

Timelike geodesic currents in the stationary, axisymmetric, force-free magnetosphere of a Kerr black hole

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The structural properties of geodesic currents in an ambient Kerr background are studied from an analytical point of view. The geodesics in this congruence may correspond to charged particles that carry energy and angular momentum from the black hole through the Blandford-Znajek mechanism. It is shown that the resulting magnetosphere naturally satisfies the Znajek regularity condition. Particular attention is paid here to the energy extracted by matter currents rather than by electromagnetic Poynting fluxes. This analytic study is motivated by the mathematical structure behind the Blandford-Znajek process, and may only have limited applications to the astrophysics of supermassive black holes.

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I. INTRODUCTION

Nearly 30 years ago, Blandford and Znajek [1] described a method by which a black hole could release its rotational energy into the magnetosphere via electromagnetic processes. They argued for the existence of a force-free magnetosphere, wherein outflowing currents and electromagnetic energy fluxes could in principle carry energy and angular momentum away from the black hole. Subsequently, a considerable amount of research has been done in trying to understand the nature of the solutions provided by a force-free electrodynamics in the vicinity of a rotating black hole (e.g., [2–6]). Such solutions are extremely important from an astrophysical point of view, as they would be leading candidates for the explanation of jets emanating from black holes. Owing to the complexity of the problem, recent efforts have been primarily numerical in nature (e.g., [7,8]).

After obtaining an approximate solution to the Blandford-Znajek mechanism of energy extraction from rotating black holes that generalizes the original perturbative split monopole solution [9], we were recently successful [10] in providing a class of exact analytic solutions to the equations of force-free electrodynamics in Kerr geometry. Although finite everywhere in the magnetosphere, these solutions were not, however, physically realistic, but provide a basis for further analytic study. For the case of stationary, axisymmetric, force-free electrodynamics in a Kerr background, we [10] showed that the exact solution permitted an electromagnetic current 4-vector of the form

$$I^\nu = -\frac{2}{a^2\sqrt{-g}} \frac{d}{d\theta} \left[\Lambda \frac{\cos\theta}{\sin^4\theta} \right] l^\nu. \quad (1)$$

Here, a is the angular momentum per unit mass of the black hole, g is the usual determinant of the Kerr spacetime metric $g_{\mu\nu}$, and Λ is an arbitrary function of θ . Also l^ν is

the infalling principle null geodesic tangent vector of the Kerr geometry, given explicitly in Boyer-Lindquist coordinates $\{t, r, \theta, \varphi\}$ by

$$l^\nu = \left(\frac{r^2 + a^2}{\Delta}, -1, 0, \frac{a}{\Delta} \right). \quad (2)$$

Here, $\Delta = r^2 + a^2 - 2Mr$, where M is the mass of the black hole. Since the geodesic in Eq. (2) is infalling, it is not possible to extract energy from the hole. More importantly, the current flows along null geodesics. While we expect the currents to flow through geodesics under force-free conditions, the fact that the geodesics are null is a less desirable feature.

In this paper, we study the general structural properties of geodesic currents of particles with mass when the magnetosphere is force-free. Instead of working with the covariant formalism, we will write all equations of electrodynamics in an intuitive 3 + 1 formalism [6]. After a brief introduction to the 3 + 1 formalism in Sec. II, we describe the relevant equations of a force-free, axisymmetric, stationary magnetosphere of a Kerr black hole in Sec. III. The structural properties of geodesic currents are described in Sec. IV, where we demonstrate that the Znajek regularity condition is satisfied. Energy and angular momentum extraction through particle currents in the force-free magnetosphere are considered in Sec. V, and we show in Sec. VI that a naked singularity cannot result from this process. We conclude in Sec. VII.

II. 3 + 1 ELECTRODYNAMICS IN KERR GEOMETRY: A PRIMER

In the 3 + 1 formalism of electrodynamics, we rewrite the covariant Maxwell equations in terms of 3-vectors \mathbf{E} , \mathbf{B} , \mathbf{D} , and \mathbf{H} . While these quantities may lose meaning under arbitrary coordinate transformations, in a particular

frame we have the advantage of working with the four familiar equations of electrodynamics. Much of the background and details presented in this section can be found in [6,9].

The 3-vectors are defined on spacelike slices of spacetime. We shall foliate the Kerr spacetime by a collection of absolute space described by surfaces of constant time t in the Boyer-Lindquist coordinates $\{t, r, \theta, \varphi\}$ of the Kerr geometry. Here, the metric takes the form

$$ds^2 = (\beta^2 - \alpha^2)dt^2 + 2\beta_\varphi d\varphi dt + \gamma_{rr}dr^2 + \gamma_{\theta\theta}d\theta^2 + \gamma_{\varphi\varphi}d\varphi^2. \quad (3)$$

The metric coefficients are given by

$$\begin{aligned} \beta^2 - \alpha^2 &= g_{tt} = -1 + \frac{2Mr}{\rho^2}, \\ \beta_\varphi &\equiv g_{t\varphi} = \frac{-2Mr a \sin^2 \theta}{\rho^2}, \\ \gamma_{rr} &= \frac{\rho^2}{\Delta}, \\ \gamma_{\theta\theta} &= \rho^2, \quad \text{and} \quad \gamma_{\varphi\varphi} = \frac{\Sigma^2 \sin^2 \theta}{\rho^2}, \end{aligned} \quad (4)$$

where

$$\begin{aligned} \rho^2 &= r^2 + a^2 \cos^2 \theta, \\ \Delta &= r^2 - 2Mr + a^2 \quad \text{and} \\ \Sigma^2 &= (r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta. \end{aligned} \quad (5)$$

Additionally

$$\begin{aligned} \alpha^2 &= \frac{\rho^2 \Delta}{\Sigma^2}, \\ \beta^2 &= \frac{\beta_\varphi^2}{\gamma_{\varphi\varphi}}, \quad \text{and} \quad \sqrt{-g} = \alpha \sqrt{\gamma} = \rho^2 \sin \theta. \end{aligned} \quad (6)$$

Parameters M and a are the mass and angular momentum per unit mass, respectively, of the Kerr black hole. The three-dimensional space for some fixed value of t is endowed with the metric

$$ds_{\text{sp}}^2 = \gamma_{rr}dr^2 + \gamma_{\theta\theta}d\theta^2 + \gamma_{\varphi\varphi}d\varphi^2. \quad (7)$$

Since the spacetime is stationary, the value of t for the absolute space will not matter in our discussion. The covariant Maxwell's equations are

$$\nabla_\beta^* F^{\alpha\beta} = 0, \quad \text{and} \quad \nabla_\beta F^{\alpha\beta} = I^\alpha. \quad (8)$$

Here $F^{\alpha\beta}$ is the Maxwell stress tensor, and I^α is the four vector of the electric current. In order to rewrite the covariant equations of electrodynamics in a familiar Maxwell-type form, three vectors \mathbf{D} and \mathbf{H} in our absolute space are defined by

$$\mathbf{D}^i = \alpha F^{ti}, \quad \text{and} \quad \mathbf{H}_i = \frac{1}{2} \alpha e_{ijk} F^{jk}. \quad (9)$$

Here, e_{ijk} is the completely antisymmetric Levi-Civita pseudotensor of the absolute space such that $e_{r\theta\varphi} = \sqrt{\gamma}$, where, γ is the determinant of ds_{sp}^2 . In addition, we define the shift 3-vector as

$$\beta = (0, 0, \beta_\varphi). \quad (10)$$

The constitutive equations relating the electromagnetic field vectors, \mathbf{E} and \mathbf{B} , and their duals, \mathbf{D} and \mathbf{H} , for regions of zero electric and magnetic susceptibilities are

$$\mathbf{E} = \alpha \mathbf{D} + \beta \times \mathbf{B}, \quad \text{and} \quad \mathbf{H} = \alpha \mathbf{B} - \beta \times \mathbf{D}, \quad (11)$$

where the curl is defined by the equation

$$(\mathbf{A} \times \mathbf{B})^i = e^{ijk} \mathbf{A}_j \mathbf{B}_k.$$

With these definitions, Maxwell's equations become

$$\bar{\nabla} \cdot \mathbf{B} = 0, \quad \text{and} \quad \partial_t \mathbf{B} + \bar{\nabla} \times \mathbf{E} = 0 \quad (12)$$

for the homogenous part and

$$\bar{\nabla} \cdot \mathbf{D} = \rho_c, \quad \text{and} \quad -\partial_t \mathbf{D} + \bar{\nabla} \times \mathbf{H} = \mathbf{J} \quad (13)$$

for the inhomogeneous part. Here,

$$\rho_c = \alpha I^t, \quad \mathbf{J}^k = \alpha \mathbf{I}^k, \quad (14)$$

are the charge and current densities, respectively, $\bar{\nabla}$ is the covariant derivative induced by the spatial metric γ_{ij} on our absolute space, and α is the function that appears in Eq. (6).

III. STATIONARY, AXISYMMETRIC, FORCE-FREE ELECTRODYNAMICS

In the Blanford-Znajek mechanism, the magnetosphere is assumed to be force-free. In addition, Blanford and Znajek impose, for simplicity, the inherent symmetry of the geometry of the rotating black hole onto the currents and fields, namely, stationarity and axisymmetry. Consequently the Lorentz force on the currents in the magnetosphere is trivial, so that

$$F_{\mu\nu} I^\nu = 0. \quad (15)$$

In our 3 + 1 language, the above condition takes the form

$$\mathbf{E} \cdot \mathbf{J} = 0, \quad (16)$$

and

$$\rho_c \mathbf{E} + \mathbf{J} \times \mathbf{B} = 0. \quad (17)$$

The consequences of Eqs. (16) and (17) along with the requirements of stationarity and axisymmetry will be summarized below. The interested reader is referred to [9] for further details. The poloidal and toroidal components (\mathbf{A}_P and \mathbf{A}_T , respectively) of a vector field are defined such that $\mathbf{A} = \mathbf{A}_P + \mathbf{A}_T$, where $\mathbf{A}_P = \mathbf{A}^r \partial_r + \mathbf{A}^\theta \partial_\theta$ and $\mathbf{A}_T = \mathbf{A}^\varphi \partial_\varphi$. Since the magnetosphere is stationary and axisymmetric,

$$\mathbf{E}_T = 0. \quad (18)$$

This is true because stationarity implies that \mathbf{E} is the gradient of a scalar, and for axisymmetric fields, all φ derivatives vanish. From Eq. (17) we have that

$$\mathbf{E} \cdot \mathbf{B} = 0. \quad (19)$$

Equations (18) and (19) imply that there exists a vector $\omega = \Omega \partial_\varphi$ such that

$$\mathbf{E} = -\omega \times \mathbf{B}. \quad (20)$$

Here, Ω is a function dependent on r and θ . Since \mathbf{B} is divergence free,

$$\mathbf{B} = \bar{\nabla} \times \mathbf{A}, \quad (21)$$

where \mathbf{A} is the 3-vector potential associated with \mathbf{B} . Therefore,

$$\mathbf{B}_P = \frac{1}{\sqrt{\gamma}} (\mathbf{A}_{\varphi,\theta} \partial_r - \mathbf{A}_{\varphi,r} \partial_\theta), \quad (22)$$

since \mathbf{A}_φ is independent of t and φ . Surfaces of constant \mathbf{A}_φ are called *equipotential surfaces*. Here \mathbf{A}_φ is the toroidal component of the 3-vector potential \mathbf{A} . Functions that are constant on equipotential surfaces are called *equipotential surface functions*, i.e., a function f is an equipotential surface function if

$$\mathbf{B}_P f = 0 = \mathbf{A}_{\varphi,\theta} \partial_r f - \mathbf{A}_{\varphi,r} \partial_\theta f. \quad (23)$$

It can be shown that Ω and \mathbf{H}_φ are equipotential surface functions, and consequently from Eq. (20) we get

$$\mathbf{E}_P = -d \int \Omega d\mathbf{A}_\varphi. \quad (24)$$

Since Ω is an equipotential surface function, it is strictly a function of \mathbf{A}_φ , and hence the above integral is well defined. The constitutive Eqs. (11) give

$$\mathbf{D} = \mathbf{D}_P = -\frac{1}{\alpha} (\Omega + \beta^\varphi) d\mathbf{A}_\varphi, \quad (25)$$

and

$$\mathbf{H}_P = (\alpha^2 - \beta^2 - \beta_\varphi \Omega) \frac{\mathbf{B}_P}{\alpha}. \quad (26)$$

The electric charge density is determined by the divergence of \mathbf{D}_P :

$$\begin{aligned} -\sqrt{\gamma} \rho_c &= \partial_r \left[\frac{1}{\alpha \sqrt{\gamma}} (\gamma_{\varphi\varphi} \Omega + \beta_\varphi) \gamma_{\theta\theta} \mathbf{A}_{\varphi,r} \right] \\ &+ \partial_\theta \left[\frac{1}{\alpha \sqrt{\gamma}} (\gamma_{\varphi\varphi} \Omega + \beta_\varphi) \gamma_{rr} \mathbf{A}_{\varphi,\theta} \right]. \end{aligned} \quad (27)$$

The toroidal current vector can be obtained by computing the φ component of the curl in Eq. (13):

$$\begin{aligned} -\sqrt{\gamma} \mathbf{J}^\varphi &= \mathbf{H}_{r,\theta} - \mathbf{H}_{\theta,r} \\ &= \partial_r \left[\frac{1}{\alpha \sqrt{\gamma}} (\alpha^2 - \beta^2 - \beta_\varphi \Omega) \gamma_{\theta\theta} \mathbf{A}_{\varphi,r} \right] \\ &+ \partial_\theta \left[\frac{1}{\alpha \sqrt{\gamma}} (\alpha^2 - \beta^2 - \beta_\varphi \Omega) \gamma_{rr} \mathbf{A}_{\varphi,\theta} \right]. \end{aligned} \quad (28)$$

The poloidal current is completely determined by \mathbf{H}_φ and is given by

$$\sqrt{\gamma} \mathbf{J}_P = \mathbf{H}_{\varphi,\theta} \partial_r - \mathbf{H}_{\varphi,r} \partial_\theta. \quad (29)$$

Finally,

$$\mathbf{B}_\varphi = \mathbf{H}_\varphi / \alpha.$$

Since \mathbf{H}_φ is an equipotential surface function, from Eq. (29) we have that

$$\mathbf{J}_P = \frac{1}{\sqrt{\gamma}} \frac{d\mathbf{H}_\varphi}{d\mathbf{A}_\varphi} (\mathbf{A}_{\varphi,\theta} \partial_r - \mathbf{A}_{\varphi,r} \partial_\theta) = \frac{d\mathbf{H}_\varphi}{d\mathbf{A}_\varphi} \mathbf{B}_P, \quad (30)$$

i.e., the currents flow along equipotential surfaces. The only remaining requirement in the explicit formulas given above for the fields and current stems from Eq. (17), and can be written in the form

$$\frac{1}{2} \frac{d\mathbf{H}_\varphi^2}{d\mathbf{A}_\varphi} = -\alpha (\rho_c \Omega \gamma_{\varphi\varphi} - \mathbf{J}_\varphi). \quad (31)$$

Since the left-hand side of the equation above is an equipotential surface function, we must pick an equipotential surface function Ω such that the right-hand side of the equation is an equipotential surface function as well. Additionally, for the fields and currents to remain well defined at the event horizon of the Kerr black hole, given by

$$r = r_+ \equiv M + \sqrt{M^2 - a^2},$$

we must have that

$$\mathbf{H}_\varphi |_{r_+} = \frac{\sin^2 \theta}{\alpha} \mathbf{B}^r (2Mr\Omega - a) |_{r_+} \quad (32)$$

in Boyer-Lindquist coordinates. This is the Znajek regularity condition derived in [11]. In the above equation, all quantities are to be evaluated at the event horizon.

The poloidal component of the Poynting vector for the electromagnetic field is given by

$$\mathbf{S}_P = (\mathbf{E} \times \mathbf{H})_P.$$

Consequently, the rate of extraction of energy from the black hole is given by

$$\frac{dE_{\text{em}}}{dt} = - \int_{r_2=\infty} \Omega \mathbf{H}_\varphi \mathbf{B}^r \sqrt{\gamma} d\theta d\varphi. \quad (33)$$

In a similar manner, using the fact that ∂_φ is a Killing

vector field of the geometry, we get the following expression for the rate of extraction of angular momentum from the black hole:

$$\frac{dL_{\text{em}}}{dt} = - \int_{r_2=\infty} \mathbf{H}_\varphi \mathbf{B}^r \sqrt{\gamma} d\theta d\varphi. \quad (34)$$

Under this circumstance, the mass and angular momentum of the black hole decreases at a rate given by

$$\frac{\delta M}{\delta t} = - \frac{dE_{\text{em}}}{dt}, \quad \text{and} \quad \frac{\delta J}{\delta t} = - \frac{dL_{\text{em}}}{dt}. \quad (35)$$

IV. GEODESIC CURRENTS

In general relativity, force-free dynamics of particles are associated with the geodesics of the curved spacetime. In the case of the Blandford-Znajek mechanism, vacuum is broken down by a sparking mechanism that results in a cascade of charges. Here we assume that the resulting current vector outside the event horizon can be written in the form

$$I^\nu = F(r, \theta) u^\nu, \quad (36)$$

where u^ν is a timelike geodesic tangent vector of the Kerr geometry. The form given in Eq. (36) has a limited range of applicability, and is not considered to be the most general form of the current vector. This form is motivated by Eq. (1). Equation (36) embodies what we mean by geodesic currents in the title of this section. The function F relates the 4-velocity of the charged particle at a point to electromagnetic current density at that point. The properties of u^ν are well understood (for example see [12,13]). In particular, u^ν can be written in the form

$$u^\nu = \left(i, \frac{\sqrt{R}}{\rho^2}, \frac{\sqrt{\Theta}}{\rho^2}, \dot{\varphi} \right). \quad (37)$$

Here, i , $\dot{\varphi}$, \sqrt{R} and $\sqrt{\Theta}$ are functions of r and θ . Also, in \sqrt{R} and $\sqrt{\Theta}$ we have not yet committed to the “+” root. The notion of time (dot) derivative is not relevant to our analysis; the notation has been simply borrowed from the existing literature on Kerr geodesics. The explicit forms of the geodesic functions are

$$i = \frac{\Sigma^2 \mathcal{E} - 2aMr\mathcal{L}}{\rho^2 \Delta}, \quad (38)$$

and

$$\dot{\varphi} = \frac{2aMr\mathcal{E}\sin^2\theta + (\rho^2 - 2Mr)\mathcal{L}}{\rho^2 \Delta \sin^2\theta}. \quad (39)$$

The functions R and Θ are given by

$$R = C^2 + \Delta(q^2 r^2 - \mathcal{K}) \quad (40)$$

and

$$\Theta = \mathcal{K} + q^2 a^2 \cos^2\theta - \frac{D^2}{\sin^2\theta}, \quad (41)$$

where

$$C = (r^2 + a^2)\mathcal{E} - a\mathcal{L} \quad \text{and} \quad D = \mathcal{L} - a\mathcal{E}\sin^2\theta. \quad (42)$$

In the study of Kerr geodesics, \mathcal{E} , \mathcal{L} , and \mathcal{K} are simply constants for a given geodesic, and

$$q^2 = 1, 0, -1$$

for spacelike, null, and timelike geodesics, respectively. The terms \mathcal{E} and \mathcal{L} can be related to the energy and angular momentum of the particle that flows along the geodesic. Explicitly,

$$\mathcal{E} = -g(K, u) = -(g_{tt}\dot{t} + g_{t\varphi}\dot{\varphi}) \quad (43)$$

and

$$\mathcal{L} = g(m, u) = (g_{t\varphi}\dot{t} + \gamma_{\varphi\varphi}\dot{\varphi}). \quad (44)$$

Here $K = \partial_t$ and $m = \partial_\varphi$. \mathcal{K} is the famous Carter’s constant [14]. In our case, since the currents flow along equipotential surfaces, \mathcal{E} , \mathcal{L} , and \mathcal{K} become equipotential surface functions, which are now subject to the condition

$$\sqrt{R}f_{,r} + \sqrt{\Theta}f_{,\theta} = 0, \quad (45)$$

for any equipotential surface function f . Also, since we are only interested in particles with mass, we will set $q^2 = -1$. Conservation of electric charge is implied by the equation $\nabla_\mu I^\mu = 0$, i.e.,

$$\partial_r \left[F \frac{\sqrt{R}}{\rho^2} \sqrt{-g} \right] + \partial_\theta \left[F \frac{\sqrt{\Theta}}{\rho^2} \sqrt{-g} \right] = 0. \quad (46)$$

This is actually an equation for F .

In writing Eq. (36), we have assumed the explicit form of the charge density and the 3-vector current density. In the remainder of this section, we will construct a formula for \mathbf{H}_φ and show that it satisfies the Znajek regularity condition, Eq. (32). But first we present a few preliminaries.

From Eq. (14) we see that

$$\rho_c = \alpha F i, \quad \mathbf{J}^\varphi = \alpha F \dot{\varphi},$$

and

$$\mathbf{J}_P = \alpha F \frac{\sqrt{R}}{\rho^2} \partial_r + \alpha F \frac{\sqrt{\Theta}}{\rho^2} \partial_\theta. \quad (47)$$

Consequently, Eq. (31) can be written as

$$- \frac{1}{2} \frac{d\mathbf{H}_\varphi^2}{d\mathbf{A}_\varphi} = \Delta \sin^2\theta F (i\Omega - \dot{\varphi}). \quad (48)$$

In the theorem that follows, we obtain an explicit expression for \mathbf{H}_φ .

Theorem 1. In force-free, stationary, axisymmetric electrodynamics in a Kerr background when the electromagnetic current vector takes the form given by Eq. (36),

$$\mathbf{H}_\varphi = \frac{\Delta \sin\theta(i\Omega - \dot{\varphi})}{\sqrt{\Theta}} \mathbf{A}_{\varphi,r}. \quad (49)$$

Proof: From Eqs. (29) and (47) we get that

$$\mathbf{H}_{\varphi,r} = -F \sin\theta \sqrt{\Theta}. \quad (50)$$

But note also that

$$-\mathbf{H}_\varphi \mathbf{H}_{\varphi,r} = -\frac{1}{2} \frac{d\mathbf{H}_\varphi^2}{d\mathbf{A}_\varphi} \mathbf{A}_{\varphi,r}.$$

Substituting Eqs. (48) and (50) into the above equation, we get the necessary result. \square

Since \mathbf{A}_φ is an equipotential surface function, \mathbf{H}_φ can also be written in the form

$$\mathbf{H}_\varphi = \frac{-\Delta \sin\theta(i\Omega - \dot{\varphi})}{\sqrt{R}} \mathbf{A}_{\varphi,\theta}. \quad (51)$$

Corollary 1. \mathbf{H}_φ as given by Eq. (49) and/or Eq. (51) will be an equipotential surface function if and only if

$$R\partial_r\left(\frac{\chi}{\sqrt{R}}\right) + \Theta\partial_\theta\left(\frac{\chi}{\sqrt{\Theta}}\right) = 0, \quad (52)$$

where

$$\chi = \Delta \sin\theta(i\Omega - \dot{\varphi}).$$

Proof: From Eqs. (49) and (51),

$$\mathbf{H}_\varphi = \frac{\chi}{\sqrt{\Theta}} \mathbf{A}_{\varphi,r} = -\frac{\chi}{\sqrt{R}} \mathbf{A}_{\varphi,\theta},$$

and since H_φ is to be an equipotential surface function

$$\sqrt{R}\mathbf{H}_{\varphi,r} + \sqrt{\Theta}\mathbf{H}_{\varphi,\theta} = 0.$$

Since \mathbf{A}_φ is an equipotential function, the above two equations immediately imply Eq. (52). \square

One of the main advantages, and justification for Eq. (36) is that the expression for \mathbf{H}_φ as derived above naturally satisfies the Znajek regularity condition.

Theorem 2. The form of \mathbf{H}_φ as given in Eq. (51) [or equivalently in Eq. (49)] satisfies the Znajek regularity condition, Eq. (32), when

$$\sqrt{R}|_{r_+} = -C.$$

Note, the only choices for $\sqrt{R}|_{r_+}$ are $\pm C$. The physical consequence of this choice will be made clear in theorem (4).

Proof: From Eq. (22),

$$\frac{\mathbf{A}_{\varphi,\theta}}{\sqrt{\gamma}} = \mathbf{B}^r.$$

Also,

$$\begin{aligned} \mathbf{H}_\varphi &= \frac{-\Delta \sin\theta(i\Omega - \dot{\varphi})}{\sqrt{R}} \mathbf{A}_{\varphi,\theta} \\ &= \frac{-\sin^2\theta}{\alpha\sqrt{R}} \rho^2 \Delta(i\Omega - \dot{\varphi}) \mathbf{B}^r. \end{aligned}$$

We choose the square root such that

$$\sqrt{R}|_{\Delta=0} = -C|_{\Delta=0}.$$

Here, C is the function defined in Eq. (42). Also,

$$\rho^2 \Delta i|_{\Delta=0} = (r_+^2 + a^2)^2 \mathcal{E} - 2aMr_+ \mathcal{L} = 2Mr_+ C|_{\Delta=0},$$

and

$$\rho^2 \Delta \dot{\varphi}|_{\Delta=0} = aC|_{\Delta=0}.$$

Putting all the above equations together, we get that

$$\begin{aligned} \mathbf{H}_\varphi|_{r_+} &= \frac{\sin^2\theta}{\alpha(-\sqrt{R})} (2Mr_+ C\Omega - aC) \mathbf{B}^r|_{r_+} \\ &= \frac{\sin^2\theta}{\alpha} (2Mr_+ \Omega - a) \mathbf{B}^r|_{r_+}, \end{aligned} \quad (53)$$

which is the required Znajek regularity condition. \square

V. THE EXTRACTION OF ENERGY AND ANGULAR MOMENTUM VIA PARTICLE JETS

In Sec. III, the terms representing the extraction of energy and angular momentum (Eqs. (33) and (34)) were derived from the conserved quantities of the electromagnetic stress tensor. In this section, we will do the same for the matter fields. This will be useful for calculating the energy released via matter jets (from possibly supermassive black holes). Separate conservation of field and particle energy and angular momentum stems from the fact that we are only considering a force-free magnetosphere. In particular, the total energy-momentum tensor of magneto-hydrodynamics can be written in the form:

$$T^{\mu\nu} = T_{\text{em}}^{\mu\nu} + T_{\text{matter}}^{\mu\nu}.$$

Since this must be divergence free, we must have that

$$\nabla_\mu T^{\mu\nu} = \nabla_\mu T_{\text{em}}^{\mu\nu} + \nabla_\mu T_{\text{matter}}^{\mu\nu} = 0.$$

But, from Eq. (15) we see that

$$\nabla_\mu T_{\text{em}}^{\mu\nu} = 0.$$

Consequently, we must have that

$$\nabla_\mu T_{\text{matter}}^{\mu\nu} = 0,$$

as mentioned above. When the magnetosphere is filled with a geodesic congruence, we can take the matter energy-momentum tensor as

$$T_{\text{matter}}^{\mu\nu} = (m/e)F(r, \theta)u^\mu u^\nu, \quad (54)$$

where e and m are the electric charge and mass of the particle species, respectively, and F and u^μ are as given in

Eq. (36). For then,

$$\nabla_{\mu} T_{\text{matter}}^{\mu\nu} = (m/e)\nabla_{\mu}(Fu^{\mu})u^{\nu} + (m/e)Fu^{\mu}\nabla_{\mu}u^{\nu} = 0.$$

In the equation above, the first term on the left-hand side vanishes due to the continuity equation for the electromagnetic current density, and the second term is trivial because of the geodesic equation. If the charge of the currents carrying particles are position dependent, we must either take the derivative of the e/m term in the above equation, or consider the regions of different values for e/m separately. Implicit in the form of the current vector congruence Eq. (36), and more so in the matter energy-momentum tensor Eq. (54), is our simplifying assumption about the nature of these geodesic currents. Such expressions may only be realized when any fixed point in the magnetosphere contains only one type of charge moving in one fixed direction. However, different points in our absolute space may have different species of charges moving in unrelated direction.

A. Matter energy flux

Since K is a Killing vector field of the Kerr geometry,

$$\nabla_{\mu}(T_{\nu}^{\mu}{}_{\text{matter}}K^{\nu}) = 0,$$

i.e.,

$$\frac{1}{\alpha\sqrt{\gamma}}\partial_t(\alpha\sqrt{\gamma}T_{\text{matter}}^t) + \frac{1}{\alpha\sqrt{\gamma}}\partial_i(\alpha\sqrt{\gamma}T_{\text{matter}}^i) = 0.$$

Let

$$e = -\alpha T_{\text{matter}}^t \quad (55)$$

be the energy density of the matter fields, and

$$\mathbf{S}^i = -\alpha T_{\text{matter}}^i \quad (56)$$

be the matter energy density flux. Therefore,

$$\frac{d}{dt}\int_V e dV = -\int_V \nabla \cdot \mathbf{S} dV = -\int_{\partial V} \mathbf{S} \cdot n dA. \quad (57)$$

Here, V is a 3-dimensional manifold in our absolute space, ∂V is the boundary of V , and n is the unit outward pointing normal on ∂V as per the divergence theorem. Consequently, the rate of matter extraction of energy from the black hole is given by

$$\frac{dE_m}{dt} = \int_{r_2=\infty} \mathbf{S}^r \sqrt{\gamma} d\theta d\varphi. \quad (58)$$

Clearly,

$$\mathbf{S}^r = -\alpha \frac{m}{e} F \frac{\sqrt{R}}{\rho^2} (g_{tt}\dot{t} + g_{t\varphi}\dot{\varphi}). \quad (59)$$

But

$$-(g_{tt}\dot{t} + g_{t\varphi}\dot{\varphi})$$

is nothing more than the poloidal function \mathcal{E} , therefore,

$$\frac{dE_m}{dt} = \frac{m}{e} \int_{r_2=\infty} F \sqrt{R} \mathcal{E} \sin\theta d\theta d\varphi. \quad (60)$$

B. Matter angular momentum flux

Using the Killing vector field $m \equiv \partial_{\varphi}$, we can deduce a similar expression for the rate of extraction of angular momentum from the black hole via particle jets. Setting

$$l = \alpha T_{\varphi}^t{}_{\text{matter}} \quad (61)$$

as the angular momentum density of the matter fields, and

$$\mathbf{L}^i = \alpha T_{\varphi}^i{}_{\text{matter}} \quad (62)$$

as the matter angular momentum density flux, we see that the rate of extraction of angular momentum from the black hole via particle jets is given by

$$\frac{dL_m}{dt} = \frac{m}{e} \int_{r_2=\infty} F \sqrt{R} \mathcal{L} \sin\theta d\theta d\varphi. \quad (63)$$

VI. THE AREA THEOREM AND THE BLANDFORD-ZNAJEK MECHANISM

In [1], it was shown that for electromagnetic extraction of energy from the horizon of the black hole, the rate of energy extraction was related to the rate of angular momentum extraction by the expression

$$\frac{dE_{\text{em}}}{dt} \leq \Omega_H \frac{dL_{\text{em}}}{dt}. \quad (64)$$

Here,

$$\Omega_H = \frac{a}{r_+^2 + a^2} \quad (65)$$

is the angular velocity of the event horizon. A black hole that is adiabatically evolving along a Kerr sequence such that its rate of energy and angular momentum extraction satisfies

$$\frac{dE}{dt} \leq \Omega_H \frac{dL}{dt} \quad (66)$$

cannot lead to the formation of a naked singularity [15]. This can be easily seen as follows. The irreducible mass M_{irr} of a Kerr Black hole is given by

$$M_{\text{irr}}^2 = \frac{1}{2}[M^2 + \sqrt{M^4 - J^2}], \quad (67)$$

where $J = aM$ is the angular momentum of the black hole. Under extraction rates subject to Eq. (66), the mass and the angular momentum of the Black hole decreases to

$$M \rightarrow M + \delta M \quad \text{and} \quad J \rightarrow J + \delta J, \quad (68)$$

where δM and δJ satisfy

$$\delta M \geq \Omega_H \delta J. \quad (69)$$

We now state the following calculation due to Christodoulou [15] without proof.

Theorem 3. When energy and angular momentum is extracted from a Kerr black hole so that Eq. (69) [or equivalently Eq. (66)] is satisfied, we have that $\delta M_{\text{irr}}^2 \geq 0$.

The area of the event horizon is given by

$$A = \int_{r_+} \sqrt{g_{\theta\theta}g_{\varphi\varphi}} d\theta d\varphi = 16\pi M_{\text{irr}}^2. \quad (70)$$

Since the area of the event horizon is proportional to M_{irr}^2 , from the theorem above, we see that it is nondecreasing, and so, we have the following corollary.

Corollary 2. A black hole that is adiabatically evolving along a Kerr sequence such that its rate of energy and angular momentum extraction satisfies Eq. (66) always contains an event horizon that cloaks the physical singularity of the Kerr geometry.

Of course, when all the angular momentum is extracted, the Kerr black hole reduces to the Schwarzschild solution, and here too we do not have a naked singularity. A dynamically evolving black hole, in general, need not follow a Kerr sequence. In this case, the powerful area theorem of Stephan Hawking [16] requires that the area of the event horizon continues to be nondecreasing.

We conclude our analysis of the geodesics currents in the force-free magnetosphere of the Kerr geometry by showing that the matter energy and angular momentum extraction rates are subject to the same inequality as their electromagnetic counterparts [Eq. (66)], and hence we are in no danger of violating the Penrose conjecture.

Theorem 4. When the currents in a force-free magnetosphere are given by Eq. (36), the extraction of energy from the matter fields will not lead to the formation of a naked singularity.

Proof: At the horizon, if charge are to flow out to the magnetosphere, from theorem (2) we need that

$$\sqrt{R}|_{r_+} = -C > 0.$$

This is possible only when the geodesics are such that

$$\mathcal{E} \leq \Omega_H \mathcal{L}, \quad (71)$$

and consequently from Eqs. (60) and (63)

$$\frac{dE_m}{dt} \leq \Omega_H \frac{dL_m}{dt}. \quad (72)$$

If extraction of energy happens outside the event horizon (but within the ergosphere), and charges actually fall inward at the event horizon, a similar argument still leads to the above equation. Therefore, corollary (2) implies the necessary result. \square

VII. CONCLUSIONS

The mechanism to form jets from black holes is a major unsolved problems in astrophysics, and its explanation is important for our understanding of nonthermal radiation form solar-mass black holes in the Milky Way, newly born black holes associated with gamma-ray bursts, and super-massive black holes at the centers of distant galaxies. One possibility is that black holes with radio-emitting jets are distinguished from radio-quiet black holes by extracting rotational energy from the black hole through the Blandford-Znajek process. Matter currents can also carry energy and angular momentum, and represent a plausible alternative to electromagnetic Poynting flux for extracting the black hole's rotational energy.

The astrophysical implications of our analytic study maybe limited, and is used mostly as a motivational tool. In this paper, we have used the 3 + 1 formulation of general relativity to analyze currents that flow in the force-free magnetosphere near a Kerr black hole. We have rewritten the constraint equation for a force-free magnetosphere, Eq. (31), to apply to particles that follow timelike geodesics, yielding Eq. (48). The toroidal magnetic susceptibility, \mathbf{H}_φ , from which the poloidal current is derived, is shown to satisfy the Znajek regularity condition. Expressions for energy and angular momentum flux associated with the matter currents were obtained. We show that extraction of energy and angular momentum cannot lead to the formation of a naked singularity. In future work, trajectories of particles in the black hole magnetosphere will be derived, and will be compared with observations of jets from black holes.

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