Fermion masses and mixings in a μ-τ symmetric $SO(10)$

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The μ-τ symmetry imposed on the neutrino mass matrix in the flavor basis is known to be quite predictive. We integrate this very specific neutrino symmetry into a more general framework based on the supersymmetric $SO(10)$ grand unified theory. As in several other models, the fermion mass spectrum is determined by Hermitian mass matrices resulting from the renormalizable Yukawa couplings of the 16-plet of fermions with the Higgs fields transforming as 10, $\overline{126}$, and 120 representations of the $SO(10)$ group. The μ-τ symmetry is spontaneously broken through the 120-plet. Consequences of this scheme are considered for fermion masses using both a type-I and a type-II seesaw mechanism. This scenario is shown to lead to a generalized CP invariance of the mass matrices and vanishing CP violating phases if the Yukawa couplings are invariant under the μ-τ symmetry. Small explicit breaking of the μ-τ symmetry is then shown to provide a very good understanding of all of the fermion masses and mixing. Detailed fits to the fermion spectrum are presented in several scenarios. One obtains a very good fit to all observables in the context of the type-I seesaw mechanism, but the type-II seesaw model also provides a good description except for the overall scale of the neutrino masses. Three major predictions on the leptonic mixing parameters in the type-I seesaw case are (i) the atmospheric mixing angle θ^l_{23} close to maximal, (ii) θ^l_{13} close to the present upper bound, and (iii) a negative but very small Dirac CP violating phase in the neutrino oscillations.

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I. INTRODUCTION

There exist a variety of theoretical frameworks/specific models [1] which try to account for the large atmospheric mixing angle observed more than a decade ago. One class of theories attribute the maximal atmospheric mixing to the presence of some underlying flavor symmetry. This would be a preferred alternative if the deviation of the atmospheric mixing angle from maximality is constrained to be very small. The simplest of such flavor symmetries is the μ-τ symmetry [2–5] which exchanges the mu and tau fields. This symmetry comes with an additional prediction that one of the three leptonic mixing angles, namely, θ^l_{13} [6], must be zero.

μ-τ symmetry is predictive and simple, but it appears to have two shortcomings. Successful predictions follow only if it is an effective symmetry of the neutrino mass matrix in a specific basis corresponding to a diagonal charged lepton mass matrix. The underlying flavor symmetry in general may not pick up this basis. Second, μ-τ symmetry has been proposed with a view of explaining the mixing angles in the leptonic sector alone. It would be more desirable to have a symmetry providing an overall understanding of the complete fermionic mass spectrum. This can be done by using the grand unified theory as the underlying framework. Various alternatives within such theories to simultaneously obtain small mixing in the quark sector and large mixing among leptons have already been proposed [1,7,8].

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The renormalizable theories based on the $SO(10)$ group are quite powerful in constraining the fermionic mass structures. The standard fermions are assigned to the 16-dimensional representation of the $SO(10)$ group, and they can obtain masses through symmetric couplings with 10 and $\overline{126}$ and antisymmetric couplings with the 120-dimensional representation of the Higgs fields. The minimal $SO(10)$ model containing 10, 126, $\overline{126}$, and 210 representations has been extensively studied [7,9–15]. In this model, the largeness of the atmospheric mixing angle gets related to the $b-\tau$ Yukawa unification if neutrinos obtain their masses through the type-II [16] seesaw mechanism [17]. This interesting observation in Ref. [7] led to many detailed investigations [12–15] which revealed the inadequacy of this simple picture. The supersymmetric version of the minimal model with the type-II seesaw mechanism is constrained by two conflicting requirements. The overall neutrino mass scale is correctly reproduced in the model if the seesaw scale is about 2–3 orders of magnitude below the grand unified theory (GUT) scale. But the spectrum of the model in this case does not allow gauge coupling unification. Moreover, the type-II contribution to neutrino masses does not always dominate over the type-I contribution in the minimal model as would be required for the mechanism in Ref. [7] to go through. The conflict with the proton decay appears in the minimal model even if the neutrinos obtain their masses through the type-I seesaw [12–14]. These problems have led to studies of the nonminimal models containing an additional 120-plet of Higgs [18–20]. Theoretical understanding of the largeness of the atmospheric mixing angle gets lost in all of these approaches, although one can choose the parameters to obtain the observed value.

It would be welcome to integrate μ-τ symmetry into the grand unified framework. This has been done for the $SU(5)$ model in Ref. [21]. We do so here in the more predictive $SO(10)$ framework. There are several motivations for unifying $SO(10)$ and μ-τ symmetry. Rather than remaining a leptonic symmetry, such symmetry would provide a constraining picture of both the quark and the lepton spectrum. The role of this symmetry in the description of the quark mixing is already discussed in Refs. [4,5]. In addition, it can provide additional constraints and reduces the number of the Yukawa couplings which describe fermion masses. Some examples of models unifying $SO(10)$ with other discrete symmetries can be found in Ref. [22].

We investigate the consequences of imposing a generalized μ-τ symmetry exchanging the second and the third generation fields on a renormalizable $SO(10)$ model. We deviate from the minimal model and add a 120-plet. This plays a crucial role in generating CP violation and the μ-τ symmetry breaking. Fermion masses with a 120-plet have been discussed in several earlier works [18–20,23–27]. Following Refs. [19,20,26,27], we impose also the parity symmetry which leads to Hermitian mass matrices for all fermions, thereby reducing the number of parameters compared to more general models. All of the fermion masses and mixing are described in our approach in terms of 14 (15) real parameters in the case of the type-II (type-I) seesaw mechanism. They provide an excellent description of fermion masses and mixing in contrast to a general model employing $10 + 120 + \overline{126}$ Higgs fields which needs [26] 31 parameters in the fermionic sector. Moreover, the (near) maximality of the atmospheric mixing and smallness of the angle θ_{13}^l get related here to the approximately broken μ-τ symmetry.

We define our model implementing μ-τ symmetry and discuss its consequences in the next section. Section III presents numerical fits in the case of both the type-II and the type-I seesaw mechanism and discusses various predictions. The last section contains a summary.

II. μ-τ SYMMETRIC $SO(10)$

If μ-τ symmetry is to be integrated with grand unification, then a more general symmetry which exchanges the second and third generations of fermions should be imposed. Consequences of this generalization were first considered in Ref. [4]. It was subsequently noted [5] that this generalization automatically leads to understanding of why the Cabibbo angle is larger than the other two angles, and a mild breaking of this symmetry was shown to lead to a correct description of the quark mixing angles and masses. Most of these works did not use the grand unified framework. Here we consider a model based on the $SO(10) \otimes Z_2^{\mu-\tau} \otimes Z_2^P$. The first Z_2 corresponds to the generalized μ-τ symmetry. The second Z_2 symmetry called [26] "parity" interchanges two components of the 16 field transforming as (4, 2, 1) and ($\bar{4}$, 1, 2) under the Pati-Salam group decomposition of $SO(10)$.

Our basic formalism is similar to Refs. [19,20,26,27]. 16-dimensional fermions obtain their masses from coupling to three Higgs multiplets transforming as 10, $\overline{126}$, and 120 representations under $SO(10)$. The $SO(10)$ breaking can be achieved with a 210-plet. An additional 126-plet of Higgs is needed in the supersymmetric context to preserve the supersymmetry at the GUT breaking scale. These Higgs multiplets contain altogether six doublets with quantum numbers of the minimal supersymmetric standard model (MSSM) field H_d and six with that of H_u. It is assumed that only two appropriate linear combinations of these Higgs doublets remain light and play the role of the H_d and H_u fields. This is achieved by the fine-tuning conditions [14]. After this fine-tuning, the resulting fermion masses can be written as

$$-\mathcal{L}_{\text{mass}} = \bar{f}_L M_f f_R + \bar{\nu}_L M_D \nu_R + \tfrac{1}{2}\bar{\nu}_L M_L \nu_L^c + \tfrac{1}{2}\bar{\nu}_R^c M_R \nu_R + \text{H.c.}, \tag{1}$$

where $f = u, d$, and l denote the up and down quarks and the charged leptons, respectively. The mass matrices appearing in the above equation can be suitably written (see [19,20,28] for details) as

$$M_d = H + F + iG, \qquad M_u = rH + sF + itG,$$
$$M_l = H - 3F + ipG, \qquad M_D = rH - 3sF + iqG,$$
$$M_L = r_L F, \qquad M_R = r_R^{-1} F. \tag{2}$$

Here M_D denotes the neutrino Dirac mass matrix. $M_L(M_R)$ is the Majorana mass matrix for the left- (right-) handed neutrinos which receives a contribution only from the vacuum expectation value (vev) of the $\overline{126}$ field. Gauge coupling unification in the minimal model requires that the vev contributing to M_R be close to the GUT scale. The dimensionless parameters r, s, t, p, q, r_L, and r_R are determined by the Clebsch-Gordan coefficients, ratios of vevs, and mixing among the Higgs fields [19].

The matrices H, F, and G originate from the fermion couplings to the 10, $\overline{126}$, and 120 fields, respectively. (G) H and F are complex (anti)symmetric matrices in general. However, generalized parity makes them real. In addition, if all vevs and (hence r, s, t, p, q, r_L, and r_R) are real, then all of the Dirac masses in Eq. (2) are Hermitian and M_L and M_R are real.

We assume that the Higgs fields in the 10 and $\overline{126}$ representations are invariant under the generalized μ-τ symmetry while the 120-dimensional representation changes sign. This assumption allows spontaneous breaking of the μ-τ symmetry. The resulting structures for H, F, and G are given by

$$H = \begin{pmatrix} h_{11} & h_{12} & h_{12} \\ h_{12} & h_{22} & h_{23} \\ h_{12} & h_{23} & h_{22} \end{pmatrix}; \qquad F = \begin{pmatrix} f_{11} & f_{12} & f_{12} \\ f_{12} & f_{22} & f_{23} \\ f_{12} & f_{23} & f_{22} \end{pmatrix};$$
$$G = \begin{pmatrix} 0 & g_{12} & -g_{12} \\ -g_{12} & 0 & g_{23} \\ g_{12} & -g_{23} & 0 \end{pmatrix}. \tag{3}$$

All of the coefficients in these matrices are real. They satisfy

$$S^T(H, F, G)S = (H, F, -G), \tag{4}$$

where

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{5}$$

exchanges the second and the third generations. The effective neutrino mass matrix $\mathcal{M}_\nu$ for the three light neutrinos follows from Eqs. (1) and (2):

$$\mathcal{M}_\nu = r_L F - r_R M_D F^{-1} M_D^T \equiv \mathcal{M}_\nu^{\mathrm{II}} + \mathcal{M}_\nu^{\mathrm{I}}. \tag{6}$$

Here $r_{L,R}$ are inversely related to the vev of the right-handed triplet component in $\overline{126}$. This vev may be identified with the GUT scale in the absence of any intermediate scale. In addition, they depend upon the details of the superpotential. Specific expressions for $r_{L,R}$ in the minimal case can be found in Refs. [13,14]. The first term corresponds to the type-II seesaw, while the second is the conventional type-I seesaw. In general, both contributions are present, but one may dominate over the other. We shall be considering two separate cases corresponding to the type-II and type-I dominance, respectively.

The relations $\theta_{23}^l = \frac{\pi}{4}$ and $\theta_{13}^l = 0$ are major predictions and motivation for imposing the μ-τ symmetry. These can arise if the effective neutrino mass matrix $\mathcal{M}_{\nu f}$ in the flavor basis possesses a μ-τ symmetry. Let us see how this can come about in our approach. It is easy to see that the fermionic mass matrices in our model satisfy

$$S^{-1} M_f S = M_f^*, \tag{7}$$

$$S^{-1} \mathcal{M}_\nu^{\mathrm{II}} S = \mathcal{M}_\nu^{\mathrm{II}}, \tag{8}$$

$$S^{-1} \mathcal{M}_\nu^{\mathrm{I}} S = \mathcal{M}_\nu^{\mathrm{I}*}. \tag{9}$$

$f = u, d, l, D$ label the (Dirac) fermionic mass matrices. The $\mathcal{M}_\nu^{\mathrm{I,II}}$ correspond to the type-I and -II contributions to the light neutrino mass matrix Eq. (6), respectively. Let us note that

(i) Eq. (8) implies an exact μ-τ symmetry for $\mathcal{M}_\nu^{\mathrm{II}}$;

(ii) Eqs. (7) and (9) correspond to an invariance under the generalized CP transformation defined [3,29] as

$$f_\alpha \to i S_{\alpha\beta} \gamma^0 C \bar{f}_\beta^T; \tag{10}$$

(iii) if Eq. (8) represents the neutrino masses in the flavor basis, then one obtains the predictions $\theta_{23}^l = \frac{\pi}{4}$ and $\theta_{13}^l = 0$;

(iv) if Eq. (9) holds in the flavor basis, then only the θ_{23}^l is maximal with definite correlations of θ_{13}^l with the CP violating phase δ_{PMNS} [29];

(v) M_l is not diagonal here, and hence these predictions do not follow immediately. It is still possible to recover these predictions even with a nondiagonal M_l.

Define

$$U_l^\dagger M_l U_l = D_l, \tag{11}$$

where D_l is the diagonal mass matrix for the charged leptons. By factoring out a diagonal phase matrix P_l, the U_l can be written as

$$U_l \equiv \tilde{U}_l P_l. \tag{12}$$

The neutrino mass matrix in the flavor basis is then given by

$$\mathcal{M}_{\nu f} = P_l^\dagger \tilde{U}_l^\dagger \mathcal{M}_\nu \tilde{U}_l^* P_l^* \equiv P_l^\dagger \tilde{\mathcal{M}}_{\nu f} P_l^*. \tag{13}$$

The predictions of the μ-τ symmetry are recovered if $\tilde{\mathcal{M}}_{\nu f}$ is μ-τ invariant. This does not require a diagonal M_l. A general μ-τ symmetric $\tilde{U}_l$ satisfying $S^{-1}\tilde{U}_l S = \tilde{U}_l$ will do the job in the case of the type-II dominance. This makes it possible to recover the predictions of the μ-τ symmetry for a nondiagonal M_l and obtain reasonably good fits to other fermion masses and mixing.

It is known [3,29] that, with an appropriate choice of P_l, $\tilde{U}_l$ can be cast into the following form if M_l satisfies Eq. (7):

$$\tilde{U}_l = \begin{pmatrix} u_{1l} & u_{2l} & u_{3l} \\ w_{1l} & w_{2l} & w_{3l} \\ w_{1l}^* & w_{2l}^* & w_{3l}^* \end{pmatrix}, \tag{14}$$

with real u_{il}. A unitary matrix with this form can be parametrized in terms of two angles and a phase:

$$\tilde{U}_l = P_\eta \begin{pmatrix} c_1 & s_1 c_2 & s_1 s_2 \\ \frac{s_1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}(c_1 c_2 - i\epsilon s_2) & -\frac{1}{\sqrt{2}}(c_1 s_2 + i\epsilon c_2) \\ \frac{s_1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}(c_1 c_2 + i\epsilon s_2) & -\frac{1}{\sqrt{2}}(c_1 s_2 - i\epsilon c_2) \end{pmatrix}, \tag{15}$$

where $\epsilon = \pm 1$, $s_{1,2} \equiv \sin\theta_{1,2}$, and $c_{1,2} = \cos\theta_{1,2}$. c_2 and s_2 can be chosen positive with an appropriate choice of P_l in Eq. (12).

$$P_\eta = \mathrm{diag}(1, e^{-i\eta}, e^{i\eta})$$

is a diagonal phase matrix. The above $\tilde{U}_l$ becomes μ-τ symmetric if $s_2 = c_2$ and $\eta = 0$. This defines a one-parameter family of the leptonic mass matrices which lead to the prediction of the μ-τ symmetry in the case of the type-II dominance. We will use this form subsequently in our numerical analysis.

There is an important but unwelcome feature associated with the generalized CP invariance of the mass matrices in Eq. (7). The Cabibbo-Kobayashi-Maskawa (CKM) matrix in this case turns out to be real. To see this explicitly, we note that, just as in the case of U_l, the matrices $U_{u,d}$ diagonalizing the up and down quark masses can be written as $\tilde{U}_{u,d}P_{u,d}$. $\tilde{U}_{u,d}$ have the same form as the right-hand side of Eq. (14) with the replacement of u_{il} with $u_{iu,id}$ and w_{il} with $w_{iu,id}$. The phase matrices $P_{u,d}$ can be absorbed in redefining the quark fields, and the remaining part of the CKM matrix is given by

$$V_{ij} \equiv (\tilde{U}_u^\dagger \tilde{U}_d)_{ij} = u_{iu}u_{jd} + 2\,\mathrm{Re}(w_{iu}w_{jd}^*),$$

which is real since $u_{iu,id}$ are real.

One can generate CP violation in the model by breaking the generalized CP invariance of the mass matrices. This can be done in two ways. Either one allows a complex vev for some of the Higgs doublets as in Ref. [19], or one retains the real vev but allows breaking of the μ-τ symmetry in the Yukawa couplings. In the following, we will discuss the second alternative.

III. FITTING FERMION SPECTRUM WITH AND WITHOUT THE μ-τ SYMMETRY

We now discuss the numerical implications of our model in detail. We assume that either the type-I or the type-II term in the neutrino mass matrix dominates and carry out analysis separately in each of these two cases. Our input parameters are r, s, t, p, and q [Eq. (2)], the real elements of the matrices G, H, and F [Eq. (3)], and the overall scales $r_{R,L}$ [Eq. (6)]. Parameter q is absent in the type-II case. An overall rotation R on G, H, and F: $(G, H, F) \rightarrow R^T(G, H, F)R$ amounts to a choice of initial basis for the 16-plet of fermions. We can use this freedom to set, say, $h_{12} = 0$. This is done with a specific choice $R = R_{23}^T(\frac{\pi}{4})R_{12}(\theta_{12}^h)R_{23}(\frac{\pi}{4})$. Here $R_{ij}(\theta)$ denotes rotation in the ijth plane by an angle θ and

$$\tan 2\theta_{12}^h = \frac{2\sqrt{2}h_{12}}{h_{11} - h_{22} - h_{23}}.$$

This rotation amounts to redefinition of elements of F and G which still retain the same form as in Eq. (3). We continue to use the same notation for the parameters of the redefined F and G. With the choice $h_{12} = 0$, we have 14 (15) input parameters in the case of type-II (type-I) seesaw dominance. These input parameters together generate 12 fermion masses and six mixing angles. As already remarked, the exact μ-τ symmetric H, F, and G are not able to generate CP violation. We introduce this CP violation by adding a small μ-τ breaking difference between the 22 and 33 elements in H. This one additional parameter now leads to four CP violating phases: one in the CKM matrix and three in the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix.

Our choice of the values of the physical observables is based on numbers given in Refs. [13,20]. We reproduce them here in Table I for convenience.

The given numbers for quark masses and mixing correspond to the respective values at the GUT scale obtained from low energy values using the MSSM and $\tan\beta = 10$. The neutrino masses and mixing that we use are the low scale values, but the effects of the evolution to M_{GUT} on the ratio of the solar to atmospheric mass scale and on the mixing angles are known to be small for the normal hierarchical spectrum that we obtain here. While fitting, we omit the parameters r_R and r_L which define the overall scales of neutrino masses in the case of the type-I and type-II seesaw, respectively. The ratio of the solar and atmospheric mass scales and neutrino mixing parameters are independent of these overall scales and are used in our definition of χ^2 function instead of the individual neutrino masses. In addition, we assume $\Delta m^2_{\mathrm{atm}}$ to be positive corresponding to the normal neutrino mass hierarchy. Parameters r_R and r_L are fixed subsequent to minimization using the atmospheric scale.

A. Numerical analysis: Type-II seesaw

We perform the minimization in three physically different cases.

(A) In this case, we impose the conditions $\theta_{23}^l = \frac{\pi}{4}$ and $\theta_{13}^l = 0$ using a μ-τ symmetric $\tilde{U}_l$. As discussed in the earlier section, this is done using parametrization in Eq. (15) with $s_2 = c_2 = 1/\sqrt{2}$. The charged lepton mass matrix is then determined completely in terms of three masses and the angle θ_1. Using the third of Eqs. (2), the real and imaginary parts of M_l can be used to determine, respectively, elements of H in terms of that of F and elements of G in terms of p, the charged lepton masses, and θ_1. f_{12} also gets determined in terms of these parameters because of the choice $h_{12} = 0$. Thus f_{22}, f_{23}, f_{11}, r, s, t, p, and θ_1 are the only free parameters which

TABLE I. Input values for quark and leptonic masses and mixing angles at $M_{\mathrm{GUT}} = 2 \times 10^{16}$ GeV and $\tan\beta = 10$ which we use in our numerical analysis.

m_d	1.03 ± 0.41	$\Delta m^2_{\mathrm{sol}}$	$(7.9 \pm 0.3) \times 10^{-5}$
m_s	19.6 ± 5.2	$\Delta m^2_{\mathrm{atm}}$	$(2.2^{+0.37}_{-0.27}) \times 10^{-3}$
m_b	$1063.6^{+141.4}_{-86.5}$	$\sin\theta^q_{12}$	0.2243 ± 0.0016
m_u	0.45 ± 0.15	$\sin\theta^q_{23}$	0.0351 ± 0.0013
m_c	$210.3273^{+19.0036}_{-21.2264}$	$\sin\theta^q_{13}$	0.0032 ± 0.0005
m_t	$82\,433.3^{+30267.6}_{-14768.6}$	$\sin^2\theta^l_{12}$	0.31 ± 0.025
m_e	0.3585 ± 0.0003	$\sin^2\theta^l_{23}$	0.5 ± 0.065
m_μ	$75.6715^{+0.0578}_{-0.0501}$	$\sin^2\theta^l_{13}$	<0.0155
m_τ	$1292.2^{+1.3}_{-1.2}$	δ_{CKM}	$60^\circ \pm 14^\circ$

determine the 11 remaining observables—six quark masses, three angles of the CKM matrix, the solar angle, and the solar to atmospheric mass ratio. The χ^2 we minimize is defined in terms of these observables using the values and errors given in Table I. The results of the minimization are shown in Tables II and III. One obtains a reasonably good fit to all observables except the down and bottom quark masses, which are, respectively, $\sim$1.5 and $\sim$2.5 sigma away from their respective mean values. All other observables are reproduced correctly with very small pulls as seen in the table.

(B) In this case, we do not impose the maximality of θ^l_{23} but include $\sin^2\theta^l_{23}$ in the χ^2 to be minimized. $\sin^2\theta^l_{13}$ is not included in the definition of χ^2, but we require it to be ≤ 0.0155 during the minimization. r, s, t, p, and elements of H, F, and G are now treated as free, and the χ^2 definition now includes the charged lepton masses as well. This results in significant improvement in the fit, and one is able to fit 15 observables in terms of 13 parameters with $\chi^2 = 3.01$. The fit to the bottom and the down quark masses also improves. $\delta_{\rm CKM}$ remains zero in this case.

TABLE II. Best fit solutions for fermion masses and mixing obtained assuming the type-II seesaw dominance. Various observables and their pulls obtained at the minimum are shown in three cases (A)–(C) defined in the text.

Quantity	A pull	B pull	C pull
m_d	−1.475 32	0.167 255	0.062 011 5
m_s	−0.8225	0.271 662	−0.054 552 3
m_b	−2.523 88	1.687 87	1.728 11
m_u	0.274 609	−0.004 466 26	−0.001 844 52
m_c	−0.012 588 7	0.000 159 604	0.007 442 92
m_t	0.001 904 76	0.009 019 41	−0.019 952 2
m_e	0	−0.000 951 761	0.000 179 815
m_μ	0	0.017 626 6	−0.000 749 102
m_τ	0	−0.019 227 4	−0.017 642
$\frac{\Delta m^2_{\rm sol}}{\Delta m^2_{\rm atm}}$	0.679 035	−0.169 337	−0.054 452 1
$\sin\theta^q_{12}$	−0.011 605 9	0.002 504 91	−0.004 123 83
$\sin\theta^q_{23}$	0.155 231	−0.007 179 26	0.040 286 1
$\sin\theta^q_{13}$	−0.070 536 2	0.000 016 398 2	0.016 396 4
$\sin^2\theta^l_{12}$	0.112 082	−0.111 783	−0.005 780 02
$\sin^2\theta^l_{23}$	0	0.129 873	−0.141 465
$\delta_{\rm CKM}$	···	···	−0.036 427 1
χ^2	9.804 73	3.009 57	3.020 19

	Predictions	Predictions	Predictions
$\sin^2\theta^l_{23}$	0.5	···	···
$\sin^2\theta^l_{13}$	0	0.000 471 537	0.000 226 908
$\delta_{\rm CKM}$	0°	0°	···
$\delta_{\rm PMNS}$	0°	0°	−12.759°
α_1	180°	180°	169.80°
α_2	0°	0°	−9.445°
r_L	2.8714×10^{-10}	1.8183×10^{-9}	1.8645×10^{-9}

TABLE III. Values of parameters of the fermionic mass matrices in Eq. (2) corresponding to the best fit solutions displayed in Table II. The cases (A)–(C) are defined in the text.

Parameters	A	B	C
h_{11}	1.959 14	−0.357 916	−0.818 923
h_{22}	466.637	−649.2	−701.354
h_{23}	283.929	−54.7552	−32.0485
h_{33}	466.637	−649.2	−598.783
f_{11}	−1.251 74	−0.176 133	−0.343 138
f_{12}	14.2058	−2.163 75	−2.072 69
f_{22}	−71.54	11.5434	11.2606
f_{23}	95.5358	−14.754	−14.3836
g_{12}	−1.666 46	3.548 11	4.198 17
g_{23}	−26.5205	614.356	617.845
r	106.129	61.8507	61.1056
s	114.802	−109.87	−121.664
t	−1.9006	67.0199	65.9824
p	22.8456	−0.989 943	−0.980 791

(C) For this case, we depart from the exact 23 symmetry and take h_{22} different from h_{33}. As already discussed, this breaks the generalized CP and results in a nontrivial CKM phase. Remarkably, a very small ($\sim 8\%$) breaking of the 23 symmetry is able to generate a nontrivial CKM phase and $\chi^2_{\rm min} = 3.02$ with 2 degrees of freedom. The bottom quark mass is the only variable which deviates from its central value considerably.

Some of the observables are not part of the χ^2, and their values get fixed at the minimum. These are shown as predictions in Table II. These include the CP violating Dirac phase $\delta_{\rm PMNS}$ and the Majorana phases $\alpha_{1,2}$ as defined in Ref. [6]. These are trivial for the cases (A) and (B) due to the generalized CP invariance, but one obtains nonzero values displayed in the table in case (C).

Before going into the more detailed predictions, let us underline some important points connected with the above fits.

(i) Detailed fits to fermion masses have been considered in a number of papers with [19,20] or without [13,15] the addition of the 120-plet to the minimal $10 + \overline{126}$ Higgs fields. The minimal model without the 120-plet but not imposing reality of the coupling has more parameters than the present case, but the fit is not better compared to here; e.g. the fit in the pure type-II case [13] with 18 parameters and 15 data points gives a minimum χ^2 around 14.5.

(ii) The best fit solutions in cases (B) and (C) give θ^l_{23} close to maximal and θ^l_{13} close to zero as seen from Table II.

(iii) We have fixed the overall scale of neutrino mass r_L in Eq. (6) by using the atmospheric scale as normalization. The resulting values are displayed in

Table II. In all three cases, r_L comes close to 10^{-10}. r_L is related to the mass of the left-handed triplet residing in the $\overline{126}$ representation and to other parameters in the superpotential. Detailed analysis [12–15] has shown that one needs this triplet mass to be at an intermediate scale $\sim 10^{12}$ GeV if the overall neutrino mass scale is to be correctly reproduced. The presence of such a light triplet conflicts with the gauge coupling unification. An additional 120-plet does not qualitatively alter the situation. One possible solution suggested [30] in the literature is to add a 54-plet of Higgs and allow $SO(10)$ to break first to $SU(5)$ leaving a complete 15-plet of Higgs light at around 10^{12} GeV. Another solution corresponds to having split supersymmetry breaking [15]. A third possibility is to allow type-I seesaw dominance [18,23]. We shall look at this in the next subsection in the present context.

We now turn to predictions in the neutrino sector. The firm predictions of the scheme can be obtained by checking the variation of χ^2 with the values of various observables. As in Refs. [13,20], we pin down a specific value p_0 of an observable P by adding a term

$$\chi_p^2 = \left(\frac{P - p_0}{0.01 p_0}\right)^2$$

to χ^2 and then minimizing

$$\hat{\chi}^2 \equiv \chi^2 + \chi_P^2.$$

If P happens to be one of the observables used in defining χ^2, then its contribution is removed from there. An artificially introduced small error fixes the value p_0 for P at the minimum of the $\hat{\chi}^2$. We then look at the variation of

$$\bar{\chi}^2_{\rm min} \equiv (\hat{\chi}^2 - \chi_p^2)|_{\rm min} \tag{16}$$

with p_0. The results are displayed in Figs. 1–3.

Figure 1 shows the variation of $\bar{\chi}^2_{\rm min}$ for various pinned-down values of $\sin^2\theta^l_{23}$. It is seen that the minimum occurs when $\sin^2\theta^l_{23}$ is fixed to around 0.46 rather than the value 0.5 obtained in the fits shown in Table II. The variation of $\bar{\chi}^2_{\rm min}$ is not drastic, and all values in the range 0.3–0.7 are allowed at 90% C.L. In comparison, variation of $\bar{\chi}^2_{\rm min}$ with $\sin^2\theta^l_{13}$ shown in Fig. 2 is a little more significant. There is a preference for values close to zero, but values up to 0.008 cannot be ruled out at 90% confidence level. Figure 3 shows the prediction for the PMNS phase in the leptonic mixing matrix. A clear prediction is the negative values for the $\sin\delta_{\rm PMNS}$. However, all negative values are allowed within the 90% confidence limit.

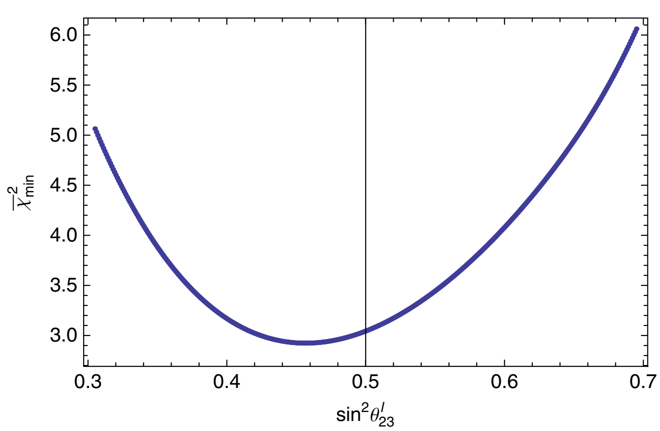

FIG. 1 (color online). Variation of $\bar{\chi}^2_{\rm min}$ with $\sin^2\theta^l_{23}$ in the type-II seesaw.

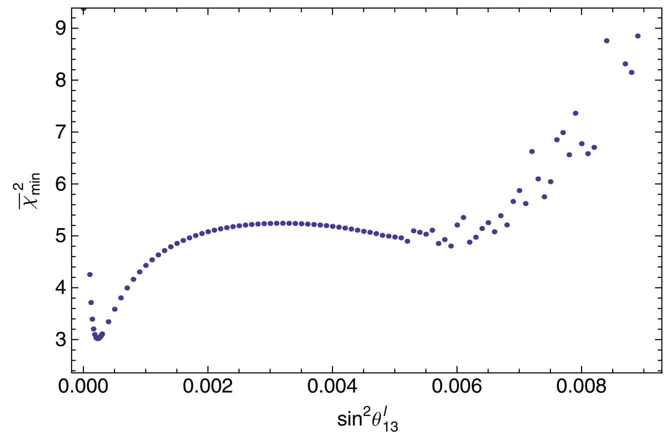

FIG. 2 (color online). Variation of $\bar{\chi}^2_{\rm min}$ with $\sin^2\theta^l_{13}$ in the type-II seesaw.

B. Numerical analysis: Type-I seesaw

The structure of the neutrino mass matrix in the type-I case is qualitatively different compared to the type-II case. Unlike $\mathcal{M}_\nu^{II}$, $\mathcal{M}_\nu^{I}$ is not μ-τ invariant in general. But it can be made approximately μ-τ symmetric if either the 120 contribution or the $10 + \overline{126}$ dominates in M_D; see Eq. (2). We discuss below fits in three qualitatively different cases as done for the type-II dominance.

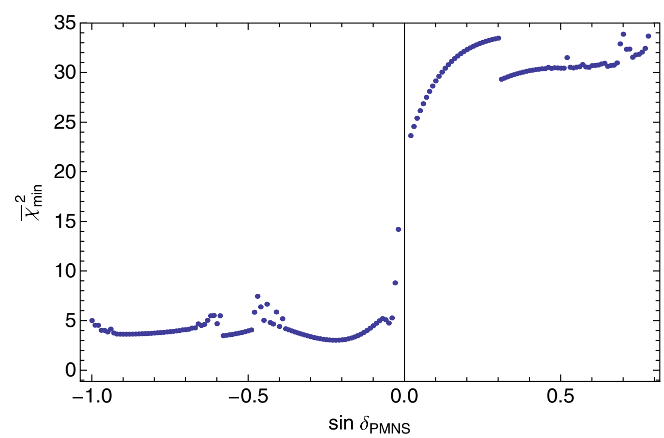

FIG. 3 (color online). Variation of $\bar{\chi}^2_{\rm min}$ with $\sin\delta_{\rm PMNS}$ in the type-II seesaw.

(A) Here we impose the exact μ-τ symmetry for $\mathcal{M}_\nu^l$ by hand, i.e. by choosing $q = 0$ in M_D. As before, U_l is also chosen μ-τ symmetric. The input parameters and observables are the same as in the case (A) of type-II seesaw. The results of the fits are displayed in the first column of Table IV. The total χ^2 involves 11 observables and is determined by 8 parameters. The minimum value is $\sim$13. While most observables can be fitted nicely, the top quark mass deviate by 3.6σ from the central value. Enforcing the exact μ-τ symmetry does not appear to be a very good choice.

(B) In this case, we do not take $q = 0$. $\mathcal{M}_\nu^l$ now satisfies Eq. (9) and is not symmetric under μ-τ symmetry. θ_{23}^l is not fixed to be maximal but is included in the definition of χ^2. As in the earlier case (B), χ^2 is defined by 15 observables and is determined in terms of 14 parameters. The *CP* violating phases are zero in this case, and the CKM phase is therefore not included in χ^2. The experimental bound on θ_{13}^l shown in Table I is imposed during the minimization. One now gets an excellent fit to all of the included variables with $\chi^2_{\min} = 0.017$.

(C) In this case we introduce a small explicit μ-τ symmetry breaking by assuming $h_{22} \neq h_{33}$ in Eq. (2). This allows *CP* violation. The χ^2 definition now includes all 16 observables and depends on 15 parameters. A bound on θ_{13}^l is imposed during minimization. Once again, we get an excellent fit to all of the observables with $\chi^2_{\min} = 0.18$. *CP* violating phases in the PMNS matrix come as predictions.

TABLE IV. Best fit solutions for fermion masses and mixing obtained assuming the type-I seesaw dominance. Various observables and their pull obtained at the minimum are shown in three cases (A)–(C) defined in the text.

Quantity	*A* pull	*B* pull	*C* pull
m_d	−0.315 69	0.034 600 7	−0.379 829
m_s	0.473 034	−0.048 377 9	−0.071 727 7
m_b	−0.108 264	−0.113 763	−0.114 314
m_u	0.502 63	0.000 263 23	0.003 446 98
m_c	−0.151 225	−0.000 606 809	−0.009 382 66
m_t	−3.607 44	−0.019 310 7	0.012 266 3
m_e	0	-4.874×10^{-6}	0.000 034 885 8
m_μ	0	0.000 480 511	0.000 783 71
m_τ	0	0.002 541 53	−0.010 606 5
$\frac{\Delta m^2_{\rm sol}}{\Delta m^2_{\rm atm}}$	−0.009 776 27	−0.001 928 56	0.012 521 8
$\sin\theta_{12}^q$	0.021 820 5	−0.000 613 12	0.007 618 17
$\sin\theta_{23}^q$	0.002 892 71	0.001 299 46	0.028 421 4
$\sin\theta_{13}^q$	−0.238 953	−0.008 233 61	0.036 641 3
$\sin^2\theta_{12}^l$	−0.012 971 2	0.000 590 904	−0.002 651 93
$\sin^2\theta_{23}^l$	0	−0.005 445 23	0.028 995 9
$\delta_{\rm CKM}$	...	...	−0.120 278
χ^2	13.6821	0.016 963 2	0.180 526

	Predictions	Predictions	Predictions
$\sin^2\theta_{23}^l$	0.5	...	...
$\sin^2\theta_{13}^l$	0	0.013 560 5	0.013 505
$\delta_{\rm CKM}$	0°	0°	...
$\delta_{\rm PMNS}$	0°	0°	−0.287 748°
α_1	180°	0°	2.156°
α_2	0°	0°	2.616°
r_R	4.1143×10^{-11}	5.2329×10^{-18}	5.0093×10^{-18}

Noteworthy features of the fits in the (B) and (C) cases above are the following:

(i) The overall neutrino mass scale is determined to be around $r_R \sim 5 \times 10^{-18}$. r_R is related to the ratio of the vev of the doublet and the right-handed triplet components in $\overline{126}$. The values of r_R obtained here are similar to the values obtained in Ref. [20] which assume a $\overline{126}$ right-handed triplet vev to be at the GUT scale. Thus one does not need an intermediate scale in order to fit the neutrino masses, and one can obtain the gauge coupling unification. This is consistent with observations in Refs. [19,20,23].

(ii) Maximality of θ_{23}^l is not imposed. But it is fixed to be very close to $\frac{\pi}{4}$ at the minimum in both cases. The departure from the μ-τ symmetry results in θ_{13}^l being nonzero and is fixed around the upper bound at the minimum as seen from Table II.

(iii) Although an explicit breaking of the μ-τ symmetry is introduced in case (C), the amount of the breaking required in order to obtain the large *CP* violating phase is extremely tiny:

$$\frac{h_{22} - h_{33}}{h_{22} + h_{33}} \sim 0.0045. \tag{17}$$

(iv) The exact μ-τ symmetry is known [5] to lead to the unwanted predictions $V_{ub} = V_{cb} = \sin^2\theta_{23}^l = 0$. Here we have two sources of breaking this symmetry: spontaneous through the vev of the 120-plet and explicit through Eq. (17), which allows one to reproduce the mixing angles correctly. In spite of the μ-τ breaking, the final fermion mass matrices display a remarkably good μ-τ symmetry. We make this explicit by giving the quark and lepton mass matrices in the case (C) above in the appendix. $M_{u,d,l}$ and $\mathcal{M}_\nu^l$ are seen to be nearly μ-τ symmetric. There is an order of magnitude difference in the imaginary parts of the 12 and 13 elements of $\mathcal{M}_\nu^l$. But these imaginary parts are much smaller than the corresponding μ-τ symmetric real parts. The only source of the large μ-τ breaking occurs as a difference between the 12 and 13 elements of the Dirac neutrino mass matrix M_D. This results from the spontaneous breakdown and rather large value of the parameter q.

(v) As in Refs. [19,20], we have concentrated here on obtaining generic fits to fermion masses rather than considering the entire parameter space of the theory given by the Yukawa couplings and basic parameters in the superpotential. Parameters in fermion mass matrices are related to the strengths of the light Higgs components in various $SO(10)$ Higgs representations. These are determined by the fine-tuning conditions and the full superpotential. Grimus and Kühböck [19] have laid down consistency constraints on these parameters following from these fine-tuning relations and from the requirement that the Yukawa couplings stay in the perturbative regime. We have checked that these conditions are satisfied by the parameters given in Tables III and V.

We follow a similar procedure as in the type-II case to obtain possible predictions on the neutrino mixing variables. We pin down an observable P to a specific value p_0 by adding a contribution χ^2_P to χ^2. We then determine the variation of $\bar{\chi}^2_{\rm min}$ defined earlier with p_0. Variations of $\bar{\chi}^2_{\rm min}$ obtained at different local minima are shown as scattered plots in Figs. 4–6.

Clear predictions emerge unlike in the type-II case. As Fig. 4 shows, the $\sin^2\theta^l_{23}$ is preferentially restricted near 0.5, and one obtains the limit ~0.42–0.63 at the 90% C.L. Figure 5 shows similar variation with respect to $\sin^2\theta^l_{13}$. Here the preferred values occur near the present limit, and one obtains $\sin^2\theta^l_{13} > 0.005$ at 90% C.L. The predicted values for $\sin\delta_{\rm PMNS}$ are displayed in Fig. 6. These are negative but very small.

All of the above solutions are obtained through an extensive search using the random search algorithm in MATHEMATICA and the MINUIT subroutine in FORTRAN, and we have shown in tables the solutions corresponding to the minimum χ^2 that we obtained. Considering the nonlinearity and complexity of the problem here, it is difficult to rule out the existence of still lower minima, and predictions may improve if they exist.

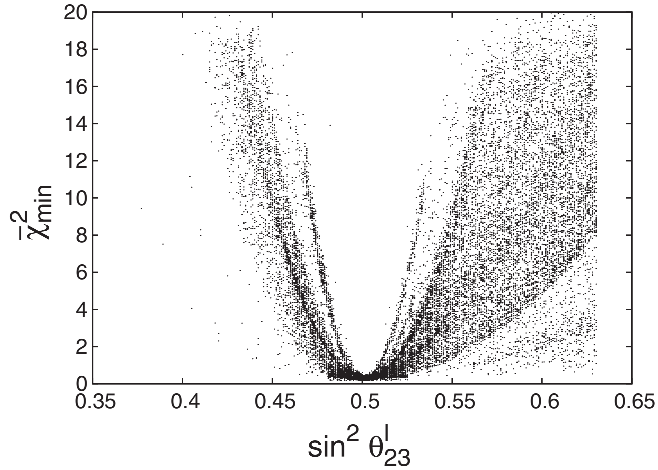

FIG. 4. Variation of $\bar{\chi}^2_{\rm min}$ with $\sin^2\theta^l_{23}$ in the type-I seesaw.

We end this section with a comment on the specific μ-τ symmetry defined by S used in Eq. (5). The definition of S is basis-dependent. One could change the original basis of the 16-plet through an arbitrary rotation R. The structure of the Yukawa couplings and the resulting fermionic mass matrices would look different in the new basis. The new Yukawa couplings would still satisfy the same equation as (4) but now with a rotated S: $S_R \equiv R^T SR$. Thus the μ-τ symmetry may appear to look different with different choices of R. Specifically, if R corresponds to a rotation by $\frac{\pi}{4}$ in the 23 plane, then the S_R assumes the form

$$S_R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (18)$$

TABLE V. Values of parameters of the fermionic mass matrices in Eq. (2) corresponding to the best fit solutions displayed in Table IV. The cases (A)–(C) are defined in the text.

Parameters	A	B	C
h_{11}	907.294	34.9749	35.0178
h_{22}	119.541	554.305	556.777
h_{23}	−119.052	554.457	554.429
h_{33}	119.541	554.305	551.775
f_{11}	74.5214	−15.7284	−15.716
f_{12}	−2.823 27	20.8852	20.8951
f_{22}	−74.237	−29.4577	−29.4636
f_{23}	74.2104	−29.5265	−29.5305
g_{12}	182.676	3.109 44	2.797 28
g_{23}	−4.5309	−3.5854	−3.213 85
r	1.245 79	83.0642	83.7973
s	0.266 298	176.883	178.571
t	0.844 656	0.450 978	1.0715
p	2.354 13	0.011 773 7	0.011 244
q	0	4042.93	4537.34

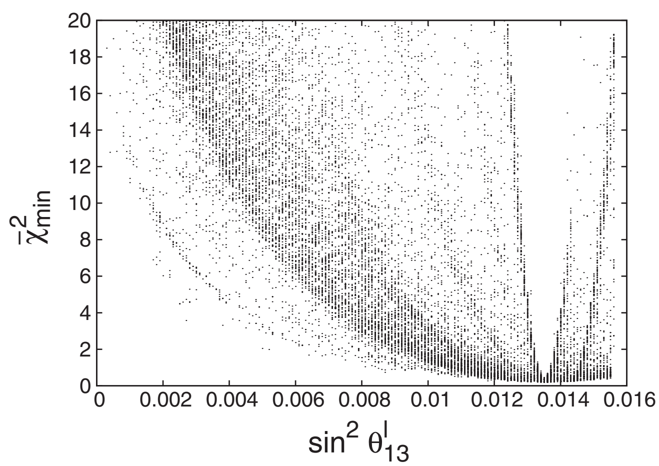

FIG. 5. Variation of $\bar{\chi}^2_{\rm min}$ with $\sin^2\theta^l_{13}$ in the type-I seesaw.

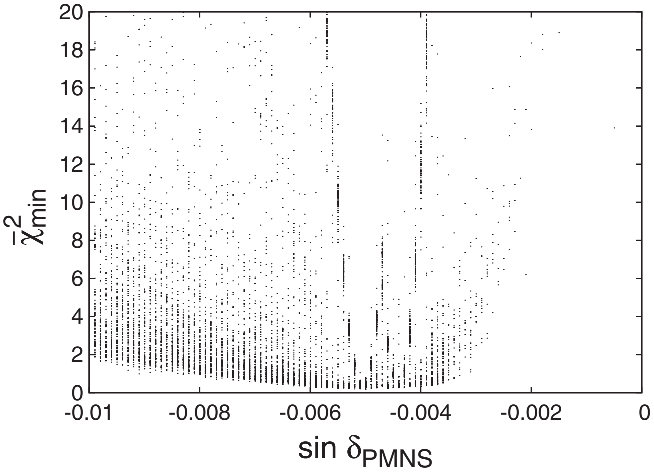

FIG. 6. Variation of $\bar{\chi}^2_{\rm min}$ with $\sin\delta_{\rm PMNS}$ in the type-I seesaw.

This is nothing but the Z_2 symmetry imposed in Ref. [19] which is thus equivalent to the generalized μ-τ symmetry considered here if both remain unbroken. A difference arises after these symmetries are broken. Reference [19] uses a complex vev to achieve Z_2 breaking as a result of which the analogues of Eqs. (7) and (8) do not hold in their case. In our approach, we introduce small explicit breaking of μ-τ symmetry in H. The model in Ref. [19] has 20 free parameters compared to 15 used here.

Note that the explicit breaking of the μ-τ symmetry is technically natural in the supersymmetric context. Alternatively, one can achieve such breaking by introducing an additional 10-plet of the Higgs field which changes sign under the μ-τ symmetry. Combined contributions of these two 10-plets would then give an explicitly μ-τ non-invariant H.

IV. SUMMARY

The aim of this paper was to integrate the successful μ-τ symmetry within the $SO(10)$ framework in order to obtain a constrained picture of fermion masses and a theoretical understanding of the largeness of the atmospheric mixing angle. The explicit model discussed here provides this integration rather well as shown by the detailed fits to fermion masses presented in Tables II and IV. Interestingly, mass matrices obtained in the model under consideration display a generalized CP invariance if Yukawa couplings are taken to be μ-τ symmetric. Small explicit breaking of this symmetry is sufficient to generate the required CP violating phase. The best scenario is obtained in the type-I seesaw model with very tiny explicit μ-τ symmetry breaking. This scenario is characterized by the predictions $\sin^2\theta^l_{23} \sim 0.42$–$0.63$ and $\sin^2\theta^l_{13} > 0.005$ and negligible CP violation in neutrino oscillations. The final quark, the charged lepton, and the light neutrino mass matrices (collected in the appendix) respect μ-τ symmetry to a very good approximation, indicating that this symmetry provides a good description of the entire fermion spectrum rather than being restricted to the neutrino sector alone.

ACKNOWLEDGMENTS

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APPENDIX

We list here the fermion mass matrices following from Eq. (2) using the best fit values of the parameters given in Table V corresponding to the type-I seesaw mechanism. The neutrino mass matrix is expressed in eV units, while all other mass matrices are expressed in MeV units.

$$M_d = \begin{pmatrix} 19.3018 & 20.8951 + 2.797\,28i & 20.8951 - 2.797\,28i \\ 20.8951 - 2.797\,28i & 527.314 & 524.898 - 3.213\,85i \\ 20.8951 + 2.797\,28i & 524.898 + 3.213\,85i & 522.311 \end{pmatrix}, \tag{A1}$$

$$M_u = \begin{pmatrix} 127.971 & 3731.25 + 2.997\,27i & 3731.25 - 2.997\,27i \\ 3731.25 - 2.997\,27i & 41395.1 & 41\,186.3 - 3.443\,63i \\ 3731.25 + 2.997\,27i & 41\,186.3 + 3.443\,63i & 40\,975.9 \end{pmatrix}, \tag{A2}$$

$$M_l = \begin{pmatrix} 82.1659 & -62.6852 + 0.031\,452\,6i & -62.6852 - 0.031\,452\,6i \\ -62.6852 - 0.0314\,526i & 645.168 & 643.02 - 0.036\,136\,5i \\ -62.6852 + 0.0314\,526i & 643.02 + 0.036\,136\,5i & 640.166 \end{pmatrix}, \tag{A3}$$

$$M_D = \begin{pmatrix} 11\,353.7 & -11\,193.7 + 12\,692.2i & -11\,193.7 - 12\,692.2i \\ -11\,193.7 - 12\,692.2i & 62\,440.4 & 62\,279.4 - 14\,582.3i \\ -11\,193.7 + 12\,692.2i & 62\,279.4 + 14\,582.3i & 62\,021.3 \end{pmatrix}, \tag{A4}$$

$$\mathcal{M}_{\nu}^{l} = \begin{pmatrix} -0.0242264 & -0.0143681 + 0.0004742i & -0.0143657 - 0.0000755678i \\ -0.0143681 + 0.0004742i & -0.0128288 + 0.00678282i & -0.0163109 + 0.000214216i \\ -0.0143657 - 0.0000755678i & -0.0163109 + 0.000214216i & -0.0127693 - 0.00629525i \end{pmatrix}. \quad \text{(A5)}$$

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