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Consistent cosmological modifications to the Einstein equations

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General relativity is a phenomenologically successful theory that rests on firm foundations, but has not been tested on cosmological scales. The deep mystery of dark energy (and possibly even the requirement of cold dark matter), has increased the need for testing modifications to general relativity, as the inference of such otherwise undetected fluids, depends crucially on the theory of gravity. In this work I outline a general scheme for constructing consistent and covariant modifications to the Einstein equations. This framework is such that there is a clear connection between the modification and the underlying field content that produces it. I argue that this is mandatory for distinguishing modifications of gravity from conventional fluids. I give two nontrivial examples, the first of which is a simple metric-based modification of the fluctuation equations for which the background is exact ΛCDM and the second has a Dvali-Gabadadze-Porrati background but differs from it in the perturbations. I present their impact on observations of the cosmic microwave background radiation.

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I. INTRODUCTION

The theory of gravity plays a fundamental role in our modelling and understanding of the Universe. If we are to know the matter constituents of the Universe, we have to be sure we understand what is the underlying gravitational theory. Einstein's general relativity (GR) has played a key role in formulating modern cosmology, first as a smooth Friedmann-Lemaître-Roberson-Walker (FLRW) spacetime, then at the level of linearized fluctuations about this spacetime.

General relativity is a very solid principle theory from the theoretical point of view, (and quite understandably the aesthetical point of view). The Lovelock-Grigore theorem [1,2] asserts that GR with a cosmological constant is unique under the following assumptions: geometry is Riemannian and the gravitational action depends only on the metric; it is local and diffeomorphism invariant and leads to second order field equations. Relaxing any of these assumptions can lead to more general gravitational theories, e.g. adding extra fields [3–12], having higher derivatives [13], having a pregeometry [14–16], or making the theory nonlocal [17–19]. See [20,21] for discussions. This is not an exhaustive list but possible theories fall into one or more categories above.

However as nice as we may think that GR is, the ultimate judge is experiment. Indeed, different aspects of GR have been vigorously tested in the lab, in the solar system, and with binary pulsars, all of which lie in the strong curvature regime (compared to cosmology).

The discovery that the expansion of the Universe is accelerating opens the possibility that general relativity breaks down on large scales or low curvatures. It may also be that the apparent missing mass in the universe is not in the form of cold dark matter but once again due to departures from general relativity. This opens the need for cosmological tests of gravity, and much work has been carried out in this direction [22–34] at various levels. More recently Hu and Sawicki [35], and Hu [36], have laid down a fully covariant formulation of modifications to gravity under well-motivated assumptions.

In this work I outline a general scheme for constructing consistent modifications to the Einstein equations. The scheme is such that, one can clearly classify the modifications according to whether they obey or violate diffeomorphism invariance, need extra fields, or stem from higher derivative theories. Indeed the advantage of this method is the direct connection between the field content and the modifications. As I argue further below, the specification of the field content is essential if we are to discriminate between a modified gravitational law or the effects of conventional matter fluids. Additional assumptions as in [24,35,36] can always be used at the very end but we shall not consider this possibility here.

I consider two examples of this method. First, I construct the most general modification for which the FLRW background is exactly ΛCDM, does not contain additional fields and leads to at most second order differential equations. I then illustrate the effects on observables in a simple subcase. As I show further below, this leads to suppression of power in the spectrum of the cosmic microwave background (CMB) on large scales up to and including the first peak. I then consider a less trivial example where the background cosmology evolves like a Dvali-Gabadadze-Porrati (DGP) model [17] and construct simple perturbed equations (different from the proper DGP model) under the assumption that the theory does not contain additional fields and leads to at most second order differential equa-

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tions. I calculate the spectrum of the CMB in the DGP-like model and show that unlike the proper DGP theory it leads to a suppression of power on large scales.

II. THE FRAMEWORK

A. Setup

Following Hu [36] we start by putting the gravitational field equations in the form

$$G_{ab} = 8\pi G T^{(\rm known)}_{ab} + U_{ab}. \tag{1}$$

Here, G_{ab} is the Einstein tensor for the universal matter metric g_{ab}, $T^{(\rm known)}_{ab}$ is the stress-energy tensor of all *known* forms of matter (like baryons, photons and neutrinos), U_{ab} is a general tensor that encapsulates all the unknown fields/modifications and can depend on T_{ab} for each field and various combinations of metric functions (such as curvature tensors). If desirable one may also include cold dark matter in $T^{(\rm known)}_{ab}$, as I also do in the examples given in this work.

The assumption of up to second order field equations, translates to having only up to first derivatives of the extra field in $U_{0\nu}$ and up to second derivatives in U_{ij}. Relaxing this assumption is possible and will simply give higher order field equations, but one has to be cautious that quite generally higher derivative theories lead to instabilities.

At this point we have to decide on the field content, i.e. whether U_{ab} depends on additional fields or the metric alone. In the former case, we must add one field at a time and ensure its energy conservation by applying the Bianchi identity, which directly translates to

$$\nabla_a U^a{}_b = 0 \tag{2}$$

and gives a field equation for the extra field. Violating the Bianchi identity leads to entirely arbitrary parameterizations and will not be considered. The caveat of this approach is that in the case that more than two independent degrees of freedom are present in U_{ab}, one would have to supply extra equations for these, not given by the Bianchi identity (see [12,37,38] for examples). Finally, there could be interactions between fields in $T^{\rm known}_{ab}$ and U_{ab}. For simplicity I do not consider this possibility further, but it is straightforward to add.

B. FLRW background dynamics

We now split the dynamics of the problem in the background FLRW dynamics and their fluctuations about that background. The FLRW metric is $ds^2 = a^2(-d\tau^2 + q_{ij}dx^i dx^j)$, where τ is the conformal time, a is the scale factor, and q_{ij} is a spatial metric of constant (dimensionless) curvature K. The FLRW assumption means that effectively we are considering a collection of scalar fields on the spatial hypersurface of homogeneity and isotropy. This boils down to requiring that the Lie derivative of the extra field vanishes for all six Killing vectors of the FLRW spacetime. Examples are a scalar field $\phi(\tau)$, a vector field with components $A^\mu = (A(\tau), 0, 0, 0)$, and a tensor field $X^a{}_b$ whose only nonvanishing components must be $X^0{}_0 = -X(\tau)$ and $X^i{}_j = Y(\tau)\delta^i{}_j$. In the special case of a unit-timelike vector field, the function $A(\tau)$ is pure gauge and the contribution of such a field to the FLRW equations is generally given in terms of functions of a, $\dot{a}$, and $\ddot{a}$ [10,11].

Lets define $E_F = -a^2 G^0{}_0$ and E_R such that $a^2 G^i{}_j = E_R \delta^i{}_j$, which are explicitly given by

$$E_F = 3\frac{\dot{a}^2}{a^2} + 3K \tag{3}$$

and

$$E_R = -2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} - K, \tag{4}$$

respectively. The FLRW equations corresponding to (1) are then simply written as

$$E_F = 8\pi G a^2 \sum_i \rho_i + a^2 X \tag{5}$$

and

$$E_R = 8\pi G a^2 \sum_i P_i + a^2 Y, \tag{6}$$

where ρ_i and P_i are the density and pressure for each known fluid, and the index i runs over all known fluids. Applying the Bianchi identity then gives

$$\dot{E}_F + \frac{\dot{a}}{a}(E_F + 3E_R) = 0, \tag{7}$$

while the fluid equation is as usual $\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + P) = 0$. Applying the Bianchi identity on U_{ab} imposes

$$\dot{X} + 3\frac{\dot{a}}{a}(X + Y) = 0. \tag{8}$$

The above equation will give the background equation for the extra field, or additional constraints on X and Y in the absence thereof. We see that any FLRW background can be modeled via one arbitrary function $Y(\tau)$ and a second function $X(\tau)$ found by solving (8). This construction tells us that it is impossible to distinguish models of modified gravity from models where the dark energy or dark matter are due to conventional fluids, by using the FLRW background equations alone.

1. Example 1: Scalar field

Adding a scalar field amounts to letting $X = X(\phi, \dot{\phi}, a, \dot{a})$ and $Y = Y(\phi, \dot{\phi}, \ddot{\phi}, a, \dot{a}, \ddot{a})$, if we are to expect at most second order field equations for the scalar. For example, a canonical scalar field corresponds to $X = \frac{1}{2a^2}\dot{\phi}^2 + V(\phi)$ and $Y = \frac{1}{2a^2}\dot{\phi}^2 - V(\phi)$.

2. Example 2: Jordan-Fierz-Brans-Dicke theory

A less trivial example is the Jordan-Fierz-Brans-Dicke theory [3–6] where in addition to g_{ab} the gravitational sector contains a scalar field ϕ and depends on a single parameter, namely, the coupling constant ω. In this case the functions $X(\tau)$ and $Y(\tau)$ correspond to

$$X = 8\pi G a^2 (e^{\phi} - 1)\rho + \frac{\omega}{2a^2}\dot{\phi}^2 + \frac{3}{a^2}\frac{\dot{a}}{a}\dot{\phi} \tag{9}$$

and

$$Y = 8\pi G a^2 (e^{\phi} - 1)P - \frac{1}{a^2}\left(\ddot{\phi} + \frac{\dot{a}}{a}\dot{\phi} - \frac{2+\omega}{2}\dot{\phi}^2\right), \tag{10}$$

respectively. Notice that these functions now depend explicitly on the total energy density and pressure of ordinary matter contained in $T_{ab}^{\rm known}$. It is easy to show that the background Bianchi identity (8) then gives the field equations for ϕ.

C. Linear perturbations

1. Gauge-form invariance

Consider now the (scalar) fluctuations about the FLRW metric as

$$ds^2 = -a^2(1 - 2\Xi)dt^2 - 2a^2(\vec{\nabla}_i \zeta)dt dx^i + a^2[(1 + \tfrac{1}{3}\chi)q_{ij} + D_{ij}\nu]dx^i dx^j, \tag{11}$$

where $D_{ij} \equiv \vec{\nabla}_i \vec{\nabla}_j - 1/3 q_{ij}\vec{\nabla}^2$ is a spatial traceless derivative operator. A perfect fluid is described at the fluctuation level by a density contrast δ, momentum θ such that its total momentum is $u_i = a\vec{\nabla}_i \theta$, dimensionless pressure perturbation Π such that $\delta T^i{}_j = \Pi \rho \delta^i{}_j$ and shear Σ, such that the shear tensor is $\Sigma_{ij} = D_{ij}\Sigma$.

We now decide whether the parameterization should obey diffeomorphism invariance. At the linearized level this is the requirement that all field equations must be gauge-form invariant. Let us demonstrate what gauge-form invariance is for the case of standard GR coupled to a fluid. Gauge transformations are infinitesimal diffeomorphisms generated by a vector field ξ^a, which can be parameterized as $\xi_\mu = a(-\xi, \vec{\nabla}_i \psi)$. All perturbations apart from Σ above are not gauge invariant but transform with ξ and ψ as in Table I. Consider the $\delta G^0{}_0$ Einstein equation, which is

$$-\frac{1}{3}(\vec{\nabla}^2 + 3K)[\chi - \vec{\nabla}^2 \nu] + \frac{\dot{a}}{a}[\dot{\chi} + 2\vec{\nabla}^2 \zeta] + 6\frac{\dot{a}^2}{a^2}\Xi = 8\pi G a^2 \rho \delta. \tag{12}$$

If we perform a gauge transformation $X \to X'$ to all perturbations above, the $\delta G^0{}_0$ Einstein equation becomes

TABLE I. Gauge transformations for the metric, fluid, and Einstein tensor variables.

$\Xi \to \Xi - \frac{\dot{\xi}}{a}$	$\zeta \to \zeta + \frac{1}{a}[\xi + \frac{\dot{a}}{a}\psi - \dot{\psi}]$
$\chi \to \chi + \frac{1}{a}[6\frac{\dot{a}}{a}\xi + 2\vec{\nabla}^2\psi]$	$\nu \to \nu + \frac{2}{a}\psi$
$V \to V + \frac{2}{a}\xi$	$J \to J + \frac{6}{a}\frac{\dot{a}}{a}\xi$
$\theta \to \theta + \frac{1}{a}\xi$	$W \to W + \frac{6}{a}[\frac{\dot{a}}{a}\dot{\xi} + (\frac{\ddot{a}}{a} - 2\frac{\dot{a}^2}{a^2})\xi$
$\delta \to \delta - \frac{3}{a}(1+w)\frac{\dot{a}}{a}\xi$	$+ \frac{1}{3}\vec{\nabla}^2\xi]$
$E_\Theta \to E_\Theta + \frac{1}{a}(E_F + E_R)\xi$	$\Pi \to \Pi + \frac{1}{a}[\dot{w} - 3w(1+w)\frac{\dot{a}}{a}]\xi$
$E_P \to E_P + \frac{3}{a}[\dot{E}_R - 2\frac{\dot{a}}{a}E_R]\xi$	$E_\Delta \to E_\Delta - \frac{3}{a}\frac{\dot{a}}{a}(E_F + E_R)\xi$

$$-\frac{1}{3}(\vec{\nabla}^2 + 3K)[\chi' - \vec{\nabla}^2 \nu'] + \frac{\dot{a}}{a}[\dot{\chi}' + 2\vec{\nabla}^2 \zeta'] + 6\frac{\dot{a}^2}{a^2}\Xi' = 8\pi G a^2 \rho \delta' + \frac{3}{a}\frac{\dot{a}}{a}[E_F + E_R - \rho - P]\xi. \tag{13}$$

The dependence on the gauge variable ξ can only be eliminated if and only if the background FLRW equations are satisfied. Any consistent diffeomorphism invariant theory must have this property.

After eliminating the gauge variable from (13) via the background FLRW equations, the only remaining difference between (12) and (13) is a simple relabeling of the perturbation variables $X \to X'$. In other words, the equation retains its exact form: it is *form invariant*. This should not be confused with general covariance where the field equation can be consistently transformed in a desired coordinate system where it may look different but still contain the same physics. At the perturbative level, general covariance is the fact that the equations written in different gauges are physically equivalent, even though they look substantially different. The reader is referred to [39–42] for the difference between general covariance and general invariance in GR. Gauge-form invariance always holds for all field equations, which stem from a diffeomorphism invariant action, no matter how complicated the theory is.

We can shortcut testing for gauge-form invariance as follows: First, define the three gauge noninvariant potentials $V \equiv \dot{\nu} + 2\zeta$, $J \equiv \chi - \vec{\nabla}^2\nu$, and $W \equiv \dot{\chi} + 2\vec{\nabla}^2\zeta = \dot{J} + \vec{\nabla}^2 V$, which are the only three combinations of metric variables appearing in the perturbed Einstein tensor. They transform only with the gauge variable ξ. Then define the two gauge-invariant potentials

$$\hat{\Phi} \equiv -\frac{1}{6}J + \frac{1}{2}\frac{\dot{a}}{a}V \tag{14}$$

and

$$\hat{\Psi} \equiv -\Xi - \frac{1}{2}\dot{V} - \frac{1}{2}\frac{\dot{a}}{a}V. \tag{15}$$

We can now split the perturbed Einstein tensor into a gauge invariant and a gauge noninvariant part that involves the variable V. For simplicity let us define $E_\Delta = -a^2 \delta G^0{}_0$, E_Θ such that $-a^2\delta G^0{}_i = \vec{\nabla}_i E_\Theta$, E_P by $E_P = a^2 \delta G^i{}_i$ and E_Σ as $a^2[\delta G^i{}_j - \frac{1}{3}\delta G^k{}_k \delta^i{}_j] = D^i{}_j E_\Sigma$. Explicitly we get

$$E_\Delta = 2(\vec{\nabla}^2 + 3K)\hat{\Phi} - 6\frac{\dot{a}}{a}\left(\dot{\hat{\Phi}} + \frac{\dot{a}}{a}\hat{\Psi}\right) - \frac{3}{2}\frac{\dot{a}}{a}(E_F + E_R)V, \tag{16}$$

$$E_\Theta = 2\left(\dot{\hat{\Phi}} + \frac{\dot{a}}{a}\hat{\Psi}\right) + \frac{1}{2}(E_F + E_R)V, \tag{17}$$

$$E_P = 6\frac{d}{dt}\left(\dot{\hat{\Phi}} + \frac{\dot{a}}{a}\hat{\Psi}\right) + 12\frac{\dot{a}}{a}\left(\dot{\hat{\Phi}} + \frac{\dot{a}}{a}\hat{\Psi}\right) - 2(\vec{\nabla}^2 + 3K) \times (\hat{\Phi} - \hat{\Psi}) - 3(E_F + E_R)\hat{\Psi} + \frac{3}{2}\left[\dot{E}_R - 2\frac{\dot{a}}{a}E_R\right]V, \tag{18}$$

and

$$E_\Sigma = \hat{\Phi} - \hat{\Psi}. \tag{19}$$

The perturbed (generalized) Einstein Eqs. (1) are then given as

$$E_\Delta = 8\pi G a^2 \rho\delta + U_\Delta, \tag{20}$$

$$E_\Theta = 8\pi G a^2(\rho + P)\theta + U_\Theta, \tag{21}$$

$$E_P = 24\pi G a^2 \rho\Pi + U_P, \tag{22}$$

$$E_\Sigma = 8\pi G a^2(\rho + P)\Sigma + U_\Sigma, \tag{23}$$

where the U_i variables are defined in the same way as for the E_i variables with $\delta G^a{}_b$ replaced by $\delta U^a{}_b$.

2. Importance of gauge-form invariance and the field content

Let us now illustrate why the procedure for establishing that the equations are gauge-form invariant is important, not at all redundant.

The majority of the parameterized schemes start by assuming the conformal Newtonian gauge. While these schemes may be consistent, it is far from obvious that they are so. In fact, from the way that they are set up it is impossible to actually test for consistency under gauge-form invariance. Writing the equations in the conformal Newtonian gauge, and then performing a gauge transformation will introduce additional terms, which will depend on the gauge variables ξ and ψ. It is not at all clear that the coefficients of the gauge variables will vanish, which is one of the requirements of gauge-form invariance.

The question that arises in the light of the above, is whether it might be possible to interpret the potentials appearing in the equations in conformal Newtonian gauge as the gauge-invariant potentials. It might seem that this solves the problem of gauge-form invariance, as all the terms would now be explicitly gauge invariant. Unfortunately, this interpretation is also incorrect. The reason is that it is impossible in general to write down the perturbed field equations (whether Einstein equations or any other set of field equations), such that all terms that appear are by construction gauge invariant. The only case that this is actually possible is when the background tensors are constant, which is forbidden in the case of an FLRW universe. This is a consequence of the well-known Stewart-Walker lemma [43]. In other words, although it is possible to write any perturbed field equation as a sum of gauge-invariant terms, each term cannot in general arise as a perturbation of a tensor constructed out of the fields of the theory. Such a construction is nothing more than a convenient mathematical construct but otherwise physically empty.

The true power of gauge-form invariance manifests in conjunction with the specification of the field content of the parameterization. Different fields transform differently under gauge transformations and this is then directly linked to the individual gauge noninvariant terms, which can be a part of $U^a{}_b$. However, it might not be directly obvious why specifying the field content of $U^a{}_b$ is by itself important. After all why not simply consider the gauge transformation of the whole of $U^a{}_b$ and ignore its composition?

Let us exemplify. The tensor $U^a{}_b$ transforms as a whole like a stress-energy tensor and this is enough for constructing consistent parameterizations of $U^a{}_b$. Such is the approach followed in [36]. However, this prohibits a direct physical interpretation of our findings in the case that a nonzero contribution from $U^a{}_b$ is detected. More specifically, it is impossible to attribute this contribution to a modification of gravity as opposed to the presence of some ordinary unknown fluid without further assumptions. For example, that the difference $\Phi - \Psi$, commonly (and incorrectly) thought of as indicating a modification of gravity could be also be sourced by a standard fluid with shear, has been mooted by Kunz and Sapone [44] and by Bertschinger and Zukin [34]. Specifying the field content of $U^a{}_b$ is the extra assumption that we need to distinguish between a modification of gravity and the gravitational effect of standard fluids. Once the fields comprising $U^a{}_b$ are specified we can proceed to answer the question, "what is the force between two well separated masses in vacuum?" We can then distinguish gravity from fluids depending on whether the field equations lead to a modification of the standard gravitational law or not. I shall not consider the details of how this last step is performed (the reader is referred to [45] for the case of GR in an expanding background) in this work but only consider the way of how such field equations can be consistently written.

3. Number of time derivatives

Once we have added the gauge noninvariant terms and correctly fixed the functions multiplying them by requiring gauge-form invariance to hold, we can proceed to add more gauge-invariant terms involving the extra fields, the known matter fields and the gauge-invariant potentials $\hat{\Phi}$ and $\hat{\Psi}$. If we want to consider only parameterizations that lead to second order field equations in all variables, then we have to be careful what terms we add and where. For example, when we add metric terms, we can add up to first derivatives in the two Einstein constraint Eqs. (20) and (21), and up to second derivatives in the two Einstein propagation Eqs. (22) and (23). Since $\hat{\Phi}$ is of first order in the metric variables, while $\hat{\Psi}$ is of second order, we can add $\hat{\Phi}$ in all four Einstein equations but $\dot{\hat{\Phi}}$ and $\hat{\Psi}$ only in the two propagation Eqs. (22) and (23). Note that although $\dot{\hat{\Phi}} + \frac{\dot{a}}{a}\hat{\Psi}$ is of first order in the metric perturbations, it contains second derivatives of the scale factor, and so it cannot be added to (20) and (21), unless we relax the second order field equations constraint. The reason it is allowed in the definition of E_Δ above is because there are also second derivatives of the scale factor appearing in the gauge non-invariant part of E_Δ proportional to V. Thus, when E_Δ is written in terms of the actual metric potentials, the $\ddot{a}$ terms cancel and the final expression contains only first derivatives in all the variables. On the contrary, when we add solely gauge-invariant terms we no longer have this luxury.

Let us also note that the gauge variable ψ is not involved in the transformation of the Einstein tensor. Thus, if any extra field does transform with ψ it will always appear in combination with ν, ζ, or χ in the field equations, in a way that the whole combination does not transform with ψ. An explicit example can be found in [38].

4. Bianchi identity

We finally utilize the Bianchi identity which at the linearized level simply translates in terms of the added variables U_Δ, U_Θ, U_P, and U_Σ as

$$\dot{U}_\Delta + \frac{\dot{a}}{a}U_\Delta - \vec{\nabla}^2 U_\Theta + \frac{1}{2}a^2(X+Y)W + \frac{\dot{a}}{a}U_P = 0 \tag{24}$$

and

$$\dot{U}_\Theta + 2\frac{\dot{a}}{a}U_\Theta - \frac{1}{3}U_P - \frac{2}{3}(\vec{\nabla}^2 + 3K)U_\Sigma + a^2(X+Y)\Xi = 0. \tag{25}$$

This provides us with the field equations for the extra fields [46], or with additional constraints on the added functions in the absence thereof.

III. EXAMPLES

A. Conventional fluid

As I have already discussed above, the requirement of gauge-form invariance severely constrains the terms involved in the perturbed field equations. Suppose that we had decided to change the right-hand side of the Einstein equations to $8\pi G a^2 f_1(\tau)\delta$, $8\pi G a^2 f_2(\tau)\theta$, and $24\pi G a^2 f_3(\tau)\Pi$, while maintaining the standard Friedmann equation $3H^2 = 8\pi G\rho$. We would then have found that by virtue of the background equations, the Einstein equations are gauge-form invariant if and only if $f_1 = f_3 = \rho$, and $f_2 = \rho + P$.

B. The extended ΛCDM model

I further illustrate the above scheme with a less trivial example than the conventional fluid. In what follows I find the most general diffeomorphism invariant modification to the Einstein equations for which the background cosmology is the plain ΛCDM model, no extra fields are present, and no higher derivative than two is present in the field equations. Since there are no extra fields and the background is unchanged from ΛCDM we can only add gauge-invariant terms to Einstein equations by setting $U_\Delta = \frac{1}{a}\mathcal{A}\hat{\Phi}$, $U_\Theta = \frac{1}{a^2}\mathcal{B}\hat{\Phi}$, $U_P = \mathcal{C}_1\hat{\Phi} + \mathcal{C}_2\dot{\hat{\Phi}} + \mathcal{C}_3\hat{\Psi}$, and $U_\Sigma = \mathcal{D}_1\hat{\Phi} + \mathcal{D}_2\dot{\hat{\Phi}} + \mathcal{D}_3\hat{\Psi}$ for spatial pseudodifferential operators $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}_i$, and $\mathcal{D}_i$. Applying the Bianchi identity we get two equations involving $\dot{\hat{\Phi}}$, $\hat{\Phi}$, and $\hat{\Psi}$, and consistency requires that these equations must be satisfied whatever the values of $\dot{\hat{\Phi}}$, $\hat{\Phi}$, and $\hat{\Psi}$. A sufficient condition is found by setting the coefficients of these terms to zero, which gives $\mathcal{C}_3 = \mathcal{D}_3 = 0$, the two constraints

$$\mathcal{A} = -\dot{a}\mathcal{C}_2, \tag{26}$$

$$\frac{1}{a^2}\mathcal{B} - \frac{1}{3}\mathcal{C}_2 = \frac{2}{3}(\vec{\nabla}^2 + 3K)\mathcal{D}_2, \tag{27}$$

and the two differential equations

$$\dot{\mathcal{A}} + \dot{a}\mathcal{C}_1 - \frac{1}{a}\vec{\nabla}^2\mathcal{B} = 0, \tag{28}$$

$$\frac{1}{a^2}\dot{\mathcal{B}} - \frac{1}{3}\mathcal{C}_1 - \frac{2}{3}(\vec{\nabla}^2 + 3K)\mathcal{D}_1 = 0. \tag{29}$$

A quick examination reveals that if $\mathcal{A}$ and $\mathcal{B}$ are both zero then we get exact GR. The same holds if $\mathcal{D}_1$ and $\mathcal{D}_2$ are also both zero, hence a generic prediction of this kind of modification to GR is that $\hat{\Phi} - \hat{\Psi}$ should deviate from the GR value. Another special case is when $\mathcal{D}_2 = \mathcal{B} = 0$ but $\mathcal{D}_1$ is not assumed at first to vanish. Using the above conditions, however, we find that all operators must vanish and once again we recover GR.

To illustrate the effect on observables lets make further assumptions and consider a simple subcase for which the spacetime is spatially flat and for which $\mathcal{B} = \mathcal{C}_1 = \mathcal{D}_1 = 0$. The only nonzero operators are $\mathcal{A} = \beta H_0^2$, $\mathcal{C}_2 = -\frac{\beta H_0^2}{\dot{a}}$, and $\mathcal{D}_2 = \frac{\beta H_0^2}{2\dot{a}}\frac{1}{\vec{\nabla}^2}$. Thus, we parameterize deviations from GR with a single dimensionless parameter β, which appears only in the perturbed equations and not in the background.

The action of $\frac{1}{\vec{\nabla}^2}$ is defined by its spectral representation, i.e.

$$\frac{1}{\vec{\nabla}^2}F(t,\vec{x}) = -\frac{1}{(2\pi)^3}\int d^3\vec{k}e^{i\vec{k}\cdot\vec{x}}\frac{1}{k^2}f(t,\vec{k}), \tag{30}$$

where F and f is a Fourier transform pair. In particular it is easy to show using the above definition that $\frac{1}{\vec{\nabla}^2}\delta^3(\vec{x}) = -\frac{1}{4\pi|\vec{x}|}$ as we would expect from the usual solution to Laplace's equation $\vec{\nabla}^2\frac{1}{|\vec{x}|} = -4\pi\delta^3(x)$ (see Appendix C). Thus, we can simply replace $\frac{1}{\vec{\nabla}^2}$ by $-\frac{1}{k^2}$ in the Fourier space field equations given in Appendix A and solve them. After we have the solution to all perturbations we can Fourier transform them back to real space without any inconsistency.

How does this modification fit in accordance with the Lovelock-Grigore theorem? It is clear that the operators above contain inverse powers of $\dot{a}$ as well as the pseudo-differential operator $\frac{1}{\vec{\nabla}^2}$. This means that if a full nonlinear theory exists that leads to an exact ΛCDM background and which deviates at the perturbative level as above, then such a theory must be nonlocal. This is in full accordance with the Lovelock-Grigore theorem.

The perturbation equations were solved numerically in both the synchronous and in the conformal Newtonian gauge for numerical consistency. These equations are displayed in the appendix. To solve the perturbed equations, we must also specify the initial conditions, and I further assume that the initial conditions are adiabatic. This introduces a β dependence in the adiabatic growing mode to order $k\tau$ in the synchronous gauge (which vanishes for $\beta = 0$), while in the conformal Newtonian gauge there is no such dependence to leading order in $k\tau$ but it arises at higher orders. The upper panel of Fig. 1 shows the CMB angular power spectrum $l(l+1)C_l$ for a ΛCDM universe ($\beta = 0$) contrasted with nonzero β. We see that for this particular model, the effect of nonzero β is to decrease power on large scales, including even the first peak. Figure 2 shows the time variation of $\hat{\Phi} - \hat{\Psi}$ for the same set of models at $k = 10^{-3}$ Mpc^{-1}. We see that like other modifications to gravity, the effect is to make $\hat{\Phi} - \hat{\Psi}$ grow. In contrast to conventional parameterizations of modified gravity [24,47], however, the difference of $\hat{\Phi} - \hat{\Psi}$ is sourced by $\dot{\hat{\Phi}}$ rather than Φ. Figure 3 shows the phase portrait in the $\{\Phi, \Psi\}$ plane for the same set of models at scales of $k = 10^{-3}$ Mpc^{-1} (upper left panel), $k = 5\times 10^{-3}$ Mpc^{-1} (upper right panel), $k = 0.01$ Mpc^{-1} (lower left panel) and $k = 0.05$ Mpc^{-1} (lower right panel). Finally, I compare this model with the Q-η_A parameterization of Amendola, Kunz, and Sapone [28] (which is directly related to the Jain-Zhang parameterization [29] with $Q = G_{\rm eff}$ and $1 + \eta_A = 1/\eta_{\rm JZ}$). To remind the reader these two parameters are defined as $\Phi - \Psi = -\eta_A(k,\tau)\Phi$ and $-2k^2\Phi = 8\pi G a^2\rho Q(k,\tau)\delta$. As shown in Fig. 4, this

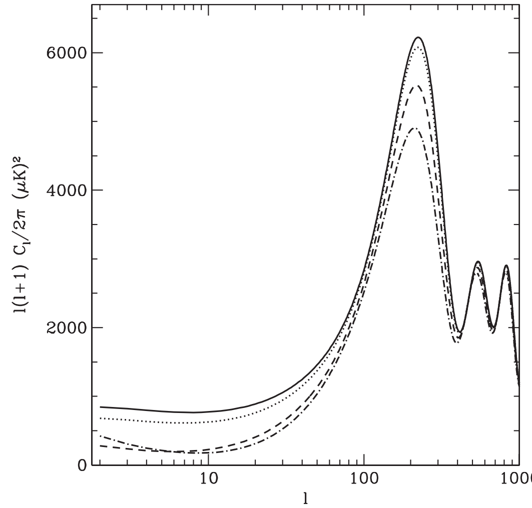

FIG. 1. The CMB spectrum for the simple modified gravity model in the text. The solid curve is the plain ΛCMD model ($\beta = 0$), while the dotted, dashed, and dotted-dashed curves are with $\beta = \{0.1, 0.5, 1\}$, respectively.

model can cover a wide range of the $\{Q, \eta_A\}$ plane rather being effectively one dimensional, as, for example, in the case of DGP [17] or clustering dark energy [44] models [48]. This can be exploited to provide for flat priors in the $\{Q, \eta_A\}$ plane [49] used to consistently probe such modifications with weak lensing.

C. DGP-like background

I now consider an even less trivial example than the extended ΛCDM model above. In particular, I assume that the background cosmology is driven by a DGP-like model for which

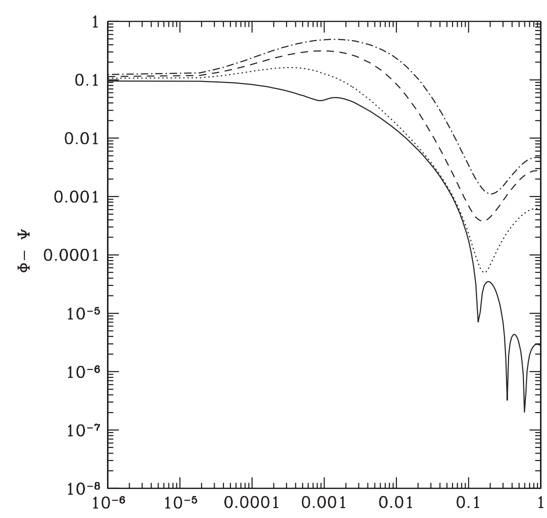

FIG. 2. The time evolution of $\hat{\Phi} - \hat{\Psi}$ at $k = 10^{-3}$ Mpc^{-1} for $\beta = 0$ (solid), $\beta = 0.1$ (dotted), $\beta = 0.5$ (dashed), and $\beta = 1$ (dotted-dashed).

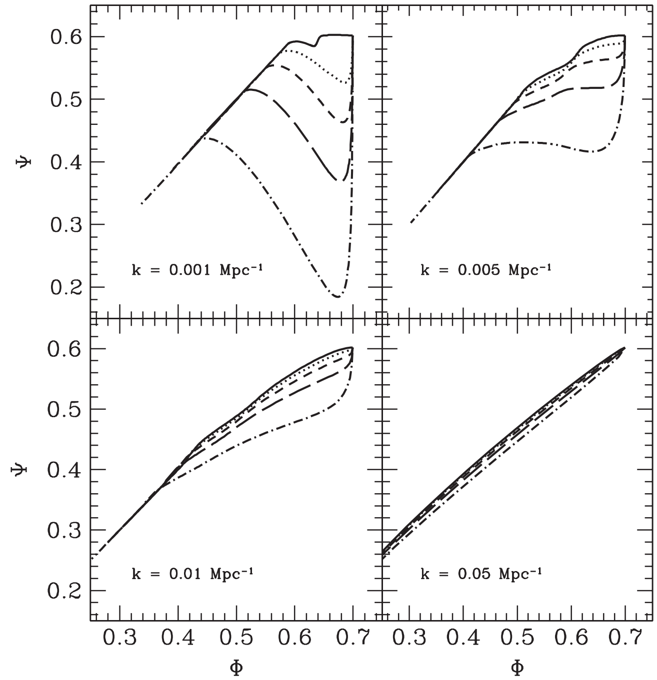

FIG. 3. The phase portrait of $\Phi(t, k)$ and $\Psi(t, k)$ at $k = \{0.001, 0.005, 0.01, 0.05\}$ Mpc^{-1} for $\beta = 0$ (solid), $\beta = 0.1$ (dotted), $\beta = 0.25$(dashed), $\beta = 0.5$ (long-dashed), and $\beta = 1$ (dotted-dashed). On small scales the phase portrait is squeezed close to a line as it would be for the case of plain ΛCDM.

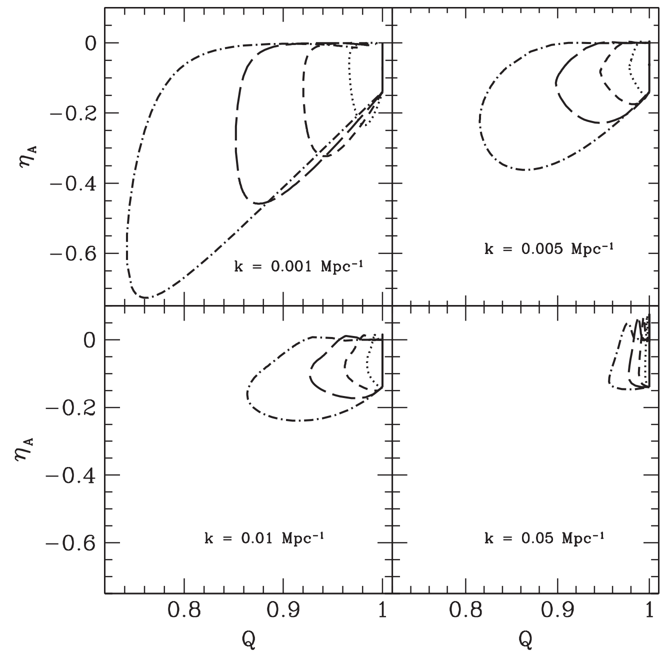

FIG. 4. The phase portrait of parameters $Q(t, k)$ and $\eta_A(t, k)$ at $k = \{0.001, 0.005, 0.01, 0.05\}$ Mpc^{-1} for $\beta = 0$ (solid), $\beta = 0.1$ (dotted), $\beta = 0.25$ (dashed), $\beta = 0.5$ (long-dashed), and $\beta = 1$ (dotted-dashed). Notice that this model can cover a wide range of the $\{Q, \eta_A\}$ plane rather being effectively one dimensional, as, for example, in the case of DGP [17] or clustering dark energy [44] models [48].

$$X = \frac{3}{ar_c}\sqrt{\frac{\dot{a}^2}{a^2} + K} = \frac{\sqrt{3E_F}}{ar_c} \tag{31}$$

for some scale r_c. The function Y is then determined as

$$Y = -\frac{1}{r_c a}\left[\frac{\ddot{a}}{\dot{a}} + \frac{\dot{a}}{a}\right] = \frac{\sqrt{3}(E_R - E_F)}{2ar_c\sqrt{E_F}} \tag{32}$$

by using the background Bianchi identity.

1. Metric-based modification

At the perturbative level we now depart from the proper DGP theory. As a first case let us assume that no additional fields are present, in which case the tensor $U^a{}_b$ is supposed to be constructed out of metric functions alone. A further assumption is that the theory does not contain higher time derivatives. Using the prescription that all field equations are gauge-form invariant one immediately finds that

$$U_\Delta = \frac{3\sqrt{3}\dot{a}}{r_c\sqrt{E_F}}\left[\frac{1}{6}\dot{J} + \frac{\dot{a}}{a}\Xi - \frac{K}{2}V\right] + \mathcal{A}\hat{\Phi}, \tag{33}$$

$$= -\frac{3\sqrt{3}\dot{a}}{r_c\sqrt{E_F}}\left[\dot{\hat{\Phi}} + \frac{\dot{a}}{a}\hat{\Psi} + \frac{E}{4}V\right] + \mathcal{A}\hat{\Phi}, \tag{34}$$

and

$$U_\Theta = -\frac{a\sqrt{3}}{r_c\sqrt{E_F}}\left[\frac{1}{6}\dot{J} + \frac{\dot{a}}{a}\Xi - \frac{K}{2}V\right] + \mathcal{B}\hat{\Phi}, \tag{35}$$

$$= \frac{a\sqrt{3}}{r_c\sqrt{E_F}}\left[\dot{\hat{\Phi}} + \frac{\dot{a}}{a}\hat{\Psi} + \frac{E}{4}V\right] + \mathcal{B}\hat{\Phi}, \tag{36}$$

where $\mathcal{A}$ and $\mathcal{B}$ are once again spatial pseudodifferential operators. The appearance of $\dot{a}$ in the *denominator* of U_Θ signifies that this theory can only come from a nonlocal modification to gravity. Of course it is also possible to use either the variable J or V to construct U_Θ but in that case, however, things become worse as we also pick up higher time derivatives in addition to nonlocalities. Notice how the gauge noninvariant part is completely fixed in terms of the background evolution, while only the gauge-invariant parts $\mathcal{A}\hat{\Phi}$ and $\mathcal{B}\hat{\Phi}$ are at this point free.

The variables U_P and U_Σ can be read off the Bianchi identities (24) and (25). Clearly both U_P and U_Σ will have at most second time derivatives, which is consistent with the our assumption. We can proceed even further by assuming the *minimal model* that $\mathcal{A} = \mathcal{B} = 0$, i.e. U_Δ and U_Θ do not contain absolutely gauge-invariant terms. In that case, one finds that U_P and U_Σ are given by

$$U_P = \frac{a\sqrt{3}}{2r_c\sqrt{E_F}}\Big\{E_P + 3\frac{\dot{a}}{a}\Big[1 - \frac{3E_R}{E_F}\Big]\Big[\dot{\hat{\Phi}} + \frac{\dot{a}}{a}\hat{\Psi} + \frac{E}{4}V\Big] + 2(\vec{\nabla}^2 + 3K)\Big[\hat{\Phi} - \hat{\Psi} + \frac{a}{\dot{a}}\Big(\dot{\hat{\Phi}} + \frac{\dot{a}}{a}\hat{\Psi}\Big)\Big]\Big\} \tag{37}$$

and

$$U_\Sigma = -\frac{a^2\sqrt{3}}{2\dot{a}r_c\sqrt{E_F}}\Big[\dot{\hat{\Phi}} + \frac{\dot{a}}{a}\hat{\Psi}\Big]. \tag{38}$$

Note that corrections vanish as $r_c \to \infty$ for both the background *and* the perturbations, hence the name *minimal model*. In other words, there is no additional parameter, apart from the background parameter r_c, appearing in the perturbations.

In Fig. 5 I show the CMB angular power spectrum for this model (dashed curve) contrasted with a ΛCDM model, which has the same angular diameter distance to recombination. The data points are the Wilkinson Microwave Anisotropy (WMAP) 5 yr data [50]. Both models overlap for $\ell > 20$ but on larger scales the DGP-like model reduces the effect of the integrated Sachs-Wolfe effect resulting to lower power than the ΛCDM model. This is in direct contrast with the proper DGP model, which increases the power on large scales, which brings it in conflict with the CMB data [51]. Finally, the growth rate for the DGP-like model is modified from both the proper DGP and standard ΛCDM model, which can be used to constrain such modifications with weak lensing measurements [52].

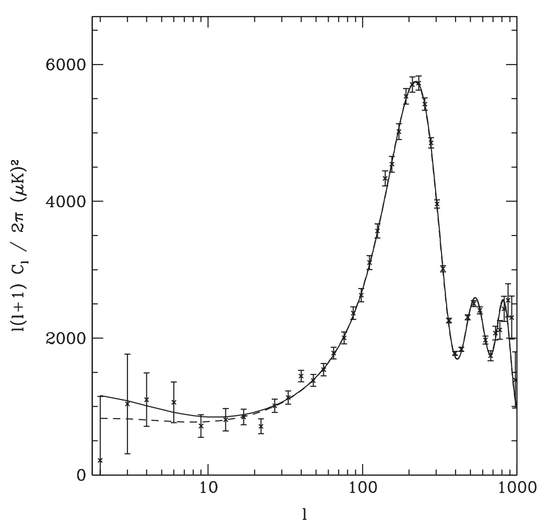

FIG. 5. The CMB spectrum for the minimal DGP-like model. The solid curve is the plain ΛCDM model with $\Omega_\Lambda = 0.721$, while the dashed curve is the DGP-like model with $\Omega_\Lambda = 0$ and $r_c = 8681$ Mpc. Both models have the same distance to recombination, however, the DGP-like model lowers the power on large scales. This is in contrast to the proper DGP model [51]. The data points are from the 5 yr data release of WMAP [50].

2. Unit-timelike vector field modification

A further option is that the modification that leads to the DGP-like background is due to a unit-timelike vector field A_a, such that $g^{ab}A_aA_b = -1$. Such a field is fixed at the background level and does not dynamically contribute to the background equations. A specific example is the generalized Einstein-Æther theory where the Friedmann equation is given as $\mathcal{F}(H^2) = 8\pi G\rho$. By choosing $\mathcal{F}$ appropriately we can recover a DGP-like background.

Since we have now postulated that the modification is due to a unit-timelike vector field, the terms coming from ${U^a}_b$ at the perturbative level would be different than the metric-based example above. The perturbed vector field contains one scalar mode α defined as $A_i = a\vec{\nabla}_i\alpha$ (the A_0 component is perturbatively fixed with respect to the metric due to the unit-timelike condition). Using gauge-form invariance we find that $U_\Delta = \frac{a}{r_c}(\frac{\ddot{a}}{a} - 2\frac{\dot{a}^2}{a^2})\alpha + \mathcal{A}\hat{\Phi}$, which would seem that in this case we pick higher time derivatives. In order to remove the higher time derivatives we must add the term $\mathcal{B}(\dot{\hat{\Phi}} + \frac{\dot{a}}{a}\hat{\Psi})$ for an operator $\mathcal{B}$, which is determined such that the second derivatives are removed. I leave the exploration of unit-timelike vector field modifications for a future work.

IV. CONCLUSIONS

I have presented a scheme that prescribes how consistent modifications of the Einstein equations can be constructed. At the heart of the scheme lies the physical requirement that the linearized field equations of any theory should be gauge-form invariant. This requires the specification of the field content of the theory and thus provides the means for distinguishing modifications of gravity from effects coming from conventional matter fluids. The resulting fluctuation equations can then be solved to obtain observable spectra on the scales of interest for any set of initial conditions. Future work would include more refinements with focus on current and future observational constraints.

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APPENDIX A: PERTURBED FIELD EQUATIONS FOR THE SIMPLIFIED EXTENDED ΛCDM MODEL IN TWO STANDARD GAUGES

1. Synchronous gauge

Letting $\gamma = \frac{\beta H_0^2}{2k^2a + \beta H_0^2}$ we find the Einstein equations in the synchronous gauge as

$$\frac{\dot{a}}{a}\dot{h} = (1-\gamma)8\pi G a^2 \rho\delta + 2k^2\eta - 6\gamma\frac{\dot{a}}{a}\dot{\eta}, \tag{A1}$$

$$2\dot{\eta} = 8\pi G a^2(\rho + P)\theta, \tag{A2}$$

$$\begin{aligned}-\ddot{h} - 2\frac{\dot{a}}{a}\dot{h} + 2k^2\eta = 24\pi G a^2\rho\Pi + \frac{\beta H_0^2}{2ak^2}(\dot{h} + 6\ddot{\eta}) \\ - \frac{\beta H_0^2}{\dot{a}}\dot{\eta} + \frac{\beta H_0^2}{\dot{a}k^2}\left(\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}\right) \\ \times(\dot{h} + 6\dot{\eta}),\end{aligned} \tag{A3}$$

and

$$\begin{aligned}\ddot{h} + 6\ddot{\eta} = a(1-\gamma)\Big\{-2\frac{\dot{a}}{a}(\dot{h} + 6\dot{\eta}) + 2k^2\eta + \frac{\beta H_0^2}{\dot{a}}\dot{\eta} \\ - \frac{\beta H_0^2}{2\dot{a}k^2}\left(\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}\right)(\dot{h} + 6\dot{\eta}) - 16\pi G a^2\rho k^2\Sigma\Big\}.\end{aligned} \tag{A4}$$

2. Conformal Newtonian gauge

In the conformal Newtonian gauge we have $\nu = \zeta = 0$, $\Xi = -\Psi$, and $\chi = -6\Phi$. The field equations become

$$\Phi = -\frac{8\pi G a^2\bar{\rho}}{2k^2 + \frac{\beta H_0^2}{a}}\left[\delta + 3(1+w)\frac{\dot{a}}{a}\theta\right], \tag{A5}$$

$$\dot{\Phi} + \frac{\dot{a}}{a}\Psi = 4\pi G a^2(\rho + P)\theta, \tag{A6}$$

$$\begin{aligned}\ddot{\Phi} + \frac{\dot{a}}{a}(2\dot{\Phi} + \dot{\Psi}) + \left(2\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}\right)\Psi + \frac{1}{3}k^2(\Phi - \Psi) \\ = 4\pi G a^2\rho\Pi - \frac{\beta H_0^2}{6\dot{a}}\dot{\Phi},\end{aligned} \tag{A7}$$

and

$$\begin{aligned}\Phi - \Psi = \frac{\beta H_0^2}{2k^2 a + \beta H_0^2}\Phi + \frac{8\pi G a^2(\bar{\rho} + \bar{P})}{1 + \frac{\beta H_0^2}{2ak^2}} \\ \times\left[\Sigma - \frac{\beta H_0^2}{4\dot{a}k^2}\theta\right].\end{aligned} \tag{A8}$$

APPENDIX B: FIELD EQUATIONS FOR THE DGP-LIKE MODEL

1. Conformal synchronous gauge

In the synchronous gauge we set $\Xi = \zeta = 0$, $\chi = h$ and $-k^2\nu = h + 6\eta$. The perturbed equations become

$$\frac{\dot{a}}{a}\dot{h} - 2k^2\eta = 8\pi G a^2\rho\delta - \frac{3a}{r_c}\dot{\eta}, \tag{B1}$$

$$2\left(1 - \frac{1}{2r_c H}\right)\dot{\eta} = 8\pi G a^2(\rho + P)\theta, \tag{B2}$$

$$\begin{aligned}-\ddot{h} - 2\frac{\dot{a}}{a}\dot{h} + 2k^2\eta = 24\pi G a^2\rho\Pi + \frac{1}{ar_c H^2}\Big[3\frac{\dot{a}}{a}\ddot{\eta} \\ + (E_F - 3E_R - k^2)\dot{\eta} - \frac{1}{4}E\dot{h}\Big],\end{aligned} \tag{B3}$$

and

$$\begin{aligned}\ddot{h} + 6\ddot{\eta} + 2\frac{\dot{a}}{a}(\dot{h} + 6\dot{\eta}) - 2k^2\eta = \\ -16\pi G a^2(\rho + P)k^2\Sigma \\ + \frac{1}{ar_c H^2}\left[\left(k^2 + \frac{3}{2}E\right)\dot{\eta} + \frac{1}{4}E\dot{h}\right],\end{aligned} \tag{B4}$$

where I have set $E = E_F + E_R$ for simplicity.

2. Conformal Newtonian gauge

In the conformal Newtonian gauge we have $\nu = \zeta = 0$, $\Xi = -\Psi$, and $\chi = -6\Phi$. The field equations become

$$-2k^2\Phi = 8\pi G a^2\rho\left[\delta + 3\frac{\dot{a}}{a}(1+w)\theta\right], \tag{B5}$$

$$\dot{\Phi} + \frac{\dot{a}}{a}\Psi = \frac{4\pi G a^2(\rho + P)}{1 - \frac{1}{2r_c H}}\theta, \tag{B6}$$

$$\begin{aligned}\ddot{\Phi} + \frac{\dot{a}}{a}(2\dot{\Phi} + \dot{\Psi}) + \left(2\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}\right)\hat{\Psi} \\ = \frac{1}{1 - \frac{1}{2r_c H}}\left[4\pi G a^2\rho\Pi - \frac{1}{3}k^2(\hat{\Phi} - \hat{\Psi})\right] \\ + \frac{1}{2aH(2r_c H - 1)}\left[\frac{\ddot{a}}{a} - \frac{1}{3}k^2\right]\left[\dot{\Phi} + \frac{\dot{a}}{a}\Psi\right],\end{aligned} \tag{B7}$$

and

$$\Phi - \Psi = 8\pi G a^2(\rho + P)\left[\Sigma - \frac{1}{2aH}\frac{1}{2r_c H - 1}\theta\right]. \tag{B8}$$

APPENDIX C: THE KERNEL OF THE PSEUDO-DIFFERENTIAL OPERATOR $\frac{1}{\vec{\nabla}^2}$

As discussed in the text the pseudodifferential operator $\frac{1}{\vec{\nabla}^2}$ can be defined by its spectral representation, which amounts to using as a symbol the function $-\frac{1}{k^2}$. The kernel of this operator is simply the function $-\frac{1}{4\pi|\vec{x}|}$. This is what we would expect by solving the conventional Laplace equation with the Dirac delta function as a source, i.e. $\vec{\nabla}^2 f(\vec{x}) = \delta^3(\vec{x})$ has solution $f(\vec{x}) = -\frac{1}{4\pi|\vec{x}|}$. Let us see how this is consistent with the spectral representation.

The Fourier transform of $\delta^3(\vec{x})$ is unity. Thus, we have

$$\frac{1}{\vec{\nabla}^2}\delta^3(\vec{x}) = -\frac{1}{(2\pi)^3}\int d^3\vec{k}e^{i\vec{k}\cdot\vec{x}}\frac{1}{k^2} = -\frac{1}{(2\pi)^3}\int dkk^2\int d\hat{k}e^{ikr\hat{k}\cdot\hat{x}}\frac{1}{k^2},$$

where $r = |\vec{x}|$. We expand the exponential using the Rayleigh formula in terms of spherical Bessel functions $j_\ell(x)$ and Legendre polynomials $P_\ell(x)$. The Legendre polynomials are further expanded in spherical harmonics $Y_{\ell m}(\hat{x})$. Setting $K(\vec{x}) = \frac{1}{\vec{\nabla}^2}\delta^3(\vec{x})$ we get

$$\begin{aligned} K(\vec{x}) &= -\frac{1}{(2\pi)^3}\int_0^\infty dk\int d\hat{k}\sum_\ell(2\ell+1)(-i)^\ell j_\ell(kr)P_\ell(\hat{k}\cdot\hat{x}) \\ &= -\frac{1}{2\pi^2}\int_0^\infty dk\int d\hat{k}\sum_\ell\sum_m(-i)^\ell j_\ell(kr)Y^*_{\ell m}(\hat{k})Y_{\ell m}(\hat{x}) \\ &= -\frac{2\sqrt{\pi}}{2\pi^2}\int_0^\infty dk\sum_\ell\sum_m(-i)^\ell j_\ell(kr)\delta_{\ell 0}\delta_{m0}Y_{\ell m}(\hat{x}) \\ &= -\frac{1}{2\pi^2}\int_0^\infty dkj_0(kr) \\ &= -\frac{1}{4\pi|\vec{x}|}\int_{-\infty}^\infty du\frac{\sin(\pi u)}{\pi u} \\ &= -\frac{1}{4\pi|\vec{x}|}. \end{aligned}$$

Thus, $\frac{1}{\vec{\nabla}^2}\delta^3(\vec{x}) = -\frac{1}{4\pi|\vec{x}|}$, which is consistent with $\vec{\nabla}^2\frac{1}{|\vec{x}|} = -4\pi\delta^3(\vec{x})$.

[1] D. Lovelock, Arch. Ration. Mech. Anal. **33**, 54 (1969).
[2] D. R. Grigore, Classical Quantum Gravity **9**, 1555 (1992).
[3] P. Jordan, Nature (London) **164**, 637 (1949).
[4] P. Jordan, Z. Phys. **157**, 112 (1959).
[5] M. Fierz, Helv. Phys. Acta **29**, 128 (1956).
[6] C. Brans and R. H. Dicke, Phys. Rev. **124**, 925 (1961).
[7] J. D. Bekenstein, Phys. Rev. D **70**, 083509 (2004).
[8] J. W. Moffat, J. Cosmol. Astropart. Phys. 03 (2006) 004.
[9] C. Skordis, Phys. Rev. D **77**, 123502 (2008).
[10] E. A. Lim, Phys. Rev. D **71**, 063504 (2005).
[11] T. G. Zlosnik, P. G. Ferreira, and G. D. Starkman, Phys. Rev. D **75**, 044017 (2007).
[12] M. Bañados, Phys. Rev. D **77**, 123534 (2008).
[13] P. D. Mannheim and D. Kazanas, Astrophys. J. **342**, 635 (1989).
[14] W.-T. Ni, Phys. Rev. D **7**, 2880 (1973).
[15] R. H. Sanders, Astrophys. J. **480**, 492 (1997).
[16] M. Visser, Gen. Relativ. Gravit. **30**, 1717 (1998).
[17] G. R. Dvali, G. Gabadadze, and M. Porrati, Phys. Lett. B **485**, 208 (2000).
[18] A. O. Barvinsky, Phys. Lett. B **572**, 109 (2003).
[19] M. E. Soussa and R. P. Woodard, Classical Quantum Gravity **20**, 2737 (2003).
[20] H. Brown and K. Brading, Diálogos (Puerto Rico) 79 (2002), http://philsci-archive.pitt.edu/archive/00000821/00/TorrettiB&B.pdf.
[21] H. Brown *Physical Relativity* (Oxford University Press, Oxford, 2005).
[22] L. Knox, Y.-S. Song, and J. A. Tyson, Phys. Rev. D **74**, 023512 (2006).
[23] M. Ishak, A. Upadhye, and D. N. Spergel, Phys. Rev. D **74**, 043513 (2006).
[24] E. Bertschinger, Astrophys. J. **648**, 797 (2006).
[25] D. Huterer and E. V. Linder, Phys. Rev. D **75**, 023519 (2007).
[26] S. Wang, L. Hui, M. May, and Z. Haiman, Phys. Rev. D **76**, 063503 (2007).
[27] T. Chiba and R. Takahashi, Phys. Rev. D **75**, 101301 (2007).
[28] L. Amendola, M. Kunz, and D. Sapone, J. Cosmol. Astropart. Phys. 04 (2008) 013.
[29] B. Jain and P. Zhang, Phys. Rev. D **78**, 063503 (2008).
[30] R. Caldwell, A. Cooray, and A. Melchiorri, Phys. Rev. D **76**, 023507 (2007).
[31] F. Schmidt, M. Liguori, and S. Dodelson, Phys. Rev. D **76**, 083518 (2007).
[32] P. Zhang, M. Liguori, R. Bean, and S. Dodelson, Phys. Rev. Lett. **99**, 141302 (2007).
[33] S. F. Daniel, R. R. Caldwell, A. Cooray, and A. Melchiorri, Phys. Rev. D **77**, 103513 (2008).
[34] E. Bertschinger and P. Zukin, Phys. Rev. D **78**, 024015 (2008).
[35] W. Hu and I. Sawicki, Phys. Rev. D **76**, 104043 (2007).
[36] W. Hu, Phys. Rev. D **77**, 103524 (2008).
[37] W. Hu, Astrophys. J. **506**, 485 (1998).
[38] M. Banados, P. G. Ferreira, and C. Skordis, Phys. Rev. D **79**, 063511 (2009).
[39] N. Straumann, *Texts and Monographs in Physics* (Springer-Verlag, Berlin, Germany, 2004).
[40] D. Giulini, Lect. Notes Phys. **721**, 105 (2007).
[41] H. Westman and S. Sonego, Found. Phys. **38**, 908 (2008).
[42] H. Westman and S. Sonego, arXiv:0711.2651.
[43] J. M. Stewart and M. Walker, Proc. R. Soc. A **341**, 49 (1974).

[44] M. Kunz and D. Sapone, Phys. Rev. Lett. **98**, 121301 (2007).
[45] J. Iliopoulos, T. N. Tomaras, N. C. Tsamis, and R. P. Woodard, Nucl. Phys. **B534**, 419 (1998).
[46] C. Skordis (unpublished).
[47] M. A. Amin, R. V. Wagoner, and R. D. Blandford, arXiv:0708.1793.
[48] S. Yong-Seon (private communication).
[49] C. Skordis and S. Yong-Seon (unpublished).
[50] M. R. Nolta *et al.* (WMAP Collaboration), Astrophys. J. Suppl. Ser. **180**, 296 (2009).
[51] W. Fang *et al.*, Phys. Rev. D **78**, 103509 (2008).
[52] C. Skordis, S. Thomas, and J. Weller (unpublished)