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Cautionary remarks on the moduli space metric for multidyon simulations

Falk Bruckmann*

Institut für Theoretische Physik, Universität Regensburg, D-93040 Regensburg, Germany, and Institut für Theoretische Physik III, Universität Erlangen, D-91058 Erlangen, Germany

Simon Dinter,† Ernst-Michael Ilgenfritz,‡ Michael Müller-Preussker,§ and Marc Wagner‖

Humboldt-Universität zu Berlin, Institut für Physik, Newtonstraße 15, D-12489 Berlin, Germany

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We perform a detailed numerical investigation of the approximate moduli space metric proposed by Diakonov and Petrov [Phys. Rev. D **76**, 056001 (2007)] for a confining model of dyons. Our findings strongly indicate that this metric is positive definite (and, therefore, a valid moduli space metric) throughout a considerable part of configuration space only for a small number of dyons at sufficiently low density. This poses strong limitations on results obtained by an unrestricted integration over collective coordinates in this model. It also indicates that strong correlations between collective coordinates will be essential for the physical content of a dyon model, which could be exhibited by a suitable simulation algorithm.

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I. INTRODUCTION

The semiclassical approximation of path integrals around nontrivial saddle points (originating from the work of Callan, Dashen, and Gross [1,2] and later adapted to finite temperature by Gross, Pisarski, and Yaffe [3]) is a prominent but not undisputed [4–6] semianalytic approach to calculate nonperturbative effects in quantum field theories. In gauge theories at zero temperature the corresponding semiclassical topological objects used to be instantons [7], four-dimensional self-dual lumps of action density carrying one unit of topological charge. Their use in a semiclassical context was opened by the seminal paper by 't Hooft [8], where the fluctuation determinant in the background of such objects was computed. However, the resulting distribution with respect to the size parameter ρ is obviously unphysical for large instantons. Empirical interactions [9–11] have been added to the first instanton gas model [1,2] eventually leading to the instanton liquid model [12], where the average size is fixed to $\bar{\rho} = 1/3$ fm (along with a density of $n^{-1/4} = 1$ fm) to roughly match phenomenological requirements.

In pure gauge theories at finite temperature the Polyakov loop is the order parameter for confinement. The fact that there is a close relation between the Polyakov loop average and the dominating classical solutions has been fully realized with the discovery of calorons with nontrivial holonomy by Kraan and van Baal [13] and Lee and Lu [14]. From this perspective, the previously known caloron solutions due to Harrington and Shepard [15] now appear as the limiting case of trivial holonomy.[1] The eigenvalues of the untraced asymptotic Polyakov loop—called holonomy—are responsible for the dissociation of calorons into constituents N for gauge group SU(N). These are Bogomol'nyi-Prasad-Sommerfield monopoles with electric charges equal to or opposite to their magnetic charges. We will refer to them as dyons (for reviews cf. [17,18]).

Hence, a semiclassical model of finite temperature gauge theory should be based on dyons. If the holonomy is related to the order parameter, all types of dyons are of equal mass in the confined phase but expected to split into light and heavy ones in the deconfined phase. Dyon gauge fields combined to form intermediate size calorons with trivial holonomy do not generate confinement; whereas forming similar calorons with maximally nontrivial holonomy creates a linearly rising potential. This has been demonstrated numerically in [19]. The actual size distribution to be used in the confinement phase, i.e. the generalization of the instanton distribution, was inspired by the evaluation of the fluctuation determinant in Refs. [20,21]. The suppression of heavy dyons due to the nonvanishing Polyakov loop above the critical temperature could be the mechanism behind the decrease of the topological susceptibility and the vanishing of the chiral condensate [22]. In contrast to this, the chiral condensate under (unphysical) periodic boundary conditions is nonvanishing, an effect seen in various studies [23–27] that is likely to reflect the presence of light dyons.

*falk.bruckmann@physik.uni-regensburg.de
†dinter@physik.hu-berlin.de
‡ilgenfri@physik.hu-berlin.de
§mmp@physik.hu-berlin.de
‖Corresponding author.
mcwagner@physik.hu-berlin.de

[1]For a dyon model based on these solutions see [16].

Here we present results of a preparatory study of SU(2) gauge theory trying to obtain numerical insight into a dyon model recently developed by Diakonov and Petrov [18,28,29]. In this work, the role of the moduli space metric was emphasized and incorporated. The authors have generalized the known form of the metric of same kind dyons and of opposite kind dyons to a general metric valid at large distances in the parameter space of an ensemble of self-dual dyons. The problem of including dyons of opposite topological charge was postponed, which means that the model is a rather crude approximation. The authors of [28] attempted to take the effects of a mixture of self-dual and anti–self-dual objects into account by multiplying some of their results by factors of 2 or $\sqrt{2}$ assuming negligible interactions between the two systems of self-dual and anti–self-dual dyons.

The resulting moduli space metric determinant consists of Coulomb-like terms and has been treated analytically by rewriting the corresponding grand-canonical ensemble into an equivalent quantum field theory both of bosons and fermions, resembling Polyakov's famous work [30] showing Abelian confinement. The relations between physical quantities such as the string tension and the critical temperature obtained from this model are rather impressive, when comparing them with lattice results. However, in order to match phenomenological values, the model needs to be pushed to rather high densities (i.e. short distances between dyons), which is difficult to reconcile with the diluteness assumption of the underlying semiclassical approach. It should be mentioned that a similar problem afflicts the instanton liquid model as well (when choosing size and density as quoted above).

Our final goal is to complement the analytical treatment of the dyon model presented in Ref. [28] by numerical simulations of the proposed moduli space metric together with interactions stemming from the gluonic fluctuation determinant, the Faddeev-Popov determinant, and the action itself. The preparatory studies to be presented here will demonstrate that the approximate moduli space metric from [28] violates the fundamental requirement of being positive definite in an overwhelming part of dyon configuration space, when explored at densities needed to match phenomenology. We conclude that this moduli space metric can only be used in a dyon model with other terms added or if correlations between the dyons are built in that lead to the positivity of the metric. A suitable simulation algorithm should guarantee that. This would result in complicated multidyon correlations and create a nontrivially related behavior of various observables and correlators.

This paper is organized as follows. In Sec. II we briefly recall the ingredients of the dyon model in the version of Diakonov and Petrov applied to SU(2) gauge theory, which is the starting point for our work. Section III deals with the spectral properties of the proposed moduli space metric. In Sec. IV we demonstrate that even a random model of dyons, i.e. a model without moduli space metric, induces confinement. Finally, we conclude and give a brief outlook.

II. DYON ENSEMBLES À LA DIAKONOV AND PETROV

A. Holonomy as external parameter

We consider pure SU(2) gauge theory at finite temperature T, which, as usual, is implemented with periodic boundary conditions in the imaginary time direction with period $\beta = 1/T$. There are four kinds of dyons, two self-dual and two anti–self-dual. The untraced Polyakov loop at spatial infinity, also called holonomy, is an element of the gauge group. It can be diagonalized everywhere[2] to $\Omega = \exp(2\pi i\omega\sigma_3)$ with eigenvalues $e^{+2\pi i\omega}$ and $e^{-2\pi i\omega}$, where $\omega \in [0, 1/2]$. Similarly to the non-Abelian adjoint Higgs system, this gives rise to complementary dyons that have actions proportional to 2ω and $1 - 2\omega$ and that become static when well separated. They are both self-dual or anti–self-dual depending on the sign of their topological charge.

Following Diakonov and Petrov [28], throughout this paper we focus on the case of maximally nontrivial holonomy, $\omega = 1/4$. This implies that all dyons have the same action. The holonomy is then traceless and matches the confinement condition $\langle P\rangle = 0$, where $P = \mathrm{Tr}(\Omega)/2$. We adopt the same restriction to purely self-dual systems, considering K dyons of the first kind and K dyons of the second kind; i.e. the total number of dyons is $n_D = 2K$. These dyons carry equal electric and magnetic charges which can take the values ± 1.

B. The approximate moduli space metric of multidyon configurations

In [28] it has been attempted to construct an approximate multidyon moduli space metric valid for dyon separations $d \gg 1/\pi T$. The starting point for this construction was the analytically known moduli space metric of a single caloron [31], which is a pair of different kind dyons and a corresponding approximation for pairs of same kind dyons at large distances. The integration over collective coordinates, which are the dyon positions $\mathbf{x}_i^m$[3] with $i = 1, \ldots, K$ denoting the dyon index and $m = 1, 2$ denoting first and second kind, respectively, is then performed with the measure

$$\left(\prod_{i=1}^{K}\prod_{m=1}^{2} d^3x_i^m\right)\sqrt{\det(g)}. \tag{1}$$

The approximate moduli space metric g is related to a matrix G,

[2] To be more precise, it can be diagonalized everywhere except for the loci of the Dirac strings, if the system is not neutral.

[3] The phases of the dyons are irrelevant in this context.

$$G_{i,j}^{m,n} = \delta^{m,n}\delta_{i,j}\left(2\pi + 2\sum_{k=1}^{K}\frac{1}{d_{i,k}^{m,m+1}} - 2\sum_{k=1,k\neq i}^{K}\frac{1}{d_{i,k}^{m,m}}\right) + 2\frac{\delta^{m,n}(1-\delta_{i,j})}{d_{i,j}^{m,m}} - 2\frac{\delta^{m,n+1}}{d_{i,j}^{m,m+1}}, \quad (2)$$

(see Appendix A) such that the determinants are related by

$$\sqrt{\det(g)} = \det(G). \quad (3)$$

The notation $d_{i,j}^{m,n} = |\mathbf{x}_i^m - \mathbf{x}_j^n|$ is used for dyon separations, and dyon type indices m are considered as cyclic, i.e. for SU(2) holds $(m = 3) \equiv (m = 1)$. We omit the temperature T in most equations, which means that distances are measured in $1/T$ units. The temperature dependence can easily be restored by making this explicit.

C. Parameters and their physical values

Within the model proposed in [28] the relation between the string tension σ and the critical temperature T_c of the confinement deconfinement phase transition can be derived analytically. It reads for SU(2) gauge theory

$$\sigma = (8\rho T)^{1/2}, \qquad T_c = (48\rho T/\pi^2)^{1/4}, \quad (4)$$

or equivalently

$$\frac{T_c}{\sqrt{\sigma}} = \left(\frac{6}{\pi^2}\right)^{1/4} = 0.883. \quad (5)$$

Equation (4) is given in parametric form, where ρ is the three-dimensional density of dyons of each kind, i.e. $\rho = K/V_3 = n_D/2V_3$ with V_3 denoting the spatial volume. Hence, ρT can be interpreted as the four-dimensional density of dyons which should actually be viewed as the fundamental parameter of the model.

In order to fix physical units, we set the scale by choosing $\beta = 1/T = 1.00$ fm $= 1/198$ MeV throughout the paper. This means we are considering a certain temperature in the confining phase of SU(2) Yang-Mills theory. Setting the string tension to its "physical value," taken here as $\sigma_{\text{physical}} = (440\ \text{MeV})^2 = 4.99/\text{fm}^2$, Eq. (4) yields a three-dimensional density $\rho_0 = 3.10/\text{fm}^3$ and $T_c = 389$ MeV. The dimensionless ratio $T_c/\sqrt{\sigma} = 0.883$ differs from that quoted in [28] by the factor $2^{1/4}$, which originates from the replacement $\sigma \to \sqrt{2}\sigma$ made in [28] in order to take the contribution of anti–self-dual dyons into account. In general, we define $\bar{d} = (1/\rho)^{1/3}$, which is proportional to and of the same order of magnitude as the average nearest neighbor dyon separation in three-dimensional space at density ρ. In particular, $\bar{d}_0 = (1/\rho_0)^{1/3} = 0.686$ fm is, according to (4), the dyon separation reproducing the physical value of the string tension for our choice of temperature $T = 198$ MeV.

III. SPECTRAL PROPERTIES OF THE APPROXIMATE MULTIDYON MODULI SPACE METRIC

We consider self-dual multidyon configurations, which are solutions of the classical Yang-Mills equations of motion and, therefore, local minima of the action functional of Yang-Mills theory. The corresponding surface of minimal action can be parametrized by a set of collective coordinates, the dyon positions $\mathbf{x}_i^m$ (which have already been introduced in Sec. II B) and phases which are unimportant for the following. This surface is embedded in a flat Euclidean space of infinite dimension spanned by the gauge field degrees of freedom $A_\mu^a(x)$. Consequently, the induced moduli space metric associated with these collective coordinates *must be positive definite*.

The approximate multidyon moduli space metric g proposed in [28] and the corresponding determinant have been given in Eqs. (1) and (2). For the investigation of positive definiteness one has to consider the eigenvalues of g. In Appendix A we show that the number of negative eigenvalues of g is equal to 4 times the number of negative eigenvalues of G. Thus it suffices to study G: if G is not positive definite, the same holds for g, which means that it fails to satisfy the fundamental property of positive definiteness inherent to any moduli space metric. In such cases the weight factor $\det(G)$ associated with the corresponding dyon configuration cannot be interpreted in a physically meaningful way.

Since G is only an approximation of a moduli space metric, the existence of loci of nonpositive definiteness are not excluded. Still, one could hope that such cases of failure of the used approximation are restricted to a small part of dyon configuration space, which might even vanish in the thermodynamic limit. Our numerical investigations, however, strongly indicate the opposite, i.e. that even at small dyon densities the percentage of configurations with positive definite G tends to 0, when the size of the system is increased. In the following we present a detailed study of the spectrum of G, particularly its positive definiteness, for various dyon numbers and densities.

A. Analytical considerations

We start by considering the simple case of a pair of different kind dyons, i.e. a caloron. The moduli space metric is exactly known [31]. At maximally nontrivial holonomy, it is given by

$$G_d = \begin{pmatrix} 2\pi + 2/d & -2/d \\ -2/d & 2\pi + 2/d \end{pmatrix}, \quad (6)$$

where d is the dyon separation. It is positive definite with eigenvalues

$$\lambda_1 = 2\pi, \qquad \lambda_2 = 2\pi + 4/d. \quad (7)$$

Hence, the weight factor $\det(G_d)$ is positive for arbitrary dyon positions.

Considering, on the other hand, a pair of same kind dyons (using the approximate moduli space metric proposed in [28]) the signs in front of the $2/d$ terms are reversed,

$$G_s = \begin{pmatrix} 2\pi - 2/d & +2/d \\ +2/d & 2\pi - 2/d \end{pmatrix}, \tag{8}$$

yielding eigenvalues

$$\lambda_1 = 2\pi, \qquad \lambda_2 = 2\pi - 4/d. \tag{9}$$

Here the weight factor $\det(G_s)$ is positive only for dyon separations $d > 2/\pi \equiv 2/\pi T$. For our choice of temperature, $T = 198$ MeV, this corresponds to $d > 0.635$ fm, which is of the same order of magnitude as the average distance of same kind dyons $\bar{d}_0 = 0.686$ fm needed to reproduce the physical value of the string tension $\sigma_{\text{physical}} = (440\ \text{MeV})^2$ (cf. Sec. II C). It has been argued that at the critical d the distance ceases to be a good collective coordinate (see [28] and references therein) as the dyons strongly overlap.

Of course, such a pair of same kind dyons has to be complemented by another pair of the other kind to arrive at the neutral situation that we are going to investigate in general (in other words two-caloron solutions need to be studied, of which a few are known [32,33]). Then the moduli space metric becomes a 4×4 matrix and the conditions, under which some of its eigenvalues are negative, are not that obvious. In Sec. III B and III C we perform a detailed numerical analysis of the dependence of the spectrum of G on the dyon number and density both for randomly sampled dyon positions and for dyon positions distributed according to $|\det(G)|$. The examples presented here already indicate that the approximate moduli space metric G might fail to fulfill the fundamental requirement of positive definiteness.

In the following we show that for a fixed dyon number n_D the percentage of configurations with nonpositive definite G becomes larger, when the dyon density is increased. The matrix G can be written in the form

$$G = 2\pi \mathbf{I}_{n_D \times n_D} + D(\mathbf{x}_m^i), \tag{10}$$

where the elements of D are linear combinations of inverse dyon distances $1/d_{i,k}^{m,n}$ [cf. Eq. (2)]. D is, therefore, inversely homogeneous in the dyon positions, i.e.

$$D(\mathbf{x}_m^i/\alpha) = \alpha D(\mathbf{x}_m^i). \tag{11}$$

We start by considering an arbitrary dyon configuration $\{\mathbf{x}_i^m\}$ at density ρ fulfilling

$$\text{Tr}\,(G) < 2\pi n_D. \tag{12}$$

For large dyon numbers n_D, roughly half of the configurations will satisfy this requirement. This is so, because $\text{Tr}(D)$ is a sum of $(n_D/2)^2$ positive and $n_D/2 \times (n_D/2 - 1)$ negative terms, i.e. it fluctuates approximately around zero, and because a traceless D gives exactly the bound of Eq. (12). Obviously, such a matrix G has at least one eigenvalue smaller than 2π, but might of course be positive definite.

Now we scale all dyon locations $\mathbf{x}_i^m$ by $1/\alpha < 1$, i.e. $\{\mathbf{x}_i^m\} \rightarrow \{\mathbf{x}_i'^m\} = \{\mathbf{x}_i^m/\alpha\}$, which amounts to increasing the dyon density by the factor α^3, i.e. $\rho \rightarrow \rho' = \alpha^3 \rho$. Because of the inverse homogeneity of D [Eq. (11)], the eigenvalues of the squeezed configuration λ_j' are related to the eigenvalues of the original configuration λ_j by multiplying their difference to 2π by α, i.e.

$$\lambda_j' = 2\pi + \alpha(\lambda_j - 2\pi). \tag{13}$$

Consequently, for any eigenvalue $\lambda_j < 2\pi$ one can choose α such that $\lambda_j' < 0$. In other words, rescaling any such generic configuration will give rise to a configuration for which G is not positive definite.

This argument can be transferred to the average spectral density of G. The latter is obtained by averaging over randomly and uniformly chosen dyon positions inside a cubic spatial volume. Comparing spectral densities for dyon densities ρ and ρ' (at fixed dyon number n_D) one finds that the latter is stretched by the factor α around a "fixed center" at 2π [cf. Eq. (13) and also Fig. 1 for numerical evidence, where the center at 2π is indicated by dashed lines]. This illustrates that all eigenvalues smaller than 2π will eventually become negative, when the dyon density is increased.

B. Multidyon configurations at fixed density: Dependence of the spectrum of G on the dyon number

In the following we consider dyon configurations with randomly and uniformly chosen positions inside a cubic spatial volume $V_3 = L^3$. At fixed density ρ we gradually increase the dyon number n_D (and consequently V_3) to investigate the effect on various quantities characterizing the positive definiteness of G:

(i) the average percentage R of negative eigenvalues of G, $R = \langle n_- \rangle / n_D$, where n_- denotes the number of negative eigenvalues for a given dyon configuration;

(ii) the probability distribution $P^-(n_-)$ of the number of negative eigenvalues of G;

(iii) the spectral density of G (obtained by averaging over sampled dyon configurations).

Results for two significantly different choices of the dyon density, $\rho = 1/\text{fm}^3$ and $\rho = 1/125\ \text{fm}^3$, are collected in Table I and Fig. 2.

At first we discuss $\rho = 1/\text{fm}^3$, a density smaller by more than a factor of 3 than the density $\rho_0 = 3.10/\text{fm}^3$, which is according to [28] needed to produce a quantitatively correct string tension, etc. [cf. also Eq. (4)]. The left columns of Table I and Fig. 2 clearly show that for dyon numbers in the range of $100 \leq n_D \leq 800$ the probability

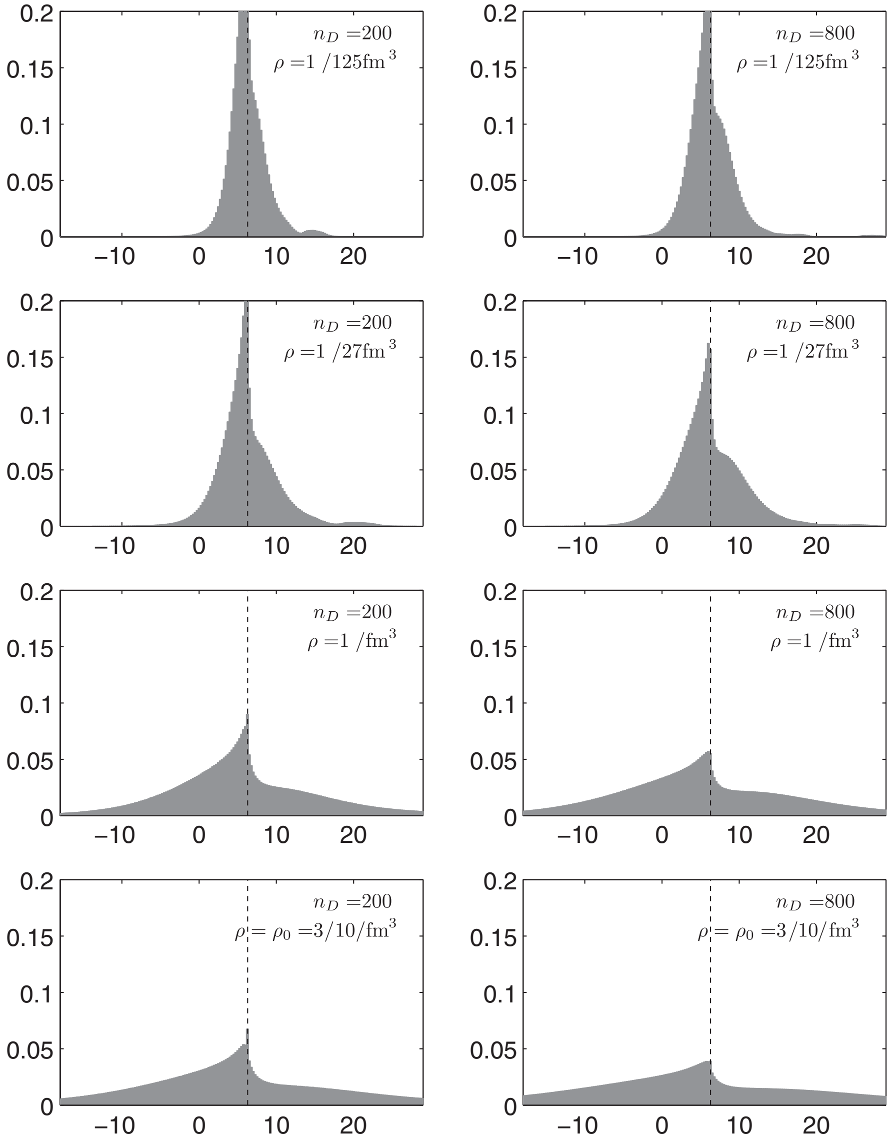

FIG. 1. Histograms obtained from 100 000 independently chosen dyon configurations representing the spectral density of the approximate moduli space metric G. Left column: $n_D = 200$. Right column: $n_D = 800$.

$P^{-}(0)$ to find configurations with positive definite G is essentially zero. In other words, among the 100 000 independently chosen dyon configurations there is not a single configuration, where the weight factor $\det(G)$ can be interpreted in a physically meaningful way, i.e. coming from a positive definite moduli space metric. Even worse, the average percentage of negative eigenvalues R increases, when we consider a larger number of dyons. This implies

TABLE I. The average percentage R of negative eigenvalues and the percentage of dyon configurations with positive definite G, $P^{-}(n_{-} = 0)$, for various dyon numbers n_D and two selected densities ρ (the averages have been computed from 100 000 independently chosen configurations).

	$\rho = 1/\text{fm}^3$			$\rho = 1/125\ \text{fm}^3$		
n_D	L in fm	R in %	$P^{-}(0)$ in %	L in fm	R in %	$P^{-}(0)$ in %
100	3.68	21.81(6)	0.000(0)	18.42	0.457(5)	64.53(35)
200	4.64	25.04(3)	0.000(0)	23.21	0.546(2)	36.72(2)
400	5.85	28.16(2)	0.000(0)	29.24	0.684(1)	10.22(3)
800	7.37	30.94(1)	0.000(0)	36.84	0.919(4)	0.60(40)

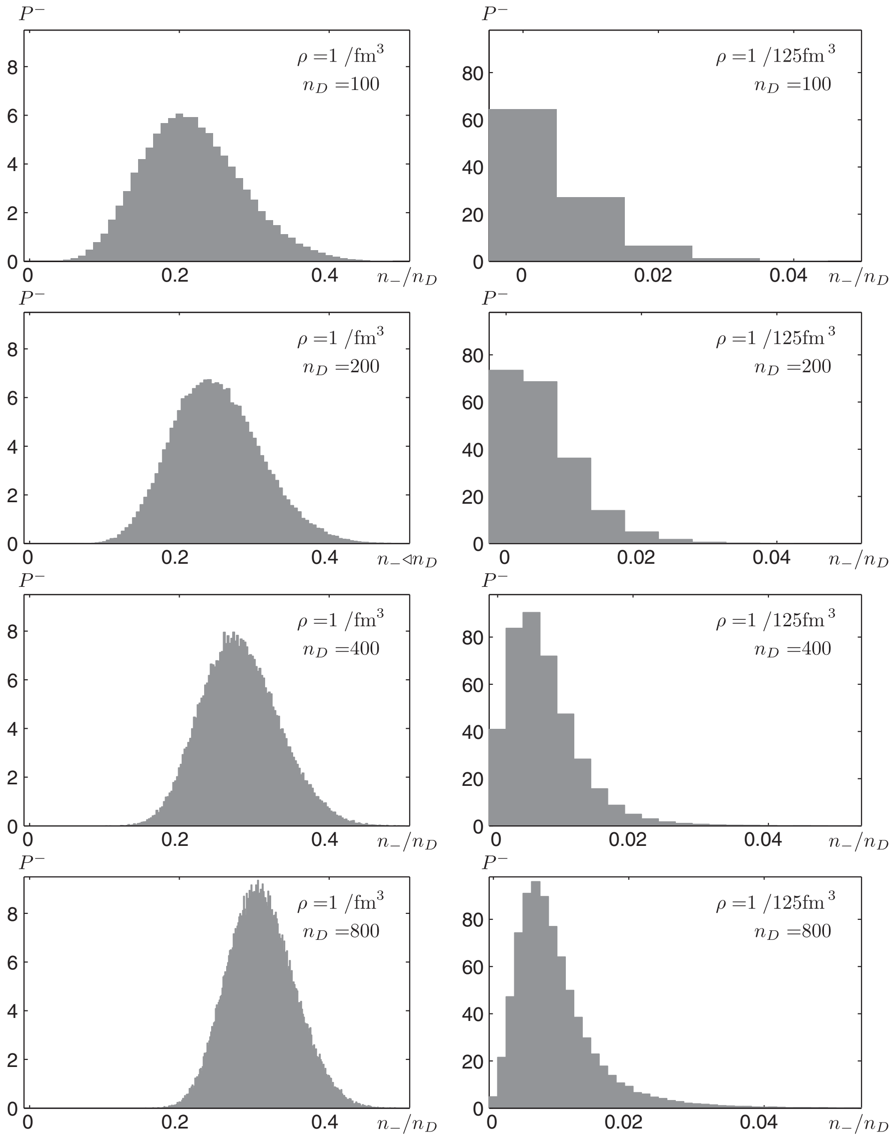

FIG. 2. Histograms obtained from 100 000 independently chosen dyon configurations representing the probability distribution $P^{-}(n_{-})$ of the number of negative eigenvalues (as function of n_{-}/n_{D}). Left column: $\rho = 1/\mathrm{fm}^3$. Right column: $\rho = 1/125\ \mathrm{fm}^3$.

that a model based on the approximate moduli space metric G and enforcing its positive definiteness contains highly nontrivial constraints determining a small admissible subspace of configuration space, and that the restrictiveness of these constraints even increases in the thermodynamic limit.

Since the construction of G in [28] is based on the assumption that all dyons are well separated, one could hope that reducing their density might cure the problem. To check this, we have repeated the analysis for a rather dilute ensemble with $\rho = 1/125\ \mathrm{fm}^3$. Note that according to the model proposed in [28] this dyon density would yield a rather low value for the string tension, $\sigma \approx 0.05 \times \sigma_{\mathrm{physical}}$ [cf. Eq. (4)]. Indeed, the average percentage of negative eigenvalues R is now significantly smaller, as can be seen in the right columns of Table I and Fig. 2. However, as before, this percentage increases, when we consider a larger number of dyons (keeping the same density), e.g. for $n_D = 800$ dyons less than 1% of all configurations have an associated matrix G, which is positive definite. This gives additional evidence that even for rather low densities the positive definiteness of the moduli space metric remains a very selective constraint.

One could still hope that results obtained by an unrestricted integration over collective coordinates might evade this problem in the sense that the integrated weight associated with dyon configurations with positive definite G, $W_G^{-}(n_{-} = 0)$, might be significantly larger than the integrated weights corresponding to "unphysical sectors," where G is not positive definite, namely $W_G^{-}(n_{-} \geq 1)$:

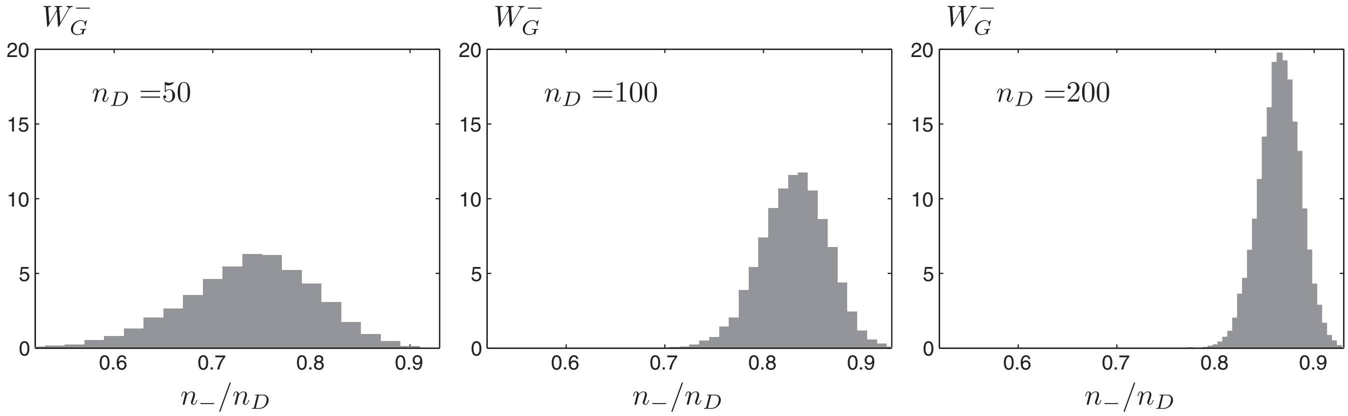

FIG. 3. Integrated weights $W_G^-(n_-)$ (shown as functions of n_-/n_D) for various dyon numbers n_D and a fixed density $\rho = 1/\mathrm{fm}^3$.

$$W_G^-(n_-) = \int \Bigg(\prod_{i=1}^{K} \prod_{m=1}^{2} d^3 x_i^m \Bigg) \Delta_{n_-}(\mathbf{x}_i^m) |\det(G(\mathbf{x}_i^m))|, \quad (14)$$

$$Z = \int \Bigg(\prod_{i=1}^{K} \prod_{m=1}^{2} d^3 x_i^m \Bigg) |\det(G(\mathbf{x}_i^m))|, \quad (15)$$

where

$$\Delta_{n_-}(\mathbf{x}_i^m) = \begin{cases} 1 & \text{if } G(\mathbf{x}_i^m) \text{ has } n_- \text{ negative eigenvalues} \\ 0 & \text{otherwise.} \end{cases} \quad (16)$$

We investigate this possibility by computing the integrated weights with the absolute value of $\det(G)$ and sorting them with respect to the number of negative eigenvalues n_-. If $W_G^-(0)$ happens to be much larger than $W_G^-(n_- \geq 1)$, then the positivity problem might be less severe in actual simulations. In Ref. [28] this was expected from the strong repulsion of dyons at a zero of the metric determinant.

Since $\det(G)$ exhibits strong fluctuations covering many orders of magnitude, we have evaluated (14) via Metropolis sampling writing $|\det(G)| = \exp(\ln|\det(G)|)$. Results at dyon density $\rho = 1/\mathrm{fm}^3$ are shown in Fig. 3. It is clearly visible, that the integrated weight $W_G^-(0)$ of the physically meaningful sector without negative eigenvalues is negligible compared to the absolute weight of all unphysical sectors, $\sum_{n_-\geq 1} W_G^-(n_-)$. Moreover, one can see that for dyon numbers in the range $50 \leq n_D \leq 200$ physical observables are dominated by dyon configurations, where G has around 80% to 90% negative eigenvalues and where $W_G^-(n_-)$ follows a smooth bell-shaped curve.

Taking the full determinant into account, i.e. taking $\det(G)$ instead of $|\det(G)|$, one has to multiply $W_G^-(n_-)$ by an alternating sign (+ if n_- is even, − if n_- is odd). Then roughly half of the dyon configurations have associated negative weights, when one computes ensemble averages ignoring the requirement of positive definiteness. Similar to the case with randomly chosen dyon positions, the average percentage R of negative eigenvalues of G increases, when the dyon number is increased at fixed density.

We conclude that results based on the approximate moduli space metric G by performing an unrestricted integration over collective coordinates (without control over positive definiteness) are physically meaningless.

C. Multidyon configurations at fixed dyon number: Dependence of the spectrum of G on the density

To investigate the dependence of the spectrum of G on the dyon density ρ, we proceed in the same way as in Sec. III B, this time keeping the dyon number n_D fixed, while varying the spatial volume V_3.

TABLE II. The average percentage R of negative eigenvalues and the percentage of dyon configurations with positive definite G, $P^-(n_- = 0)$, for two dyon numbers n_D and various values of the density ρ (the averages have been computed from 100 000 independently chosen configurations).

		$n_D = 200$			$n_D = 800$	
ρ	L in fm	R in %	$P^-(0)$ in %	L in fm	R in %	$P^-(0)$ in %
$1/125\ \mathrm{fm}^3$	23.21	0.546(2)	36.72(2)	36.84	0.919(4)	0.60(40)
$1/64\ \mathrm{fm}^3$	18.57	1.225(8)	13.28(12)	29.47	2.305(8)	0.006(4)
$1/27\ \mathrm{fm}^3$	13.92	3.241(3)	1.183(12)	22.10	5.997(7)	0.000(0)
$1/8\ \mathrm{fm}^3$	9.28	9.382(18)	0.003(1)	14.74	14.60(2)	0.000(0)
$1/\mathrm{fm}^3$	4.64	25.04(3)	0.000(0)	7.37	30.94(1)	0.000(0)
$\rho_0 = 3.10/\mathrm{fm}^3$	3.18	33.08(2)	0.000(0)	5.05	38.10(1)	0.000(0)

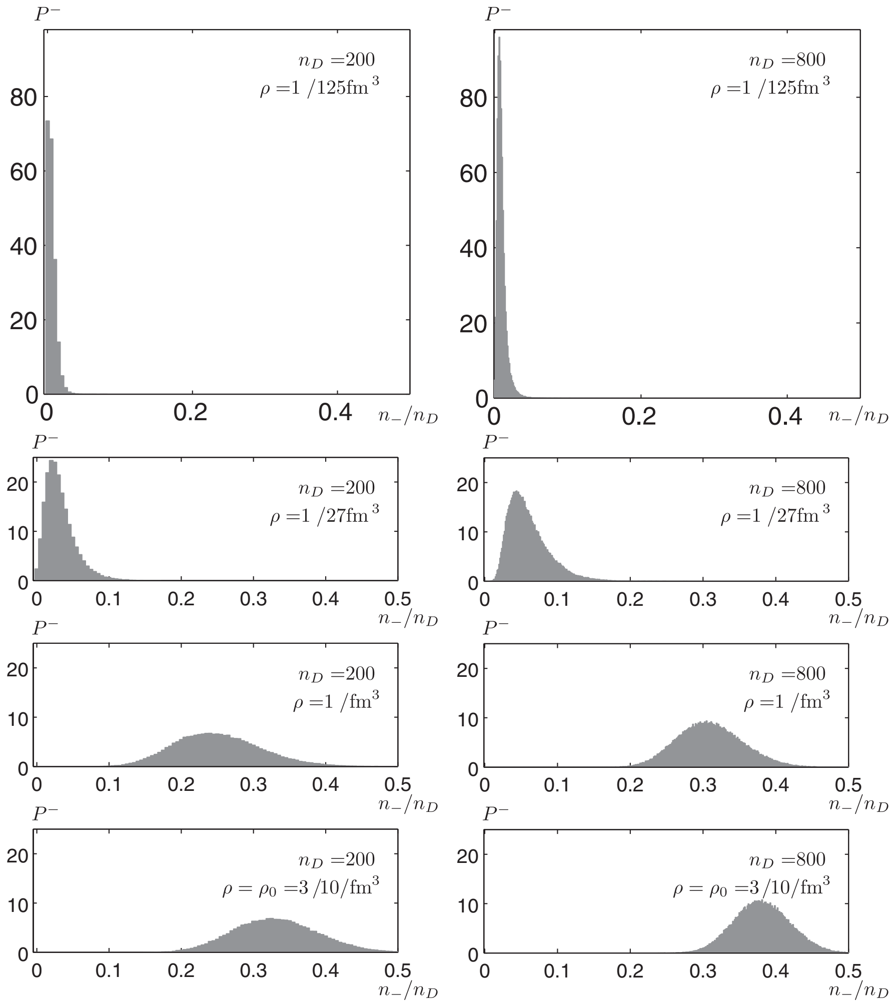

FIG. 4. Histograms obtained from 100 000 independently chosen dyon configurations representing the probability distribution $P^-(n_-)$ of the number of negative eigenvalues (as function of n_-/n_D). Left column: $n_D = 200$. Right column: $n_D = 800$.

For randomly and uniformly chosen dyon positions, our results show that the average percentage of negative eigenvalues R becomes larger, when the density is increased (cf. Table II and Fig. 4). This is in agreement with the analytical argument given in Sec. III A. This result was expected, since the moduli space metric G is an approxi-

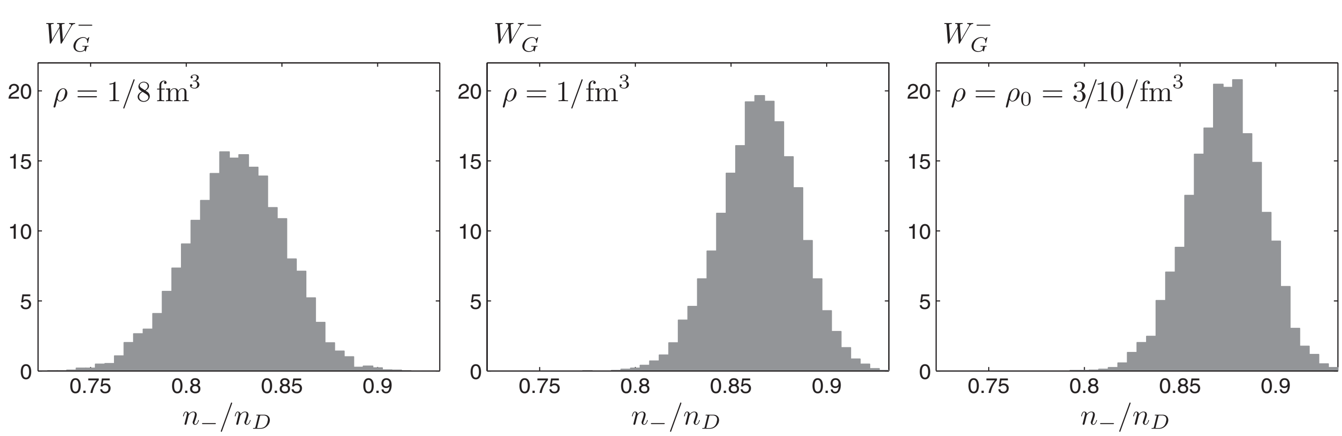

FIG. 5. Integrated weights $W_G^-(n_-)$, shown as functions of n_-/n_D, as obtained for a number $n_D = 200$ of dyons and various densities.

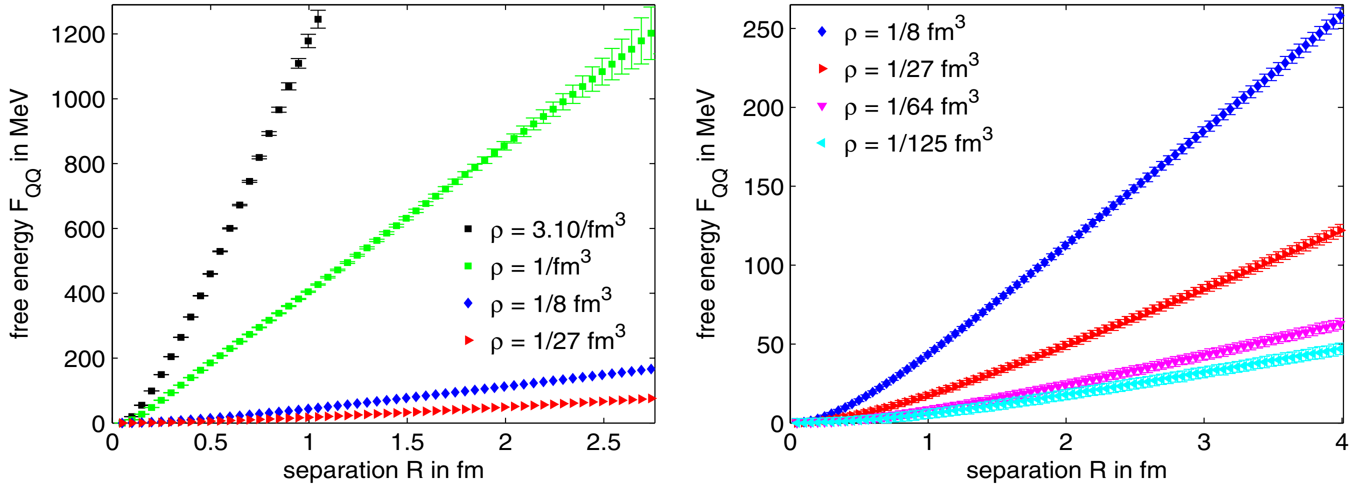

FIG. 6 (color online). The free energy of a pair of static charges $F_{Q\bar{Q}}$ as a function of the separation R for various dyon densities ρ.

mation valid only for large dyon separations. Even when considering rather dilute ensembles ($\rho = 1/125\ \mathrm{fm}^3$), still less then 40% of all dyon configurations for $n_D = 200$ and less than 1% of all dyon configurations for $n_D = 800$ possess a positive definite G.

To complement the picture, we refer to Fig. 1 where we have shown the spectral density of G both for $n_D = 200$ and for $n_D = 800$. From these plots one can clearly see that large dyon numbers or densities inevitably yield configurations, where the associated matrices G are not positive definite. The scaling of the spectrum predicted in Sec. III A is nicely confirmed.

We have also studied the dependence of the integrated weights $W_G^-(n_-)$ [cf. Eq. (14)] on the dyon density ρ at fixed dyon number $n_D = 200$. Results are shown in Fig. 5. Again one can see that for $\rho \geq 1/8\ \mathrm{fm}^3$ observables are dominated by dyon configurations with associated matrices G, which have between 80% and 90% negative eigenvalues. Moreover, the maximum of the integrated weights is shifted towards larger numbers of negative eigenvalues, when the dyon density is increased towards ρ_0 that is considered a realistic value.

IV. CONFINEMENT FROM A RANDOM DYON GAS

In the following we consider dyon ensembles without a moduli space metric (or other interactions), i.e. we perform a uniform sampling of dyon positions. Although this is a rather extreme simplification of a dyon model (see also [22]), such ensembles might be worth studying, because confinement still persists, as we will shortly demonstrate. This investigation might also be helpful to better understand the impact of a moduli space metric on ensembles of dyons.

We consider $n_D = 1,600$ dyons at the same densities as in Sec. III. We compute the Polyakov loop correlator yielding the free energy of a pair of static charges:

$$F_{\bar{Q}Q}(R) = -T \ln\langle P(\mathbf{x}) P^\dagger(\mathbf{y})\rangle, \qquad R = |\mathbf{x} - \mathbf{y}|. \quad (17)$$

The Polyakov loop,

$$P(\mathbf{x}) = \frac{1}{2}\,\mathrm{Tr}\Big(\exp\Big(i\int_0^{1/T} dx_0 A_0(\mathbf{x})\Big)\Big) = \cos(A_0^3(\mathbf{x})/2T), \quad (18)$$

can be evaluated analytically, because the gauge field is treated in the Abelian far field limit (see Appendix B). In the algebraic gauge it is given by

$$a_0^3(\mathbf{x}; q) = \frac{q}{r}, \qquad a_1^3(\mathbf{x}; q) = -\frac{qy}{r(r-z)}, \qquad a_2^3(\mathbf{x}; q) = +\frac{qx}{r(r-z)}, \quad (19)$$

where $\mathbf{x} = (x, y, z)$ and $r = |\mathbf{x}|$. All other gauge field components are zero. The multidyon configurations we use are linear superpositions of gauge potentials (19) determined by dyon positions $\mathbf{x}_i^m$ and charges q^m,

$$A_\mu^a(\mathbf{x}) = \delta^{a3}\delta_{\mu 0}\pi T + \sum_{i=1}^{K}\sum_{m=1}^{2} a_\mu^a(\mathbf{x} - \mathbf{x}_i^m; q^m), \quad (20)$$

where $q^m = \pm 1$ for the $m = 1, 2$ dyons plus an additional term in A_0^3 generating (in periodic gauge) the nontrivial holonomy $\Omega = \exp(i\pi/2\sigma_3)$ sufficiently far away from all dyons. We regularize the singularities at the dyon centers by factors $1 - \exp(-r/\epsilon)$ with $\epsilon = 0.25$ fm.[4]

As is shown in Fig. 6 the free energy rises linearly up to ≈ 1.5 fm for $\rho = 3.10/\mathrm{fm}^3$ and up to ≈ 2.7 fm for $\rho = 1/\mathrm{fm}^3$, where statistical noise starts to dominate and up to ≈ 4 fm for $\rho \leq 1/8\ \mathrm{fm}^3$, which is the maximum separation considered.

[4]We have checked that numerical results remain essentially unaltered, when ϵ is further decreased.

TABLE III. String tensions in units of the physical string tension $\sigma_{\rm physical} = (440\ {\rm MeV})^2$ for various dyon densities ρ.

ρ	L in fm	$\sigma/\sigma^{\rm physical}$
$1/125\ {\rm fm}^3$	46.42	0.015(1)
$1/64\ {\rm fm}^3$	37.13	0.020(1)
$1/27\ {\rm fm}^3$	27.85	0.037(1)
$1/8\ {\rm fm}^3$	18.57	0.074(1)
$1/{\rm fm}^3$	9.28	0.466(13)
$\rho_0 = 3.10/{\rm fm}^3$	6.37	1.434(19)

The corresponding string tensions are summarized in Table III. We conclude that even noninteracting dyons give rise to confinement, when considering the Polyakov loop correlator. Quantitatively there is a strong dependence of the string tension σ on the dyon density ρ. Note that for dyon density $\rho = \rho_0 = 3.10/{\rm fm}^3$ (which is according to the model proposed in [28] that density yielding approximate agreement with lattice results) the extracted value of the string tension even slightly overshoots its physical value: $\sigma/\sigma^{\rm physical} \approx 1.4$.

On the other hand it is hardly surprising that confinement is already present on such a simple level. Because of the long range nature of the dyon far fields, the models studied in [28] and also in this paper exhibit certain similarities to ensembles of regular gauge instantons and merons [34–36] and to the pseudoparticle approach [37–41], for which it is well known that confinement and the long range nature of the ''gauge field building blocks'' are closely related (in particular cf. [36], where it has been demonstrated that even a random ensemble of merons yields a linearly rising static potential, as well as [38,39], where the relation between the long range nature of the gauge field building blocks and confinement has been established). In a semiclassical approach at finite temperature, however, dyons are rather the natural building blocks.

V. CONCLUSIONS

The results obtained in [28] are based on an approximation of the multidyon moduli space metric, as shown in Eq. (2). This metric G—as we have demonstrated above—is positive definite only at small dyon numbers n_D and small dyon densities ρ. However, it is used at values of n_D and ρ, where it does not satisfy the fundamental requirement of positive definiteness.

In detail our findings are the following:

(i) At dyon numbers $100 \leq n_D \leq 800$ and typical densities ($\rho \approx 1/{\rm fm}^3$), practically all dyon configurations correspond to matrices G, which are not positive definite.

(ii) Roughly half of the dyon configurations have odd numbers of negative eigenvalues. This implies that every second dyon configuration receives a negative weight factor, when the metric determinant is taken into account as weight of an unrestricted integration over collective coordinates.

(iii) All attempts to approach the thermodynamic limit by increasing the dyon number, while keeping their density fixed, have lead to more severe violations of positive definiteness.

(iv) Decreasing the temperature (throughout this work we have used $\beta = 1/T = 1.00$ fm), while keeping the physical values of the string tension and the critical temperature fixed [according to Eq. (4)], amounts to increasing the three-dimensional density. As we have demonstrated this makes the situation even worse.

(v) Dyons generically induce confinement; already dyon ensembles with randomly chosen positions do so.

We expect that the problems encountered for SU(2) are generic also for higher gauge groups.

We consider these findings a challenge. Since positive definiteness is a fundamental property of any moduli space metric, it seems doubtful that results without restriction to the ''positive definite subset'' of dyon configurations can be interpreted in a physically meaningful way. If the metric determinant is taken into account as weight, averages of physical observables are computed from alternating sums over the number of negative eigenvalues of G. We conclude that, in order to obtain physically meaningful results, it is necessary either to modify the dyon model in a way that the (corrected) weight factor is always positive, or to restrict the integration over dyon positions to those parts of configuration space, where G is positive definite.

In a subsequent publication we plan to present numerical simulations of dyon models with interactions derived from the approximate moduli space metric G, which do not suffer from ''unphysical sign problems.'' One appealing possibility is the following integration over collective coordinates:

$$\bigg(\prod_{i=1}^{K}\prod_{m=1}^{2} d^3x_i^m\bigg)W, \qquad W = \begin{cases} \det(G) & \text{if } G \text{ is positive definite} \\ 0 & \text{otherwise.} \end{cases} \tag{21}$$

Since the constraints imposed on the dyon coordinates $\mathbf{x}_i^m$ by this measure are highly nontrivial, it seems unlikely that this model can be treated analytically. Therefore, we are currently developing efficient Monte Carlo algorithms, which are respecting the positive definiteness of G, i.e. which are designed to avoid ''forbidden'' multidyon configurations. In this respect our finding that already dyons with randomly chosen positions generate confinement is encouraging in the sense that models based on dyons seem to capture the relevant degrees of freedom of SU(2) Yang-Mills theory.

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APPENDIX A: RELATION BETWEEN THE SPECTRUM OF *G* AND THE SPECTRUM OF *g*

In the following we show that the number $n_-(g)$ of negative eigenvalues of g is 4 times the number $n_-(G)$ of negative eigenvalues of G.

The relation between G and g is

$$g = \begin{pmatrix} G + W_x^T G^{-1} W_x & W_x^T G^{-1} W_y & W_x^T G^{-1} W_z & W_x^T G^{-1} \\ W_y^T G^{-1} W_x & G + W_y^T G^{-1} W_y & W_y^T G^{-1} W_z & W_y^T G^{-1} \\ W_z^T G^{-1} W_x & W_z^T G^{-1} W_y & G + W_z^T G^{-1} W_z & W_z^T G^{-1} \\ G^{-1} W_x & G^{-1} W_y & G^{-1} W_z & G^{-1} \end{pmatrix}$$
$$= \underbrace{\begin{pmatrix} 1 & 0 & 0 & W_x^T \\ 0 & 1 & 0 & W_y^T \\ 0 & 0 & 1 & W_z^T \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{=\tilde{W}^T} \underbrace{\begin{pmatrix} G & 0 & 0 & 0 \\ 0 & G & 0 & 0 \\ 0 & 0 & G & 0 \\ 0 & 0 & 0 & G^{-1} \end{pmatrix}}_{=\tilde{G}} \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ W_x & W_y & W_z & 1 \end{pmatrix}}_{=\tilde{W}}; \tag{A1}$$

cf. Eq. (20) in [28].

Obviously, $\tilde{G}$ has $4n_-(G)$ negative eigenvalues. We denote the eigenvectors of $\tilde{G}$ by $\mathbf{v}^{(k,\pm)}$ and the eigenvalues by $\lambda^{(k,\pm)}$, i.e. $\tilde{G}\mathbf{v}^{(k,\pm)} = \lambda^{(k)}\mathbf{v}^{(k,\pm)}$. $+$ denotes a positive eigenvalue, i.e. $\lambda^{(k,+)} > 0$, and $-$ denotes a negative eigenvalue, i.e. $\lambda^{(k,-)} < 0$. Because $\tilde{G}$ is symmetric, the eigenvectors can be chosen orthonormal. Moreover, all eigenvalues are real and all eigenvectors can be chosen real.

With arbitrary real coefficients $\alpha(k)$ one has

$$0 > \sum_k \alpha(k)^2 \lambda^{(k,-)} = \Big(\sum_k \alpha(k)\mathbf{v}^{(k,-)}\Big)^T \tilde{G} \Big(\sum_{k'} \alpha(k')\mathbf{v}^{(k',-)}\Big) = \Big(\tilde{W}^{-1}\sum_k \alpha(k)\mathbf{v}^{(k,-)}\Big)^T \underbrace{\tilde{W}^T \tilde{G}\, \tilde{W}}_{=g} \Big(\tilde{W}^{-1}\sum_{k'} \alpha(k')\mathbf{v}^{(k',-)}\Big). \tag{A2}$$

The $4n_-(G)$ dimensional subspace

$$S = \tilde{W}^{-1}\sum_k \alpha(k)\mathbf{v}^{(k,-)} \tag{A3}$$

can be expanded in terms of the orthonormal and real eigenvectors $\mathbf{x}^{(k,\pm)}$ of g (g is also symmetric), i.e.

$$S = \sum_k \beta(k,-)\mathbf{x}^{(k,-)} + \sum_k \beta(k,+)\mathbf{x}^{(k,+)}. \tag{A4}$$

If g would have less than $4n_-(G)$ negative eigenvalues, one could chose a combination of $\alpha(k)$ such that $\beta(k,-) = 0$. This, however, would be a contradiction to (A2). Therefore, g has at least $4n_-(G)$ negative eigenvalues, i.e. $n_-(g) \geq 4n_-(G)$.

Analogously one can show that $n_+(g) \geq 4n_+(G)$. This proves that $n_-(g) = 4n_-(G)$.

APPENDIX B: THE DYON GAUGE FIELD IN THE FAR FIELD LIMIT

The gauge field of a single dyon can be obtained by considering the gauge field of a caloron at maximal holonomy in the limit of infinite dyon separation.

When the distance to the dyon center $r = (x^2 + y^2 + z^2)^{1/2}$ is large, the gauge field $A_\mu = A_\mu^a \sigma^a/2$ is Abelian:

$$A_\mu^1 = 0, \qquad A_\mu^2 = 0, \qquad A_\mu^3 = \delta_{\mu 0}\pi T + q\bar{\eta}_{\mu\nu}^3 \partial_\nu \ln(\phi), \tag{B1}$$

where

$$\phi = \frac{1}{r - z}, \tag{B2}$$

and $\bar{\eta}_{\mu\nu}^a = \epsilon_{a\mu\nu} - \delta_{a\mu}\delta_{0\nu} + \delta_{a\nu}\delta_{0\mu}$ is the 't Hooft symbol. The charge q is either $+1$ (dyons of the first kind) or -1 (dyons of the second kind). The coordinate system has been chosen such, that the singular Dirac string points in positive z-direction.

The nonzero components of the gauge field read

$$A_0^3 = \pi T + \frac{q}{r}, \qquad A_1^3 = -\frac{qy}{r(r-z)}, \qquad A_2^3 = +\frac{qx}{r(r-z)}. \tag{B3}$$

With the definitions $E_j = F_{0j}$, $B_j = -\epsilon_{jkl}F_{kl}/2$, and

$F_{\mu\nu} = \partial_\mu A^3_\nu - \partial_\nu A^3_\mu$ the corresponding electric and magnetic monopole fields are given by

$$\mathbf{E} = \frac{q\mathbf{r}}{r^3}, \qquad \mathbf{B} = \frac{q\mathbf{r}}{r^3}. \tag{B4}$$

Throughout this paper we approximate dyons always by these Abelian far fields.

[1] C. G. Callan, Jr., R. Dashen, and D. J. Gross, Phys. Rev. D **17**, 2717 (1978).
[2] C. G. Callan, Jr., R. Dashen, and D. J. Gross, Phys. Rev. D **19**, 1826 (1979).
[3] D. J. Gross, R. D. Pisarski, and L. Yaffe, Rev. Mod. Phys. **53**, 43 (1981).
[4] E. Witten, Nucl. Phys. **B149**, 285 (1979).
[5] I. Horvath *et al.*, Phys. Rev. D **67**, 011501 (2003).
[6] S. Ahmad, J. T. Lenaghan, and H. B. Thacker, Phys. Rev. D **72**, 114511 (2005).
[7] A. A. Belavin, A. M. Polyakov, A. S. Shvarts, and Yu. S. Tyupkin, Phys. Lett. **59B**, 85 (1975).
[8] G. 't Hooft, Phys. Rev. D **14**, 3432 (1976); **18**, 2199(E) (1978).
[9] E.-M. Ilgenfritz and M. Müller-Preussker, Nucl. Phys. **B184**, 443 (1981).
[10] G. Münster, Z. Phys. C **12**, 43 (1982).
[11] D. Diakonov and V. Y. Petrov, Nucl. Phys. **B245**, 259 (1984).
[12] E. V. Shuryak, Nucl. Phys. **B203**, 93 (1982).
[13] T. C. Kraan and P. van Baal, Nucl. Phys. **B533**, 627 (1998).
[14] K. M. Lee and C. h. Lu, Phys. Rev. D **58**, 025011 (1998).
[15] B. J. Harrington and H. K. Shepard, Phys. Rev. D **17**, 2122 (1978).
[16] Yu. A. Simonov, arXiv:hep-ph/9509403.
[17] F. Bruckmann, Eur. Phys. J. Special Topics **152**, 61 (2007).
[18] D. Diakonov, Acta Phys. Pol. B **39**, 3365 (2008).
[19] P. Gerhold, E.-M. Ilgenfritz, and M. Müller-Preussker, Nucl. Phys. **B760**, 1 (2007).
[20] D. Diakonov, N. Gromov, V. Petrov, and S. Slizovskiy, Phys. Rev. D **70**, 036003 (2004).
[21] K. Zarembo, Nucl. Phys. **B463**, 73 (1996).
[22] F. Bruckmann, Proc. Sci., Confinement8 (2009) 179.
[23] C. Gattringer and S. Schaefer, Nucl. Phys. **B654**, 30 (2003).
[24] V. G. Bornyakov, E. V. Luschevskaya, S. M. Morozov, M. I. Polikarpov, E.-M. Ilgenfritz, and M. Müller-Preussker, Phys. Rev. D **79**, 054505 (2009).
[25] V. G. Bornyakov, E.-M. Ilgenfritz, B. V. Martemyanov, and M. Müller-Preussker, Phys. Rev. D **79**, 034506 (2009).
[26] M. A. Stephanov, Phys. Lett. B **375**, 249 (1996).
[27] E. Bilgici, F. Bruckmann, C. Gattringer, and C. Hagen, Phys. Rev. D **77**, 094007 (2008).
[28] D. Diakonov and V. Petrov, Phys. Rev. D **76**, 056001 (2007).
[29] D. Diakonov and V. Petrov, AIP Conf. Proc. **1134**, 190 (2009).
[30] A. M. Polyakov, Nucl. Phys. **B120**, 429 (1977).
[31] T. C. Kraan, Commun. Math. Phys. **212**, 503 (2000).
[32] F. Bruckmann and P. van Baal, Nucl. Phys. **B645**, 105 (2002).
[33] F. Bruckmann, D. Nogradi, and P. van Baal, Nucl. Phys. **B698**, 233 (2004).
[34] F. Lenz, J. W. Negele, and M. Thies, Phys. Rev. D **69**, 074009 (2004).
[35] J. W. Negele, F. Lenz, and M. Thies, Nucl. Phys. B, Proc. Suppl. **140**, 629 (2005).
[36] F. Lenz, J. W. Negele, and M. Thies, Ann. Phys. (N.Y.) **323**, 1536 (2008).
[37] M. Wagner and F. Lenz, Proc. Sci., LAT2005 (2006) 315.
[38] M. Wagner, Phys. Rev. D **75**, 016004 (2007).
[39] M. Wagner, AIP Conf. Proc. **892**, 231 (2007).
[40] C. Szasz and M. Wagner, Phys. Rev. D **78**, 036006 (2008).
[41] C. Szasz and M. Wagner, arXiv:0810.4689.