sigma \hat{*} \cdot k, \tag{7}$$

$$m = -\int_\Sigma \hat{*} \cdot j, \tag{8}$$

where $\hat{*}$ denotes the volume form times unit normal of a spatial hypersurface with regular center. The charges are found to be area volume

$$V = \tfrac{4}{3}\pi r^3 \tag{9}$$

and the active gravitational mass m [19]:

$$1 - 2m/r = g^{-1}(dr, dr), \tag{10}$$

where spatial metrics are positive definite and the Newtonian gravitational constant is unity. Evidently $r > 2m$, $r = 2m$, or $r < 2m$, respectively, on untrapped, marginal, or trapped spheres. Various other properties illuminating the physical meaning of m have been derived [17,18,20].

Surface gravity was defined as [18]

$$\kappa = \tfrac{1}{2} * d * dr, \tag{11}$$

where d is the exterior derivative in the normal space, i.e. $*d*d$ is a two-dimensional wave operator. It can be shown [18] to satisfy

$$k \cdot (\nabla \wedge g(k)) = \kappa dr \tag{12}$$

and therefore

$$k \cdot (\nabla \wedge g(k)) \cong \pm \kappa g(k), \tag{13}$$

where $\cong$ henceforth denotes evaluation on a trapping horizon $r \cong 2m$, similarly to the usual Killing identity. Then a trapping horizon is said to be *outer, degenerate*, or *inner*, respectively, if $\kappa > 0$, $\kappa = 0$, or $\kappa < 0$. Examples of all types are provided by Reissner-Nordström solutions.

The Einstein equation implies

$$\kappa = \frac{m}{r^2} - 4\pi r w, \tag{14}$$

where the work density is

$$w = -\tfrac{1}{2}\,\mathrm{tr}\, T \tag{15}$$

and the trace is in the normal space. *In vacuo*, κ is m/r^2 and therefore reduces to the Newtonian surface gravity in the Newtonian limit, since m reduces to the Newtonian mass [17,18].

Another invariant of T is the energy flux

$$\psi = T \cdot g^{-1}(dr) + w dr. \tag{16}$$

The Einstein equation implies

$$dm = A\psi + w dV \tag{17}$$

which was dubbed the unified first law [18], as it encodes first laws of both thermodynamics and black-hole dynamics. Essentially, it expresses energy conservation, with the terms on the right-hand side being interpreted, respectively, as energy supply and work.

The fields $(A, r, k, j, V, m, \kappa, w, \psi)$ have been introduced above in a manifestly invariant way. For calculations, it is often useful to use dual-null coordinates $x^\pm$, in terms of which any spherically symmetric metric can locally be written as

$$ds^2 = r^2 d\Omega^2 - 2e^{2\varphi} dx^+ dx^-, \tag{18}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ for spherical polar coordinates (θ, ϕ), and (r, φ) are functions of (x^+, x^-). There is still the freedom to rescale functionally $x^\pm \to \tilde{x}^\pm(x^\pm)$, under which φ transforms by additive functions of x^+ and x^-. Then the following explicit expressions can be obtained:

$$k = e^{-2\varphi}(\partial_+ r \partial_- - \partial_- r \partial_+), \tag{19}$$

$$2m/r - 1 = 2e^{-2\varphi}\partial_+ r \partial_- r, \tag{20}$$

$$\kappa = -e^{-2\varphi}\partial_+\partial_- r, \tag{21}$$

$$w = e^{-2\varphi}T_{+-}, \tag{22}$$

$$\psi = -e^{-2\varphi}(T_{++}\partial_- r dx^+ + T_{--}\partial_+ r dx^+). \tag{23}$$

This also shows how ψ encodes radiative components of T, while w encodes the Coulomb-like component, e.g. $w = E^2/8\pi$ for the electric field $E = q/r^2$ of a Reissner-Nordström black hole with charge q.

The Einstein equations in these coordinates are

$$\partial_\pm\partial_\pm r - 2\partial_\pm\varphi\partial_\pm r = -4\pi r T_{\pm\pm}, \tag{24}$$

$$r\partial_+\partial_- r + \partial_+ r\partial_- r + \tfrac{1}{2}e^{2\varphi} = 4\pi r^2 T_{+-}, \tag{25}$$

$$r^2\partial_+\partial_-\varphi - \partial_+ r\partial_- r - \tfrac{1}{2}e^{2\varphi} = -4\pi r^2(T_{+-} + e^{2\varphi}p), \tag{26}$$

where $p = T^\theta_\theta = T^\phi_\phi$ is the transverse pressure. It follows that

$$\partial_\pm m = 4\pi r^2 e^{-2\varphi}(T_{+-}\partial_\pm r - T_{\pm\pm}\partial_\mp r), \tag{27}$$

which is a coordinate version of the unified first law (17).

III. WORMHOLE MOUTHS

A *wormhole mouth*, previously called horizon [4], is defined as a temporal outer trapping horizon. It is temporal in order to be two-way traversable, while the outer condition

$$\kappa > 0 \tag{28}$$

is proposed as the generalization of the minimality condition. Since $g^{-1}(dr, dr) = -2e^{-2\varphi}\partial_+ r\partial_- r$, either $\partial_+ r$ or $\partial_- r$ must vanish on the mouth, and $\partial_+ r \cong 0$ will be assumed henceforth.

Introducing a generating vector ξ of the marginal surfaces composing the mouth, its defining property is

$$\xi \cdot d(\partial_+ r) \cong 0. \tag{29}$$

Writing $\xi = \xi^+\partial_+ + \xi^-\partial_-$, one has

$$\xi^+ > 0, \qquad \xi^- > 0 \tag{30}$$

for future-pointing ξ. Then (29) expands as $\xi^+\partial_+\partial_+ r + \xi^-\partial_-\partial_+ r \cong 0$, then (21) shows that

$$\partial_+\partial_+ r > 0 \tag{31}$$

which expresses strict minimality of the sphere in the null hypersurface generated in the ∂_+ direction.

Hochberg and Visser [21,22] gave an alternative definition using a nonstrict version of (31) rather than $\kappa > 0$. However, they specifically allowed spatial ξ, which does not give local two-way traversability and for which (31) can select maximal rather than minimal surfaces. For instance, consider any Robertson-Walker space-time with a bounce and a maximal surface in that time-symmetric hypersurface. There are two trapping horizons intersecting at the maximal surface, which if spatial satisfy (31) there.

Strict minimality has been assumed for simplicity. For a minimal surface in a null hypersurface, one has merely $\kappa \geq 0$, which suffices for many purposes but also includes surfaces which are not minimal. The analysis to follow will be of a single wormhole mouth, though it should be stressed that two-way traversability requires two mouths with opposite senses, i.e. marginal in opposite null directions, in mutual causal contact.

IV. STATIC WORMHOLES

Locally one can always introduce coordinates (t, r_*) defined by $\sqrt{2}x^\pm = t \pm r_*$, where r_* is a generalization of the Regge-Wheeler "tortoise" coordinate [23]. Then the metric takes the form

$$ds^2 = r^2 d\Omega^2 + e^{2\varphi}(dr_*^2 - dt^2). \tag{32}$$

The metric in either (x^+, x^-) or (t, r_*) coordinates is manifestly regular if (r, φ) are finite, assumed henceforth, and r is nonzero.

In a static case with static Killing vector ∂_t, so that (r, φ) are independent of t, transforming further from r_* to r yields

$$ds^2 = r^2 d\Omega^2 + (1 - 2m/r)^{-1}dr^2 - e^{2\varphi}dt^2 \tag{33}$$

which is essentially the Morris-Thorne form of the metric. They introduced new jargon which has been enthusiastically adopted by wormhole aficionados, namely, "shape function" for $2m$ and "redshift function" for φ. Indeed φ is related to redshift, but better understood as a gravitational potential, reducing to the Newtonian potential in the Newtonian limit, if t reduces to Newtonian time. This and the fact that m is active gravitational mass are useful in physically interpreting such metrics.

The Morris-Thorne metric is singular at the wormhole throat $r \cong 2m$, so they used an embedding method to express minimality, as verified in the book of Visser [2]. Actually, there is no need to use a fictitious embedding space, as minimality is an intrinsic property. While the Morris-Thorne paper is still in many ways an excellent read, this is one unfortunate aspect which continues to inspire confusion. In particular, it is not recommended to generalize by naively inserting time-dependent or angular-dependent factors into a metric which is singular precisely at the object of interest.

In static cases, a wormhole mouth as defined above must be a double trapping horizon, $\partial_+ r \cong \partial_- r \cong 0$, since $\partial_\pm = \sqrt{2}(\partial_t \pm \partial_*)$ and $\partial_t r = 0$. Then

$$\partial_* r \cong 0 \tag{34}$$

so the surface is extremal in the static hypersurface. Since $\partial_t\partial_t r = 0$, one finds $\partial_*\partial_* r = 2\partial_+\partial_+ r = 2\partial_-\partial_- r = -2\partial_+\partial_- r$, so the minimality condition (31) implies that the surface is strictly minimal in the static hypersurface,

$$\partial_*\partial_* r > 0. \tag{35}$$

Thus the proposed definition of wormhole mouth recovers the Morris-Thorne definition appropriately. One may equivalently use proper radius $\int e^\varphi dr_*$ instead of r_* [2].

A calculation shows that

$$2\kappa = \frac{m}{r^2} - \frac{\partial_r m}{r} + \left(1 - \frac{2m}{r}\right)\partial_r\varphi. \tag{36}$$

In particular,

$$2\kappa \cong (m - r\partial_r m)/r^2 \tag{37}$$

so that $\kappa > 0$ reduces to the flaring-out condition of Morris and Thorne, their Eq. (54). Thus their embedding method correctly expresses strict minimality in static cases.

Another calculation shows that

$$\kappa \cong 2\pi r(\tau - \rho), \tag{38}$$

where $\rho = -T_t^t$ is the energy density and $\tau = -T_r^r$ the radial tension. This confirms that the weak energy condition must be violated [1],

$$\tau > \rho. \tag{39}$$

Moreover, it provides a simple measure of the violation, $\tau - \rho$ equalling surface gravity over circumference [4].

One might object to referring to κ as surface gravity in the case of a static wormhole; it is rather an invariant measure of the radial curvature or flaring-out. Quoting tetrad Riemann curvature components in Eqs. (8) of Morris and Thorne:

$$R^{\hat{r}}{}_{\hat{\theta}\hat{r}\hat{\theta}} = R^{\hat{r}}{}_{\hat{\phi}\hat{r}\hat{\phi}} \cong -2\kappa/r. \tag{40}$$

While Riemann curvature, like Gaussian curvature, has units of inverse length squared, κ, like principal curvature, has units of inverse length, as appropriate for measuring radial curvature. Actually, anywhere in a static space-time, the same expressions show that

$$R^{\hat{r}}{}_{\hat{\theta}\hat{r}\hat{\theta}} + R^{\hat{t}}{}_{\hat{\theta}\hat{t}\hat{\theta}} = R^{\hat{r}}{}_{\hat{\phi}\hat{r}\hat{\phi}} + R^{\hat{t}}{}_{\hat{\phi}\hat{t}\hat{\phi}} = -2\kappa/r \tag{41}$$

which indicates how κ generally also includes temporal curvature.

V. LAWS OF WORMHOLE DYNAMICS

This section derives some basic laws of wormhole dynamics which were stated previously [4]. Applications to dynamical processes involving wormholes have been made concrete in various examples, so will be only briefly mentioned and cited.

Negative-energy density: the null energy condition is necessarily violated on a wormhole mouth. Proof: use the focussing equations (24) and the minimality property (31): $T_{++} < 0$.

This confirms that "exotic" matter is required even in dynamic cases, which might also be called phantom or ghost matter, generalizing terminology from cosmology or quantum field theory, respectively. Claims that this need not be so in Einstein gravity either do not involve traversable wormholes as defined here, or evaluate energy density somewhere other than a mouth, e.g. at a center of symmetry described as a throat, or involve calculational errors. In dynamic cases, there are actually two independent constraints on energy density [24].

Since a black hole may be locally defined by a future outer trapping horizon [14,18], this allows interconversion of black holes and traversable wormholes. A future outer trapping horizon characterizes a black hole if achronal, equivalently $T_{++} \geq 0$, and a wormhole if temporal, equivalently $T_{++} < 0$. Thus a traversable wormhole can collapse to a black hole if its negative-energy source fails, or if enough positive-energy matter or radiation is pumped in [4–9,11]. Conversely, a black hole can be converted into a traversable wormhole by beaming in enough negative-energy radiation [4,5,8–13]. Wormhole construction from disjoint regions of flat space-time has recently been demonstrated by Maeda [25], albeit with a singularity at the topology change.

Zeroth law: κ is constant on a static wormhole throat. Proof: obvious.

Second law: future, past, or static wormhole mouths, respectively, have decreasing, increasing, or constant area. Proof: the expansion of the mouth is

$$A'/A = 2r'/r, \tag{42}$$

where the prime henceforth denotes $\xi \cdot d$. One can expand $\xi \cdot dr = \xi^+ \partial_+ r + \xi^- \partial_- r \cong \xi^- \partial_- r$, while $\partial_- r$ is negative, positive, or zero, respectively, for future, past, or static wormhole mouths.

This is like the second law of black-hole dynamics [14], but with reversed sign, reflecting the causal character of the mouth, or equivalently, the reversed null energy condition. It follows that a static wormhole is enlarged or reduced, respectively, by opening then closing a region of past or future trapped surfaces. To enlarge, this can be done by beaming in negative energy, balanced by subsequent positive energy, while the opposite order would reduce the area [4,6,7,9–13]. For some matter models, this can be done in an apparently stable way [5,6,8–13], while others are unstable, leading either to collapse to a black hole as above, or to inflationary expansion [7,11,26–28].

First law:

$$m' \cong \frac{\kappa A'}{8\pi} + wV'. \tag{43}$$

Proof: one can calculate this by a few steps as the projection of the unified first law (17) along ξ, but the easiest way is to multiply the expression (14) for κ by A', then use the fact that $r \cong 2m$ and $r' \cong 2m'$.

This has the same form as the first law of black-hole dynamics [18]. These three laws therefore suggest a genuine connection with thermodynamics. Indeed, it has recently been shown that any future outer trapping horizon has a local Hawking temperature $\kappa/2\pi$ [29], which therefore applies to future wormhole mouths. Thus there is a new field of wormhole thermodynamics waiting to be explored.

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