PHYSICAL REVIEW D **79,** 121301(R) (2009)

Coexistence of matter dominated and accelerating solutions in $f(\mathcal{G})$ gravity

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(Received 17 April 2009; published 5 June 2009)

Working within the theory of modified Gauss-Bonnet gravity, we show that Friedmann-Lemaître-Robertson-Walker–like power-law solutions only exist for a very special class of $f(\mathcal{G})$ theories. Furthermore, we point out that any transition from decelerated to accelerated expansion must pass through $\mathcal{G} = 0$, and no function $f(\mathcal{G})$ that is differentiable at this point can admit both a decelerating power-law solution and any accelerating solution. This strongly constrains the cosmological viability of $f(\mathcal{G})$ gravity, since it may not be possible to obtain an expansion history of the Universe which is compatible with observations. We explain why the same issue does not occur in $f(R)$ gravity and discuss possible caveats for the case of $f(\mathcal{G})$ gravity.

DOI: 10.1103/PhysRevD.79.121301 PACS numbers: 98.80.Cq

I. INTRODUCTION

One of the most remarkable developments in cosmology over the past decade has been the confrontation of the standard model of cosmology with observations, leading to the conclusion that the Universe is accelerating in the current epoch. This rather counterintuitive result has led to one of the most challenging theoretical puzzles in 21st century physics, namely, to provide an explanation for this late-time cosmic acceleration. Currently the most popular idea is that the energy density of the Universe is at the present time dominated by some mysterious dark component known as dark energy and there have been many proposals offered to explain its nature. At the moment, the one which appears to fit all available observations (supernovae Ia [1], cosmic microwave background anisotropies [2], large scale structure formation [3], baryon oscillations [4], and weak lensing [5]), turns out to be the so-called *concordance model* in which a tiny cosmological constant is present [6] and ordinary matter is dominated by cold dark matter (CDM). However, despite its success, the ΛCDM model is affected by significant fine-tuning problems related to the vacuum energy scale, so other exotic negative-pressure fluids, often described in terms of scalar fields [7], have been proposed to address these issues. The problem remains, however, that at the present time there is no direct experimental evidence for the existence of the scalar fields responsible for the late-time (and also the early-time) accelerated expansion rate of the Universe and this has led to a search for other viable theoretical schemes, many of which are based on the idea that the "dark sector" originates from modifications of the gravitational interaction itself.

Currently, one of the most popular alternatives to the concordance model is based on modifications of the Einstein-Hilbert action. Such models first became popular in the 1980s because it was shown that they naturally admit a phase of accelerated expansion which could be associated with an early Universe inflationary phase [8]. The fact that the phenomenology of dark energy requires the presence of a similar phase (although only a late-time one) has recently revived interest in these models. In particular, the idea that dark energy may have a geometrical origin, i.e., that there is a connection between dark energy and a nonstandard behavior of gravitation on cosmological scales, is currently a very active area of research. Additionally, such modifications are appealing due to the fact that they can evade constraints on the strength of the gravitational field which are restricted to below solar system scales.

One of the most promising modifications to date are those based on gravitational actions which are nonlinear in the Ricci curvature R and/or contain terms involving combinations of derivatives of R [9–12]. These theories have provided a number of very interesting results on both cosmological [13–17] and astrophysical [15,18] scales. One of the nice features of these theories is that the field equations can be recast in a way that the higher order corrections are written as an energy-momentum tensor of geometrical origin describing an "effective" source term on the right-hand side of the standard Einstein field equations [13,14]. In this *curvature quintessence* scenario, the cosmic acceleration can be shown to result from such a new geometrical contribution to the cosmic energy density budget, due to higher order corrections of the Hilbert-Einstein Lagrangian.

More recently modifications based on a Lagrangian density which is some general function of the Ricci scalar and the Gauss-Bonnet term $f(R, \mathcal{G})$ [19,20] have been studied in the context of cosmology. In particular, a simple extension of Einstein gravity: $f(R, \mathcal{G}) = R/2 + f(\mathcal{G})$ has been investigated by a number of authors [21–27]. Corrections of this type can be motivated from low-energy string effective actions and compactification of other higher dimensional theories [28]. Uddin, Lidsey, and Tavakol [29] considered the existence and stability of power-law scaling solutions in $f(\mathcal{G})$ models and found that scaling solutions exist in the model $f(\mathcal{G}) = \pm 2\sqrt{\alpha\mathcal{G}}$, where α is an arbitrary constant. Inspired by the work on $f(R)$ gravity by Carloni *et al.* [30] and Amendola *et al.* [31], Zhou *et al.* [32,33] used dynamical systems techniques to study the cosmology of $f(\mathcal{G})$ models and argued that one could find cosmologically viable trajectories that mimic the ΛCDM cosmic history in the radiation and matter dominated periods, but also have a distinctive signature at late times

In this paper, we investigate the cosmological viability of $f(\mathcal{G})$ models by investigating the conditions under which one can find power-law solutions that mimic the standard Friedmann-Lemaître-Robertson-Walker (FLRW) expansion history of the Universe. We discover that such solutions only exist for a very special class of $f(\mathcal{G})$ theories. Furthermore, we show that for there to be a transition between decelerated and accelerated expansion phases, $\mathcal{G}$ must pass through zero, and there are no functions $f(\mathcal{G})$ that are differentiable at this point that admit both a power-law decelerating solution and any accelerating solution. This seriously constrains the cosmological viability of $f(\mathcal{G})$ gravity, since it may not be possible to find an expansion history of the Universe compatible with observations.

II. FIELD EQUATIONS FOR GENERAL $f(\mathcal{G})$

We consider the following action within the context of four-dimensional homogeneous and isotropic spacetimes, i.e., the FLRW universes with no spatial curvature:

$$\mathcal{A} = \int d^4x\sqrt{-g}[R + f(\mathcal{G}) + \mathcal{L}_m], \tag{1}$$

where R is the Ricci scalar f is a general differentiable (at least C^2) function of the Gauss-Bonnet term,

$$\mathcal{G} = R^2 - 4R_{\alpha\beta}R^{\alpha\beta} + R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}, \tag{2}$$

and $\mathcal{L}_m$ represents the matter contribution. As we know, in four dimensions, the Gauss-Bonnet term is a total differential, and hence for $f(\mathcal{G}) = \mathcal{G}$, the field equations remain invariant. However for other functions this term has non-trivial contributions to the field equations. For the homogeneous and isotropic spacetimes, the field equations are

(i) the Raychaudhuri equation

$$\dot{\Theta} + \frac{1}{3}\Theta^2 = -\frac{\kappa^2}{2}(\rho + 3P) + \frac{4}{9}\Theta^3 f_{\mathcal{G}\mathcal{G}}\dot{\mathcal{G}} - (f - \mathcal{G}f_{\mathcal{G}}) - \frac{3}{\Theta}\mathcal{G}\dot{f}_{\mathcal{G}} - \frac{4}{3}\Theta^2\ddot{f}_{\mathcal{G}}, \tag{3}$$

where Θ is the volume expansion which defines a scale factor $a(t)$ along the fluid flow lines via the standard relation $\Theta = 3\dot{a}/a$, and $f_{n\mathcal{G}}$ abbreviates $\partial^n f/(\partial\mathcal{G})^n$ for $n = 1, 2$;

(ii) the Friedmann equation

$$\Theta^2 + \tfrac{8}{3}\Theta^3 f_{\mathcal{G}\mathcal{G}}\dot{\mathcal{G}} + 3f = 3\kappa^2\rho + 3\mathcal{G}f_{\mathcal{G}}; \tag{4}$$

(iii) the total trace of the Einstein equations

$$-4\Theta^2 - 6\dot{\Theta} = 3\kappa^2(3P - \rho) + 12(f - \mathcal{G}f_{\mathcal{G}}) + \frac{8}{3}\Theta^3 f_{\mathcal{G}\mathcal{G}}\dot{\mathcal{G}} + \frac{18}{\Theta}\mathcal{G}\dot{f}_{\mathcal{G}} + 8\Theta^2\ddot{f}_{\mathcal{G}}; \tag{5}$$

(iv) the conservation equation for standard matter

$$\dot{\rho} = -\Theta(\rho + P). \tag{6}$$

For FLRW spacetimes, the Gauss-Bonnet term is given by

$$\mathcal{G} = \frac{8}{9}\Theta^2\Big[\dot{\Theta} + \frac{1}{3}\Theta^2\Big] = 24\frac{\dot{a}^2\ddot{a}}{a^3}. \tag{7}$$

We can easily see that accelerating models have $\mathcal{G} > 0$, while decelerating models have $\mathcal{G} < 0$. In particular, any expansion history evolving from deceleration to acceleration must pass through $\mathcal{G} = 0$. This observation will be of importance in the following discussion.

A. Requirements for the existence of power-law solutions

Let us now assume there exists an *exact* power-law solution to the field equations; i.e., the scale factor behaves as

$$a(t) = a_0 t^m. \tag{8}$$

From now on, we assume that m is a *fixed* real number. If $0 < m < 1$, then the required power-law solution is *decelerating*, while for $m > 1$ it is *accelerating*. Since we know that within the standard paradigm, the expansion history of the Universe underwent a power-law decelerating phase, it is important to study these kinds of exact solutions in our modified gravity models.

We further assume that matter can be described by a barotropic perfect fluid such that $P = w\rho$, with $w \in [-1, 1]$. From the energy conservation equation, we obtain

$$\rho(t) = \rho_0 t^{-3m(1+w)}, \tag{9}$$

and the Gauss-Bonnet term becomes

$$\mathcal{G} = 24m^3(m - 1)t^{-4} \equiv \alpha_m t^{-4}. \tag{10}$$

The negative sign of $\mathcal{G}$ for all decelerating models is reflected by $\alpha_m < 0$ for the power-law models with $0 < m < 1$.

Using the background solutions above, we can write the Friedmann, Raychaudhuri and trace equations in terms of functions of time t only. We can assume with no loss of generality that $t > 0$. We then solve (10) for t and rewrite these equations in terms of $\mathcal{G}$, $f(\mathcal{G})$ and its derivatives. The Friedmann equation, for example, becomes

$$f - f_{\mathcal{G}}\mathcal{G} + 3m^2\sqrt{\frac{\mathcal{G}}{\alpha_m}} - \frac{96 f_{\mathcal{G}\mathcal{G}}\mathcal{G}^2 m^3}{\alpha_m} - K\left(\frac{\mathcal{G}}{\alpha_m}\right)^{(3/4)m(1+w)} = 0, \tag{11}$$

where $K = \rho_0 a_0^{3(1+w)}$. Note that for the power-law solution (8), $\mathcal{G}/\alpha_m$ is positive at all times by definition (10), and therefore Eq. (11) is real-valued over the range of $\mathcal{G}$.

Since we want (8) to be a solution at all times, i.e., $\mathcal{G}$ spans over an entire branch of the real axis, we can interpret (11) as a differential equation for the function f in $\mathcal{G}$ space. This differential equation has the general solution

$$f(\mathcal{G}) = A_m\sqrt{\tilde{\mathcal{G}}} + B_{mw}\tilde{\mathcal{G}}^{(3/4)m(1+w)} + C_1\mathcal{G} + C_2\mathcal{G}^{(1/4)-(m/4)}. \tag{12}$$

We have abbreviated $\tilde{\mathcal{G}} \equiv 24\mathcal{G}/\alpha_m$, which we repeat is positive for the power-law solution (8), and

$$A_m = -\sqrt{\frac{3}{2}}\frac{m^2(m-1)}{m+1}, \tag{13}$$

$$B_{mw} = \frac{-2^{2-(9/4)m(1+w)}3^{-(3/4)m(1+w)}(m-1)K}{4 - m(19+15w) + 3m^2(4+7w+3w^2)}. \tag{14}$$

The constants $C_{1,2}$ are constants of integration. Since we know that the term linear in $\mathcal{G}$ does not change the field equations, we can without any loss of generality assume $C_1 = 0$. Furthermore, if we wish to ensure that for $m = 2/[3(1+w)]$ and $K = 4/[3(1+w)^2]$, the theory reduces to general relativity (GR), i.e., $f(\mathcal{G}) = 0$, then this constrains $C_2 = 0$. Hence, for an exact power-law solution to exist, the required form of the function f becomes

$$f(\mathcal{G}) = A_m\sqrt{\tilde{\mathcal{G}}} + B_{mw}\tilde{\mathcal{G}}^{(3/4)m(1+w)}. \tag{15}$$

We note that the above form of f identically satisfies the other field equations, if we similarly convert them as differential equations in $\mathcal{G}$ space. The coefficients A_m and B_{mw} are real-valued and nonzero unless $m = 1$, in which case $a(t) \propto t$. In general, the function $f(\mathcal{G})$ is real-valued only if $\mathcal{G}/\alpha_m > 0$, which is true by construction for the exact power-law solution (8). Similar results were found in [29] using a scalar-tensor approach to $f(\mathcal{G})$ gravity.

It is interesting to note that an exact GR-like solution $[m = 2/3(1+w)]$ is possible with nonzero C_2 for $f(\mathcal{G}) = C_2\mathcal{G}^{(1+3w)/4(1+w)}$ for values of w for which $f(\mathcal{G})$ is real-valued.

B. Coexistence of decelerating power-law solutions and accelerating solutions

To mimic the standard expansion history of the Universe in $f(\mathcal{G})$ gravity, we assume there exists an exact decelerating power-law solution, and that the Universe was well described by this solution in the past, before coming to the accelerated phase. As argued in the previous section, the existence of the exact solution fixes the form of $f(\mathcal{G})$ as in (15), and the deceleration fixes the power m as $0 < m < 1$, implying $\alpha_m < 0$.

Now, if any *additional* accelerating solution exists in the whole solution space of the model, then $\mathcal{G} > 0$ for this solution as evident from (7). However, we can see from the form of $f(\mathcal{G})$ that this is not possible, as for $\mathcal{G} > 0$ the function is no longer real-valued.

This problem can be artificially remedied by including absolute values of $\tilde{\mathcal{G}}$ in (15) (similarly to [32]), i.e., redefining

$$\tilde{f}(\mathcal{G}) = A_m\sqrt{|\tilde{\mathcal{G}}|} + B_{mw}|\tilde{\mathcal{G}}|^{(3/4)m(1+w)}. \tag{16}$$

This function $\tilde{f}(\mathcal{G})$ now seems to allow for both a decelerating power-law solution and accelerating solutions. However, $\tilde{f}(\mathcal{G})$ is not differentiable at $\mathcal{G} = 0$ and hence no longer a C^2 function, which is required for any Lagrangian in the Einstein-Hilbert action. Any expansion history evolving from deceleration to acceleration must pass through $\mathcal{G} = 0$, and we can easily see that the field equations are no longer defined at this point. Hence we can conclude that no *well-defined C^2 action in $f(\mathcal{G})$ gravity can allow for exact decelerating power-law solutions to coexist with accelerating solutions.*

C. Comparison to $f(R)$ gravity

One important difference between $f(R)$ and $f(\mathcal{G})$ gravity is that in $f(\mathcal{G})$ gravity the energy-momentum tensor decouples from the correction terms. In $f(R)$ gravity on the other hand, the correction term modifies the matter energy-momentum tensor, which becomes rescaled to $\tilde{T}^M_{ab} = \frac{1}{f'(R)}T^M_{ab}$. Since in $f(\mathcal{G})$ gravity matter decouples from the correction terms any matter dominated era must strictly obey the corresponding GR type power-law behavior. However, in $f(R)$ gravity a similar constraint on the form of $f(R)$ can be found by requiring the existence of exact power-law solutions; this is currently under investigation [34].

The analogue also breaks down in the sense that any cosmologically viable model must make a transition from deceleration to acceleration, and therefore go through $\mathcal{G} = 0$ [see (7)]. This poses a problem for the simple models including terms of the form $\mathcal{G}^n$ with $n < 1$, which are not differentiable at $\mathcal{G} = 0$. The value of the Ricci scalar R, on the other hand, does not necessarily pass through $R = 0$, and therefore models including terms like R^n may still be viable. In fact if $f(R)_{,R}|_{R\to 0\pm} = 0$ for any $f(R)$-gravity model, then the plane $R = 0$ is actually an invariant submanifold. This means that no solution can cross that plane in phase space. For example, for any $n < 1$, R^n is not differentiable at $R = 0$, but any solution can only approach this plane from either side and not reach it in finite time. In fact, as shown in [35,36], in R^n gravity a Friedmann-like matter dominated decelerated solution coexists with a de Sitter-like accelerated solution. These two fixed points can be linked without R changing sign; i.e., the field equations are well defined and fully differentiable along the entire orbit.

III. DISCUSSION AND CONCLUSION

The expansion history of the Universe is thought to have undergone a phase of decelerated power-law expansion followed by late-time acceleration. Therefore, power-law solutions play an important role in cosmology as matter dominated phases that later connect to an accelerating phase.

In this work we have shown that *exact* power-law solutions in $f(\mathcal{G})$ gravity only exist for the very special class of models given in (15). In particular, many of the popular $f(\mathcal{G})$ models, e.g., all examples considered in [37], do not allow for any exact power-law solutions. This means that for these models, there exists no exact matter dominated solution, and therefore these models may not be of cosmological interest.

Furthermore, we have shown that decelerating power-law solutions cannot coexist with *any* accelerating solutions, since no differentiable function $f(\mathcal{G})$ can admit both decelerating power-law solutions and accelerating solutions. The problem may be remedied by including absolute values in (15). However, this makes $f(\mathcal{G})$ nondifferentiable at $\mathcal{G} = 0$, which is the value $\mathcal{G}$ takes when the scale factor evolves from deceleration to acceleration, and therefore cannot be ignored. This result seriously constrains the cosmological viability of $f(\mathcal{G})$ gravity, since it may not be possible to obtain an expansion history similar to the ΛCDM model in this context.

This issue has not been addressed in recent papers [32], where dynamical systems methods were used to study certain classes of $f(\mathcal{G})$-gravity models. It must be emphasized that even if equilibrium points corresponding to both decelerating power-law and accelerating solutions are found to coexist as in [32], there cannot exist any trajectories connecting these points, since these solutions can only coexist for actions that are nondifferentiable at the transition from deceleration to acceleration.

One possible way around this problem stems from the fact that our conclusions are based on the requirement that there exists an *exact* power-law solution. If however the scale factor behaves like $a(t) \propto e^{\Lambda t} t^m$, then at early times $a(t) \propto t^m$. In this case the basic assumption of our analysis is not satisfied.

We therefore conclude that when looking for cosmologically viable solutions or fixed points in the dynamical systems state space in $f(\mathcal{G})$ gravity, one should not look for exact power-law solutions, but rather exact solutions that approximate decelerating power-law solutions at early times. Furthermore, we must restrict ourselves to functions $f(\mathcal{G})$ that are at least C^2 functions, and, in particular, differentiable at $\mathcal{G} = 0$.

ACKNOWLEDGMENTS

The authors would like to thank the National Research Foundation (South Africa) for financial support. R. G. would like to thank the University of Cape Town for support.

[1] S. Perlmutter *et al.*, Astrophys. J. **517**, 565 (1999); A. G. Riess *et al.*, Astron. J. **116**, 1009 (1998); J. L. Tonry *et al.*, Astrophys. J. **594**, 1 (2003); R. A. Knop *et al.*, Astrophys. J. **598**, 102 (2003); A. G. Riess *et al.*, Astrophys. J. **607**, 665 (2004); S. Perlmutter *et al.*, Astrophys. J. **517**, 565 (1999); Astron. Astrophys. **447**, 31 (2006).

[2] D. N. Spergel *et al.*, Astrophys. J. Suppl. Ser. **148**, 175 (2003); **170**, 377 (2007).

[3] M. Tegmark *et al.*, Phys. Rev. D **69**, 103501 (2004); U. Seljak *et al.*, Phys. Rev. D **71**, 103515 (2005); S. Cole *et al.*, Mon. Not. R. Astron. Soc. **362**, 505 (2005).

[4] D. J. Eisenstein *et al.*, Astrophys. J. **633**, 560 (2005)

[5] B. Jain, A. Taylor, Phys. Rev. Lett. **91**, 141302 (2003).

[6] P. Astier *et al.*, Astron. Astrophys. **447**, 31 (2006).

[7] C. Wetterich, Nucl. Phys. **B302**, 668 (1988); B. Ratra and J. Peebles, Phys. Rev. D **37**, 3406 (1988); T. Chiba, T. Okabe, and M. Yamaguchi, Phys. Rev. D **62**, 023511 (2000); R. R. Caldwell, Phys. Lett. B **545**, 23 (2002); B. Feng, X. L. Wang, and X. M. Zhang, Phys. Lett. B **607**, 35 (2005).

[8] A. A. Starobinsky, Phys. Lett. **91B**, 99 (1980); K. S. Stelle, Gen. Relativ. Gravit. **9**, 353 (1978).

[9] S. M. Carroll, V. Duvvuri, M. Trodden, and M. S. Turner, Phys. Rev. D **70**, 043528 (2004); S. Nojiri and S. D. Odintsov, Phys. Rev. D **68**, 123512 (2003); S. Capozziello, Int. J. Mod. Phys. D **11**, 483 (2002); S. Mustafa, arXiv:gr-qc/0607116; V. Faraoni, Phys. Rev. D **72**, 124005 (2005); M. L. Ruggiero and L. Iorio, J. Cosmol. Astropart. Phys. 01 (2007) 010; A. de la Cruz-Dombriz and A. Dobado, Phys. Rev. D **74**, 087501 (2006); N. J. Poplawski, Phys. Rev. D **74**, 084032 (2006); Classical Quantum Gravity **24**, 3013 (2007); A. W.

Brookfield, C. van de Bruck, and L. M. H. Hall, Phys. Rev. D **74**, 064028 (2006); Y. Song, W. Hu, and I. Sawicki, Phys. Rev. D **75**, 044004 (2007); B. Li, K. Chan and M. Chu, Phys. Rev. D **76**, 024002 (2007); X. Jin, D. Liu, and X. Li, arXiv:astro-ph/0610854; T. P. Sotiriou and S. Liberati, Ann. Phys. (N.Y.) **322**, 935 (2007); T. P. Sotiriou, Classical Quantum Gravity **23**, 5117 (2006); R. Bean, D. Bernat, L. Pogosian, A. Silvestri, and M. Trodden, Phys. Rev. D **75**, 064020 (2007); I. Navarro and K. Van Acoleyen, J. Cosmol. Astropart. Phys. 02 (2007) 022; A. J. Bustelo and D. E. Barraco, Classical Quantum Gravity **24**, 2333 (2007); G. J. Olmo, Phys. Rev. D **75**, 023511 (2007); J. Ford, S. Giusto, and A. Saxena, Nucl. Phys. **B790**, 258 (2008); F. Briscese, E. Elizalde, S. Nojiri, and S. D. Odintsov, Phys. Lett. B **646**, 105 (2007); S. Baghram, M. Farhang, and S. Rahvar, Phys. Rev. D **75**, 044024 (2007); D. Bazeia, B. Carneiro da Cunha, R. Menezes, and A. Petrov, Phys. Lett. B **649**, 445 (2007); P. Zhang, Phys. Rev. D **76**, 024007 (2007); B. Li and J. D. Barrow, Phys. Rev. D **75**, 084010 (2007); T. Rador, Phys. Lett. B **652**, 228 (2007); Phys. Rev. D **75**, 064033 (2007); L. M. Sokolowski, Classical Quantum Gravity **24**, 3391 (2007); V. Faraoni, Phys. Rev. D **75**, 067302 (2007); O. Bertolami, C. G. Boehmer, T. Harko, and F. S. N. Lobo, Phys. Rev. D **75**, 104016 (2007); S. K. Srivastava, Int. J. Theor. Phys. **47**, 1966 (2008); S. Capozziello, V. F. Cardone, and A. Troisi, J. Cosmol. Astropart. Phys. 08 (2006) 001; A. A. Starobinsky, arXiv:0706.2041; M. Ishak and J. Moldenhauer, J. Cosmol. Astropart. Phys. 01 (2009) 024; D. A. Easson, F. P. Schuller, M. Trodden, and M. N. R. Wohlfarth, Phys. Rev. D **72**, 043504 (2005).

[10] R. Kerner, Gen. Relativ. Gravit. **14**, 453 (1982); J. P. Duruisseau and R. Kerner, Classical Quantum Gravity **3**, 817 (1986).

[11] P. Teyssandier, Classical Quantum Gravity **6**, 219 (1989).

[12] G. Magnano, M. Ferraris, and M. Francaviglia, Gen. Relativ. Gravit. **19**, 465 (1987).

[13] S. Capozziello, V. F. Cardone, S. Carloni, and A. Troisi, Int. J. Mod. Phys. D **12**, 1969 (2003).

[14] S. Capozziello, S. Carloni, and A. Troisi, Recent Res. Dev. Astron. Astrophys. **1**, 625 (2003).

[15] S. Capozziello, V. F. Cardone, and A. Troisi, J. Cosmol. Astropart. Phys. 08 (2006) 001.

[16] K. i. Maeda and N. Ohta, Phys. Lett. B **597**, 400 (2004); Phys. Rev. D **71**, 063520 (2005); N. Ohta, Int. J. Mod. Phys. A **20**, 1 (2005); K. Akune, K. i. Maeda, and N. Ohta, Phys. Rev. D **73**, 103506 (2006).

[17] Kishore N. Ananda, Sante Carloni, and Peter K. S. Dunsby, arXiv:0812.2028; arXiv:0809.3673.

[18] S. Capozziello, V. F. Cardone, A. Troisi, Mon. Not. R. Astron. Soc. **375**, 1423 (2007).

[19] S. M. Carroll, A. De Felice, V. Duvvuri, D. A. Easson, M. Trodden, and M. S. Turner, Phys. Rev. D **71**, 063513 (2005).

[20] S. Nojiri and S. D. Odintsov, Gen. Relativ. Gravit. **36**, 1765 (2004).

[21] S. Nojiri and S. D. Odintsov, Phys. Lett. B **631**, 1 (2005).

[22] A. De Felice and M. Hindmarsh, J. Cosmol. Astropart. Phys. 06 (2007) 028.

[23] B. Li, J. D. Barrow, and D. F. Mota, Phys. Rev. D **76**, 044027 (2007).

[24] S. C. Davis, arXiv:0709.4453.

[25] S. Nojiri, S. D. Odintsov, and P. V. Tretyakov, Prog. Theor. Phys. Suppl. **172**, 81 (2008).

[26] S. Nojiri, S. D. Odintsov, and O. G. Gorbunova, J. Phys. A **39**, 6627 (2006).

[27] G. Cognola, E. Elizalde, S. Nojiri, S. D. Odintsov, and S. Zerbini, Phys. Rev. D **73**, 084007 (2006).

[28] S. Nojiri, S. D. Odintsov, and M. Sami, Phys. Rev. D **74**, 046004 (2006).

[29] K. Uddin, J. E. Lidsey, and R. Tavakol, arXiv:0903.0270.

[30] S. Carloni, A. Troisi, and P. K. S. Dunsby, arXiv:0706.0452; M. Abdelwahab, S. Carloni, and P. K. S. Dunsby, Classical Quantum Gravity **25**, 135002 (2008).

[31] L. Amendola, R. Gannouji, D. Polarski, and S. Tsujikawa, Phys. Rev. D **75**, 083504 (2007).

[32] Shuang-Yong Zhou, Edmund J. Copeland, and Paul M. Saffin, arXiv:0903.4610.

[33] G. Cognola, E. Elizalde, S. Nojiri, S. D. Odintsov, L. Sebastiani, and S. Zerbini, Phys. Rev. D **77**, 046009 (2008).

[34] N. Goheer, J. Larena, P. Dunsby, and R. Goswami (to be published).

[35] S. Carloni, P. Dunsby, S. Capozziello, and A. Trois, Classical Quantum Gravity **24**, 5689 (2007).

[36] N. Goheer, J. Leach, and P. Dunsby, Classical Quantum Gravity **22**, 4839 (2005).

[37] A. De Felice and S. Tsujikawa, Phys. Lett. B **675**, 1 (2009).