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Charged seven-dimensional spacetimes with spherically symmetric extra dimensions

Antonio De Felice* and Christophe Ringeval†

Theoretical and Mathematical Physics Group, Centre for Particle Physics and Phenomenology, Louvain University, 2 Chemin du Cyclotron, 1348 Louvain-la-Neuve, Belgium

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We derive exact solutions of the seven-dimensional Einstein-Maxwell equations for a spacetime exhibiting Poincaré invariance along four dimensions and spherical symmetry in the extra dimensions. Such topology generically arises in the context of braneworld models. Our solutions generalize previous results on Ricci-flat spacetimes admitting the two-sphere and are shown to include wormhole configurations. A regular coordinate system suitable to describe the whole spacetime is singled out, and we discuss the physical relevance of the derived solutions.

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I. INTRODUCTION

Extra dimensions are considered to be a key ingredient for explaining the quantum behavior of fundamental theories. Starting from the work of Kaluza and Klein [1–3]), the embedding of our Universe into a higher-dimensional space has been invoked to give an explanation of apparently unrelated four-dimensional phenomena. The unification of forces at first, but also more recently the hierarchy problem, the acceleration of the Universe are different issues which have been addressed by using the features of the extra dimensions.

It is therefore of interest to look how gravity behaves in a higher-dimensional world, that is, to find exact nonperturbative solutions for the classical equations of motion. Many works have been devoted to this task, especially in the framework of superstring theories in which supersymmetry and compactification play a major role [4]. Various black brane configurations have been studied in the literature as exact solutions of superstring low energy effective actions. For this reason, most of them still exhibit some amount of supersymmetry, as well as nontrivial configuration of higher-dimensional form fields [5–10]. In this paper, motivated by cosmology in the presence of extra dimensions, we adopt a (very) low energy effective approach and remain in the framework of standard general relativity. In this context, it is clear that the simplest braneworld model one can think of consists of a four-dimensional Minkowski manifold embedded in a higher-dimensional spacetime (bulk). For a five-dimensional anti–de Sitter bulk, one would recover Randall-Sundrum constructions [11,12], whereas asymptotically flat extra dimensions would be reminiscent with the Dvali-Gabadadze-Porrati (DGP) braneworld models [13,14]. From a classical field theory approach, it has recently been shown in Ref. [15] that the DGP gravity confinement mechanism along a four-dimensional world volume can be realized in the core of a 't Hooft-Polyakov seven-dimensional hypermonopole. In fact, seven dimensions is the minimal number of spacetime dimensions for which the trapping of gravitons by curvature effects may occur, and this is closely related to the existence of a foliation of the extra dimensions by positively curved hypersurfaces [16]. Although the full system of Einstein-Yang-Mills equations have been numerically solved in Ref. [15], under the above-mentioned symmetry, one may wonder if some exact solutions could not be derived. In fact, far from the core of a 't Hooft-Polyakov hypermonopole, that is to say, where the $SO(3)$ Higgs and gauge fields reach their vacuum expectation values, the remaining unbroken gauge symmetry is $U(1)$. As a result, at large distances, the stress-tensor content of the theory is similar to a higher-dimensional Dirac monopole [17]. Motivated by this picture, we derive in this paper exact analytical solutions of the Einstein-Maxwell equations and study the resulting curved spacetime in the presence of an electrical/magnetic field seeded by an $U(1)$ 2-form. In the braneworld framework, we are looking for static Einstein-Maxwell solutions for a compactification of a seven-dimensional spacetime into $M^4 \times R \times S^2$. The solutions studied can therefore be viewed as the spatially extended generalization of the Dirac monopole in four dimensions, where a gauge field, carrying a magnetic charge $1/q$, introduces a nonvacuum structure for the spacetime. Notice that, since Poincaré invariance is not broken along the brane, such a topology differs from higher-dimensional black hole solutions on the brane [18–23]. However the electrically dual action would involve a dual 5-form field which is reminiscent with higher-dimensional generalization of Reissner-Nordström solutions [24]. More specifically, such extra-dimensional topology has been studied for the vacuum case in five dimensions in Refs. [25,26] and recently revisited in Refs. [27–29]. Apart from the different number of dimensions, our approach generalizes these results for nonvacuum spacetimes. The electric dual counterpart of our configuration has been discussed in Refs. [30,31] for even spacetime dimensions only and spherically symmet-

*antonio.defelice@uclouvain.be
†christophe.ringeval@uclouvain.be

ric in all spatial dimensions. As discussed in the following, we will encounter the same kind of problem of apparently ‘‘truncated’’ solutions described in Ref. [29], where the coordinates were not able to explore the whole manifold. This can be solved with a choice of a suitable coordinate system that we describe in the following. Although not using these coordinates is convenient for the study of exterior spacetimes, as the one discussed in Ref. [15], this could lead to misinterpreting the physical content of the theory. In our regular coordinate system, we show, in particular, that some solutions of the Einstein-Maxwell system simply correspond to charged wormhole configurations [32–34]. Such exact solutions, and especially the treatment of the coordinates introduced here, may shed some light into other similar cases.

In Sec. II, we introduce the model. In Sec. III, we rederive the vacuum solutions within our coordinate system and in seven dimensions. We moreover recap the merging of different and apparently unrelated patches. In Sec. IV, we move on to the general solution we are interested in, namely, in the presence of a magnetic field in the extra dimensions. We then conclude in the last section.

II. THE MODEL

The model we consider is defined by the standard Einstein-Maxwell action

$$S = \int \sqrt{-g}\mathrm{d}^7 x\left(\frac{R}{2\kappa^2} - \frac{1}{4}F_{AB}F^{AB}\right), \tag{1}$$

where the field strength 2-form $\boldsymbol{F}$ can be defined in terms of a 1-form $\boldsymbol{A}$, as $\boldsymbol{F} = \mathbf{d}\boldsymbol{A}$, so that the Lagrangian is invariant under a $U(1)$ gauge transformation $\boldsymbol{A} \rightarrow \boldsymbol{A} + \mathbf{d}\chi$. Capital Latin indices run from 0 to 6, whereas Greek indices from 0 to 3. The dimensions of the gauge field are $[C^{aM}] = M^{5/2}$, and those of the seven-dimensional Newton constant are $[\kappa^2] = M^{-5}$, for an electric charge $[q] = M^{-3/2}$. The stress-energy tensor is therefore

$$T^A{}_B = F^{AN}F_{BN} - \tfrac{1}{4}F^2\delta^A{}_B. \tag{2}$$

Finally, the Einstein equations are

$$G^A{}_B = \kappa^2 T^A{}_B, \tag{3}$$

while the Maxwell equations for the 2-form read as $\mathbf{d}\boldsymbol{F} = 0$, or

$$F_{AB,C} + F_{BC,A} + F_{CA,B} = 0, \tag{4}$$

and $\nabla_B F^{AB} = 0$.

A. Background metric

We are looking for background solutions with spherical symmetry in the extra dimensions, i.e., a compactification into $M^4 \times R \times S^2$. Therefore, we impose the following ansatz for the metric:

$$\mathrm{d}s^2 = A(r)(-\mathrm{d}t^2 + \mathrm{d}\vec{x}^2) + \xi(r)\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2, \tag{5}$$

with

$$\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2. \tag{6}$$

Furthermore, the 2-form field is assumed to be of the form

$$\boldsymbol{F} = f(r, \theta, \phi)\mathbf{d}\theta \wedge \mathbf{d}\phi. \tag{7}$$

The Maxwell equations $\mathbf{d}\boldsymbol{F} = 0$ immediately imply that $\partial f/\partial r = 0$, or $f = f(\theta, \phi)$. Then we have

$$\nabla_A F_\theta{}^A = \frac{1}{r^2\sin^2\theta}\frac{\partial f}{\partial\phi} = 0, \tag{8}$$

$$\nabla_A F^A_\phi = -\frac{1}{r^2\sin\theta}\left[\frac{\partial f}{\partial\theta}\sin\theta - f\cos\theta\right] = 0, \tag{9}$$

which have the nontrivial solution

$$f = -\frac{\sin\theta}{q}, \tag{10}$$

where the integration constant q is the electric charge.

This ansatz automatically solves the Maxwell equations for the given metric, independently of the profiles for the fields $A(r)$ and $\xi(r)$. In the three extra dimensions, it is now possible to define a three-vector, with components B^i, which is the magnetic field associated to the 2-form. With

$$B^i = -\varepsilon^{ijk}F_{jk}/\sqrt{{}^{(3)}g}, \tag{11}$$

where $i, j, k \in \{4, 5, 6\}$, one gets

$$B_r = \frac{1}{qr^2}. \tag{12}$$

Therefore $1/q$ appears as a magnetic charge, the magnetic field is purely radial, and we are in the presence of a spatially extended Dirac magnetic monopole at the origin of the coordinate system. The associated stress tensor is nonvanishing and reads

$$T^0{}_0 = T^x{}_x = T^r{}_r = -T^\theta{}_\theta = -T^\phi{}_\phi = -\frac{1}{2q^2r^4}. \tag{13}$$

In fact, this ansatz for the 2-form is the only one consistent with the choice of the metric ansatz, as the other components for F_{AB} have no solutions for the Einstein equations.

B. Equations of motion

From the above ansatz, the Einstein-Maxwell equations read

$$\frac{3}{2}\frac{A'^2}{A^2\xi} + \frac{4A'}{A\xi r} - \frac{1}{r^2} + \frac{1}{\xi r^2} + \frac{\kappa^2}{2q^2r^4} = 0, \tag{14}$$

$$\frac{2A'}{A\xi r} - \frac{1}{2}\frac{\xi'}{\xi^2 r} + \frac{1}{2}\frac{A'^2}{A^2\xi} + \frac{2A''}{A\xi} - \frac{A'\xi'}{\xi^2 A} - \frac{\kappa^2}{2q^2r^4} = 0, \tag{15}$$

$$\frac{3}{2}\frac{A''}{A\xi} - \frac{3}{4}\frac{A'\xi'}{\xi^2 A} + \frac{3A'}{A\xi r} - \frac{\xi'}{\xi^2 r} - \frac{1}{r^2} + \frac{1}{\xi r^2} + \frac{\kappa^2}{2q^2r^4} = 0, \tag{16}$$

where the prime denotes the derivative with respect to r. From Eq. (15) it is possible to solve for A''. One can then eliminate A'' from Eq. (16) and solve for A'^2 in terms of A', ξ', A, and ξ. Then, using this, one can eliminate A'^2 from Eq. (14). Finally, one gets

$$\frac{A'}{A} = \frac{1}{4}\frac{\xi'}{\xi} - \frac{2}{5}\frac{\xi\kappa^2}{r^3q^2} + \frac{1}{2}\frac{\xi - 1}{r}. \tag{17}$$

One can use this equation for A, together with Eq. (14), to obtain

$$\xi' = -\frac{2\xi^2}{r} - \frac{10}{3}\frac{\xi}{r} + \frac{16}{5}\frac{\xi^2 {r_0}^2}{r^3} \pm \frac{4}{3} \times \frac{\xi\sqrt{(6\xi + 10)r^2 - 6{r_0}^2\xi}}{r^2}, \tag{18}$$

where r_0 stands for

$$r_0 \equiv \frac{\kappa}{\sqrt{2}q}. \tag{19}$$

The vacuum case is obtained in the limit $r_0 \rightarrow 0$. If $r_0 \neq 0$, we can define the dimensionless variable $\rho = r/r_0$, and the previous equation becomes

$$\xi' = -\frac{2\xi^2}{\rho} - \frac{10}{3}\frac{\xi}{\rho} + \frac{16}{5}\frac{\xi^2}{\rho^3} \pm \frac{4}{3}\frac{\xi\sqrt{(6\xi + 10)\rho^2 - 6\xi}}{\rho^2}, \tag{20}$$

where now a prime denotes differentiation with respect to ρ, and one can see that there are two branches. In order to study the presence of singularities, it is convenient to study both the Ricci curvature R and the Kretschmann scalar defined as

$$K = \tfrac{1}{4}R_{ABCD}R^{ABCD}. \tag{21}$$

This second scalar invariant will be especially useful for the vacuum case, when R vanishes on the solutions of the equations of motion. Their expression for the metric (5) are given in the appendix.

One can use Eq. (17) in order to express R and K only in terms of ξ and its derivatives. Moreover, for the solutions of the Einstein equations one finds that $R \propto T$, where T is the trace of the stress tensor. As a result, for the nonvacuum case, one has on-shell

$$r_0^2 R = \frac{6}{5\rho^4}, \tag{22}$$

which states that only for $\rho = 0$ the Ricci scalar blows up. Before facing the problem of solving Eq. (20), it is worth illustrating the method and the choice of a suitable coordinate system for the vacuum solutions. This will allow us to make contact with previous works, albeit in a different coordinate system [25–29].

III. VACUUM SOLUTIONS

The vacuum equations are obtained in the limit $q \rightarrow \infty$ (or $r_0 \rightarrow 0$). Defining in this section $\rho \equiv r$, Eq. (17) simplifies to

$$\frac{A'}{A} = \frac{\xi'\rho + 2\xi^2 - 2\xi}{4\rho\xi}, \tag{23}$$

which can be integrated into

$$A(\rho) = \exp\left(\int^{\rho} \frac{\xi' r + 2\xi^2 - 2\xi}{4r\xi} \mathrm{d}r\right). \tag{24}$$

This expression can be further simplified into

$$\frac{A\sqrt{\rho}}{\xi^{1/4}} = \exp\left(\frac{1}{2}\int^{\rho} \frac{\xi(r)}{r}\mathrm{d}r\right). \tag{25}$$

Finally, as for ξ, Eq. (18) reads

$$\rho\frac{\xi'}{\xi} = -\frac{10}{3} - 2\xi \pm \frac{4}{3}\sqrt{10 + 6\xi}. \tag{26}$$

The plus and minus signs indicate that there are two branches for the $\xi(\rho)$ solutions. This equation can be integrated by a separation of variables. The above expression requires $\xi \geq -5/3$, and the limiting case $\xi = -5/3$ is a particular solution of Eq. (26), for which $A = A_0\rho^{-4/3}$. This solution corresponding to a timelike ρ coordinate will not be considered as a physical one. Notice that switching the sign of A_0 does not cure the problem, as the three coordinates x, y, and z would all become timelike.

A. First branch

After separating variables in Eq. (26) and choosing the plus sign for the square root, one gets

$$\int^{\xi} \frac{\mathrm{d}\tilde{\xi}}{\tilde{\xi}(-\frac{10}{3} - 2\tilde{\xi} + \frac{4}{3}\sqrt{10 + 6\tilde{\xi}})} = \ln\rho, \tag{27}$$

whose solution reads

$$\frac{|\sqrt{3\xi + 5} - \sqrt{5}|^{2\sqrt{2/5}}}{|\xi|^{\sqrt{2/5}-1/2}|\sqrt{3\xi + 5} - 2\sqrt{2}|} = \frac{\rho}{\rho_*}, \tag{28}$$

where $\rho_* > 0$ is an integration constant. The solution is defined only on three intervals: $-5/3 < \xi < 0$, $0 < \xi < 1$, and $\xi > 1$. Among them, the first one again corresponds to a timelike spatial coordinates and will not be considered as a physical one.

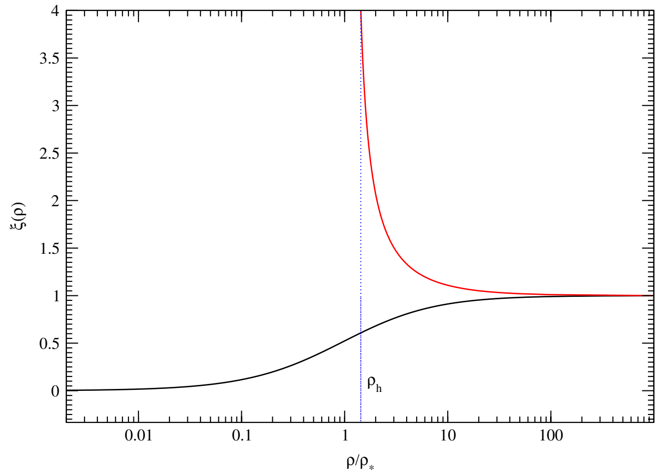

FIG. 1 (color online). Metric coefficient $\xi(\rho)$ for the first branch "+" of the vacuum solution. There are two solutions: one exhibiting a null surface at $\rho_h = \rho/\rho_* = 3^{\sqrt{2/5}-1/2}$ and the other a naked singularity at $\rho_s = 0$. In this coordinate system, the solution exhibiting the null surface is incomplete.

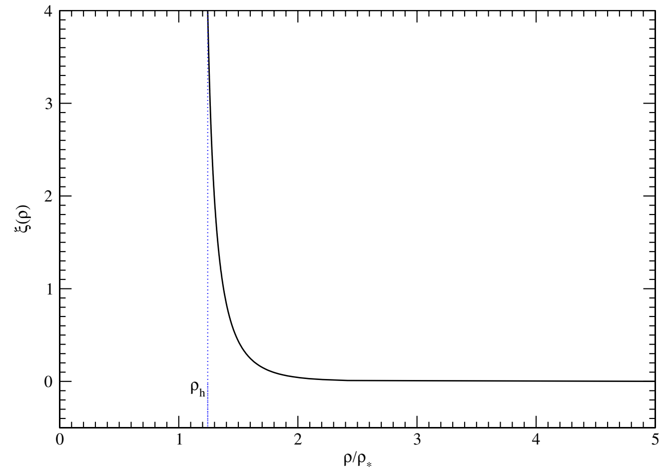

FIG. 2 (color online). Metric coefficient $\xi(\rho)$ for the "−" branch of the vacuum solution. The solution has a null surface at $\rho/\rho_* = 3^{\sqrt{2/5}-1/2}$ and a naked singularity as $\rho \to \infty$. In this coordinate system, the solution is incomplete.

As can be seen in Eq. (28), $\xi(\rho) \to \infty$ for

$$\rho = \rho_h = 3^{\sqrt{2/5}-1/2}\rho_*. \tag{29}$$

Since the Kretschmann scalar given in Eq. (A2) remains finite at that point, we are in the presence of a coordinate singularity only. In fact, as can be checked from Eq. (5), since $\xi(\rho)$ is divergent at that point, the hypersurface $\rho = \rho_h$ is actually a null surface. Notice however that the redshift counterpart $A(\rho_h)$ remains finite. As we discuss in Sec. III D, this null surface is not a horizon. On the other hand, there is a singularity, i.e., $K \to \infty$, at the point where $\xi \to 0^+$:

$$\rho = \rho_s = \rho_* \lim_{\xi\to 0^+} \frac{(\sqrt{3\xi+5}-\sqrt{5})^{2\sqrt{2/5}}}{\xi^{\sqrt{2/5}-1/2}(2\sqrt{2}-\sqrt{3\xi+5})} = 0, \tag{30}$$

which is the origin of the coordinate system. Finally, for noncompact spacetime, the coordinate ρ should reach $\rho \to +\infty$. This is possible if $\xi \to 1^+$ or if $\xi \to 1^-$. As a result, the spacetime is necessarily asymptotically Minkowski. In Fig. 1, we have plotted the metric coefficient $\xi(\rho)$ given by Eq. (28). Notice that we find two disjoint solutions; even so, we are here considering only the "+" branch. As discussed in Ref. [29], it is not clear what happens for $\rho < \rho_h$ since the ρ coordinates are not able to describe the whole spacetime, leaving an apparently empty spot in the manifold (see Fig. 1).

B. Second branch

Similarly to the previous discussion, but choosing now the minus sign for the square root in Eq. (20), one gets, with $\xi > 0$,

$$\frac{(\sqrt{3\xi+5}+\sqrt{5})^{2\sqrt{2/5}}}{\xi^{\sqrt{2/5}-1/2}(\sqrt{3\xi+5}+2\sqrt{2})} = \frac{\rho}{\rho_*}. \tag{31}$$

The position of the null surface is again given by the point at which $\xi \to +\infty$, i.e., for

$$\rho_h = 3^{\sqrt{2/5}-1/2}\rho_*. \tag{32}$$

Asymptotically, we want $\rho \to \infty$ which is obtained for the vanishing denominator in Eq. (31), i.e., for $\xi \to 0^+$. Contrary to the "+" branch, there is therefore only one solution for the "−" branch. Notice that the point for which $\xi \to 0^+$ is also a singularity since $K \to \infty$. As a result, this solution has a null surface at a finite value of ρ and a singularity for $\rho \to \infty$. The corresponding solution has been plotted in Fig. 2. Once more, the situation for $\rho < \rho_h$ is unclear. As we shall see in the next section, this issue is only due to a bad choice of the coordinate system.

C. Filling the gaps

The variable ρ does not seem to clarify the situation for the whole space of solutions. Some solutions look truncated at a finite distance, and others seem to describe different behaviors for the same ρ. It is possible that the different patches may be joined through a different and more suitable choice of coordinates. Mostly the behavior of ξ is quite unclear for $\rho < \rho_h$. Pursuing the idea that all of this situation is due to a bad choice of coordinates, we introduce a new coordinate system joining the two branches in the next sections.

1. First branch

Let us introduce the new coordinate $v(\rho)$ such that

$$\xi(\rho) = \tfrac{5}{3}(u^2 - 1), \tag{33}$$

where

$$u = \frac{5\gamma_0 - 2\sqrt{10}\rho v}{5\gamma_0 - 5\rho v} \quad \text{and} \quad \gamma_0 \equiv \frac{13 + 4\sqrt{10}}{3}. \tag{34}$$

In terms of v, the equation of motion (20) becomes

$$\frac{\mathrm{d}v}{\mathrm{d}\rho} = \frac{v}{\rho[v(\rho)\rho - \gamma_0]}. \tag{35}$$

This equation can be inverted to obtain $\rho(v)$:

$$\frac{\mathrm{d}\rho}{\mathrm{d}v} = \rho^2 - \gamma_0 \frac{\rho}{v}. \tag{36}$$

This is a Riccati equation, which can be linearized with another change of variable. Introducing $s(v)$ such that

$$\rho(v) = -\frac{1}{s}\frac{\mathrm{d}s}{\mathrm{d}v}, \tag{37}$$

one can rewrite Eq. (36) as

$$\frac{\mathrm{d}^2 s}{\mathrm{d}v^2} + \frac{\gamma_0}{v}\frac{\mathrm{d}s}{\mathrm{d}v} = 0. \tag{38}$$

This equation has the symmetry $v \to -v$, and the general solution is

$$s = c_1 |v|^{1-\gamma_0} + c_2. \tag{39}$$

Therefore for every solution $s(v)$ with $v > 0$ there is another solution $s(-v) = s(v)$. Without loss of generality, we can study only the region $v > 0$. It is interesting to notice that under this symmetry $\rho(-v) = -\rho(v)$ and $\xi(-v) = \xi(v)$. Finally, the general solution for ρ reads

$$\rho_1(v) = \frac{\gamma_0 - 1}{v(1 - C|v|^{\gamma_0 - 1})}, \tag{40}$$

where C is a dimensionful integration constant. It is clear that ρ_1 does not seem to be necessarily positive for all value of v and C.

2. Second branch

As for the first branch, the new coordinate $v(\rho)$ is defined by

$$\xi(\rho) = \tfrac{5}{3}(u^2 - 1), \tag{41}$$

where

$$u = -\frac{5\gamma_0 + 2\sqrt{10}\rho v}{5\gamma_0 + 5\rho v}, \tag{42}$$

and the differential equation for v becomes

$$\frac{\mathrm{d}v}{\mathrm{d}\rho} + \frac{v}{\rho[v(\rho)\rho - \gamma_0]} = 0 \tag{43}$$

and is inverted into

$$\frac{\mathrm{d}\rho}{\mathrm{d}v} = -\rho^2 - \gamma_0 \frac{\rho}{v}. \tag{44}$$

Denoting by $\rho_2(v)$ the solution of the above equation and comparing Eqs. (36) and (44), one finds $\rho_2(v) = -\rho_1(v)$. Therefore, the solution reads

$$\rho_2(v) = -\frac{\gamma_0 - 1}{v(1 - C|v|^{\gamma_0 - 1})}. \tag{45}$$

3. Joining the branches

For any value of C and v, either $\rho_1(v)$ or $\rho_2(v)$ will be positive, as $\rho_2 = -\rho_1$. Hence, for all $v \neq 0$, we can set

$$\rho(v) \equiv |\rho_1(v)| = |\rho_2(v)|. \tag{46}$$

Similarly, for the two branches, the metric factor verifies $\xi_1(v) = \xi_2(v)$ for all $v \neq 0$, and we define

$$\xi(v) \equiv \xi_1(v) = \xi_2(v). \tag{47}$$

In terms of the v coordinate, the two branches are therefore unified, and, assuming from now that $v > 0$, we have

$$\rho(v) = \frac{\gamma_0 - 1}{|v - Cv^{\gamma_0}|}, \qquad \xi(v) = \frac{20C\gamma_0 v^{\gamma_0 + 1}}{3[C\gamma_0 v^{\gamma_0} - v]^2}. \tag{48}$$

4. Singularities

For all $C \neq 0$ these solutions give

$$\lim_{v \to +\infty} \rho = 0, \qquad \lim_{v \to +\infty} \xi = 0, \tag{49}$$

and, as can be checked with the Kretschmann scalar, there is a singularity for $v \to \infty$ ($\rho \to 0$). For vanishing v, we recover the asymptotic singularity

$$\lim_{v \to 0} \rho = +\infty, \qquad \lim_{v \to 0} \xi = 0. \tag{50}$$

For intermediate values of v, we recover the null surface at the values of v such that $\xi \to \infty$. As can be checked in Eq. (48), the existence of coordinate singularities in the intermediate region depends on the values of the integration constant C. Denoting by v_{h} the location of this surface, Eq. (48) yields

$$v_{\mathrm{h}} = \frac{1}{(\gamma_0 C)^{1/(\gamma_0 - 1)}}, \tag{51}$$

provided $C > 0$, and no solution if $C < 0$.

We can also separate the different regions of spacetime by looking at the behavior of $\rho(v)$ for finite values of v. Denoting by v_∞ the point at with $\rho \to \infty$, from Eq. (48), one gets

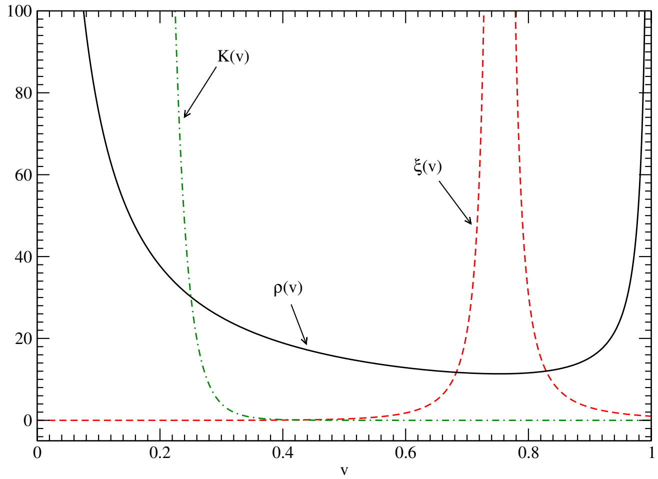

FIG. 3 (color online). Wormhole patch: A singularity is present at $v \to 0$, together with a null surface at $\rho = \rho_h$ and an asymptotically flat region for $v \to v_\infty = 1$. The behavior of $\rho(v)$ is multivalued and describes a wormholelike configuration. Notice that the divergence of ξ is simply the results of the ill-defined ρ-coordinate system. The null surface is not a horizon and corresponds to the wormhole throat. Although the wormhole has no horizon, it exhibits a singularity on the brane side and at a finite proper distance from the throat.

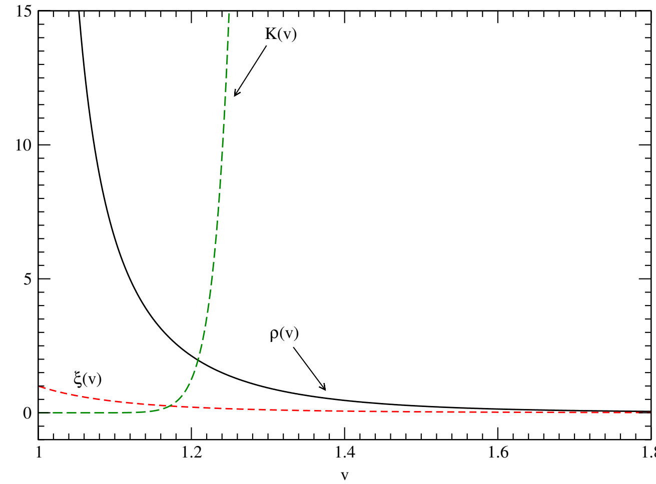

FIG. 4 (color online). Naked singularity patch: An asymptotically flat region is reached as $v \to v_\infty = 1$ but there is a naked singularity as $\xi \to 0$.

$$v_\infty = \frac{1}{C^{1/(\gamma_0 - 1)}}. \tag{52}$$

Also in this case, there is a solution only for $C > 0$. Moreover, for any nonvanishing values of v_∞ one has $\xi \to 1$, and the spacetime is asymptotically flat.

From Eqs. (51) and (52), it is clear that we always have $v_h < v_\infty$. As a result, for positive C, there are two different patches for the spacetime

(i) $0 < v < v_\infty$.—A singularity is present for $v \to 0$, together with a null surface at v_h, and an asymptotically flat region for $v \to v_\infty$. We refer to this patch as the wormhole solution (see Fig. 3). Notice the multivalued and unbounded behavior of $\rho(v)$ which is reminiscent with a wormhole solution. As we show in the next section, the divergence of $\xi(\rho)$ comes from the bad choice of the coordinate system: The null surface is not a horizon and traces instead a wormhole throat. This wormhole, however, connects the asymptotically flat region to a singularity, which is at a finite proper distance from the throat.

(ii) $v > v_\infty$.—The asymptotically flat region occurs for $v \to v_\infty$, and there is a singularity for $v \to \infty$ (see Fig. 4). There is no horizon nor a wormhole configuration in that patch, and the singularity is naked.

For completeness, let us consider the case $C < 0$. From Eq. (48), two singularities are now present for $v \to 0$ and $v \to \infty$, without any horizon and with $\xi < 0$. This solution exhibits two naked singularities and two timelike coordinates. Therefore it is pathological and will not be discussed any longer in the following.

D. Wormhole solution

All solutions discussed in the previous section exhibit singularities, and we discuss in this section the "wormhole patch." As can be seen in Fig. 3, it is a nontrivial merging of the first and second branches of the ρ-coordinate system.

1. Nature of the null surface

Clearly, the v-coordinate system is nonsingular for this solution (except at most the physical singularity where $\xi \to 0$), and the metric becomes

$$\mathrm{d}s^2 = A[\rho(v)](-\mathrm{d}t^2 + \mathrm{d}\vec{x}^2) + \xi(v)\left(\frac{\mathrm{d}\rho}{\mathrm{d}v}\right)^2 \mathrm{d}v^2 + [\rho(v)]^2 \mathrm{d}\Omega^2. \tag{53}$$

From Eq. (48), one can express the metric coefficient g_{vv} as

$$g_{vv} = \xi(v)\left(\frac{\mathrm{d}\rho}{\mathrm{d}v}\right)^2 = \frac{20\gamma_0(\gamma_0 - 1)^2 C v^{\gamma_0 - 1}}{3(Cv^{\gamma_0} - v)^4}. \tag{54}$$

This quantity does not blow up at v_h, namely, for the points at which $\xi \to \infty$, so that in these variables the coordinate singularities are no longer present. In order to determine the behavior of $A(v)$, we still need to integrate Eq. (25) in terms of v and, in particular, the integral

$$I_1 \equiv \frac{1}{2}\int \frac{\xi(v)}{\rho(v)}\frac{\mathrm{d}\rho}{\mathrm{d}v}\mathrm{d}v. \tag{55}$$

Explicitly, I_1 reads

$$I_1 = -\frac{10}{3C}\int \frac{v^{\gamma_0 - 2}\mathrm{d}v}{[v^{\gamma_0 - 1} - C^{-1}][v^{\gamma_0 - 1} - (C\gamma_0)^{-1}]}, \tag{56}$$

which can be integrated exactly into

$$I_1 = -\frac{1}{2}\ln\left|\frac{Cv^{\gamma_0} - v}{C\gamma_0 v^{\gamma_0} - v}\right| + \text{const.} \tag{57}$$

Therefore $A(v)$ simplifies to

$$A = A_0|C|^{1/4}v^{(\gamma_0+1)/4}, \tag{58}$$

where $A_0^2 = |C|^{1/(\gamma_0-1)}$ is a constant of integration fixed according to the clock of an asymptotically flat observer. The "redshift function" $A(v)$ is therefore regular at the $v = v_h$ location, and, at this point, it does not vanish.

In this coordinate system, one can immediately check that the constant radial hypersurfaces still correspond to constant v hypersurfaces. From Eq. (53), it is then clear that the lightlike geodesics can intercept and cross the null hypersurface located along $\rho = \rho_h$. This hypersurface is therefore not a horizon.

The nature of the coordinate singularity at $\rho = \rho_h$ can be further clarified by studying the convergence θ_r of a congruence of radial null geodesics. In fact, although the v coordinates cure the coordinate singularity at the null hypersurface location, it is not well defined asymptotically, and one may question the existence of an asymptotically flat observer at infinity. Therefore, let us define a new radial variable as

$$\zeta = \int_0^v \sqrt{\frac{g_{vv}(v')}{A(v')}}\mathrm{d}v', \tag{59}$$

so that the singularity is located at $\zeta = 0$. Then it is straightforward to find a null coordinate system which is regular everywhere and asymptotically flat by defining

$$u = t - \zeta, \qquad w = t + \zeta. \tag{60}$$

In this coordinate system, one gets

$$\mathrm{d}s^2 = -A[v(u,w)]\mathrm{d}u\mathrm{d}w + A\mathrm{d}\vec{x}^2 + \rho[v(u,w)]^2\mathrm{d}\Omega^2, \tag{61}$$

as $v = v(\zeta) = v[\zeta(u,w)]$. For the outgoing radial light geodesic $u = \text{const}$, we can define then $\theta_r = \nabla^A k_A$, where $k_A = -\partial_A u$ is the normal to the surface. We find that θ_r can be written as

$$\theta_r = \frac{1}{4\sqrt{Ag_{vv}}}\left[\frac{3}{A}\frac{\mathrm{d}A}{\mathrm{d}v} + \frac{4}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}v}\right], \tag{62}$$

so that, giving the expressions of $A(v)$ and $\rho(v)$, one can check that, in $v_s < v < v_\infty$, one has

$$\theta_r > 0. \tag{63}$$

In particular, this implies that there are no trapped surfaces for $v \le v_h$, so that the surface v_h is not a horizon [35].

In fact, as the $\rho(v)$ behavior suggests, such a spacetime is of the wormhole kind, the throat of which is precisely located at $v = v_h$ [36,37]. This wormhole does not possess any horizon and may seem to be traversable. Usually, traversable wormholes require exotic matter to connect two asymptotically flat regions of spacetime in four dimensions [36]. Here we are only in vacuum, and our wormhole does not connect two asymptotically flat regions: A singularity is still present on one side.

In order to understand the physical meaning of C, it is instructive to study the asymptotic limit. In fact, $\rho \to \infty$ at $v \to v_\infty$ ($C > 0$) with

$$\rho(v) \sim \frac{1}{v_\infty - v}, \qquad \xi(v) \sim 1 + (\gamma_0 + 1)\left(1 - \frac{v}{v_\infty}\right), \tag{64}$$

or, in terms of ρ,

$$\xi(\rho) \sim 1 + \frac{(\gamma_0+1)C^{1/(\gamma_0-1)}}{\rho} + \mathcal{O}\left(\frac{1}{\rho^2}\right). \tag{65}$$

Similarly, expanding Eq. (58) in terms of ρ, one gets

$$A(\rho) \sim 1 - \frac{(\gamma_0+1)C^{1/(\gamma_0-1)}}{4\rho} + \mathcal{O}\left(\frac{1}{\rho^2}\right). \tag{66}$$

Therefore, for an observer at rest at infinity, C is simply encoding the tension T of a three-brane associated with the Poincaré invariance

$$C = \left(\frac{\gamma_0+1}{4G_7T}\right)^{1-\gamma_0}. \tag{67}$$

Notice that C (or T) also fixes the size ρ_h of the wormhole throat

$$C = \frac{1}{\gamma_0^{\gamma_0}\rho_h^{1-\gamma_0}}. \tag{68}$$

2. Embedding function

At a fixed four-dimensional location $(t, \vec{x})$, the spatial metric reads

$$\mathrm{d}\ell^2 = \xi(\rho)\mathrm{d}\rho^2 + \rho^2\mathrm{d}\Omega^2. \tag{69}$$

Following Ref. [36], this hypersurface can be embedded in a four-dimensional Euclidian space of metric

$$\mathrm{d}s_e^2 = \mathrm{d}\omega^2 + \mathrm{d}\rho^2 + \rho^2\mathrm{d}\Omega^2. \tag{70}$$

This hypersurface has spherically symmetric sections and is described by the function $\omega(\rho)$ such that

$$\left(\frac{\mathrm{d}\omega}{\mathrm{d}\rho}\right)^2 = \xi(\rho) - 1. \tag{71}$$

The embedding function is no longer defined for $\xi < 1$, the hypersurface being no longer embeddable into an Euclidian space (but instead it is inside a Minkowski space). In Fig. 5, we have represented $\omega(\rho)$ in the domains for which $\xi \ge 1$. In the asymptotically flat region, from Eq. (65), one gets

$$\omega(\rho) \propto \sqrt{\rho}. \tag{72}$$

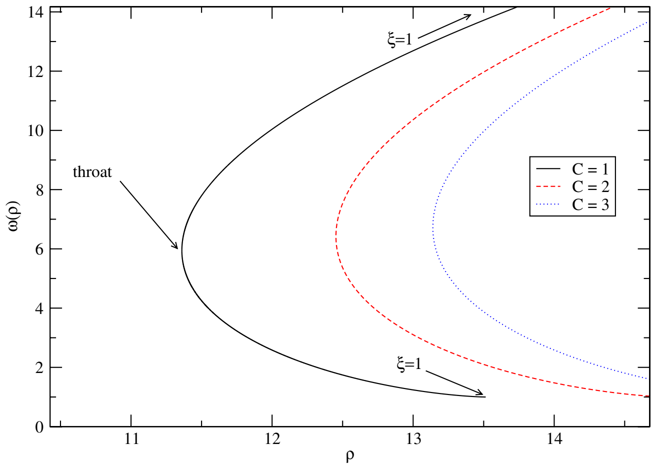

FIG. 5 (color online). Embedding function $\omega(\rho)$ of the wormhole configuration for three values of C. The plot has been truncated to the ρ values for which $\xi \geq 1$ and does not show the singularity.

3. Radial motion

It is instructive to study the radial geodesics around the wormhole solution (see Fig. 3). Assuming without loss of generality that $C = 1$, the spacetime is asymptotically flat for $v \to v_\infty = 1$. In this case, if t is the proper time of the asymptotic observer, one has $A \to 1$, and thus $A_0 = 1$ and

$$A(v) = v^{(\gamma_0+1)/4}. \tag{73}$$

The metric for a purely radial motion simplifies into

$$\mathrm{d}s^2 = -v^{(\gamma_0+1)/4}\mathrm{d}t^2 + \frac{20\gamma_0(\gamma_0-1)^2 v^{\gamma_0-5}}{3(v^{\gamma_0-1}-1)^4}\mathrm{d}v^2. \tag{74}$$

For a timelike geodesics, denoting by τ the proper time, a constant of motion is

$$A\frac{\mathrm{d}t}{\mathrm{d}\tau} \equiv \sqrt{A_*}. \tag{75}$$

The meaning of A_* is clear in the asymptotic region: $\mathrm{d}t/\mathrm{d}\tau$ corresponds to the energy per unit mass, so that

$$A_* = \frac{1}{1-V_\infty^2} \geq 1, \tag{76}$$

where V_∞ stands for the radial velocity of the particle at infinity. Conservation of energy along the geodesics now implies

$$\frac{1}{2}\left(\frac{\mathrm{d}v}{\mathrm{d}\tau}\right)^2 + U_{\mathrm{eff}}(v) = 0, \tag{77}$$

where

$$U_{\mathrm{eff}} = \frac{1}{2g_{vv}} - \frac{A_*}{2g_{vv}A} = -\frac{3(1-v^{\gamma_0-1})^4(A_* - v^{1/4+\gamma_0/4})}{40(\gamma_0-1)^2\gamma_0 v^{5\gamma_0/4-19/4}} \tag{78}$$

is the effective potential. Since $A_* \geq 1$, then $U_{\mathrm{eff}} \leq 0$ and $\lim_{v\to 0^+} U_{\mathrm{eff}} = -\infty$. In this case any radial geodesics will fall into the singularity. Although the coordinate ρ goes to infinity as v vanishes, the proper distance d_{sh} between the singularity and the wormhole throat is actually finite. From Eq. (54), one indeed obtains

$$d_{\mathrm{sh}} = \int_0^{v_\infty} \sqrt{g_{vv}}\mathrm{d}v \simeq 9.988. \tag{79}$$

On the other hand, one can check that the asymptotically flat region is at an infinite proper distance from the throat.

The proper time of a free-falling particle along a radial geodesics is given by

$$\Delta\tau = -\int_{v_i}^0 \mathrm{d}v\sqrt{\frac{g_{vv}A}{A_* - A}}. \tag{80}$$

Since $A_* \geq 1$, the geodesics is well defined for all $0 < v < 1$. Similarly, the asymptotic time t reads

$$\Delta t = -\int_{v_i}^0 \mathrm{d}v\sqrt{\frac{g_{vv}A_*}{A(A_* - A)}}. \tag{81}$$

As an example, considering $A_* = 2$ and $v_i = 0.9$, one finds that the singularity $v = 0$ is reached in the finite proper time $\Delta\tau \simeq 12.1$ but also for a finite $\Delta t \simeq 43$. For any $v_i < 1$, it takes a finite amount of proper time $\Delta\tau$ and of Δt (the time measured from an observer in the flat region) to hit the singularity.

Having clarified the coordinate system artifacts for the vacuum case, we derive in the next section exact solutions for the generic charged case. In fact, the situation is qualitatively similar, and one has to introduce a new regular coordinate system to fully describe the manifold.

IV. CHARGED SOLUTIONS

In this section, we derive the charged solutions generated by the monopole. Instead of discussing the solutions in terms of the radial coordinate ρ, we directly introduce a generalized version of our new radial coordinate v, in a way similar to the vacuum case.

A. First branch

As in Sec. III, this branch refers to choosing a plus sign for the square root in Eq. (20), and we define v such that

$$\xi(\rho) = \frac{5\rho^2}{3(\rho^2 - 1)}\left(\left\{\frac{[4\rho^2 + (4+\sqrt{10})\rho + \sqrt{10}]v - (8+2\sqrt{10})\rho}{[\sqrt{10}\rho^2 + (4+\sqrt{10})\rho + 4]v - (8+2\sqrt{10})\rho}\right\}^2 - 1\right). \tag{82}$$

The coordinate v is now obtained in terms of ρ by using Eq. (20), and one finds an Abel differential equation of the second kind, namely,

$$\frac{\mathrm{d}v}{\mathrm{d}\rho} = \frac{2(4-\sqrt{10})(v-1)v}{2(4+\sqrt{10})\rho - (\rho+1)(\sqrt{10}\rho+4)v}. \tag{83}$$

As for the vacuum case, we can invert the previous relation such that now ρ is a function of v, and one gets

$$\frac{\mathrm{d}\rho}{\mathrm{d}v} = -\frac{\sqrt{10}\rho^2}{2(4-\sqrt{10})(v-1)} - \frac{(4+\sqrt{10})(v-2)\rho}{2(4-\sqrt{10})(v-1)v} - \frac{2}{(4-\sqrt{10})(v-1)}. \tag{84}$$

This equation is again a Riccati differential equation which can be linearized by defining $s(v)$ such that

$$\rho(v) = \left(\frac{4}{5}\sqrt{10} - 2\right)\frac{v-1}{s}\frac{\mathrm{d}s}{\mathrm{d}v}. \tag{85}$$

One can use Eq. (84) to finally get

$$\frac{\mathrm{d}^2 s}{\mathrm{d}v^2} + \frac{(19+4\sqrt{10})v - 8\sqrt{10} - 26}{6(v-1)v}\frac{\mathrm{d}s}{\mathrm{d}v} + \frac{40+13\sqrt{10}}{18(v-1)^2}s = 0. \tag{86}$$

As in the vacuum case, there is still a symmetry in this differential equation, that is, $v \to v/(v-1)$. As a result, each solution in the range $0 < v < 1$ can be mapped to the region $v < 0$, each solution in the range $1 < v < 2$ to the region $v > 2$, and conversely. Therefore it is sufficient to study the interval $0 < v < 2$. Along this range, the general solutions of Eqs. (85) and (86) are

(i) $0 < v < 1$.—The function $s(v)$ is given by

$$s = c_1 s_1 + c_2 s_2, \tag{87}$$

where s_1 and s_2 stand for

$$s_1 = \left[\frac{1-v}{(\sqrt{1-v}+1)^4}\right]^{\gamma_1}, \quad s_2 = \left[\frac{1-v}{(\sqrt{1-v}-1)^4}\right]^{\gamma_1}, \tag{88}$$

and

$$\gamma_1 = \tfrac{5}{6} + \tfrac{1}{3}\sqrt{10}. \tag{89}$$

Hence, we obtain $\rho(v)$ as

$$\rho = \left(\frac{4}{5}\sqrt{10} - 2\right)(v-1)\frac{\frac{\mathrm{d}s_1}{\mathrm{d}v} - C\frac{\mathrm{d}s_2}{\mathrm{d}v}}{s_1 - Cs_2}, \tag{90}$$

where the integration constant $C = -c_2/c_1$. From the above-mentioned symmetry $\rho(v, C) = -\rho[v/(v-1), C]$, and $\xi(v, C) = \xi[v/(v-1), C]$. Using this property, it is possible to find the solution for $v < 0$ from the one in $0 < v < 1$.

(ii) $1 < v < 2$.—For this region, one can first extend the previous solution to the complex domain, and then, by choosing real linear combinations of the complex mode functions, one finds

$$s = c_1 s_1 + c_2 s_2, \tag{91}$$

where s_1 and s_2 are now given by

$$s_1 = \left[\frac{v-1}{v^2}\right]^{\gamma_1}\cos[\Theta(v)], \quad s_2 = \left[\frac{v-1}{v^2}\right]^{\gamma_1}\sin[\Theta(v)], \tag{92}$$

with

$$\Theta(v) \equiv \gamma_1\left[\pi - 4\arccos\left(\sqrt{1-\frac{1}{v}}\right)\right]. \tag{93}$$

Notice that $\Theta(2) = 0$ so that s_1 and s_2 are, respectively, even and odd under the symmetry $v \to v/(v-1)$. The solution for $\rho(v)$ is still given by Eq. (90) in terms of the above s_1 and s_2 functions and therefore explicitly exhibits the above-mentioned symmetry. Because of the antisymmetry of s_2, we now have $\rho(v, C) = -\rho[v/(v-1), -C]$, and $\xi(v, C) = \xi[v/(v-1), -C]$.

B. Second branch

This branch corresponds to the plus sign in Eq. (20), and, this time, we define $v(\rho)$ such that

$$\xi(\rho) = \frac{5\rho^2}{3(\rho^2 - 1)}\left(\left\{\frac{[-4\rho^2 + (4+\sqrt{10})\rho - \sqrt{10}]v - (8+2\sqrt{10})\rho}{[\sqrt{10}\rho^2 - (4+\sqrt{10})\rho + 4]v + (8+2\sqrt{10})\rho}\right\}^2 - 1\right). \tag{94}$$

The Abel equation for $v(\rho)$ reads

$$\frac{\mathrm{d}v}{\mathrm{d}\rho} = \frac{2(4-\sqrt{10})(v-1)v}{2(4+\sqrt{10})\rho + (\rho-1)(\sqrt{10}\rho - 4)v}, \tag{95}$$

which, as for the first branch, can be written as a Riccati equation, and one gets

$$\frac{\mathrm{d}\rho}{\mathrm{d}v} = \frac{\sqrt{10}\rho^2}{2(4-\sqrt{10})(v-1)} - \frac{(4+\sqrt{10})(v-2)\rho}{2(4-\sqrt{10})(v-1)v} + \frac{2}{(4-\sqrt{10})(v-1)}. \tag{96}$$

Comparing this expression to Eq. (84) shows that the solutions $\rho_2(v) = -\rho_1(v)$, where $\rho_1(v)$ denotes the solution of the first branch. As for the vacuum case, for every $v \neq 0$ and $v \neq 1$, there is always a positive solution for ρ: If ρ is negative for a branch, it is positive in the other and vice versa. We also have $\xi_2(v) = \xi_1(v)$ in between the two branches.

As for the vacuum case, we join the two branches by defining $\rho(v) \equiv |\rho_1(v)| = |\rho_2(v)|$ and $\xi(v) \equiv \xi_1(v) = \xi_2(v)$ for all $0 < v < 2$. In the following, we discuss the nature of the solution over the coordinate v.

C. Singularities

At the boundary of the intervals $0 < v < 1$ and $1 < v < 2$, the metric coefficient and coordinates behave as

$$\lim_{v\to 0^+} \rho = +\infty, \qquad \lim_{v\to 0^+} \xi = 0, \tag{97}$$

$$\lim_{v\to 0^+} K = +\infty, \qquad \lim_{v\to 0^+} R = 0, \tag{98}$$

and a singularity is present for $v = 0$ (and $\rho \to \infty$). As $v \to 1$ we have

$$\lim_{v\to 1} K < \infty, \qquad \lim_{v\to 1^\pm} \rho = 1, \tag{99}$$

and one may wonder if the manifold patches can be matched at $v = 1$. However, the metric factor $\xi(v)$ is not continuous at that point, and one has

$$\lim_{v\to 1^-} \xi = \frac{160C}{(13-6\gamma_1)^2(C+1)^2}, \tag{100}$$

$$\lim_{v\to 1^+} \xi = \frac{40(D^2+1)(13-6\gamma_1)^{-2}}{[\sin(\pi\gamma_1) - D\cos(\pi\gamma_1)]^2}, \tag{101}$$

where we have allowed a different integration constant D for the regions $v > 1$. In fact, there is no real solution for the constant D to match the value of ξ for all C. As a result, these two patches cannot be prolonged one onto another.

For $v \to 2$, and assuming $C \neq 0$, there is no singularity, and one has

$$\lim_{v\to 2} \rho = |C|, \qquad \lim_{v\to 2} \xi = \frac{40C^2(C^2+1)}{[C^2(6\gamma_1 - 5) + 8]^2}. \tag{102}$$

For the intermediate regions, as for the vacuum case, the metric in the ρ coordinate system possesses a null surface, which ends up being the throat of a wormhole, if there is a value v_{h} for which

$$\lim_{v\to v_{\mathrm{h}}} \xi = \infty. \tag{103}$$

A singularity is present when the Kretschmann scalar diverges, i.e., for the values v_{s} such that

$$\lim_{v\to v_{\mathrm{s}}} \xi = 0. \tag{104}$$

We will also separate the patches explored by the ρ coordinates according to the values v_∞ for which

$$\lim_{v\to v_\infty} \rho = \infty. \tag{105}$$

The values of v_{h}, v_{s}, and v_∞ are obtained by taking the corresponding limits in Eqs. (82) and (90). In general, these equations do not have any obvious analytical solution, and they have been numerically solved as a function of C on the two domains $0 < v < 1$ and $1 < v < 2$.

1. Over $0 < v < 1$

Taking the limit $\rho \to \infty$ in Eq. (90) gives

$$v_\infty = 1 - \left[\frac{1 - C^{1/(4\gamma_1)}}{1 + C^{1/(4\gamma_1)}}\right]^2, \tag{106}$$

which is such that $0 < v_\infty < 1$ for $0 < C < 1$ only.

From Eq. (82), the null surface position is the solution of

$$f_{\mathrm{h}}^{(1)}(v_{\mathrm{h}}, C) = 0, \tag{107}$$

where $f_{\mathrm{h}}^{(1)}(v, C)$ is given by Eq. (A3). Finally, except for some particular cases, the singularity coincides with the points where $\rho(v)$ vanishes, so that v_{s} is the solution of

$$f_{\mathrm{s}}^{(1)}(v_{\mathrm{s}}, C) = 0, \tag{108}$$

$f_{\mathrm{s}}^{(1)}(v, C)$ being given in Eq. (A4).

The solutions of the above equations have been plotted in Fig. 6, where v_{h}, v_{s}, and v_∞ are functions of C. From this plot, one can distinguish three cases:

(i) For $C < 0$, we have a singularity for $v \to 0$, a null surface (for $-1 < C < 0$) and $\xi < 0$ for all v in $0 < v < 1$. We will therefore not consider any longer this pathological case.

(ii) For $0 < C < 1$ and $0 < v < v_\infty$, we have a wormhole configuration whose solution is shown in Fig. 8. This case is discussed in more detail in the next section.

(iii) For $0 < C < 1$ and $v_\infty < v < 1$, ξ vanishes without changing its sign and a naked singularity appears.

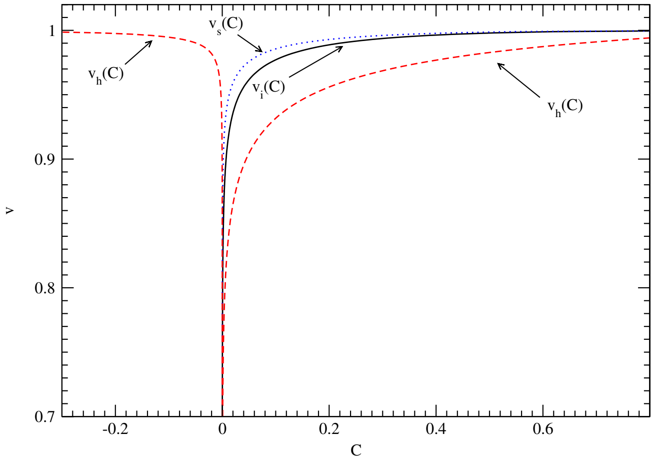

FIG. 6 (color online). Location of the singular points and asymptotic ρ domains as functions of C in the region $0 < v < 1$. For $v \to 0$, ξ vanishes and a singularity appears (not represented; see text).

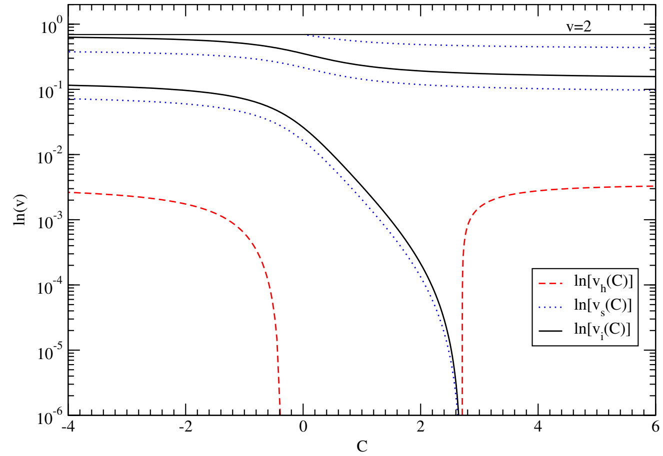

FIG. 7 (color online). Position of the singularities and asymptotic ρ domains as a function of C in the region $1 < v < 2$. These patches exhibit naked singularities.

Notice that the proper distance in this interval tends to infinity, although $\rho \to 1$ as $v \to 1^-$.

(iv) For $C > 1$, there is a naked singularity in $v_s = 0$.

In the next section, we perform the same analysis on the domain $1 < v < 2$.

2. *Over* $1 < v < 2$

It is interesting to notice the difference with respect to the previous case, as now ξ is positive definite. The existence of coordinate singularities is still given by the condition $\xi \to \infty$, and v_h is the solution of

$$f_h^{(2)}(v_h, C) = 0, \qquad (109)$$

where $f_h^{(2)}(v, C)$ is derived in the appendix. In the domain $1 < v < 2$, Eq. (A5) is well defined only for

$$1 < v < \frac{-(6 + 24\gamma_1) + 4\sqrt{78\gamma_1 + 15}}{12\gamma_1 - 23} \simeq 1.003. \qquad (110)$$

The singularities are given by the values for v at which ξ vanishes and are the zeros of

$$f_s^{(2)}(v_s, C) = 0, \qquad (111)$$

where $f_s^{(2)}(v, C)$ is given by Eq. (A6).

The conditions for the points v_∞ at which $\rho \to \infty$ can be simplified into

$$v_\infty = \frac{1}{1 - \cos^2[\frac{\pi}{4} - \frac{\arctan(C^{-1})}{4\gamma_1}]}. \qquad (112)$$

The condition of existence and dependency of v_h, v_s, and v_∞ with respect to the integration constant C have been represented in Fig. 7.

For a given value of C, this plot shows whether and where there are singularities, bounded or unbounded ρ domains, and the null surfaces. Among the many possibilities, none of them seems to represent physically interesting regions, as there are naked singularities and no horizon connected to asymptotically flat spacetime. Therefore we will not consider further this part of the full solution, although some of these configurations may have some interest once regularized.

In the next section, we study in more detail the charged wormhole solution exhibited for $0 < v < 1$.

D. Charged wormhole configuration

We denote by this charged wormhole configuration the solution obtained in the region $0 < v < v_\infty$. By definition

$$\lim_{v \to v_\infty^-} \rho(v) = +\infty. \qquad (113)$$

Defining the following polynomials:

$$P_1(\rho) = 4\rho^2 + (\sqrt{10} + 4)\rho + \sqrt{10}, \qquad (114)$$

$$P_2(\rho) = \sqrt{10}\rho^2 + (\sqrt{10} + 4)\rho + 4, \qquad (115)$$

$$P_3(\rho) = 2(\sqrt{10} + 4)\rho, \qquad (116)$$

one gets from Eq. (82)

$$\xi(v) = \frac{10\rho^2 v}{(P_2 v - P_3)^2}[(\rho + 1)^2 v - 4\rho] \qquad (117)$$

and from Eq. (84)

$$\frac{d\rho}{dv} = \frac{P_2 v - P_3}{2(4 - \sqrt{10})v(1 - v)}. \qquad (118)$$

Therefore, in the v variable, the metric component g_{vv} becomes

$$g_{vv} = \xi\left(\frac{\mathrm{d}\rho}{\mathrm{d}v}\right)^2 = \frac{5}{2(4-\sqrt{10})^2}\frac{\rho^2[(\rho+1)^2 v - 4\rho]}{v(1-v)^2}, \tag{119}$$

where $\rho(v)$ is explicitly given by Eq. (90). It is already clear using the v variable that the g_{vv} component of the metric is no longer divergent at the $\xi(\rho_{\mathrm{h}})$ null surface. As for the uncharged case studied in Sec. III, one can look for the presence of horizons by studying Eq. (62). We find that $\theta_r > 0$ for any $v < v_{\mathrm{h}}$, so that this field configuration is in fact a wormhole which does not possess any horizon. The proper distance from the singularity ($v_{\mathrm{s}} = 0$) to the throat is actually finite, and this configuration is reminiscent with the uncharged one:

$$\Delta s = \int_0^{v_{\mathrm{h}}} \sqrt{g_{vv}}\mathrm{d}v < \infty. \tag{120}$$

Also in this case, v grows continuously from the singularity point to an asymptotically flat region, whereas ρ shows a multivalued typical behavior of a wormhole configuration. The various geometrical quantities associated with this solution have been represented in Fig. 8.

We have plotted in Fig. 9 the embedding function of the charged wormhole solution, and, as can be checked in this figure, the topology is similar to the uncharged one. In fact, Eq. (19) shows that changing the charge simply rescales the radial coordinate and therefore modifies the throat width. However, the wormhole solution exists only for $0 < C < 1$, and in the limit $C \to 1$ the throat width collapses to zero and a naked singularity appears. The physical meaning of the constant C can be inferred from the asymptotic behavior of $\xi(r)$. In the limit $r \to \infty$, from Eq. (117) one has

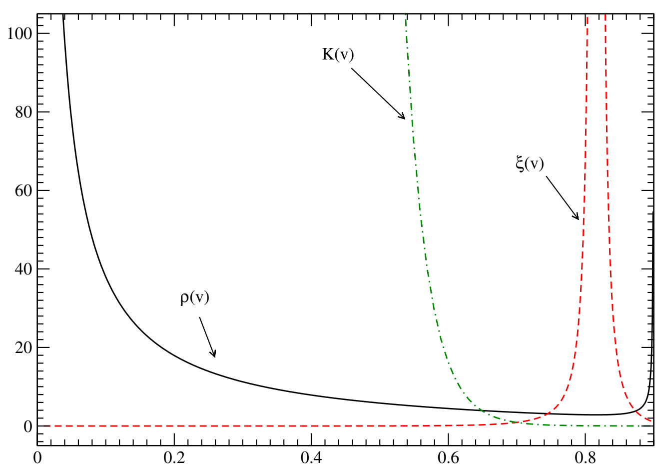

FIG. 8 (color online). Charged wormhole solution. There is a singularity for $v \to 0$, the wormhole throat is located in $v = v_{\mathrm{h}} < v_\infty$, at a finite proper distance from the singularity, and the space is asymptotically flat as $v \to v_\infty$. For convenience, the brane tension has been chosen such that $v_\infty = 9/10$.

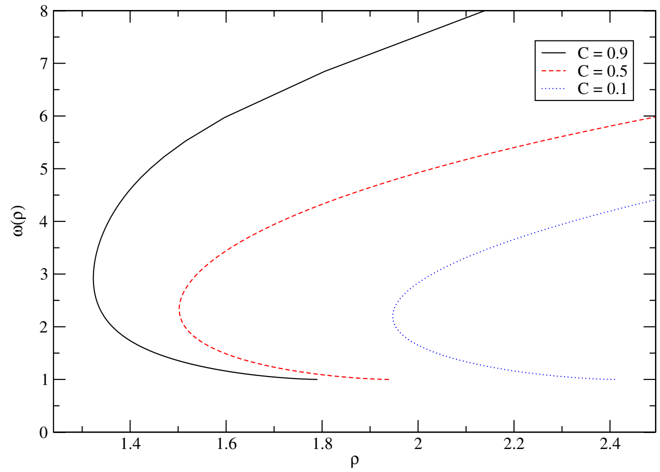

FIG. 9 (color online). Embedding function $\omega(\rho)$ of the charged wormhole configuration for three values of C. If $C = 1$, then the throat of the wormhole collapses to zero and a pure naked singularity appears.

$$\xi(\rho) \sim 1 + \frac{8}{\sqrt{10}}\frac{2 - v_\infty}{v_\infty}\frac{r_0}{r} + \mathcal{O}\left(\frac{1}{r^2}\right). \tag{121}$$

By using Eqs. (19) and (106), C is related to the brane tension T and magnetic charge $1/q$ by

$$C = (\sqrt{5}x - \sqrt{5x^2 - 1})^{4\gamma_1}, \tag{122}$$

where $x = qG_7T/\kappa$. As a result, the throat collapses now when the magnetic charge equals the brane tension, i.e., for $G_7T/\kappa = 1/(\sqrt{5}q)$, and there is no longer a wormhole solution for T less than this value.

V. CONCLUSIONS

We have presented exact solutions of the Einstein equations in seven dimensions in the presence of matter in the form of a monopolar magnetic field. The solutions have been derived for compactified spacetimes of the form $M^4 \times R \times S^2$ for which the extra dimensions exhibit spherical symmetry. The form field considered is generated by a spatially extended Dirac monopole present in the origin and found to generate a charged brane configuration. This charged brane will be in general a singularity of the spacetime and may or not be naked. In the pseudoregular case, a null surface is situated at a finite proper distance from the brane and corresponds to the throat of a wormhole configuration. In order to derive such an explicit solution that covers the whole spacetime, we have proposed a new coordinate system which cures the incompleteness of the natural spherical coordinates (ρ coordinate). Such a choice of coordinates clarifies the issues of the apparent "holes" in the manifold, and our approach may be applied to other similar braneworld models. Notice that the solution described here corresponds to the asymptotic behavior of more complicated models exhibiting a residual $U(1)$ symmetry, such as the ones described in Refs. [15,16]. Let us also notice that we have not discussed the stability of our solutions. Since they correspond to wormhole configurations with spatially unbounded traverse sections, one may be concerned about instabilities. It is indeed possible that such systems, in the way presented here, may be unstable against breakdown into standard seven-dimensional black holes [38–40]. However, since such solutions also match with exterior spacetime of higher-dimensional topological defects, the stability of the latter remains an open question for future works.

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APPENDIX: GEOMETRICAL QUANTITIES

1. Scalar invariants

From the metric of Eq. (5), the Ricci and Kretschmann scalars are given by

$$R = -\frac{4A''}{\xi A} + \frac{2A'\xi'}{A\xi^2} - \frac{8A'}{r\xi A} + \frac{2\xi'}{r\xi^2} + \frac{2}{r^2} - \frac{2}{r^2\xi} - \frac{A'^2}{\xi A^2}, \tag{A1}$$

$$K = -\frac{A''A'^2}{\xi^2 A^3} + \frac{1}{4}\frac{A'^2\xi'^2}{\xi^4 A^2} + \frac{1}{2}\frac{A'^3\xi'}{\xi^3 A^3} + \frac{A''^2}{\xi^2 A^2} + \frac{2A'^2}{\rho^2\xi^2 A^2} + \frac{1}{\rho^4} + \frac{1}{2}\frac{\xi'^2}{\rho^2\xi^4} - \frac{A''A'\xi'}{\xi^3 A^2} - \frac{2}{\rho^4\xi} + \frac{5}{8}\frac{A'^4}{\xi^2 A^4} + \frac{1}{\rho^4\xi^2}. \tag{A2}$$

2. Singularities

As discussed in Sec. IV, the location of singularities in the ρ-coordinate system for the charged solution are given by, respectively, considering the limit $\xi \to \infty$ and $\xi \to 0$ in Eq. (82).

a. Region $0 < v < 1$

For $0 < v < 1$, the null surface location v_h is given by the root of the limit $\xi(v) \to \infty$ in Eq. (82) where $\rho(v)$ is obtained from Eq. (90) with the mode functions s_1 and s_2 of Eq. (88). After some calculations, v_h is found to be the root of

$$f_h^{(1)}(v, C) = -C + \frac{2(\sqrt{1-v}+1)^{6-8\gamma_1}v^{4\gamma_1}}{6\gamma_1 - 13} \times \frac{6\gamma_1(v-1) \pm \sqrt{2}\sqrt{(v-1)\{12\gamma_1[v(v+4)-4] + (12-23v)v - 12\}} + 3v - 3}{v\{v[(\sqrt{1-v}+8)v^2 - 8(4\sqrt{1-v}+11)v + 160\sqrt{1-v} + 272] - 64(4\sqrt{1-v}+5)\} + 128(\sqrt{1-v}+1)}. \tag{A3}$$

Similarly, the singularities location v_s are given by the root of

$$f_s^{(1)}(v, C) = -C + \frac{v + 2\sqrt{1-v} - 2}{v - 2\sqrt{1-v} - 2}\left[\frac{(\sqrt{1-v}-1)^4}{v^2 - 8v + 4(1-v)^{3/2} + 4\sqrt{1-v} + 8}\right]^{\gamma_1}. \tag{A4}$$

b. Region $1 < v < 2$

As for the previous domain, one obtains v_h and v_s from the corresponding limits of Eq. (82), with $\rho(v)$ given by Eq. (90) but with now the mode functions s_1 and s_2 of Eq. (92). The locations v_h end up being the root of

$$f_h^{(2)}(v, C) = -C + \frac{-6(2\gamma_1+1)\sqrt{v-1} - 2(6\gamma_1-13)\sqrt{v-1}\cos[2\Theta(v)] + (6\gamma_1-13)(v-2)\sin[2\Theta(v)]}{(6\gamma_1-13)(v-2)\cos[2\Theta(v)] \pm 2\sqrt{-24\gamma_1(v^2+4v-4) + 46v^2 - 24v + 24} + 2(6\gamma_1-13)\sqrt{v-1}\sin[2\Theta(v)]}. \tag{A5}$$

The singularities positions v_s are obtained by the zeros of

$$f_{\mathrm{s}}^{(2)}(v, C) = C + \frac{(v-2)\sqrt{v-1}\cos[\Theta(v)] + 2(v-1)\sin[\Theta(v)]}{2(v-1)\cos[\Theta(v)] - (v-2)\sqrt{v-1}\sin[\Theta(v)]}. \tag{A6}$$

In both of the previous expressions, the function $\Theta(v)$ is given by Eq. (93).

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