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Axial and pseudoscalar current correlators and their couplings to η and η' mesons

Janardan P. Singh[1] and J. Pasupathy[2]

[1]*Physics Department, Faculty of Science, The Maharaja Sayajirao University of Baroda, Vadodara-390002, India*
[2]*Center for High Energy Physics, Indian Institute of Science, Bangalore-560012, India*

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Correlators of singlet and octet axial currents, as well as anomaly and pseudoscalar densities have been studied using QCD sum rules. Several of these sum rules are used to determine the couplings f_η^8, f_η^0, $f_{\eta'}^8$ and $f_{\eta'}^0$. We find mutually consistent values which are also in agreement with phenomenological values obtained from data on various decay and production rates. While most of the sum rules studied by us are independent of the contributions of direct instantons and screening correction, the singlet-singlet current correlator and the anomaly-anomaly correlator improve by their inclusion.

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I. INTRODUCTION

The determination of the decay constants of the η and η' mesons for the octet and singlet axial vector currents is of great interest both experimentally and theoretically. The constants are defined by

$$\langle 0|J_{\mu 5}^a|P(p)\rangle = if_P^a p_\mu, \tag{1}$$

where the index $a = 8$, 0 denotes the octet and singlet currents, respectively. In terms of u, d and s quark fields the currents are defined by

$$J_{\mu 5}^8 = \frac{1}{\sqrt{6}}(\bar{u}\gamma_\mu\gamma_5 u + \bar{d}\gamma_\mu\gamma_5 d - 2\bar{s}\gamma_\mu\gamma_5 s) \tag{2a}$$

$$J_{\mu 5}^0 = \frac{1}{\sqrt{3}}(\bar{u}\gamma_\mu\gamma_5 u + \bar{d}\gamma_\mu\gamma_5 d + \bar{s}\gamma_\mu\gamma_5 s). \tag{2b}$$

The pseudoscalar meson state P of momentum p can be either η or η'. The four couplings f_η^8, $f_{\eta'}^8$, f_η^0, and $f_{\eta'}^0$ occur in the determination of a number of production and decay amplitudes involving η and η'. Among the nonet of pseudoscalars π, K, η and η', the isosinglets η and η' are of special interest because of the so-called U(1) problem [1–7] and the presence of an anomaly in the divergence of the axial singlet current. Thus, one has

$$\partial^\mu J_{\mu 5}^0 = \frac{2i}{\sqrt{3}}(m_u\bar{u}\gamma_5 u + m_d\bar{d}\gamma_5 d + m_s\bar{s}\gamma_5 s) - \frac{\sqrt{3}}{4} \times \frac{\alpha_s}{\pi}G\tilde{G} \tag{3}$$

$$\partial^\mu J_{\mu 5}^8 = \frac{2i}{\sqrt{6}}(m_u\bar{u}\gamma_5 u + m_d\bar{d}\gamma_5 d - 2m_s\bar{s}\gamma_5 s) \tag{4}$$

where

$$G\tilde{G} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}G_{\mu\nu}^a G_{\rho\sigma}^a, \qquad \varepsilon^{0123} = +1. \tag{5}$$

Following current literature we write the four constants f_P^a ($a = 0, 8$; $P = \eta, \eta'$) defined in Eq. (1) in the matrix form

$$\begin{pmatrix} f_\eta^8 & f_\eta^0 \\ f_{\eta'}^8 & f_{\eta'}^0 \end{pmatrix} = \begin{pmatrix} f_8\cos\theta_8 & -f_0\sin\theta_0 \\ f_8\sin\theta_8 & f_0\cos\theta_0 \end{pmatrix} \tag{6}$$

in terms of two mixing angles θ_8 and θ_0.

A number of theoretical approaches have been used to compute the four constants f_8, f_0, θ_8, and θ_0. After the initial works in a chiral Lagrangian approach with $1/N_C$ expansion, where two mixing angles were introduced by Schechter *et al.* [8] and Moussallam [9], they have been calculated in chiral perturbation theory by Kaiser and Leutwyler [10–12]; Shore [13] has computed them using the so-called generalized Dashen-Gell-Mann-Oakes-Renner (DGMOR) program. A number of theoretical papers based on QCD sum rules have also appeared [14–19].

The topological susceptibility $\chi(q^2)$ is defined by

$$\chi(q^2) = i\int d^4x e^{iqx}\langle 0|T\{Q_5(x), Q_5(0)\}|0\rangle \tag{7}$$

where

$$Q_5(x) = (\alpha_s/8\pi)G_{\mu\nu}^a(x)\tilde{G}^{a\mu\nu}(x). \tag{8}$$

In an earlier work [20], we had computed the derivative of the topological susceptibility at zero momentum $\chi'(0)$ and determined the mass of η' in the chiral limit as well as singlet decay constant in the same limit. For $\chi'(0)$ we obtained a value $\approx 1.82 \times 10^{-3}$ GeV^2. This is close to the value 1.9×10^{-3} GeV^2 obtained in Ref. [21] using only the axial vector current sum rules. Further it was used to determine the isosinglet axial vector coupling $\langle p, s|\bar{u}\gamma_\mu\gamma_5 u + \bar{d}\gamma_\mu\gamma_5 d|p, s\rangle$ which along with G_A, and the octet coupling G_8 successfully account for the Bjorken sum rule. In a complimentary approach Ioffe and his collaborators [16,17] used the experimental data on Bjorken sum rule to determine $\chi'(0)$ and found

$$\chi'(0) = (2.3 \pm 0.6) \times 10^{-3}\ \text{GeV}^2. \tag{9}$$

Dorokhov and Broniowski [22] using a nonlocal chiral quark model found $\chi'(0) = 2.5 \times 10^{-3}$ GeV^2. Further, Leutwyler [23] in chiral perturbation theory found that

$$\chi'(0) = 2.2 \times 10^{-3}\ \text{GeV}^2 + \frac{\tilde{H}_0}{6}. \tag{10}$$

This suggests that the $\tilde{H}_0$ term in the effective Lagrangian is indeed small.

In Ref. [20], we had estimated the mass of η' in the chiral limit to be

$$m_{\eta'}(m_q = 0) = 723\ \text{MeV} \tag{11a}$$

$$F(m_q = 0) = 178\ \text{MeV}. \tag{11b}$$

Many years ago, Witten [3] and Veneziano [5], using $1/N_C$ expansion where N_C is the number of colors, obtained the relation

$$m_{\eta'}{}^2(m_q = 0) = 12\chi(0)|_{\text{GD}}/F^2 \tag{12}$$

where $\chi(0)|_{\text{GD}}$ is the topological susceptibility in gluodynamics (GD), i.e., in SU(N_c) gauge theory without any quark fields. From Eqs. (11a), (11b), and (12), one gets

$$\chi(0)|_{\text{GD}} = (193\ \text{MeV})^4, \tag{13a}$$

which is in excellent agreement with the lattice value [24]

$$\chi(0)|_{\text{GD}} = (191 \pm 5)^4\ \text{MeV}^4. \tag{13b}$$

The above discussion on $\chi'(0)$ and Eqs. (13a) and (13b) suggests that despite the various approximations involved, the QCD sum rule method can be a useful tool to determine the values of f_8, f_0, θ_8, and θ_0, which is the main theme of the current work.

The paper is organized as follows: In the next section we introduce several functions which *a priori* can be useful to compute the four constants f_8, f_0, θ_8, and θ_0. We briefly discuss the various low-energy theorems and briefly discuss the QCD sum rule method and point out that replacement of the correlators by the operator product expansion can violate low-energy theorems, and therefore introduce poles at $q^2 = 0$ while the exact function has none. In Sec. III, we write down the OPE for the various correlators and corresponding sum rules for the various functions of interest. In Sec. IV, we analyze the fits for the sum rules and extract the values of f_8, f_0, θ_8, and θ_0 from the residues from the sum rules that are not dependent on direct instanton contributions. In Sec. V sum rules for the $\chi(q^2)$ and the singlet-singlet current correlators are studied, which improve with inclusion of direct instantons with screening. Section VI. contains a summary and brief comments on other authors' works. An Appendix gives brief details of low-energy theorems.

II. FORMALISM

Following Ioffe [15–18], we introduce the correlator of axial vector currents

$$\Pi^{ab}_{\mu\nu}(q) = i \int d^4x e^{iqx} \langle 0|T\{J^a_{\mu 5}(x), J^b_{\nu 5}(0)\}|0\rangle; \quad (a, b = 8, 0). \tag{14}$$

The general form of the polarization tensor $\Pi^{ab}_{\mu\nu}(q)$ is

$$\Pi^{ab}_{\mu\nu}(q) = -P^{ab}_L(q^2)g_{\mu\nu} + P^{ab}_T(q^2)(-q^2 g_{\mu\nu} + q_\mu q_\nu). \tag{15}$$

The functions $P^{ab}_L(q^2)$ and $P^{ab}_T(q^2)$ are free from kinematic singularities. On forming the divergence with the momentum, we get

$$q^\mu \Pi^{ab}_{\mu\nu}(q) q^\nu = -P^{ab}_L(q^2) q^2. \tag{16}$$

On the other hand from Eq. (14) we have the Ward identity [17]

$$\begin{aligned} q^\mu \Pi^{00}_{\mu\nu}(q) q^\nu = {} & i12 \int d^4x e^{iqx} \langle 0|T\{Q_5(x), Q_5(0)\}|0\rangle \\ & - i2 \int d^4x e^{iqx} \langle 0|T\{Q_5(x), D(0)\}|0\rangle \\ & - i2 \int d^4x e^{iqx} \langle 0|T\{D(x), Q_5(0)\}|0\rangle \\ & + i\frac{1}{3} \int d^4x e^{iqx} \langle 0|T\{D(x), D(0)\}|0\rangle \\ & + \frac{4}{3} \sum_{i=u,d,s} m_i \langle o|\bar{q}_i q_i|0\rangle. \end{aligned} \tag{17}$$

In Eq. (17) we have introduced the notation

$$D(x) = 2i \sum_{i=u,d,s} m_i \bar{q}_i(x) \gamma_5 q_i(x). \tag{18}$$

Following Ioffe we note

$$\lim_{q\to 0} q^\mu \Pi^{00}_{\mu\nu}(q) q^\nu = \lim_{q\to 0} - P^{00}_L(q^2) q^2 = 0 \tag{19}$$

since the invariant $P^{00}_L(q^2)$ is regular at $q^2 = 0$. This low-energy theorem, namely, the vanishing of the left-hand side of Eq. (17), has been studied in detail by Ioffe [17]. In particular, he noted that the contributions of the Goldstone states, which are linear in quark masses, must vanish separately in the right-hand side of Eq. (17) for zero momentum. The special nature of the matrix elements of the anomaly $Q_5(x)$ between the vacuum and Goldstone states plays a crucial role. One has the following results:

$$\langle 0|Q_5|\pi\rangle = -\frac{1}{2\sqrt{2}} \frac{m_u - m_d}{m_u + m_d} f_\pi m_\pi^2 \tag{20}$$

$$\langle 0|Q_5|\eta\rangle = \frac{1}{2}\sqrt{\frac{1}{6}} f_\pi m_\eta^2 \tag{21}$$

which shows that the anomaly matrix elements are far from flavor symmetric and linear in quark masses. In Eq. (20)

$$\frac{m_u - m_d}{m_u + m_d} = O(1)$$

and we have the GMOR relation

$$m_\pi^2 = -2(m_u + m_d)\langle 0|\bar{q}q|0\rangle/f_\pi^2$$
$$m_\eta^2 = -\frac{8}{3}m_s\left(1 - \frac{1}{4}\frac{m_u + m_d}{m_s}\right)\langle 0|\bar{q}q|0\rangle/f_\pi^2$$

so that the matrix elements (20) and (21) are linear in quark masses. The intermediate states other than the π and η occurring in Eq. (21) have nonzero masses in the chiral limit. We can therefore separately consider terms linear in the light quark masses in analyzing the low-energy theorem Eq. (19) which lead Ioffe to obtain the result [17]:

$$\chi(0) = m_{\rm red}\langle 0|\bar{q}q|0\rangle + \text{higher order terms in quark masses.} \quad (22)$$

Let us briefly consider the method of QCD sum rules. Denoting generically

$$F(q^2) = \frac{i}{\pi}\int d^4x e^{iqx}\langle 0|T\{A(x), B(0)\}|0\rangle$$

where $A(x)$ and $B(x)$ are the local fields that connect the vacuum to the hadronic state of interest, one considers the dispersion relation

$$F(q^2) = \frac{1}{\pi}\int \frac{{\rm Im}F(s)}{s - q^2}ds + \text{subtractions}$$

and Borel transforms it to obtain

$$\hat{B}F(q^2) = \frac{1}{\pi}\int {\rm Im}F(s)e^{-s/M^2}ds \quad (23)$$

where the Borel transform is defined by

$$\hat{B}F(q^2) = \lim_{-q^2\to\infty, n\to\infty}\left[\frac{(-q^2)^{n+1}}{n!}\left(\frac{d}{dq^2}\right)^n F(q^2)\right]_{-q^2/n=M^2}.$$

Now the left-hand side of Eq. (23) is computed using the operator product expansion while the right-hand side is written in the form

$$\frac{1}{\pi}\int {\rm Im}F(s)e^{-s/M^2}ds = \lambda_H e^{-m_H^2/M^2} + \frac{1}{\pi}\times\int_{W^2}^\infty {\rm Im}F(s)e^{-s/M^2}ds \quad (23a)$$

where λ_H is the coupling involving the lowest mass state H in the dispersion representation:

$${\rm Im}\,F(s) = \pi\lambda_H\delta(s - m_H^2) + \text{contributions from higher mass states.}$$

This leads to

$$\lambda_H e^{-m_H^2/M^2} = \hat{B}F(q^2) - \frac{1}{\pi}\int_{W^2}^\infty {\rm Im}F(s)e^{-s/M^2}ds. \quad (23b)$$

One matches the left-hand side and the right-hand side over some M^2 interval to determine λ_H and m_H. There are several issues to be addressed here: (1) which function $F(q^2)$ should one choose where there is more than one choice—in our case instead of P_L^{88} we could have chosen the function describing the correlator

$$i\int d^4x e^{iqx}\langle 0|T\{D(x), D(0)\}|0\rangle,$$

(2) what W^2 one should choose for the second term in Eq. (23a), and (3) what is the M^2 region over which we should match the left-hand side and right-hand side in Eq. (23b)? These are all related questions. The choice of $F(q^2)$ is dictated by its asymptotic behavior for large q^2. If the choice is between, say

$$F_1(q^2) \sim q^4\ln(-q^2)$$

and

$$F_2(q^2) \sim q^2\ln(-q^2),$$

$F_2(q^2)$ is to be preferred since higher mass states in ${\rm Im}F_2(s)$ are less dominant as compared to higher mass states in ${\rm Im}F_1(s)$. One can at best make an estimate of the higher mass state contribution by using duality; that is, one equates the sum over excited states by the smeared average as given by the perturbative loop in $F_2(q^2)$. Clearly W^2 should be close to the squared mass of the first excited state which one expects to be in the range 2 to 2.5 GeV^2. Using a significantly higher value of W^2 invalidates Eqs. (23a) and (23b). Similarly the interval in M^2 is dictated by the following. In computing $\hat{B}F(q^2)$ using OPE, we are usually able to calculate only a small number of higher dimensional operators. The smaller the M^2 is the more important are the higher dimensional operators which put a lower limit on M^2, while the larger the M^2 is the more important are the excited states in Eqs. (23a) and (23b) which put an upper limit on M^2. This, therefore, determines the M^2 interval over which Eqs. (23a) and (23b) can be expected to be valid. The constants m_H and λ_H are then obtained by looking for the best fit for Eqs. (23a) and (23b). It is easy to see that if one fits m_H at the experimental mass, this leads to a better determination of the coupling since m_H appears in the exponential. It should be borne in mind that the sum rule results are subject also to the errors in values of the vacuum expectation values for the various condensates.

We shall consider several functions: $P_L^{88}(q^2)$, $P_L^{08}(q^2)$, $P_L^{00}(q^2)$, $\frac{\chi(q^2)}{q^2}$, $\frac{\chi'(q^2)}{q^2}$, $-P_L^{00}(q^2) - 12\frac{\chi(q^2)}{q^2}$, and $S(q^2)$ which are discussed in the Secs. IV and V. Before that we turn to the OPE for the various T-products that are needed.

III. OPERATOR PRODUCT EXPANSION AND DIRECT INSTANTON CONTRIBUTION

We will be using the following operator product expansion, cf. Refs. [14,25,26]

$$i\int d^4xe^{iqx}\langle 0|T\{Q_5(x),Q_5(0)\}|0\rangle = -\left(\frac{\alpha_s}{8\pi}\right)^2\frac{2}{\pi^2}q^4\ln\left(\frac{-q^2}{\mu^2}\right)\left[1+\frac{\alpha_s}{\pi}\left(\frac{83}{4}-\frac{9}{4}\ln\left(-\frac{q^2}{\mu^2}\right)\right)\right]-\frac{1}{16}\frac{\alpha_s}{\pi}\langle 0|\frac{\alpha_s}{\pi}G^2|0\rangle \times\left(1-\frac{9}{4}\frac{\alpha_s}{\pi}\ln\left(\frac{-q^2}{\mu^2}\right)\right)+\frac{1}{8q^2}\frac{\alpha_s}{\pi}\langle 0|\frac{\alpha_s}{\pi}g_sG^3|0\rangle-\frac{15}{128}\frac{\pi\alpha_s}{q^4}\langle 0|\frac{\alpha_s}{\pi}G^2|0\rangle^2 + 16\left(\frac{\alpha_s}{4\pi}\right)^3\sum_{i=u,d,s}m_i\langle\bar{q}_iq_i\rangle\left[\ln\left(-\frac{q^2}{\mu^2}\right)+\frac{1}{2}\right]-\frac{1}{2}\int d\rho n(\rho)\rho^4q^4K_2^2(Q\rho) + \text{screening correction to the direct instanton.} \tag{24}$$

The perturbative term above is taken from Kataev *et al.* [26]. The so-called direct instanton (DI) terms and their screening are described in detail by Forkel [25]. In the constant density or spike approximation

$$n(\rho) = n_0\delta(\rho-\rho_c)$$

one gets

$$i\int d^4xe^{iqx}\langle 0|T\{Q_5(x),Q_5(0)\}|0\rangle|_{\rm DI} \simeq \frac{1}{4}n_0M^3\rho_c^3\sqrt{\pi}e^{-M^2\rho_c^2}\left(M^2\rho_c^2+\frac{13}{4}+\frac{165}{32}\frac{1}{M^2\rho_c^2}\right) \tag{25}$$

while in the Gaussian-tail approximation [25]

$$n_G(\rho)=\frac{2^{18}}{3^6\pi^3}\frac{\bar{n}}{\bar{\rho}}\left(\frac{\rho}{\bar{\rho}}\right)^4\exp\left(-\frac{2^6}{3^2\pi}\frac{\rho^2}{\bar{\rho}^2}\right), \quad N_f=N_c=3,\ \bar{\rho}^{-1}\simeq 0.6\ \text{GeV}, \quad \bar{n}\cong 7.53\times 10^{-4}\ \text{GeV}^4, \tag{26}$$

$$i\int d^4xe^{iqx}\langle 0|T\{Q_5(x),Q_5(0)\}|0\rangle|_{\rm DI} = -\frac{1}{2}\int d\rho n_G(\rho)\rho^4q^4K_2^2(Q\rho) \tag{26a}$$

and the integration has to be performed numerically. Forkel has also extensively described the screening corrections to the above, caused by correlations between instantons, which are very important. We shall return to this point later.

For the crossed correlation between the anomaly and psuedoscalar density, we have

$$i(-2im_s)\int d^4xe^{iqx}\langle 0|T\{Q_5(x),\bar{s}\gamma_5s(0)\}|0\rangle = m_s^2q^2\ln\left(-\frac{q^2}{\mu^2}\right)\left\{\frac{\gamma}{2}-\frac{7}{4}+\frac{1}{4}\ln\left(\frac{-q^2}{\mu^2}\right)\right\} + \left(\frac{\alpha_s}{\pi}\right)^2m_s\langle\bar{s}s\rangle\ln\left(-\frac{q^2}{\mu^2}\right)-\frac{1}{4}\frac{\alpha_s}{\pi}\left\langle\frac{\alpha_s}{\pi}G^2\right\rangle\frac{m_s^2}{q^2} \times\ln\left(-\frac{q^2}{\mu^2}\right)+\frac{\alpha_s}{2\pi}m_s\langle\bar{s}g_s\sigma\cdot Gs\rangle\frac{1}{q^2}. \tag{27}$$

The expressions in Eq. (27) are the result of our independent calculations. For the psuedoscalar density–density correlation we have the OPE [14]

$$i(2im_s)^2\int d^4xe^{iqx}\langle 0|T\{\bar{s}\gamma_5s(x),\bar{s}\gamma_5s(0)\}|0\rangle = -\frac{3}{2\pi^2}m_s^2q^2\left[\ln\left(-\frac{q^2}{\mu^2}\right)-2+\left\{-\frac{131}{12}+\frac{17}{3}\ln\left(-\frac{q^2}{\mu^2}\right)-\frac{11}{3}\ln^2\left(-\frac{q^2}{\mu^2}\right)\right\}\right]+8m_s^3\langle\bar{s}s\rangle\frac{1}{q^2} - m_s^2\left\langle\frac{\alpha_s}{\pi}G^2\right\rangle\frac{1}{q^2}+4m_s^2\left\{\frac{16}{3}\pi^2\frac{\alpha_s}{\pi}\langle\bar{s}s\rangle^2 + m_s\langle\bar{s}g_s\sigma\cdot Gs\rangle\right\}\frac{1}{q^4}. \tag{28}$$

The last dimension five and six terms above have been computed by us.

IV. ANALYSIS AND DISCUSSION

Our interest in this work is to determine the couplings listed in Eq. (6). The seven functions listed in the last paragraph of Sec. II contain differing combinations of these four couplings. They have differing asymptotic q^2 behavior, and differ in the remaining nonperturbative terms given the various vacuum condensates. We fix the η and η' masses at their experimental values

$$m_\eta = 0.547\ \text{GeV}, \qquad m_{\eta'} = 0.958\ \text{GeV}.$$

For the other quantities needed in the sum rules, we shall use the values (cf. Refs. [17,20])

$$\begin{aligned}\alpha_s(1\ \text{GeV}) &= 0.5,\\ a &= -(2\pi)^2\langle\bar{q}q\rangle = 0.55\ \text{GeV}^3,\\ b &= \langle g_s^2G^2\rangle = 0.5\ \text{GeV}^4,\\ \langle\bar{q}g_s\sigma\cdot Gq\rangle &= m_0^2\langle\bar{q}q\rangle \quad \text{with } m_0^2 = 0.8\ \text{GeV}^2,\\ \langle\bar{s}s\rangle &= 0.8\langle\bar{u}u\rangle,\\ \left\langle g_s\frac{\alpha_s}{\pi}G^3\right\rangle &= \frac{\varepsilon}{2}\left\langle\frac{\alpha_s}{\pi}G^2\right\rangle \quad \text{with } \varepsilon = 1.0\ \text{GeV}^2,\\ \text{and}\quad m_s &= 0.153\ \text{GeV}.\end{aligned} \tag{29}$$

We first begin with the octet-octet correlator, $P_L^{88}(q^2)$ obtained from Eq. (16) with both $a=b=8$. We have from Eq. (4)

$$-q^2 P_L^{88}(q^2) = i q^\mu q^\nu \int d^4x e^{iqx} \langle 0|T\{J^8_{\mu 5}(x), J^8_{\nu 5}(0)\}|0\rangle$$
$$= i \int d^4x e^{iqx} \langle 0|T\{\partial^\mu J^8_{\mu 5}(x), \partial^\nu J^8_{\nu 5}(0)\}|0\rangle$$
$$+ \text{ETCR} \tag{30}$$

where the equal time commutation relation ETCR has the value $8m_s\langle\bar{s}s\rangle/3$. The Borel transformed sum rule is obtained following the procedure of Eqs. (23), (23a), and (23b). Since we must include η, η' mixing, the ground state hadrons consist of both η and η'. In Sec. II it was pointed out that in $P_L^{88}(q^2)$, because of division by q^2 in the left-hand side of Eq. (16), suppression of the excited state contributions will lead to a better sum rule than, for example, the one obtained from $S(q^2)$ in Eq. (35) below. However, care is needed. First note that the left-hand side of Eq. (16) is zero when q^2 is zero. This means that the exact function $P_L^{88}(q^2)$ is regular at $q^2 = 0$. As noted by Ioffe [17], the vanishing at $q^2 = 0$ of the right-hand side of Eq. (30) results from the cancellation of Goldstone state contribution at $q^2 = 0$ and ETCR. As explained in detail in the Appendix, the replacement of the T product

$$T\{D_s(x), D_s(0)\} \tag{31}$$

by its operator product expansion valid for large q^2 can lead to a violation of the low-energy theorem

$$\lim_{q_\mu \to 0} q^\mu \Pi^{88}_{\mu\nu}(q) q^\nu = \lim_{q_\mu \to 0} -q^2 P_L^{88}(q^2) = 0. \tag{32}$$

Hence the approximate $P_L^{88}(q^2)$ obtained by OPE used in the sum rule introduces a spurious pole in $P_L^{88}(q^2)$ at $q^2 = 0$ whose residue we denote by K^{88}. With this, the sum rule reads

$$K^{88} + m^2_{\eta'}(f^8_{\eta'})^2 \exp(-m^2_{\eta'}/M^2) + m^2_\eta (f^8_\eta)^2 \exp(-m^2_\eta/M^2)$$
$$= \frac{1}{\pi^2} m_s^2 M^2 \left\{1 + \frac{\alpha_s}{\pi}\left(\frac{17}{3} + 2\gamma - 2\ln\frac{M^2}{\mu^2}\right)\right\} E_0(W^2/M^2)$$
$$- \frac{8}{3} m_s \langle\bar{s}s\rangle + \frac{16}{3} m_s^3 \frac{1}{M^2}\langle\bar{s}s\rangle - \frac{2}{3} m_s^2 \frac{1}{M^2}\left\langle\frac{\alpha_s}{\pi}G^2\right\rangle$$
$$- \frac{64}{9}\pi^2 \frac{\alpha_s}{\pi} m_s^2 \frac{1}{M^4}\langle\bar{s}s\rangle^2 - \frac{4}{3} m_s^3 \frac{1}{M^4}\langle\bar{s}g_s\sigma\cdot Gs\rangle. \tag{33}$$

To extract $(f^8_{\eta'})^2$ and $(f^8_\eta)^2$ we need to specify the range of M^2 over which the left-hand side and the right-hand side match and the value of the continuum threshold W^2. In this and the following sum rules we use the criterion that at the lower end of M^2 the contribution of the highest dimensional term to the OPE side be less than 5% and, at the higher end of M^2, the continuum state contributions be less than 32% of the sum of all terms in the right-hand side. In Eq. (33) we use a value $W^2 = 2.3$ GeV^2 and the results of fitting Eq. (33) in the range 1.0 GeV$^2 \le M^2 \le 1.7$ GeV^2 are displayed in Fig. 1. We find

$$K^{88} = 1.097 \times 10^{-3}\ \text{GeV}^4,$$
$$(f^8_\eta)^2 m^2_\eta = 8.20 \times 10^{-3}\ \text{GeV}^4,$$
$$(f^8_{\eta'})^2 m^2_{\eta'} = 3.55 \times 10^{-3}\ \text{GeV}^4 \tag{34}$$

leading to the values

$$f_8 = 176.8\ \text{MeV} \quad \text{and} \quad |\theta_8| = 20.6°. \tag{34a}$$

Next, we write the pseudoscalar density correlator

$$S(q^2) = i \int d^4x e^{iqx} \langle 0|T\{m_s\bar{s}(x)\gamma_5 s(x), m_s\bar{s}(0)\gamma_5 s(0)\}|0\rangle. \tag{35}$$

We have the sum rule

$$m^4_{\eta'}(f^8_{\eta'})^2 \exp(-m^2_{\eta'}/M^2) + m^4_\eta (f^8_\eta)^2 \exp(-m^2_\eta/M^2)$$
$$= \frac{8}{3} m_s^2 \left[\frac{3}{8\pi^2} M^4 \left[1 - \frac{\alpha_s}{\pi}\left\{\frac{5}{3} - \frac{22}{3}\left(\gamma - \ln\frac{M^2}{\mu^2}\right)\right\}\right]\right.$$
$$\times E_1\left(\frac{W^2}{M^2}\right) - 2m_s\langle\bar{s}s\rangle + \frac{1}{4}\left\langle\frac{\alpha_s}{\pi}G^2\right\rangle$$
$$\left. + \frac{1}{M^2}\left\{\frac{16}{3}\pi^2 \frac{\alpha_s}{\pi}\langle\bar{s}s\rangle^2 + m_s\langle\bar{s}g_s\sigma\cdot Gs\rangle\right\}\right]. \tag{36}$$

Notice that there is no division by q^2 as in the case of $P_L^{88}(q^2)$ and therefore there is no spurious pole. For the same reason $S(q^2)$ grows faster at large q^2 than $P_L^{88}(q^2)$ which means excited states are more significant in $S(q^2)$ than in $P_L^{88}(q^2)$. Also the residues at η and η' have additional m^2_η and $m^2_{\eta'}$ respectively as compared to Eq. (33). We use $W^2 = 2.3$ GeV^2 and the values of the parameters

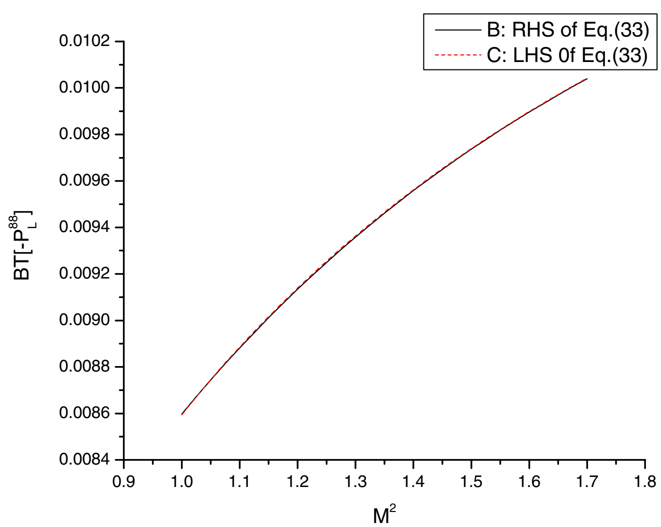

FIG. 1 (color online). Plots of the two sides of Eq. (33) (called BT[$-P_L^{88}$]), with a constant included on the left-hand side, as a function of the Borel mass squared. The best fit corresponds to $K^{88} = 1.10 \times 10^{-3}$ GeV^4, $m^2_\eta(f^8_\eta)^2 = 8.20 \times 10^{-3}$ GeV^4, and $m^2_{\eta'}(f^8_{\eta'})^2 = 3.55 \times 10^{-3}$ GeV^4.

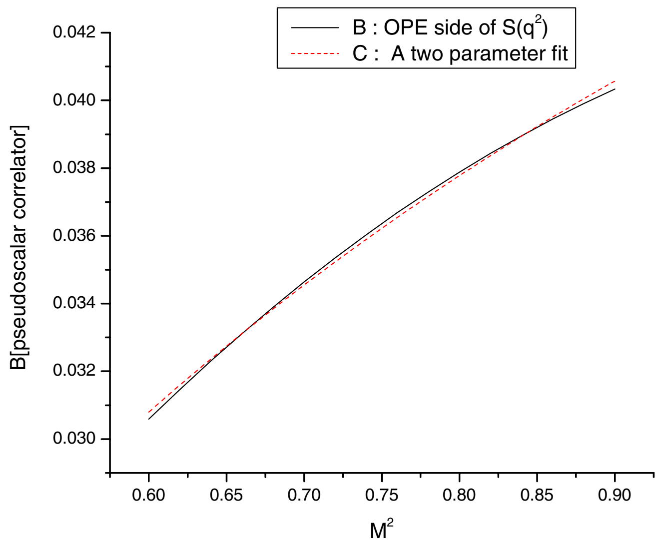

FIG. 2 (color online). The plots of Borel transforms of pseudoscalar correlator $S(q^2)$: right-hand side of Eq. (36) $\times \frac{3}{8m_s^2}$ (curve B) and a two-parameter fit (curve C). The fit corresponds to 3.64×10^{-2} GeV^4 and 4.02×10^{-2} GeV^4 as residues at η- and η'-poles. This gives $f_8 = 168.4$ MeV and $\theta_8 = \pm 18.9°$.

the same as in Eq. (29). In order to reduce the contributions of excited states to a reasonable limit ($\leq 32\%$), the limits on the Borel parameter were taken somewhat lower in this case: 0.6 GeV$^2 \leq M^2 \leq 0.9$ GeV^2. Our results of fits are displayed in Fig. 2. We have

$$3m_\eta^4 (f_\eta^8)^2/(8m_s^2) = 3.64 \times 10^{-2}\ \text{GeV}^4, \quad 3m_{\eta'}^4 (f_{\eta'}^8)^2/(8m_s^2) = 4.02 \times 10^{-2}\ \text{GeV}^4. \tag{37}$$

This corresponds to

TABLE I. List of $\chi = \sqrt{\chi^2}$ of curves which are independent of instanton contribution. χ^2 has been defined in Eq. (38). F has been defined in the text below Eq. (40a).

Fig. no.	Plot of	χ	n
1	P_L^{88}	2.0×10^{-4}	28
2	S	1.6×10^{-3}	28
3	P_L^{08}	1.1×10^{-3}	28
4	F	2.8×10^{-3}	28
5	χ'	5.1×10^{-3}	28

$$f_8 = 168.4\ \text{MeV} \quad \text{and} \quad |\theta_8| = 18.9°, \tag{37a}$$

very close to the values listed in Eq. (34a). This confirms that our introduction of the spurious pole in Eq. (33) is correct. In order to compare the quality of fits obtained from various curves, we define χ^2 by the relation

$$\chi^2 = \left(\sum_{i=0}^{n} [f(x_i) - f_{\text{fit}}(x_i)]^2/[f(x_i) + f_{\text{fit}}(x_i)]^2\right)/(1+n). \tag{38}$$

The values corresponding to Figs. 1 and 2 are given in Table I. It is seen that as expected $P_L^{88}(q^2)$ fits better than $S(q^2)$. We can check the effect of changing the lower and higher M^2 ends on χ^2. For Eq. (36), the interval 0.5 GeV$^2 \leq M^2 \leq 0.9$ GeV^2 gives $f_8 = 165.5$ MeV and $|\theta_8| = 20.2°$ with $\chi = 2.5 \times 10^{-3}$, while 0.6 GeV$^2 \leq M^2 \leq 1.0$ GeV^2 gives $f_8 = 171.7$ MeV and $|\theta_8| = 17.2°$ with $\chi = 3.1 \times 10^{-3}$ and excited states contribution rising to the level of 42% for the last case. For further discussion, we consider only the values given in Eq. (37a).

We next consider the sum rule for $P_L^{08}(q^2)$. We have

$$\begin{aligned} K^{08} &+ m_{\eta'}^2 f_{\eta'}^0 f_{\eta'}^8 \exp(-m_{\eta'}^2/M^2) + m_\eta^2 f_\eta^0 f_\eta^8 \exp(-m_\eta^2/M^2) \\ &= \frac{3}{\sqrt{2}\pi^2}\left(\frac{\alpha_s}{\pi}\right)^2 m_s^2 M^2 \left(\frac{7}{4} - \frac{1}{2}\ln\frac{M^2}{\mu^2}\right) E_0(W^2/M^2) - \frac{1}{\sqrt{2}\pi^2} m_s^2 M^2 \left\{1 + \frac{\alpha_s}{\pi}\left(\frac{17}{3} + 2\gamma - 2\ln\frac{M^2}{\mu^2}\right)\right\} E_0(W^2/M^2) \\ &\quad + \frac{4\sqrt{2}}{3} m_s \langle\bar{s}s\rangle - 2\sqrt{2}\left(\frac{\alpha_s}{\pi}\right)^2 m_s \langle\bar{s}s\rangle\left(\gamma - \ln\frac{M^2}{\mu^2}\right) + \frac{\sqrt{2}}{3} m_s^2 \frac{1}{M^2}\left\langle\frac{\alpha_s}{\pi}G^2\right\rangle + \frac{1}{\sqrt{2}}\frac{\alpha_s}{\pi} m_s^2 \frac{1}{M^2}\left\langle\frac{\alpha_s}{\pi}G^2\right\rangle\left(1 - \gamma + \ln\frac{M^2}{\mu^2}\right) \\ &\quad - \frac{8\sqrt{2}}{3} m_s^3 \frac{1}{M^2}\langle\bar{s}s\rangle - \sqrt{2}\frac{\alpha_s}{\pi} m_s \frac{1}{M^2}\langle\bar{s}g_s\sigma\cdot Gs\rangle + \frac{32\sqrt{2}}{9}\pi^2 \frac{\alpha_s}{\pi} m_s^2 \frac{1}{M^4}\langle\bar{s}s\rangle^2 + \frac{2\sqrt{2}}{3} m_s^3 \frac{1}{M^4}\langle\bar{s}g_s\sigma\cdot Gs\rangle. \end{aligned} \tag{39}$$

We again take $W^2 = 2.3$ GeV^2 with parameter values the same as in Eq. (29). The details of the fit in the interval 0.8 GeV$^2 \leq M^2 \leq 1.5$ GeV^2, are displayed in Fig. 3 with the result

$$K^{08} = -3.7 \times 10^{-3}\ \text{GeV}^4, \quad m_\eta^2 f_\eta^0 f_\eta^8 = 1.36 \times 10^{-3}\ \text{GeV}^4 \quad \text{and} \quad m_{\eta'}^2 f_{\eta'}^0 f_{\eta'}^8 = -7.97 \times 10^{-3}\ \text{GeV}^4. \tag{40}$$

Since we have $f_\eta^0 = -f_0 \sin\theta_0$ positive and $f_{\eta'}^8 = f_8 \sin\theta_8$ negative, it follows that both θ_0 and θ_8 are negative. Combining with Eq. (34a) we find

$$f_0 = 142.3\ \text{MeV} \quad \text{and} \quad \theta_0 = -11.1°. \tag{40a}$$

Let us now consider the combination $F(q^2) = -P_L^{00}(q^2) - 12\chi(q^2)/q^2$ where $\chi(q^2)$ is defined by Eq. (7). This has the effect of removing $\chi(q^2)$ from $-P_L^{00}(q^2)$, and has the advantage that the $F(q^2)$ receives no contribution for direct instantons. We can write the sum rule corresponding to $F(q^2)$ as

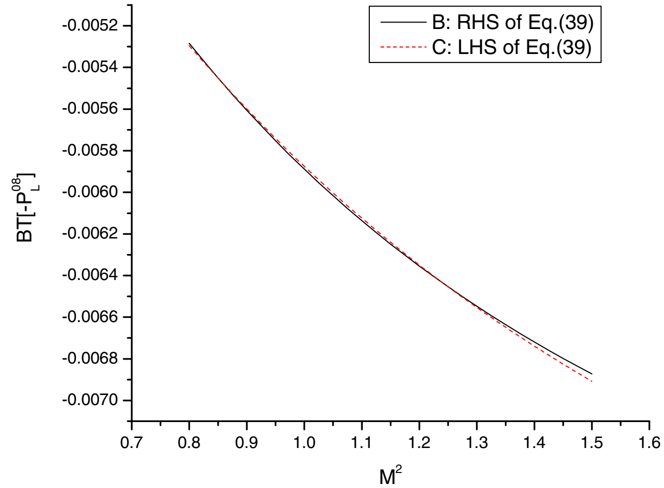

FIG. 3 (color online). Plots of the two sides of Eq. (39) (called $\mathrm{BT}[-P_L^{08}]$), with a constant included on the left-hand side, as a function of the Borel mass squared. The best fit corresponds to $K^{08} = -3.7 \times 10^{-3}$ GeV^4, $m_\eta^2 f_\eta^0 f_\eta^8 = 1.36 \times 10^{-3}$ GeV^4, and $m_{\eta'}^2 f_{\eta'}^0 f_{\eta'}^8 = -7.97 \times 10^{-3}$ GeV^4.

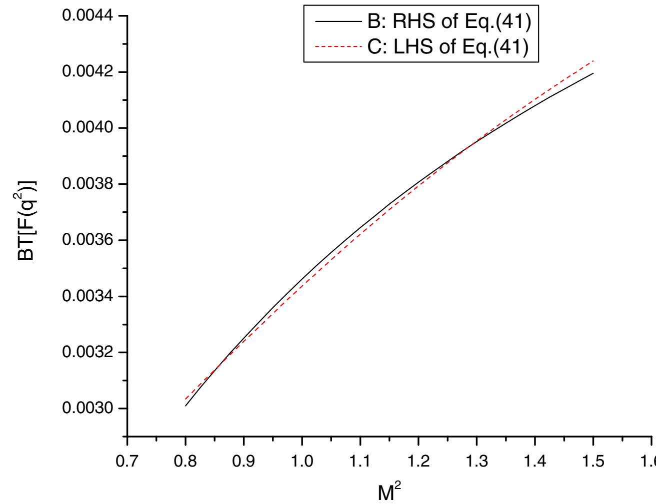

FIG. 4 (color online). Plots of $\mathrm{BT}[-P^{00}{}_L(q^2) - 12\chi(q^2)/q^2]$ and its three-parameter fit. The fit corresponds to a constant $K = 0.00464$ GeV^4 and residues as -0.00658 GeV^4 and 0.0092 GeV^4 at η- and η'- poles.

$$K - \frac{1}{2} m_{\eta'}^2 [(f_8 \sin\theta_8)^2 + 2\sqrt{2} f_0 f_8 \cos\theta_0 \sin\theta_8] e^{-m_{\eta'}^2/M^2} - \frac{1}{2} m_\eta^2 [(f_8 \cos\theta_8)^2 - 2\sqrt{2} f_0 f_8 \sin\theta_0 \cos\theta_8] e^{-m_\eta^2/M^2}$$
$$= -\frac{3}{\pi^2}\left(\frac{\alpha_s}{\pi}\right)^2 m_s^2 M^2 \left(\frac{7}{4} - \frac{1}{2}\ln\frac{M^2}{\mu^2}\right) E_0(W^2/M^2) + \frac{1}{2\pi^2} m_s^2 M^2 \left\{1 + \frac{\alpha_s}{\pi}\left(\frac{17}{3} + 2\gamma - 2\ln\frac{M^2}{\mu^2}\right)\right\} E_0(W^2/M^2) - \frac{4}{3} m_s \langle \bar{s}s \rangle$$
$$+ 4\left(\frac{\alpha_s}{\pi}\right)^2 m_s \langle \bar{s}s \rangle \left(\gamma - \ln\frac{M^2}{\mu^2}\right) - \frac{1}{3} m_s^2 \frac{1}{M^2}\left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle - \frac{\alpha_s}{\pi} m_s^2 \frac{1}{M^2}\left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle \left(1 - \gamma + \ln\frac{M^2}{\mu^2}\right) + \frac{8}{3} m_s^3 \frac{1}{M^2} \langle \bar{s}s \rangle$$
$$+ 2\frac{\alpha_s}{\pi} m_s \frac{1}{M^2} \langle \bar{s} g_s \sigma \cdot G s \rangle - \frac{32}{9}\pi^2 \frac{\alpha_s}{\pi} m_s^2 \frac{1}{M^4} \langle \bar{s}s \rangle^2 - \frac{2}{3} m_s^3 \frac{1}{M^4} \langle \bar{s} g_s \sigma \cdot G s \rangle. \tag{41}$$

Using $W^2 = 2.3$ GeV^2 and the same values of other parameters as in Eq. (29), we have fitted Eq. (41) in the range $0.8\ \mathrm{GeV}^2 \le M^2 \le 1.5\ \mathrm{GeV}^2$ and displayed it in Fig. 4. This gives

$$K = 4.64 \times 10^{-3}\ \mathrm{GeV}^4,$$
$$-\frac{1}{2} m_\eta^2 [(f_8 \cos\theta_8)^2 - 2\sqrt{2} f_0 f_8 \sin\theta_0 \cos\theta_8] = -6.58 \times 10^{-3}\ \mathrm{GeV}^4, \quad \text{and}$$
$$-\frac{1}{2} m_{\eta'}^2 [(f_8 \sin\theta_8)^2 + 2\sqrt{2} f_0 f_8 \cos\theta_0 \sin\theta_8] = 9.2 \times 10^{-3}\ \mathrm{GeV}^4. \tag{42}$$

Combining with Eq. (34a) we find

$$f_0 = 140.5\ \mathrm{MeV} \quad \text{and} \quad \theta_0 = -14.6°. \tag{42a}$$

We reconsider the sum rule for $\chi'(q^2)/q^2 - \chi'(0)/q^2$ from a slightly different perspective than in our earlier work [20] where we determined $\chi'(0)$ using the empirical values of the f_8, f_0, θ_8, θ_0 for residues of poles at η and η'. Here, we shall regard $\chi'(0)$ as well as the pole residues as unknowns to be determined by the sum rule. Writing it in the form (we set $m_u = m_d = 0$ so that the pion pole is absent)

$$\chi'(0) - \frac{1}{24}(f_8 \cos\theta_8 - \sqrt{2} f_0 \sin\theta_0)^2 \left(1 + \frac{m_\eta^2}{M^2}\right) e^{-m_\eta^2/M^2} - \frac{1}{24}(f_8 \sin\theta_8 + \sqrt{2} f_0 \cos\theta_0)^2 \left(1 + \frac{m_{\eta'}^2}{M^2}\right) e^{-m_{\eta'}^2/M^2}$$
$$= -\left(\frac{\alpha_s}{4\pi}\right)^2 \frac{1}{\pi^2} M^2 E_0\left(\frac{W^2}{M^2}\right)\left[1 + \frac{\alpha_s}{\pi}\frac{74}{4} + \frac{\alpha_s}{\pi}\frac{9}{2}\left(\gamma - \ln\frac{M^2}{\mu^2}\right)\right] - 16\left(\frac{\alpha_s}{4\pi}\right)^3 \frac{1}{M^2} m_s \langle \bar{s}s \rangle - \frac{9}{64}\frac{1}{M^2}\left(\frac{\alpha_s}{\pi}\right)^2 \left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle$$
$$+ \frac{1}{16}\frac{1}{M^4}\frac{\alpha_s}{\pi}\left\langle g_s \frac{\alpha_s}{\pi} G^3 \right\rangle - \frac{5}{128}\frac{\pi^2}{M^6}\frac{\alpha_s}{\pi}\left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle^2. \tag{43}$$

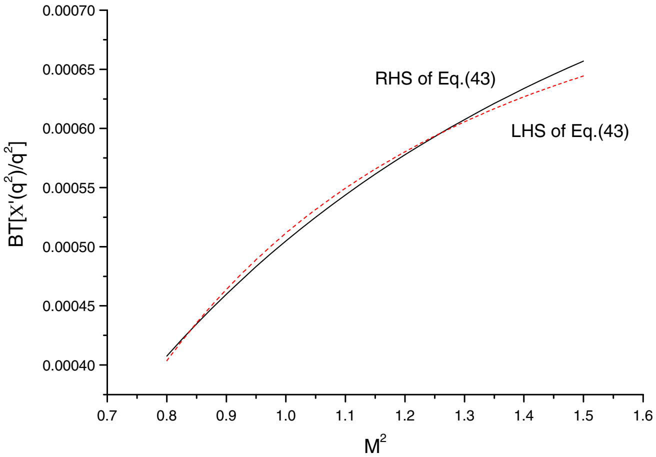

FIG. 5 (color online). Plots of $\mathrm{BT}[\chi'(q^2)/q^2]$ and its three-parameter fit as a function of Borel mass squared. The fit corresponds to $0.00165\ \mathrm{GeV}^2$, $0.00147\ \mathrm{GeV}^2$, and $0.000967\ \mathrm{GeV}^2$ as the constant $[\chi'(0)]$ and the coefficients of $(1+m_\eta{}^2/M^2)\exp(-m_\eta{}^2/M^2)$ and $(1+m_{\eta'}{}^2/M^2)\times\exp(-m_{\eta'}{}^2/M^2)$ respectively.

We have ignored the possible contribution from direct instantons given in the last term in Eq. (24). In Ref. [20] we had already pointed out that adding the direct instanton (DI) term without screening [25] gives an absurdly large contribution in Eq. (43) and completely destroys the sum rule. This point will be discussed below later, but for now, discard the plausible DI terms in Eq. (43). With $W^2 = 2.3\ \mathrm{GeV}^2$ and other parameters the same as in Eq. (29), fitting Eq. (43) in the range $0.8\ \mathrm{GeV}^2 \le M^2 \le 1.5\ \mathrm{GeV}^2$, we have from Fig. 5

$$\chi'(0) = 1.65\times10^{-3}\ \mathrm{GeV}^2, \quad (44a)$$

$$\frac{1}{24}(f_8\cos\theta_8 - \sqrt{2}f_0\sin\theta_0)^2 = 1.47\times10^{-3}\ \mathrm{GeV}^4, \quad (44b)$$

$$\frac{1}{24}(f_8\sin\theta_8 + \sqrt{2}f_0\cos\theta_0)^2 = 9.67\times10^{-4}\ \mathrm{GeV}^4. \quad (44c)$$

On combining with the results from Eq. (34a) we get

$$f_0 = 152.5\ \mathrm{MeV} \quad \text{and} \quad \theta_0 = -5.9^\circ. \quad (44d)$$

Not surprisingly, when the values of f_8, f_0, θ_8, and θ_0 used in Ref. [20] are used in Eqs. (44b) and (44c), the numbers obtained here are recovered. However, there is a small difference in the value of $\chi'(0)$, which can be accounted for by including pion pole contribution which was done in Ref. [20] but is ignored here.

We list in Table II the values for f_8, f_0, θ_8, and θ_0 from the results of Eqs. (34a), (37a), (40a), (42a), and (44d). We also note that from Table I, with values used in Eq. (29), the quality of fit is best for P_L^{88} followed by P_L^{08}, S, F, and χ'.

V. SINGLET CHANNEL AND INSTANTON CONTRIBUTIONS

For completeness we also consider the sum rule for $P_L^{00}(q^2)$. We have

$$\begin{aligned} &K^{00} + m_{\eta'}^2(f_{\eta'}^0)^2\exp(-m_{\eta'}^2/M^2) + m_\eta^2(f_\eta^0)^2\exp(-m_\eta^2/M^2)\\ &= \frac{3}{8\pi^2}\left(\frac{\alpha_s}{\pi}\right)^2 M^4 E_1\left(\frac{W^2}{M^2}\right)\left[1+\frac{\alpha_s}{\pi}\left\{\frac{65}{4}+\frac{9}{2}\left(\gamma-\ln\frac{M^2}{\mu^2}\right)\right\}\right] - \frac{3}{\pi^2}\left(\frac{\alpha_s}{\pi}\right)^2 m_s^2M^2\left(\frac{7}{4}-\frac{1}{2}\ln\frac{M^2}{\mu^2}\right)E_0\left(\frac{W^2}{M^2}\right)\\ &+\frac{1}{2\pi^2}m_s^2M^2\left[1+\frac{\alpha_s}{\pi}\left(\frac{17}{3}+2\gamma-2\ln\frac{M^2}{\mu^2}\right)\right]E_0\left(\frac{W^2}{M^2}\right) - \frac{4}{3}m_s\langle\bar{s}s\rangle + \frac{3}{4}\frac{\alpha_s}{\pi}\left\langle\frac{\alpha_s}{\pi}G^2\right\rangle\left[1-\frac{9}{4}\frac{\alpha_s}{\pi}\left(\ln\frac{M^2}{\mu^2}-\gamma\right)\right]\\ &+4\left(\frac{\alpha_s}{\pi}\right)^2 m_s\langle\bar{s}s\rangle\left(\gamma-\ln\frac{M^2}{\mu^2}\right) - \frac{\alpha_s}{\pi}\frac{m_s^2}{M^2}\left\langle\frac{\alpha_s}{\pi}G^2\right\rangle\left(1-\gamma+\ln\frac{M^2}{\mu^2}\right) - \frac{1}{3}\frac{m_s^2}{M^2}\left\langle\frac{\alpha_s}{\pi}G^2\right\rangle + \frac{3}{2}\frac{\alpha_s}{\pi}\frac{1}{M^2}\left\langle g_s\frac{\alpha_s}{\pi}G^3\right\rangle\\ &+2\frac{\alpha_s}{\pi}\frac{m_s}{M^2}\langle\bar{s}g_s\sigma\cdot Gs\rangle + \frac{8}{3}\frac{m_s^3}{M^2}\langle\bar{s}s\rangle - 3\left(\frac{\alpha_s}{\pi}\right)^3 m_s\langle\bar{s}s\rangle\left(\frac{1}{2}-\gamma+\ln\frac{M^2}{\mu^2}\right) + \frac{45}{64}\frac{\pi^2}{M^4}\frac{\alpha_s}{\pi}\left\langle\frac{\alpha_s}{\pi}G^2\right\rangle^2\\ &-\frac{32}{9}\pi^2\frac{\alpha_s}{\pi}m_s^2\frac{1}{M^4}\langle\bar{s}s\rangle^2 - \frac{2}{3}\frac{m_s^3}{M^4}\langle\bar{s}g_s\sigma\cdot Gs\rangle + \text{direct instanton} + \text{screening terms}. \end{aligned} \quad (45)$$

TABLE II. Determination of the coupling constants of η and η' mesons for the octet and singlet axial vector current with one or a combination of two equations out of Eqs. (34a), (37a), (40a), (42a), and (44d).

Eqs. used	Sum rule/ sum rule pair	$f_8\cos\theta_8$ (MeV)	$-f_8\sin\theta_8$ (MeV)	$f_0\cos\theta_0$ (MeV)	$-f_0\sin\theta_0$ (MeV)	f_8 (MeV)	f_0 (MeV)	$-\theta_8$ (degree)	$-\theta_0$ (degree)
(34a)	P_L^{88}	165.6	62.2	⋯	⋯	176.8	⋯	20.6	⋯
(37a)	S	159.3	54.6	⋯	⋯	168.4	⋯	18.9	⋯
(34a) and (44d)	P_L^{88}, χ'	165.6	62.2	151.7	15.7	176.8	152.5	20.6	05.9
(34a) and (40a)	P_L^{88} P_L^{08}	165.6	62.2	139.6	27.5	176.8	142.3	20.6	11.1
(34a) and (42a)	P_L^{88}, F	165.6	62.2	136.0	35.4	176.8	140.5	20.6	14.6

A brief discussion of the last two terms of Eq. (45) is now necessary. Although there is no universally accepted description of the QCD vacuum, the model based on instanton fluid, which regards the ground state as a collection of instanton–anti-instanton pairs has been widely used to study a number of vacuum correlation functions [27]. As is well known, an instanton of size ρ located at x_0 (see Eq. (65) of Forkel [25]) corresponds to the field strength

$$G^{(I),a}_{\mu\nu}(x) = \frac{-4\rho^2}{g_s}\frac{\eta_{a\mu\nu}}{[(x-x_0)^2+\rho^2]^2}$$

where $\eta_{a\mu\nu}$ is a 't Hooft symbol. This means the anomaly-anomaly correlation has a contribution directly from the distribution of instantons in the vacuum. In the picture in which instantons are noninteracting, the calculation is simple as in statistical mechanics of a noninteracting gas. In Eq. (25) the DI contribution for the constant density case is displayed, and in Eq. (26) the density function in the Gaussian tail approximation [25] for using in the numerical integration in Eq. (24) to get the DI contribution is displayed. Forkel has pointed out that this contribution has to be corrected for screening caused by exchanges of the Goldstone fields. We then begin first ignoring both the direct instanton term and screening in the right-hand side of Eq. (45); this and the contribution of DI with density in the Gaussian tail approximation of Eq. (26a) are shown in Fig. 6. It is easily seen that the DI term is much too large. Forkel has estimated the screening corrections arising from η, η' exchange as

$$i\int d^4x e^{iqx}\langle 0|T\{Q_5(x), Q_5(0)\}|0\rangle_{\mathrm{DISC}} = (8\pi)^2\left(\frac{F^2_{\eta'}}{Q^2+m^2_{\eta'}}+\frac{F^2_{\eta}}{Q^2+m^2_{\eta}}\right), \qquad (Q^2=-q^2) \quad (46)$$

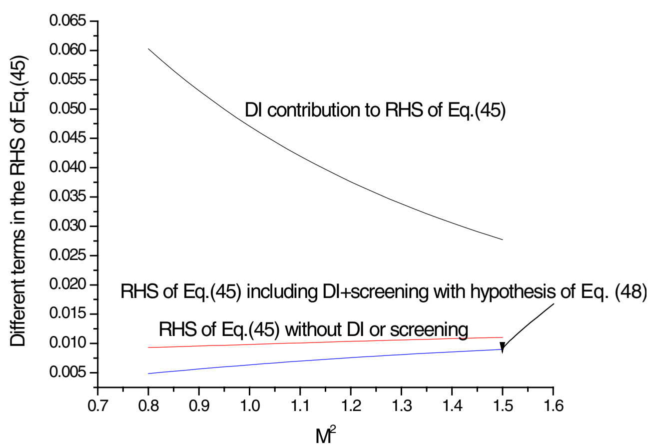

FIG. 6 (color online). Plots of direct instanton contribution (DI) to the OPE side of Eq. (45), the OPE side of Eq. (45), and a combination of the two with DI contribution included with a factor of −0.074. The last curve is separately plotted in Fig. 8 also. All quantities are in GeV units. Note the difference in scale in Fig. 8.

where the subscript in the left-hand side refers to the screening and

$$F^2_\eta = 0.0886\ \mathrm{GeV}^6, \qquad F^2_{\eta'} = 0.543\ \mathrm{GeV}^6. \quad (47)$$

In Fig. 7, the DI term and the screening term are displayed after Borel-transformation to M^2, which shows that the screening is comparable to the DI. We have already seen that the sum rule for $\chi'(q^2)/q^2$, Eqs. (43) and (44d), works very well by discarding the DI and screening and more importantly yields values for the couplings consistent with values obtained from Eqs. (34a) and (40a) which have no direct instanton terms at all. Encouraged by this, we can consider the possibility of the screening term in $P^{00}_L(q^2)$ being even larger than DI. To be specific we tried the form

$$\mathrm{DI} + \mathrm{Screening} = \delta \times \mathrm{RHS}[\mathrm{Eq.}(26a)] \quad (48)$$

where δ is some numerical factor to be determined by fitting Eq. (45). We find the value $\delta = -0.074$ fits the sum rule well as can be seen from Figs. 8 (curves B and C).

Taking $W^2 = 2.5\ \mathrm{GeV}^2$ and the values of the parameters as in Eq. (29), we fit the sum rule in the range $0.8\ \mathrm{GeV}^2 < M^2 < 1.5\ \mathrm{GeV}^2$ with results

$$K^{00} = -9.22\times 10^{-4}\ \mathrm{GeV}^4, \quad (49a)$$
$$m^2_\eta (f^0_\eta)^2 = 7.2\times 10^{-5}\ \mathrm{GeV}^4, \quad (49b)$$
$$m^2_{\eta'} (f^0_{\eta'})^2 = 1.812\times 10^{-2}\ \mathrm{GeV}^4. \quad (49c)$$

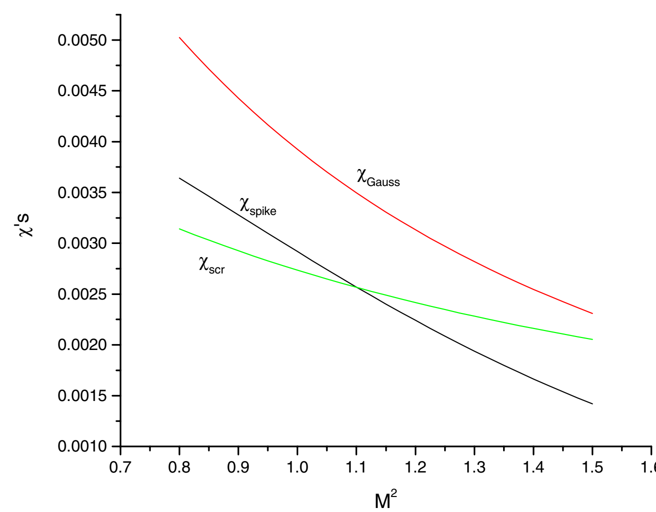

FIG. 7 (color online). Plots of Borel transforms of $\chi_{\mathrm{DI}}(\chi_{\mathrm{Gauss}}$ and $\chi_{\mathrm{spike}})$ and χ_{scr} as a function of Borel mass parameter squared. χ_{scr} is from Forkel [25] cf. our Eqs. (46) and (47). All quantities are in GeV unit.

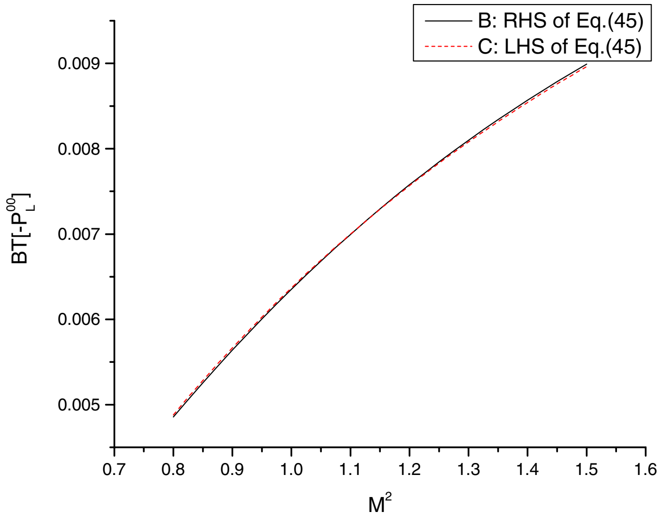

FIG. 8 (color online). Plots of two sides of Eq. (45) (called BT$[-P_L^{00}]$): OPE side with DI contribution (fraction $= -0.074$) included is curve B. Curve C is a three-parameter fit with $K^{00} = -9.22 \times 10^{-4}$ GeV^4 as a constant and 7.2×10^{-5} GeV^4 and 1.812×10^{-2} GeV^4 as residues at η- and η'-poles.

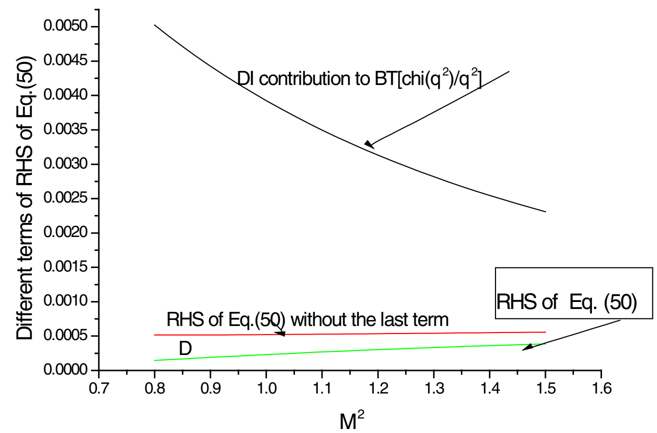

FIG. 9 (color online). Plots of BT$[\chi(q^2)/q^2]$, DI contribution to BT$[\chi(q^2)/q^2]$ and the combination of the two with the DI contribution appearing with a factor of -0.074 as a function of Borel mass squared. All the quantities are in GeV units. The last curve is plotted separately in Fig. 10 also. Note the scale is different in Fig. 10.

This corresponds to $f_0 = 141.4$ MeV and $\theta_0 = \pm 6.4°$. To see the sensitivity of the physical parameters on the coefficients of the DI term taken, we find that a change of δ from -0.074 to -0.1 changes the results substantially,

$$K^{00} = -6.872 \times 10^{-3}\ \mathrm{GeV}^4,$$
$$m_\eta^2 (f_\eta^0)^2 = 6.433 \times 10^{-3}\ \mathrm{GeV}^4,$$
$$m_{\eta'}^2 (f_{\eta'}^0)^2 = 1.866 \times 10^{-2}\ \mathrm{GeV}^4,$$

which corresponds to $f_0 = 204.5$ MeV and $\theta_0 = \pm 45.8°$.

We now turn to the sum rule for $\chi(q^2)/q^2 - \chi(0)/q^2$. We have

$$\begin{aligned} -\chi(0) &+ \frac{m_\eta^2}{24}(f_8\cos\theta_8 - \sqrt{2}f_0\sin\theta_0)^2 e^{-m_\eta^2/M^2} + \frac{m_{\eta'}^2}{24}(f_8\sin\theta_8 + \sqrt{2}f_0\cos\theta_0)^2 e^{-m_{\eta'}^2/M^2} \\ &= \left(\frac{\alpha_s}{4\pi}\right)^2 \frac{1}{2\pi^2} M^4 E_1\left(\frac{W^2}{M^2}\right)\left[1 + \frac{\alpha_s}{\pi}\left(\frac{65}{4} + \frac{9}{2}\left(\gamma - \ln\frac{M^2}{\mu^2}\right)\right)\right] + \frac{1}{16}\frac{\alpha_s}{\pi}\left\langle\frac{\alpha_s}{\pi}G^2\right\rangle\left[1 - \frac{9}{4}\frac{\alpha_s}{\pi}\left(\ln\frac{M^2}{\mu^2} - \gamma\right)\right] \\ &\quad + \frac{1}{8}\frac{1}{M^2}\frac{\alpha_s}{\pi}\left\langle g_s\frac{\alpha_s}{\pi}G^3\right\rangle + \frac{15}{256}\frac{\pi^2}{M^4}\frac{\alpha_s}{\pi}\left\langle\frac{\alpha_s}{\pi}G^2\right\rangle^2 - 16\left(\frac{\alpha_s}{4\pi}\right)^3 m_s\langle\bar{s}s\rangle\left(\frac{1}{2} + \ln\frac{M^2}{\mu^2} - \gamma\right) \\ &\quad - 0.074 \times \hat{B}\left[-\frac{1}{2}q^2\int d\rho\, n(\rho)\rho^4 K_2^2(Q\rho)\right]. \end{aligned} \tag{50}$$

For $P_L^{00}(q^2)$, Eq. (45) above, we accounted for screening by multiplying the DI by $\delta = -0.074$. As explained already with other sum rules, replacing the T-product by OPE means $\chi(0)$ may not satisfy Eq. (A2) as demanded by low-energy theorems. As in the analysis of sum rule (45), we again take $W^2 = 2.5$ GeV^2 and fit in the range 0.8 GeV$^2 \leq M^2 \leq 1.5$ GeV^2. We get (cf. Figs. 9 and 10):

$$\chi(0) = 4.3 \times 10^{-4}\ \mathrm{GeV}^4, \tag{51a}$$

$$m_\eta^2 \frac{1}{24}(f_8\cos\theta_8 - \sqrt{2}f_0\sin\theta_0)^2 = 4.4 \times 10^{-4}\ \mathrm{GeV}^4, \tag{51b}$$

$$m_{\eta'}^2 \frac{1}{24}(f_8\sin\theta_8 + \sqrt{2}f_0\cos\theta_0)^2 = 8.5 \times 10^{-4}\ \mathrm{GeV}^4 \tag{51c}$$

as the constant and residues at the η- and η'-poles, respectively. This corresponds to $f_0 = 150.2$ MeV and $\theta_0 = -6.0°$ assuming the values of f_8 and θ_8 as given by the $P_L^{88}(q^2)$- sum rule.

It is instructive to compare the results for the $\chi'(q^2)/q^2$, $P_L^{00}(q^2)$ and $\chi(q^2)/q^2$ as given in Figs. 5, 8, and 10. First, although the addition of small DI with a negative coefficient -0.074 is *ad hoc*, the same factor fits both Eqs. (45) and (50) reasonably. Moreover, η and η' residues are in the ratio 4.4:8.5 in Eqs. (51b) and (51c), that is, roughly a factor of 2. In the $P_L^{00}(q^2)$ sum rule, the η residue is very small compared to η' residue as seen in Eqs. (49b) and (49c), with values 7.2×10^{-5} GeV^4 and 1.812×10^{-2} GeV^4 respectively; that is, differing roughly by a factor of 250. Despite this, the estimates for f_0 and θ_0

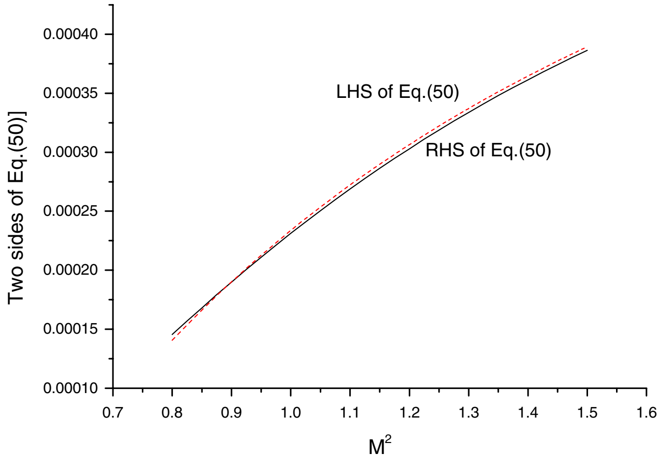

FIG. 10 (color online). Plots of $\mathrm{BT}[\chi(q^2)/q^2]$ and a three-parameter fit with a constant $-\chi(0) = -4.3 \times 10^{-4}\ \mathrm{GeV}^4$ and residues $4.4 \times 10^{-4}\ \mathrm{GeV}^4$ and $8.5 \times 10^{-4}\ \mathrm{GeV}^4$ at η- and η'-poles as a function of Borel mass squared.

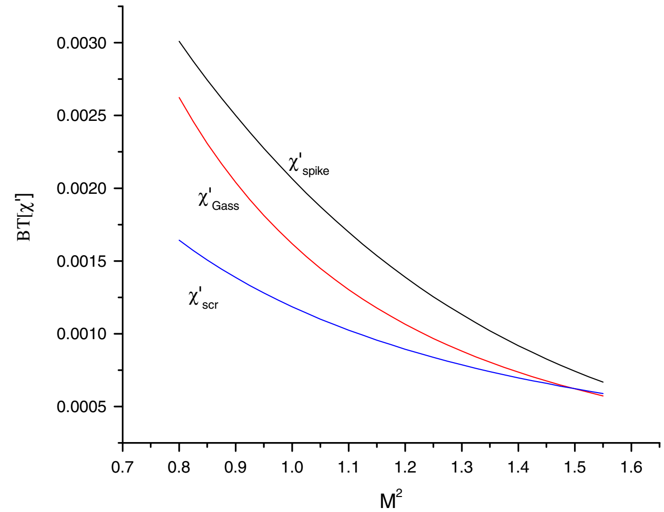

FIG. 11 (color online). Plots of $\mathrm{BT}[\chi'_{\mathrm{DI}}]$ ($\mathrm{BT}[\chi'_{\mathrm{Gauss}}]$ and $[\mathrm{BT}\chi'_{\mathrm{spike}}]$) and $\mathrm{BT}[\chi'_{\mathrm{scr}}]$ as a function of Borel mass parameter squared (M^2). χ'_{scr} is from Forkel [25], cf. our Eqs. (46) and (47).

from the two sum rules are close in values. Let us now compare the residue results of $\chi'(q^2)$ and $\chi(q^2)$. In the former we have discarded the DI and screening assuming them to cancel each other, while in the latter, the screening is slightly larger as reflected by the factor -0.074. We have from Eqs. (44b) and (44c)

$$\frac{1}{24}(f_8 \cos\theta_8 - \sqrt{2} f_0 \sin\theta_0)^2 = 1.47 \times 10^{-3}\ \mathrm{GeV}^4,$$

$$\frac{1}{24}(f_8 \sin\theta_8 + \sqrt{2} f_0 \cos\theta_0)^2 = 9.67 \times 10^{-4}\ \mathrm{GeV}^4,$$

while from Eqs. (51b) and (51c)

$$\frac{1}{24}(f_8 \cos\theta_8 - \sqrt{2} f_0 \sin\theta_0)^2 = 1.47 \times 10^{-3}\ \mathrm{GeV}^4,$$

$$\frac{1}{24}(f_8 \sin\theta_8 + \sqrt{2} f_0 \cos\theta_0)^2 = 9.26 \times 10^{-4}\ \mathrm{GeV}^4.$$

which are remarkably close to values from Eqs. (44b) and (44c). It therefore appears to conclude: a) screening corrections to DI are vital to obtain consistent results from sum rules and b) for the derivative of the topological susceptibility, the screening is almost complete, at least given the uncertainties inherent in the sum rule approach. It is useful to compare the coefficients $F_\eta^2 = 0.0886\ \mathrm{GeV}^6$ and $F_{\eta'}^2 = 0.543\ \mathrm{GeV}^6$ used by Forkel [25] with the matrix element of the anomaly between the vacuum and pseudoscalar states. From the equations [28]

$$\langle 0| \frac{3\alpha_s}{4\pi} G\tilde{G} |\eta\rangle = \sqrt{\frac{3}{2}} m_\eta^2 (f_8 \cos\theta_8 - \sqrt{2} f_0 \sin\theta_0),$$

$$\langle 0| \frac{3\alpha_s}{4\pi} G\tilde{G} |\eta'\rangle = \sqrt{\frac{3}{2}} m_{\eta'}^2 (f_8 \sin\theta_8 + \sqrt{2} f_0 \cos\theta_0),$$

we have from Eq. (44b) and (44c)

$$\langle 0|\alpha_s G\tilde{G}|\eta\rangle\langle\eta|\alpha_s G\tilde{G}|0\rangle = 0.083\ \mathrm{GeV}^6$$

and

$$\langle 0|\alpha_s G\tilde{G}|\eta'\rangle\langle\eta'|\alpha_s G\tilde{G}|0\rangle = 0.515\ \mathrm{GeV}^6$$

which are close to the numbers of Forkel [25] used by us in our Eqs. (46) and (47) above. We must also add that since u and d quark masses are different, $\langle 0|\alpha_s G\tilde{G}|\pi^0\rangle \neq 0$ and pion exchange contributes to screening. In Figs. 7 and 11, we have displayed the specific values of Forkel [25], who ignores the pion. While accepting the general picture, we can not be quantitatively accurate. We emphasize that screening effects require more study.

VI. SUMMARY AND CONCLUDING REMARKS

We have considered seven functions consisting of axial current correlators and pseudoscalar current correlators: $P_L^{88}(q^2)$, $S(q^2)$, $P_L^{08}(q^2)$, $-P_L^{00}(q^2) - 12\frac{\chi(q^2)}{q^2}$, $\frac{\chi'(q^2)}{q^2}$, $P_L^{00}(q^2)$, and $\frac{\chi(q^2)}{q^2}$ and corresponding sum rules. The first four have no contributions from direct instantons, while the last three would have possible contributions. The octet current couplings are well determined by the first two functions, and as expected, $P_L^{88}(q^2)$ sum rule works better with a better χ^2 than $S(q^2)$. As displayed in Table II, both sum rules give nearly the same values for the octet coupling and the mixing angle. With the knowledge of the octet couplings, we have seen that the octet-singlet correlator $P_L^{08}(q^2)$ works better than the hybrid function F. While the feature that the sign of both angles is negative and that singlet coupling and the magnitude of the singlet angle are smaller than the octet counterparts is true,

$P_L^{08}(q^2)$ sum rule results are closer to phenomenological values than the F sum rule results.

As noted in our earlier work [20], we have that the sum rule for $\frac{\chi'(q^2)}{q^2}$ without any direct instantons works very well. We used this observation and a semiquantitative discussion of the screening of the direct instanton to find a simple multiplicative factor to get reasonable fits of $P_L^{00}(q^2)$ and $\frac{\chi(q^2)}{q^2}$ sum rules. We have pointed out that while division by q^2 improves asymptotic behavior and therefore gives better sum rules, it can introduce a spurious pole at $q^2 = 0$ and should be accounted for in the analysis. We found that constants K^{88}, K^{08}, etc. are not zero as demanded by low-energy theorems. This caveat also applies to $\chi'(0)$. However, as discussed in the Introduction, the $\frac{\chi'(q^2)}{q^2}$ sum rule value is close to values of three other determinations, namely, axial current sum rules [21], Bjorken sum rule [17,21], and chiral perturbation theory [23].

As pointed out earlier, sum rule determinations are subject to errors arising from the uncertainties in the vacuum expectation values of various operators, the values of W^2, the continuum threshold, variation in match region of the Borel mass variable, and the ignored higher dimensional terms in the OPE and higher order terms in the Wilson coefficients. It is usual to expect that the errors are in the (10–15)% range. Nevertheless, we can rely on our results since these are mutually consistent and are also in agreement with phenomenological values as seen from Table III. We emphasize that we stayed with the rules for the Borel mass range, which is limited at the lower end by the contribution of the highest dimensional terms on OPE, and at the higher end by the contributions of the excited states which we have limited by about 32% or less. We have uniformly used a value of 2.3 GeV^2 for W^2, except in the sum rules for χ and P_L^{00}, where we have used $W^2 = 2.5$ GeV^2, a slightly higher value to get a better fit. There are suggestions [33,34] that the increase in W^2 is necessitated for richer crops of resonances in the singlet channel as compared to the octet. Alternatively, violations of duality for singlet continuum states may be more important compared to the octet.

We summarize our results as follows. As noted in Table II, the values of f_8 and θ_8 obtained from Eqs. (34a) and (37a) listed in the first two rows are close and certainly within the errors of the sum rule method. In Table II, we have given in the fifth and sixth columns the values for $f_0 \cos\theta_0 = f^0_{\eta'}$ and $-f_0 \sin\theta_0 = f^0_\eta$ obtained from using the results of the equations listed in the first column. We note that the value of $f_0 \cos\theta_0$ is better determined than $f_0 \sin\theta_0$. Part of the reason is due to the different functional relations of the couplings at the η and η' poles as seen from Eqs. (40), (42), (44b), and (44c). Despite this, the general feature that f_0 is smaller than f_8 and the numerical values of θ_0 are significantly smaller than θ_8 clearly emerges. In Table III we have listed the values of f_8, θ_8, f_0, and θ_0 from our work, the simple average of Eqs. (34a) and (37a) for f_8 and θ_8 namely 172.6 MeV and $-19.8°$ and these numbers are, in turn, used in Eqs. (40), (42), (44b), and (44c) to obtain the average values for $f_0 = 149.1$ MeV and $\theta_0 = -10.9°$.

In Table III, we have listed some of the results obtained in the current literature. Feldman and Kroll [30,31], using two-angle parametrization, have achieved a simultaneous description of the two-photon decays of η and η' and the transition form factors of $\eta\gamma$ and $\eta'\gamma$ at large momentum transfer. Shore [13] has derived the QCD formula for the two-photon decays of η and η' and the corresponding DGMOR relations by generalizing conventional PCAC to include the effect of the anomaly in a way which is consistent with the renormalization group and $1/N_C$ expansion. In Refs. [11,12], the reader will find $1/N_C$ expansion results in the context of chiral perturbation theory. It will be

TABLE III. Comparison of our results on couplings and mixing angles with those obtained by other authors. The values in the first row give the average of Eqs. (34a) and (37a) for f_8 and θ_8. This, in turn, is used in Eqs. (40), (42), (44b), and (44c) to obtain the average values of f_0 and θ_0.

Ref.	Specification	f_8 (MeV)	f_0 (MeV)	θ_8 (Degree)	θ_0 (Degree)
This work	Sum rules, averaged results	172.6	149.1	-19.8	-10.9
[29]	f_8 from [11]	$1.28 f_\pi = 167.3$	154.23 ± 5.2	$-(22.2 \pm 1.8)$	$-(8.7 \pm 2.1)$
	f_8 from [10]	$1.34 f_\pi = 175.5$	156.84 ± 5.2	$-(22.9 \pm 1.8)$	$-(6.9 \pm 2.0)$
	best fit phen.	$(1.51 \pm 0.05) f_\pi = 197.4 \pm 6.5$	168.60 ± 5.2	$-(23.8 \pm 1.4)$	$-(2.4 \pm 1.9)$
[30,31]	Theory	155.53 ± 7.8	143.77 ± 5.2	$-(19.4 \pm 1.4)$	$-(6.8 \pm 1.4)$
	Phen.	164.68 ± 7.8	152.92 ± 5.2	$-(21.2 \pm 1.4)$	$-(9.2 \pm 1.4)$
[11]	ChPT	167.30	143.77	-20.5	-4.0
[32]	ChPT	172.53	164.05	-20.0	-1.0 ± 1.5
[19]	Sum rules	188.21	176.45	-8.4	-13.8
[18]	Sum rules		178 ± 17	$-(17.0 \pm 5.0)$	
[13]	Extended current algebra	148.0	150.7	-20.1	-12.3

interesting to study the comparison of sum rule results with chiral perturbation theory. To conclude the discussion of Table III, we comment briefly on the results of Refs. [13,18,19] which are listed on the last three rows of Table III. De Fazio and Pennington [19] had used a somewhat oversimplified approach to sum rules. To calculate η couplings they use the perturbative term without radiative corrections and only the $\langle\bar{q}q\rangle$ for the nonperturbative term with a low-energy value for W^2. Moreover, to find the η' coupling they simply increase the value of W^2. No details of combined fits to η and η' are given by them. Their results for the octet and singlet angles, which we have listed in Table III, are in disagreement with the earlier rows in Table III. We may add that the results of Ref. [19] is also internally inconsistent as the value obtained for θ_8 from pseudoscalar densities is $-23°$ (not mentioned by them) is different from $-8.4°$ quoted and obtained by them using axial vector current correlators, unlike our results displayed in the first two rows of Table II. Turning to Ref. [18], we have already commented extensively in Ref. [20]. Briefly, the authors in Ref. [18] erroneously use physical η' mass instead of its value in the chiral limit, besides the fact that in their sum rule, the two sides hardly match with each other. Coming to the work of Shore in Ref. [13], we note that it is based on generalized current algebra and is different from others listed in Table III. Reference [13] has used the De Vechhia-Veneziano [35] formula (their Eq. (A4))

$$\chi(q^2) = -\frac{aF_\pi^2}{2N_c}\left[1 - \frac{a}{N_c}\sum_i \frac{1}{q^2-\mu_i^2}\right]^{-1} \tag{52}$$

where the Goldstone boson mass squared μ_i^2 is related to the quark condensate by

$$\mu_i^2 = -2m_i\frac{1}{F_\pi^2}\langle 0|\bar{q}q|0\rangle \tag{53}$$

and a is some constant. Shore [13] uses Eq. (52) to get

$$\chi(0) = -A\left(1 - A\sum_{q=u,d,s}\frac{1}{m_q\langle\bar{q}q\rangle}\right)^{-1}$$

where the constant A is

$$A = \frac{aF_\pi^2}{2N_c} \tag{54}$$

and uses it in the DGMOR relation for the singlet sector to obtain

$$(f_\eta^0 m_\eta)^2 + (f_{\eta'}^0 m_{\eta'})^2 = -\frac{2}{3}(m_u\langle\bar{u}u\rangle + m_d\langle\bar{d}d\rangle + m_s\langle\bar{s}s\rangle) + 6A. \tag{55}$$

Ioffe has correctly pointed out that Eq. (52) has a wrong pole structure at the Goldstone states. The reader can easily check that Eq. (52) used by Shore [13] has zeros at the Goldstone states while $\chi(q^2)$ should have poles at the Goldstone states (cf. Leutwyler [23]). Consequently Eqs. (6.1), (6.4) and (6.5) of Ref. [13] used by Shore [13] cannot be trusted.

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APPENDIX: LOW-ENERGY THEOREMS AND REGULARITY OF FUNCTIONS USED IN SUM RULES AT $q^2 = 0$

We first note that the anomaly-anomaly correlator or the topological susceptibility

$$\chi(q^2) = i\int d^4x e^{iqx}\langle 0|T\{Q_5(x), Q_5(0)\}|0\rangle \tag{A1}$$

satisfies a low-energy theorem. It was pointed out by Crewther [4] that $\chi(0)$ vanishes in any theory which has at least one massless quark. The large N_C (number of colors) limit was considered by Veneziano [5]. They showed that in a theory with N_f light quarks with masses $m_i \ll M$, where M is the mass of strong interaction

$$\chi(0) = \langle 0|\bar{q}q|0\rangle\left(\sum_{i=1}^{N_f}\frac{1}{m_i}\right)^{-1}. \tag{A2}$$

Here $\langle 0|\bar{q}q|0\rangle$ is the flavor symmetric value of the quark condensate and corrections of the order (m_i/M) have been neglected in Eq. (A2). Clearly the reduced mass

$$m_{\rm red} = \left(\sum_{i=1}^{N_f}\frac{1}{m_i}\right)^{-1} \tag{A3}$$

vanishes when any one of the m_i is zero, consistent with Crewther's theorem [4]. Leutwyler and Smilga [36] were able to show that for the case of two light quarks Eq. (A2) is valid at any N_c. This was further extended for three flavors by Smilga [37].

Ioffe [17] has derived the result (A2) above from yet another perspective; we briefly outline his derivation since it is useful in the context of understanding the regularity of the functions used for sum rules at $q^2 = 0$. Consider the singlet-singlet current correlator $\Pi_{\mu\nu}^{00}(q)$. Since there are no poles at $q^2 = 0$ in the physical correlator, we have

$$\lim_{q_\mu\to 0} q^\mu\Pi_{\mu\nu}^{00}(q)q^\nu = \lim_{q_\mu\to 0} -P_L^{00}(q^2)q^2 \tag{A4}$$

$$= 0 \tag{A5}$$

which implies that $P_L^{00}(q^2)$ is regular at $q^2 = 0$. On the other hand

$$\lim_{q_\mu\to 0} q^\mu \Pi^{00}_{\mu\nu}(q) q^\nu = i12\int d^4x e^{iqx}\langle 0|T\{Q_5(x), Q_5(0)\}|0\rangle - i2\int d^4x e^{iqx}\langle 0|T\{Q_5(x), D(0)\}|0\rangle - i2\int d^4x e^{iqx}\langle 0|T\{D(x), Q_5(0)\}|0\rangle + i\frac{1}{3}\int d^4x e^{iqx}\langle 0|T\{D(x), D(0)\}|0\rangle + \frac{4}{3}\sum_{i=u,d,s} m_i\langle 0|\bar{q}_i q_i|0\rangle = 0. \quad (A6)$$

A plausible Schwinger term $[J^0_{05}(x), Q_5(0)]\delta(x_0)$ can be shown to be zero [17]. Similarly by considering the correlator

$$P_\mu(q) = i\int d^4x e^{iqx}\langle 0|[J^0_{\mu 5}(x), Q_5(0)]|0\rangle \quad (A7)$$

and the fact that

$$\lim_{q_\mu\to 0} q^\mu P_\mu(q) = 0, \quad (A8)$$

one derives

$$i\int d^4x\langle 0|T\{2Q_5(x), Q_5(0)\}|0\rangle - i\frac{1}{3}\int d^4x\langle 0|T\{D(x), Q_5(0)\}|0\rangle = 0. \quad (A9)$$

Combining (A6) and (A9) one gets [17]

$$i12\int d^4x\langle 0|T\{Q_5(x), Q_5(0)\}|0\rangle - i\frac{1}{3}\int d^4x\langle 0|T\{D(x), D(0)\}|0\rangle - \frac{4}{3}\sum_{i=u,d,s} m_i\langle 0|\bar{q}_i q_i|0\rangle = 0. \quad (A10)$$

Ioffe rewrites Eq. (A10) in the form

$$i12\int d^4x\langle 0|T\{Q_5(x), Q_5(0)\}|0\rangle = i\frac{1}{3}\int d^4x\langle 0|T\{D(x), D(0)\}|0\rangle + \frac{4}{3}\sum_{i=u,d,s} m_i\langle 0|\bar{q}_i q_i|0\rangle. \quad (A11)$$

The term linear in the quark masses in the first term in the right-hand side of Eq. (A11) can be found from the matrix elements of $\langle 0|D(x)|\pi^0\rangle$ and $\langle 0|D(x)|\eta\rangle$ and leads back to Eq. (A2). Complete details can be found in Ioffe [17]. For the purpose of the present paper, the important question is how the various terms in the right-hand side of Eq. (A6) conspire to keep their sum zero, so that $P^{00}_L(q^2)$ is regular at $q^2 = 0$. We have seen above that the low-energy theorem (A2), $|\pi^0\rangle$ and $|\eta\rangle$ contributions and the equal time commutator add to give the zero. On the other hand, to derive the QCD sum rule for $P^{00}_L(q^2)$ we have operator product expansion for various terms like $i\int d^4x e^{iqx}\langle 0|T\{Q_5(x), Q_5(0)\}|0\rangle$ and $i\int d^4x e^{iqx}\times \langle 0|T\{Q_5(x), D(0)\}|0\rangle$ as given in Eqs. (24) and (27) respectively, which is a good approximation at high q^2. We cannot, therefore, expect $q^2 P^{00}_L(q^2) = \mathrm{OPE} + \mathrm{ETCR}$ to vanish at $q^2 = 0$ as demanded by the low-energy theorem. So in dividing by q^2 to derive an expression for $P^{00}_L(q^2)$ valid at large q^2, we introduce a spurious pole at $q^2 = 0$.

We can explicate this by considering the octet-octet correlator in some detail. To simplify matters we set $m_u = m_d = 0$ but keep $m_s \neq 0$. Isospin is exact in this limit, so that the current $J^8_{\mu 5}(x)$ does not couple to the pion and therefore the correlator $\Pi^{88}_{\mu\nu}(q)$ is still regular at $q^2 = 0$. Consider now the analog of Eq. (A6). We have

$$\lim_{q_\mu\to 0} q^\mu \Pi^{88}_{\mu\nu}(q) q^\nu = i\frac{4}{6}\int d^4x\langle 0|T\{D_s(x), D_s(0)\}|0\rangle + \frac{8}{3}m_s\langle 0|\bar{s}s|0\rangle. \quad (A12)$$

Here $D_s(x) = 2im_s\bar{s}(x)\gamma_5 s(x)$. To see how the two terms in Eq. (A12) cancel up to linear order in m_s we note

$$\langle 0|D_s|\eta\rangle = -\sqrt{\frac{3}{2}} f_\pi m^2_\eta \quad (A13)$$

so that

$$\frac{4}{6}i\int d^4x\langle 0|T\{D_s(x), D_s(0)\} = f^2_\pi m^2_\eta + \text{higher order terms}. \quad (A14)$$

Now by GMOR relation

$$f^2_\pi m^2_\eta = -\frac{8}{3}m_s\langle 0|\bar{s}s|0\rangle, \quad (A15)$$

so the right-hand side of Eq. (A12) adds to zero, thus preserving the regularity of $P^{88}_L(q^2)$ at $q^2 = 0$. Returning to the QCD sum rule for $P^{88}_L(q^2)$, we have from Eq. (30)

$$-q^2 P^{88}_L(q^2) = \frac{4}{6}i\int d^4x e^{iqx}\langle 0|T\{D_s(x), D_s(0)\}|0\rangle + \frac{8}{3}m_s\langle 0|\bar{s}(0)s(0)|0\rangle \quad (A16)$$

$$|q^2|\to\infty \approx \left\{-\frac{1}{\pi^2}m^2_s q^2\left[\ln\left(\frac{-q^2}{\mu^2}\right) - 2\right] + \ldots\right\} + \frac{8}{3}m_s\langle 0|\bar{s}s|0\rangle \quad (A17)$$

where we have used the operator expansion for $T\{D_s(x), D_s(0)\}$. Unlike the right-hand side of Eq. (A16) which vanishes as $q_\mu \to 0$, we cannot expect the right-hand side of Eq. (A17) to vanish at $q^2 = 0$. Apart from the fact that in Eq. (A17) we are using large q^2 approximation in the actual sum rule evaluation we also use numerical estimates for m_s and $\langle 0|\bar{s}s|0\rangle$ while in Eq. (A16) we used an algebraic identity using current algebra. Therefore in the process of dividing by q^2 in Eq. (A17) we introduce a spurious pole at $q^2 = 0$ in $P_L^{88}(q^2)$ which must be accounted for. On Borel transformation the spurious pole at $q^2 = 0$ becomes an M^2 independent constant term which we have denoted by K^{88} in our sum rule analysis.

Similar considerations hold for $P_L^{08}(q^2)$. We have again taken $m_u = m_d = 0$, $m_s \neq 0$, the Ward identity

$$\lim_{q_\mu \to 0} q^\mu \Pi^{08}_{\mu\nu}(q) q^\nu = i \frac{12}{\sqrt{18}} \int d^4x \langle 0|T\{Q_5(x), D_s(0)\}|0\rangle - \frac{2}{\sqrt{18}} \int d^4x \langle 0|T\{D_s(x), D_s(0)\}|0\rangle - \frac{8}{3\sqrt{2}} m_s \langle 0|\bar{s}s|0\rangle. \quad (A18)$$

Now from Eq. (A9) we have

$$i \int d^4x \langle 0|T\{D_s(x), Q_5(0)\}|0\rangle = 0 (m_u = m_d = 0) \quad (A19)$$

since $\chi(0) = 0$ and $D(x) = D_s(x)$ when $m_u = m_d = 0$. We can then drop the first term in the right-hand side of Eq. (A18), in which case, neglecting higher order terms and apart from an overall factor of $-\sqrt{2}$, it is identical to Eqs. (A14) and (A15). Thus we see that the sum rules for $P_L^{00}(q^2)$, $P_L^{08}(q^2)$, and $P_L^{88}(q^2)$ can have spurious poles and must be accounted for while extracting the coefficients of the η and η' poles.

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