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Generalizing the Ginsparg-Wilson relation: Lattice supersymmetry from blocking transformations

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The Ginsparg-Wilson relation is extended to interacting field theories with general linear symmetries. Our relation encodes the remnant of the original symmetry in terms of the blocked fields and guides the construction of invariant lattice actions. We apply this approach in the case of lattice supersymmetry. An additional constraint has to be satisfied because of the appearance of a derivative operator in the symmetry transformations. The solution of this constraint leads to nonlocal SLAC-type derivatives. We investigate the corresponding kinetic operators on the lattice within an exact solution of supersymmetric quantum mechanics. These solutions—analogs of the overlap operator for supersymmetry—can be made local through a specific choice of the blocking kernel. We show that the symmetry relation allows for local lattice symmetry operators as well as local lattice actions. We argue that for interacting theories the lattice action is polynomial in the fields only under special circumstances, which is exemplified within an exact solution.

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I. INTRODUCTION

Lattice simulations of the path integral are a powerful tool to study quantum field theories, especially their nonperturbative properties. The first step in this program is to find a correct discretization of the continuum action and its symmetries (and appropriate discrete observables). In that respect a lot of experience has been collected in the case of gauge and chiral theories.

There have also been efforts to simulate supersymmetric (SUSY) theories on the lattice for many years. For a quadratic, i.e. free theory, a supersymmetric lattice action can be constructed. However, one encounters great difficulties in finding more general lattice actions that are invariant under the (naively) discretized supersymmetry transformations. One of them can be traced back to the fact that SUSY transformations contain derivatives of fields and that the continuum SUSY actions are invariant up to total derivative terms. In general the corresponding terms on the lattice do not vanish because the Leibniz and chain rule of differentiation is violated by any lattice derivative operator [1]. The optimal choice in this respect is the SLAC derivative [2], which preserves the Leibniz rule in the first Brillouin zone (BZ) (see [3] for recent works), but leads to nonlocalities [4].

Simulations without a realization of supersymmetry on the lattice [5] generically suffer from fine-tuning problems in the continuum limit. In higher SUSY theories it is possible to realize a part of the supersymmetry, and one can hope that this partial realization already ensures the correct continuum limit. Many approaches to lattice supersymmetry rely on such constructions, as e.g. [6]. Another possibility is to use a (nonlinear) deformation of the continuum transformations on the lattice [7]. One then still has to show that such lattice transformations resemble the SUSY transformations in the continuum limit even in the presence of quantum corrections to all orders of perturbation theory, as done for a specific model in [8]. This can ensure the correct continuum limit of these deformed transformations, but a nonperturbative argument would be desirable. A further attempt to keep the full SUSY by representing it on noncommutative objects or link objects [9] has been criticized in [10]. For more details on lattice supersymmetry we refer the reader to [11]. The conclusion of all these studies seems to be that there is a fundamental obstacle against interacting local lattice theories with exact SUSY implemented in the naive way.

This situation resembles very much that of chiral theories on the lattice. The Nielsen-Ninomiya theorem [12] forbids exact chiral symmetry on the lattice under a few very reasonable assumptions like locality. Ginsparg and Wilson have derived a modified symmetry relation for a chiral lattice theory [13]; in particular, the lattice Dirac operator anticommutes with γ_5 except for a term that vanishes in the continuum limit $a \to 0$. A solution to this relation was found by Neuberger [14], which is not ultralocal [15], but local if the background gauge field is smooth (the plaquette is close to the identity) [16].

The Ginsparg-Wilson (GW) relation is not a mere modification of the naive lattice symmetry with terms that formally vanish in the limit $a \to 0$; it is derived from an analysis of the Wilsonian renormalization group [13,17]. For the general setting see also [18–21] and for a recent blocking derivation of the overlap operator [22]. Thus it is a nonperturbative construction of a deformed symmetry transformation. Our strategy in putting supersymmetry on the lattice is therefore to revisit the corresponding procedure (this was already suggested in [23] and some earlier but incomplete attempts can be found in [24]). It leads to a lattice theory in an effective Wilsonian sense. This can be viewed as integrating out quantum fluctuations up to a finite number of lattice degrees of freedom, the integration over which is performed numerically. It is well known that in such a process of quantization classical symmetries get deformed. The aim of the present work is not only to extend the Ginsparg-Wilson approach to supersymmetry. We will also generalize the investigations of Ginsparg and Wilson for an arbitrary linear symmetry. The reduction of the degrees of freedom is done with an appropriate blocking transformation and the results will depend on the choice of a blocking kernel.

If only a quadratic theory is considered the effective lattice action for a given continuum action can be calculated. This was done in [25] for a supersymmetric model. Here we are not attempting to find such explicit solutions, since this is possible only for a free theory. Rather we investigate the implications of a continuum symmetry for the effective lattice action. If these implications are fulfilled a symmetric theory should be approached in the continuum limit.

As our main result, we derive an exact relation that a lattice theory has to obey in order to represent the continuum symmetry. Also interacting theories are included in this generalization of the GW relation to a general linear symmetry. By contrast, the GW relation deals only with the chiral symmetry acting on the quadratic fermion part of the action; the gauge fields are mere spectators. As in the GW case the naive lattice symmetry will be modified by terms that are proportional to the inverse blocking kernel. If this relation should represent a proper lattice symmetry, certain conditions, especially locality, must be satisfied. These conditions exclude e.g. the Wilson operator as a solution for chiral symmetry.

Applying the formalism to supersymmetry with its derivative transformations, an additional constraint has to be fulfilled in order to derive a lattice remnant of the continuum symmetry. The solution of this constraint generically leads to the nonlocal SLAC operator in the lattice transformations. We also investigate the continuum limit of the blocking kernel, which is less restrictive.

We are able to solve the relation for the quadratic (kinetic plus mass) sector of supersymmetric quantum mechanics (SUSYQM). In the case of this one-dimensional toy model the relation already yields nontrivial difference operators—analogs of the overlap operator for the case of supersymmetry.

The locality of these derivative operators can be improved using the freedom in the blocking kernel. This is an important result of our strategy: The blocking kernel helps in achieving the desired properties of the lattice theory, "at the expense of" introducing a right-hand side (rhs) in the lattice symmetry relation rendering it different from the naive one.

Concerning interacting SUSY theories, the symmetry relation generically couples different powers of the fields in the action beyond second order. It is therefore very intricate to truncate the interaction in the power of fields. As an example displaying these difficulties we solve the case of constant fields in SUSYQM. For general theories we give a necessary criterion for the construction of polynomial interactions.

The paper is organized as follows: First we introduce the blocking procedure in detail and derive the symmetry relation. It is particularly simple when a quadratic action is considered, and for chiral symmetry we recover the GW relation. The next section is devoted to the additional constraint and its solutions. Their continuum limit as well as the continuum limit of the blocking kernel is analyzed carefully in Sec. IV.

In Sec. V we start to apply the approach to SUSYQM, giving the general solution for the quadratic case and discussing its properties, especially locality. Afterwards in Sec. VI we solve the constant field case and investigate the possibility of polynomial actions fulfilling the symmetry relation, both in general and for SUSY. We end with a summary of our results and a discussion of their implications.

II. WILSONIAN EFFECTIVE ACTION

In this chapter the Wilsonian effective action computed from a general blocking transformation is introduced. The key results of the following investigation are the blocked symmetry transformations. These are specified later for the case of supersymmetry. The simple form of the symmetry transformations in case of a quadratic action is derived in Sec. II D. With this result the general concepts are elucidated at the example of chiral symmetry where the well-known Ginsparg-Wilson relation is reproduced.

A. Blocking

The starting point is the generating functional of a general quantum theory

$$Z[j] = \frac{1}{\mathcal{N}} \int d\varphi e^{-S_{\rm cl}[\varphi] + \int j\varphi}, \qquad (1)$$

with classical action $S_{\rm cl}[\varphi]$ and fields $\varphi(x) = (\varphi^1(x), \ldots, \varphi^A(x))$ comprising both, bosonic and fermionic degrees of freedom.

The theory is reduced to a finite number of degrees of freedom by introducing averaged or blocked fields $\phi_f(an)$ at lattices sites $n \in N_1 \times N_2 \times \cdots \times N_d$ by a linear blocking procedure,

$$\phi_f(an) := \int d^d x f(an - x)\varphi(x), \tag{2}$$

where a is the lattice distance, and a blocking function f peaked at 0. f should have the dimension inverse to the d-dimensional integral, such that the original and blocked fields have the same dimension.

This blocking can be easily generalized to finite blocking steps and fields φ^i where i comprises internal indices and species of fields. Then a general blocking reads

$$(\phi_f)^i_n = f^{ij}_{nx}\varphi^j_x, \tag{3}$$

where a summation/integration over indices is understood. The blocking matrix f^{ij}_{nx} is rectangular, and usually relates only the same species, but in general it also mixes the internal indices. In (3) x can be a discrete index, then φ_x is already understood as a blocked continuum field $\varphi_x = (\phi_{f_0})_x$. In that case the blocking maps a lattice with a smaller lattice spacing and consequently more degrees of freedom onto a smaller one with a larger lattice spacing.

The blocking procedure amounts to rewriting the generating functional Z in Eq. (1) in terms of a path integral on the lattice defined by the image of f. For the sake of simplicity we concentrate on the generating functional at vanishing sources. For now, the general case, important for the discussion of the continuum limit, will be discussed in Sec. IV. We proceed with rewriting the generating functional $Z[0]$ as

$$Z[0] = \frac{1}{\mathcal{N}}\int d\phi e^{-S[\phi]}, \tag{4}$$

where the ϕ^i_n live on the lattice given by the blocking f, and S is the Wilsonian effective action with

$$e^{-S[\phi]} = \mathrm{SDet}^{1/2}\alpha \int d\varphi e^{-(1/2)(\phi-\phi_f)\alpha(\phi-\phi_f)-S_{\mathrm{cl}}[\varphi]}. \tag{5}$$

SDet is the superdeterminant, i.e. the determinant for bosons and its inverse for fermions.[1] In (5) we have introduced a quadratic smearing with a blocking kernel

$$(\phi - \phi_f)\alpha(\phi - \phi_f) = (\phi - \phi_f)^i_n \alpha^{ij}_{nm}(\phi - \phi_f)^i_m. \tag{6}$$

By inserting (5) into (4) and performing the Gaussian integration over ϕ it can be straightforwardly checked that (4) gives $Z[0]$ in (1). Note that the above smearing also encompasses α's with diverging or vanishing determinants. This and further details will be discussed in Sec. IV. The explicit examples of quadratic actions and chiral symmetry are shown in Secs. II D and II E respectively.

For illustration of the smearing procedure we briefly discuss a specifically simple case with $\alpha^{ij}_{nm} = \alpha\delta^{ij}\delta_{mn}$ with $\alpha \to \infty$. Then the smearing term turns into a δ function in field space,

$$\mathrm{SDet}^{1/2}\alpha e^{-(1/2)(\phi-\phi_f)\alpha(\phi-\phi_f)} = \delta(\phi - \phi_f), \tag{7}$$

and the smearing is removed. Removing the smearing in the continuum limit, $a \to 0$, is necessary in order to recover the original action in this limit, $S \to S_{\mathrm{cl}}$, apart from the blocking f becoming the delta distribution, that is $\phi_f \to \varphi$. Furthermore, for achieving (7), α/a^d has to diverge. More general restrictions for the continuum limit will be addressed in detail in Sec. IV.

The blocking kernel shall connect bosons and fermions only among themselves and in these subspaces it obeys

$$\alpha^{ij}_{nm} = \alpha^{ji}_{mn}(-1)^{|\phi^i||\phi^j|}, \tag{8}$$

where

$$|\phi^i| = \begin{cases} 1 & \phi^i \text{ fermionic,} \\ 0 & \phi^i \text{ bosonic.} \end{cases} \tag{9}$$

In other words, α is antisymmetric for fermions and symmetric for bosons, $\alpha = \pm\alpha^T$, where the minus sign applies whenever fermionic indices are interchanged in the transposition of the matrix.

B. Symmetries

The primary concern of this construction is the blocking transformation of symmetries of the classical action. We now investigate in what form the lattice action inherits this symmetry. Our main result will be the relation (18) that corresponds to the Ginsparg-Wilson relation, but is valid for a general linear symmetry. Continuum implementations of the related ideas close to the present line of arguments can be found in e.g. [20,21]; for reviews see [18,19].

Let the classical action S be invariant under a linear transformation

$$\varphi \to \varphi + \tilde{\delta}\varphi, \qquad (\tilde{\delta}\varphi)^i_x = \epsilon \tilde{M}^{ij}_{xy}\varphi^j_y, \tag{10}$$

where $\tilde{M}$ in general relates different field species (i, j), but may also act nontrivially on the coordinates (x, y). ϵ is the small parameter of the transformation. In the application to SUSY, ϵ is Grassmann valued as $\tilde{M}$ mixes bosons and fermions, and $\tilde{M}$ also contains derivatives. We have introduced the notation that for a given lattice quantity $\mathcal{O}$ the $\tilde{\mathcal{O}}$ refers to the corresponding continuum quantity.

In combination with the averaging function f, see Eqs. (2) and (3), this symmetry transformation induces a corresponding transformation on the blocked field ϕ_f

$$(\tilde{\delta}\phi_f)^i_n = f^{ij}_{ny}(\tilde{\delta}\varphi)^j_y = \epsilon f^{ij}_{ny}\tilde{M}^{jk}_{yx}\varphi^k_x. \tag{11}$$

Of course we want to represent this transformation solely

[1]Strictly speaking $\mathrm{SDet}^{1/2}\alpha$ in our case means the inverse of the Pfaffian of α for fermions and $\det^{1/2}\alpha$ for the bosons.

on the lattice fields. Indeed, the transformation can be lifted to ϕ as

$$\phi \to \phi + \delta\phi, \qquad (\delta\phi)^i_n = \epsilon M^{ij}_{nm}\phi^j_m, \tag{12}$$

with a lattice transformation M, if

$$M^{ik}_{nm} f^{kj}_{mx} = f^{ik}_{ny}\tilde{M}^{kj}_{yx} \tag{13}$$

holds.

This property can be viewed as a constraint ensuring the compatibility of the lattice symmetry transformation with the blocking. One might be tempted to use it to define M, but f has no right inverse, since it maps onto fewer degrees of freedom. This constraint has been mentioned without further investigation in [24]. We will analyze it in full detail in Sec. III and argue that it has severe consequences, if $\tilde{M}$ contains a derivative. Since the transformations defined by M act on lattice fields they can be regarded as a naive realization of the symmetry transformations on the lattice.

According to Eq. (12) this naive transformation δ has the operator representation

$$\delta = \epsilon M^{ij}_{nm}\phi^j_m \frac{\delta}{\delta\phi^i_n} \tag{14}$$

on the space of fields ϕ. In the fermionic sector the left derivative is used, e.g. $\frac{\delta}{\delta\psi}\bar{\psi}\psi = -\bar{\psi}$.

Now we can transform the classical symmetry into a relation of the effective theory on the lattice. To that end we apply in Eq. (5) the naive symmetry transformation $\phi \to \phi + \delta\phi$ defined above to the Wilsonian action as well as the classical symmetry transformation $\varphi \to \varphi + \tilde{\delta}\varphi$ to the integration variable using the invariance of the classical action, $\delta S_{\mathrm{cl}}[\varphi] = 0$. To linear order in ϵ we arrive at

$$\begin{aligned} M^{ij}_{nm}\phi^j_m \frac{\delta}{\delta\phi^i_n} S[\phi] &= -\mathrm{STr}\tilde{M} - e^{S[\phi]}\int d\varphi e^{-S_{\mathrm{cl}}[\varphi]} M^{ij}_{nm}(\phi - \phi_f)^j_m \\ &\quad \times \frac{\delta}{\delta\phi^i_n} e^{-(1/2)(\phi-\phi_f)\alpha(\phi-\phi_f)}. \end{aligned} \tag{15}$$

The supertrace term $\mathrm{STr}\tilde{M}$ on the rhs comes from the expansion of the Jacobi determinant and comprises the possible anomaly of the symmetry transformation $\tilde{\delta}$.

The difference $(\phi - \phi_f)$ in this equation can be expressed as a ϕ derivative using

$$\begin{aligned} (\phi - \phi_f)^j_m \frac{\delta}{\delta\phi^i_n} e^{-(1/2)(\phi-\phi_f)\alpha(\phi-\phi_f)} &= -\bigg((-1)^{|\phi^i|}\delta_{mn}\delta^{ij} + \alpha^{-1jk}{}_{mr}\frac{\delta}{\delta\phi^k_r} \\ &\quad \times \frac{\delta}{\delta\phi^i_n}\bigg) e^{-(1/2)(\phi-\phi_f)\alpha(\phi-\phi_f)}. \end{aligned} \tag{16}$$

When inserted into (15), the first term on the rhs of this identity contracts to the supertrace $+\epsilon\mathrm{STr}M$ of the lattice symmetry M, while the rest only contains derivatives with respect to the blocked field ϕ and can be pulled outside the φ integral. Then (15) turns into

$$\begin{aligned} M^{ij}_{nm}\phi^j_m \frac{\delta S[\phi]}{\delta\phi^i_n} &= \mathrm{STr}M - \mathrm{STr}\tilde{M} + e^{S[\phi]}(M\alpha^{-1})^{ij}_{nm} \\ &\quad \times \frac{\delta}{\delta\phi^j_m}\frac{\delta}{\delta\phi^i_n} e^{-S[\phi]}. \end{aligned} \tag{17}$$

Finally, performing the derivatives leads to a nonlinear relation for the blocked action S containing up to second order derivatives in ϕ,

$$\begin{aligned} M^{ij}_{nm}\phi^j_m \frac{\delta S}{\delta\phi^i_n} &= (M\alpha^{-1})^{ij}_{nm}\bigg(\frac{\delta S}{\delta\phi^j_m}\frac{\delta S}{\delta\phi^i_n} - \frac{\delta^2 S}{\delta\phi^j_m\delta\phi^i_n}\bigg) \\ &\quad + (\mathrm{STr}M - \mathrm{STr}\tilde{M}). \end{aligned} \tag{18}$$

This is the key relation for the Wilsonian or lattice action $S[\phi]$, the naive symmetry transformation M, and the blocking kernel α. While the left-hand side of this relation is just the naive symmetry variation of the action S, the rhs constitutes some nontrivial modification of it that has been derived in the blocking procedure. The behavior of this term with respect to the continuum limit will be investigated in Sec. IV. Note furthermore that Eq. (18) represents the lattice version of the (modified) quantum master equation, see e.g. [18–21,26].

A few comments are in order here. The supertraces of $\tilde{M}$ and M terms in this relation carry the noninvariance of the measures $d\varphi$ and $d\phi$, respectively, and hence comprise possible (integrated) anomalies of the theory. More precisely, $\mathrm{STr}\tilde{M}$ carries the full anomaly related to the measure $d\varphi$. The blocking removes a part of the integrations from the path integral leaving only the field ϕ to be integrated. The related part of the anomaly leads to $\mathrm{STr}\tilde{M} - \mathrm{STr}M$.

The above derivation also works for a finite blocking step where φ is already a blocked field and x, y are lattice coordinates. Then, however, the starting point must be regarded as a Wilsonian action for a fine lattice ($S_{\mathrm{cl}}[\varphi] = S[\varphi]$) that already satisfies the relation (18), which e.g. brings in the continuum anomaly. This again leads to (18) for the blocked effective action on the coarser lattice.

If the right-hand side of (18) vanishes, we are left with the invariance of the action under the naive symmetry transformations,

$$M^{ij}_{nm}\phi^j_m \frac{\delta S}{\delta\phi^i_n} = 0. \tag{19}$$

This happens for symmetric blocking matrices α_S fulfilling

$$M\alpha_S^{-1} \pm (M\alpha_S^{-1})^T = 0, \tag{20}$$

since only the (anti)symmetric part of $M\alpha^{-1}$ enters the rhs of the relation. The minus sign appears only if the matrix M connects fermions with fermions, i.e. if fermionic fields are transformed into themselves by the symmetry. The above condition just means that the blocking kernel is invariant under the naive symmetry variation. More generally, $\alpha^{-1} + \alpha_S^{-1}$ leads to the same symmetry relation (18) for all α_S^{-1}. This defines a family of equivalent blocking kernels $\alpha^{-1}(\alpha_S^{-1})$.

For the chiral case $\alpha = \alpha_S$ is excluded by the vector symmetry as we will elaborate on in Sec. II E. For the case of supersymmetry, there is in general not such an argument and, indeed, such a matrix has been used e.g. in [24]. However, we shall show below that the naive symmetry M in the systematic blocking approach to SUSY is inherently nonlocal and hence excluded. Instead, a nonsymmetric blocking kernel α must be used.

Then the relevant symmetry can be written in terms of a modified field-dependent symmetry operator, $M_{\rm def}(\phi)$, defined as

$$(M_{\rm def})^{ij}_{nm}\phi^j_m := M^{ij}_{nm}\bigg(\phi^j_m - (\alpha^{-1})^{jk}_{mr}\frac{\delta S}{\delta\phi^k_r}\bigg). \tag{21}$$

Inserting this definition into the symmetry relation (18), we are led to the relation

$$(M_{\rm def})^{ij}_{nm}\phi^j_m \frac{\delta S}{\delta\phi^i_n} = (-1)^{|\phi^i||\phi^j|}\frac{\delta}{\delta\phi^i_n}[(M_{\rm def})^{ij}_{nm}\phi^j_m] - {\rm STr}\tilde{M}, \tag{22}$$

the right-hand side being related to a total field derivative. The above derivation of $M_{\rm def}$ closely follows the analogous continuum arguments as used in [18–21]. A discussion of various representations of (22) and their use can be found in [18].

C. Local lattice symmetries

It is important to emphasize that (22) in general does not comprise a symmetry, as the above construction applies to any blocking kernel α. Thus, in general (22) only disguises an explicit symmetry breaking induced by the blocking. We shall exemplify this statement in Sec. II E at the standard Wilson-Dirac operator that explicitly breaks chiral symmetry, but still satisfies (22).

The question arises what are the additional conditions on $M_{\rm def}$ that make it a deformed symmetry. For local continuum symmetries it is important that the corresponding lattice version of the symmetry carries this locality. More generally, for a given symmetry the blocking should only induce a local symmetry breaking or deformation generated by f and α. Consequently we are led to two conditions:

(1) A mandatory condition for a deformed lattice symmetry is the locality of $M_{\rm def}$. This guarantees a well-defined continuum limit in which the lattice artifacts related to the deformation tend to zero in a controlled way as they are local. Hence, in order to have a deformed symmetry the family of blockings $\alpha^{-1}(\alpha_S^{-1})$ must contain at least one blocking $\alpha^{-1}_{\rm local}$ that leads to a local symmetry operator $M_{\rm def}$. We emphasize that this does not necessarily imply that $\alpha^{-1}_{\rm local}$ is local. The locality of $M_{\rm def}$ reads

$$\lim_{|x-y|\to\infty}|M_{\rm def}(x,y)| < e^{-c|x-y|} \tag{23}$$

for some $c > 0$. In the present investigation we shall relax (23), and demand

$$|x^r M_{\rm def}(x,y)| < \infty \quad \forall\ r\in\mathbb{N},\ x,y\in a\mathbb{N}. \tag{24}$$

For explanations, see Appendix H. Clearly operators $M_{\rm def}$ with (23) satisfy (24) but (24) also allows for softer decay, e.g. polynomial times exponential decay. Moreover, for interacting theories the locality conditions (23) and (24) involve field-dependent terms as $M_{\rm def}(\phi)$ in (21) is field dependent.

(2) $M_{\rm def}$ has to carry the original continuum symmetry related to the symmetry operator $\tilde{M}$. This condition excludes e.g. the trivial solution $M_{\rm def}\equiv 0$. For this solution it is clear that the symmetry pattern of the lattice action is not entailed in $M_{\rm def}$, and $M_{\rm def}$ does not tend toward the continuum symmetry $\tilde{M}$ in the continuum limit. We can summarize this condition in the demand that $M_{\rm def}$ is identical to the continuum symmetry operator $M_{\rm cont}$ up to lattice artifacts at $p = 0$, where the continuum limit is located. Hence, the condition that $M_{\rm def}$ carries the continuum symmetry can be formulated as

$$\lim_{p\to 0} M_{\rm def} = M_{\rm cont}(\mathbb{1} + O(ap)). \tag{25}$$

Note that $M_{\rm cont} = \tilde{M}$ only for $\alpha^{-1} = 0$ in the continuum; see the discussion in Sec. IV B.

The above conditions (21)–(25) should be seen as a definition of a deformed symmetry, and put constraints on the blocking kernel α. In order to formally obtain a symmetric continuum limit, one might use actions without such a symmetry, but the above considerations guarantee the existence of a local lattice symmetry for every finite lattice spacing that converges locally toward the continuum symmetry. The latter property is very important for a successful numerical implementation.

In the case of lattice supersymmetry the question is whether such a deformed symmetry operator $M_{\rm def}$ according to this definition can be constructed.

D. Quadratic action

The general results above simplify greatly for quadratic actions

$$S = \tfrac{1}{2}\phi^i_n K^{ij}_{nm}\phi^j_m, \tag{26}$$

with the kernel K comprising kinetic and mass terms. At first sight this case seems trivial as it describes a free field theory. Nonetheless, it already includes the nontrivial case of Ginsparg-Wilson fermions [13] with background gauge fields; see the next section. Moreover, locality of a symmetry operator $M_{\rm def}$ of an interacting theory relates directly to the locality of its kinetic noninteracting part.

With this action the general symmetry relation (18) simplifies to

$$\phi M^T K\phi = \phi K^T(M\alpha^{-1})^T K\phi - \mathrm{tr}(M\alpha^{-1})K^T + (\mathrm{STr}M - \mathrm{STr}\tilde{M}). \quad (27)$$

In many cases the second line vanishes. In case of field-independent transformation matrices M and kinetic operators K it anyway is just an irrelevant constant. However, in the case of anomalous symmetries it contributes to the anomaly. If one considers nonquadratic actions the corresponding term in general becomes ϕ dependent.

The first line in (27) has to be valid for general fields ϕ and hence we conclude that

$$M^T K \pm (M^T K)^T = K^T(M\alpha^{-1})^T K \pm (K^T(M\alpha^{-1})^T K)^T. \quad (28)$$

Again the minus signs appear on the left- and right-hand sides only if fermions are transformed into fermions by the naive symmetry M.

The interesting information in the symmetry relation is that of the propagation of symmetry breaking on the lattice. This propagation can be seen from

$$(K^{-1})^T M^T \pm MK^{-1} = (\alpha^{-1})^T M^T \pm M\alpha^{-1}. \quad (29)$$

This equation highlights how the breaking of the symmetry by the blocking matrix α and the breaking by the kernel K must compensate each other. It also enables us to read off the general solution K,

$$K^{-1} = \alpha^{-1} - \alpha_S^{-1}. \quad (30)$$

Here, α_S^{-1} is a general symmetry-preserving term fulfilling (20). We emphasize that (30) can also be used for determining a family $\alpha(K)$ for a given K. We conclude that pairs (K^{-1}, α^{-1}) are unique up to symmetry-preserving terms α_S^{-1}. The symmetry relation can be also rewritten by introducing the deformed symmetry matrix $M_{\rm def}$ as defined in (21). Here we find a ϕ-independent $M_{\rm def}$ with

$$M_{\rm def} := M(\mathbb{1} - \alpha^{-1}K) = -M\alpha_S^{-1}K. \quad (31)$$

$M_{\rm def}$ may, however, now depend on background fields via K and α_S^{-1}, e.g. link variables if K is the Dirac operator. Note that (31) defines a family of symmetry matrices, as α_S^{-1} is a general symmetric matrix satisfying (20). For the modified symmetry the relation (28) reads

$$M_{\rm def}^T K \pm (M_{\rm def}^T K)^T = 0. \quad (32)$$

As already mentioned in the previous section, (32) in general does not comprise a symmetry, as the above construction applies to any kinetic operator. Thus, in general (32) only disguises an explicit symmetry breaking induced by the blocking kernel. We also clearly see the necessity of the second condition (25): for $\alpha^{-1} = K^{-1}$ we have $\alpha_S^{-1} = 0$ and hence $M_{\rm def} \equiv 0$. Then the modified symmetry relation (32) carries no information about the symmetry at hand.

In turn, only $M_{\rm def}$'s in (31) with (32) and the locality and continuum limit properties (24) and (25) respectively define deformed lattice symmetries.

Note that for a quadratic action, and only in this case, there is also a simpler way to derive a symmetry relation. In this specific case the saddle point approximation for the path integral is exact. Thus, instead of a solution of the path integral, one can also discuss the symmetries of the saddle point solutions as done in [27].

E. Chiral symmetry

We shall first discuss the above construction and conditions at the example of the chiral symmetry. Consider an action out of the field multiplet of two fermionic fields: $\phi = (\psi, \bar{\psi}^T)$. The related kinetic operator is given in terms of the Dirac operator

$$\frac{K}{a^d} = \begin{pmatrix} 0 & -\mathcal{D}^T \\ \mathcal{D} & 0 \end{pmatrix}, \quad (33)$$

and the action (26) reads $S = a^d \bar{\psi}\mathcal{D}\psi$. As explained above, in our units the quantities relevant for the continuum must get additional factors of a^d to account for the integral. The continuum action is invariant under symmetry transformations generated by

$$\tilde{M}\varphi = \begin{pmatrix} \gamma_5 & 0 \\ 0 & \gamma_5^T \end{pmatrix}\begin{pmatrix} \psi \\ \bar{\psi}^T \end{pmatrix}, \quad (34)$$

with $\gamma_5^\dagger = \gamma_5$. Since the transformation acts only algebraically on spinor indices it is easy to fulfill the constraint (13). The naive transformation is just the same as the continuum transformation. A general blocking matrix α carries the fermionic antisymmetry and reads

$$\frac{\alpha}{a^d} = \begin{pmatrix} 0 & -\alpha_1^T \\ \alpha_1 & 0 \end{pmatrix}, \quad (35)$$

with a general α_1. Note that in order to get a real action both $\mathcal{D}$ and α_1 must be Hermitian. Inserting the kinetic operator (33), the chiral transformation matrix (34), and the general blocking (35) into (27) we are led to

$$\{\mathcal{D}, \gamma_5\} = \mathcal{D}\{\gamma_5, \alpha_1^{-1}\}\mathcal{D}, \quad (36)$$

which comes from the field-dependent part of (27). It can be rewritten in terms of a deformed symmetry, cf. [23], which is according to the general definition of $M_{\rm def}$ in (31) given as

$$M_{\rm def} = \begin{pmatrix} \gamma_{5,\rm def} & 0 \\ 0 & (\bar{\gamma}_{5,\rm def})^T \end{pmatrix} \quad (37)$$

with

$$\gamma_{5,\mathrm{def}} = \gamma_5(1 - \alpha_1^{-1}\mathcal{D}), \qquad \bar{\gamma}_{5,\mathrm{def}} = (1 - \mathcal{D}\alpha_1^{-1})\gamma_5. \tag{38}$$

In terms of the deformed γ_5's the symmetry relation reads

$$\bar{\gamma}_{5,\mathrm{def}}\mathcal{D} + \mathcal{D}\gamma_{5,\mathrm{def}} = 0. \tag{39}$$

For Hermitian α_1^{-1} and $\mathcal{D}$ we arrive at $\bar{\gamma}_{5,\mathrm{def}} = \gamma_{5,\mathrm{def}}^{\dagger}$.

In case of a theory with vector symmetry the blocking should respect it. Hence the simplest α_1 is a fermionic mass term with mass $1/a$, $\alpha_1 = 1/a\mathbb{1}$, where $\mathbb{1}$ is diagonal with respect to the lattice sites and the identity in Dirac space. The result,

$$\{\mathcal{D}, \gamma_5\} = 2a\mathcal{D}\gamma_5\mathcal{D}, \tag{40}$$

is the Ginsparg-Wilson relation [13].

The deformed symmetry operator M_{def} from (37) and (38) is local due to the ultralocality of α^{-1} and the locality of $\mathcal{D}$. It should, however, be noted that $\gamma_{5,\mathrm{def}}$ is not normalized, $\gamma_{5,\mathrm{def}}^2 \neq \mathbb{1}$, and even vanishes at the doublers. We conclude that $\gamma_{5,\mathrm{def}}$ does not define a chiral projection. Indeed no such normalized $\gamma_{5,\mathrm{def}}$ can be constructed for a single Weyl fermion, see [28,29], as a consequence of the Nielsen-Ninomiya no-go theorem. In the given example the normalization of $\gamma_{5,\mathrm{def}}$ fails at the doublers; it is neither smooth nor local.

The part of Eq. (27), that is independent of ϕ, carries the integrated chiral anomaly,

$$\mathrm{Tr}\{\gamma_5, \alpha_1^{-1}\}\mathcal{D} - 2(\mathrm{Tr}\gamma_5 - \mathrm{Tr}_{\mathrm{cont}}\gamma_5) = 0, \tag{41}$$

where the trace Tr sums over the lattice. Hence the second term vanishes, $\mathrm{Tr}\gamma_5 = 0$. The first term in (41) only has to be summed over the nonzero eigenfunctions of $\mathcal{D}$, denoted by Tr′, and we arrive at

$$\mathrm{Tr}'\{\gamma_5, \alpha_1^{-1}\}\mathcal{D} = 2\,\mathrm{Tr}'\gamma_5 = -2(n_+ - n_-)_{\mathrm{lattice}}, \tag{42}$$

where we have used (36). Here, $n_{\pm\mathrm{lattice}}$ are the numbers of fermionic zero modes with positive and negative chirality, respectively. Note that (42) implies that $(n_+ - n_-)_{\mathrm{lattice}}$ vanishes, if we choose a blocking compatible with axial symmetry. Then, however, vector symmetry is broken, and we would lose (background) gauge symmetry. This analysis is reflected in the well-known fact that $\mathrm{Tr}_{\mathrm{cont}}\gamma_5$ is regularization dependent. Choosing a vector symmetric regularization of the trace, e.g.

$$\mathrm{Tr}_{\mathrm{cont}}\gamma_5 := \lim_{\epsilon\to 0}\mathrm{Tr}_{\mathrm{cont}}\gamma_5 e^{\epsilon\mathcal{D}_{\mathrm{cont}}^2}, \tag{43}$$

we are led to $\mathrm{Tr}_{\mathrm{cont}}\gamma_5 = (n_+ - n_-)_{\mathrm{cont}}$. Here, $n_{\pm\mathrm{cont}}$ are the numbers of fermionic zero modes with positive and negative chirality, respectively. In turn, an axially symmetric regularization leads to $\mathrm{Tr}_{\mathrm{cont}}\gamma_5 = 0$.

In summary we arrive at

$$(n_+ - n_-)_{\mathrm{lattice}} = (n_+ - n_-)_{\mathrm{cont}}, \tag{44}$$

which constrains the continuum regularization in terms of the lattice blocking and vice versa. We conclude that full chiral symmetry in the presence of a background gauge field is maintained if and only if the lattice gauge field permits the same difference of positive and negative chirality zero modes as for the continuum gauge field.

Note that for the standard GW-relation (40), i.e. with $\alpha_1 = 1/a\mathbb{1}$, one can rewrite the lattice terms in (41) as

$$\mathrm{Tr}\,\gamma_5(1 - aD) = \sum_x\sum_n(1 - a\lambda_n)\psi_n^{\dagger}(x)\gamma_5\psi(x) =: \sum_x Q_{\mathrm{top}}(x), \tag{45}$$

which is the fermionic definition of the topological charge density $Q_{\mathrm{top}}(x)$ introduced by Niedermayer [30]. For more general α_1 it can be found in [31].

As an example for an explicit breaking of chiral symmetry we consider Wilson fermions with Dirac operator $\mathcal{D}_W$,

$$a\mathcal{D}_W = i\gamma_\mu\sin(ap_\mu) + r\sum_\mu(1 - \cos(ap_\mu)). \tag{46}$$

In this case chiral symmetry is explicitly broken due to the momentum-dependent Wilson mass. We start with the relation (30) for general Dirac operators $\mathcal{D}$. The corresponding blocking kernel [cf. (35)] is given by

$$\alpha_1^{-1} = \mathcal{D}^{-1} + \alpha_{1,S}^{-1}. \tag{47}$$

The singularity of $\mathcal{D}^{-1}$ at the center of the Brillouin zone has to be removed from $\alpha_{1,S}^{-1}$ in order to guarantee the continuum limit of $M_{\mathrm{def}} \to M_{\mathrm{cont}}$, (25). This is achieved with

$$\alpha_{1,S}^{-1} = \gamma_\mu\frac{1}{d}\mathrm{tr}\gamma_\mu\mathcal{D}^{-1} + \Delta\alpha_{1,S}^{-1}, \tag{48}$$

with $\mathrm{tr}\gamma_\mu\gamma_\nu = -d\delta_{\mu\nu}$, and d is the space-time dimension. We conclude that α_1^{-1} is given by

$$\alpha_1^{-1} = \mathbb{1}\frac{1}{d}\mathrm{tr}\mathcal{D}^{-1} - \Delta\alpha_{1,S}^{-1}, \tag{49}$$

with a scalar first term proportional to $\mathbb{1}$, and a symmetric contribution $\Delta\alpha_{1,S}^{-1}$ proportional to $\gamma_\mu f_\mu(p)$. Note that the first term cannot be changed by $\Delta\alpha_{1,S}^{-1}$ and hence carries the unique information about the symmetry-breaking part of the kinetic operator K. This part is the same for all members of the family $\alpha(K)$ of blockings corresponding to a given K. Restrictions on this part will hence constrain the class of possible lattice actions.

For Ginsparg-Wilson fermions we have $\alpha_1^{-1} = a\mathbb{1}$ and $\Delta\alpha_{1,S}^{-1} = 0$, that is α^{-1} has no symmetric part. For Wilson fermions the choice $\Delta\alpha_{1,S}^{-1} = 0$ leads to a nonlocal α_1^{-1}: some higher derivative of $\mathrm{tr}\mathcal{D}_W^{-1}(p)$ is not bounded at the origin and this contradicts locality; see Appendix H. Furthermore this nonlocality cannot be changed by the symmetric term $\Delta\alpha_{1,S}^{-1}$ except in one dimension. We conclude that the blocking α^{-1} related to the Wilson-Dirac operator is inherently nonlocal. Still *a priori* this does not entail that the corresponding $M_{\rm def}$ is nonlocal. However, the product of this α^{-1} (including $\Delta\alpha_{1,S}^{-1}$) with the Wilson-Dirac operator is also inherently nonlocal and enters $M_{\rm def}$. Actually, the nonlocality of $M_{\rm def}$ is most easily seen from the nonlocality of the left-hand side of (29) in a Taylor expansion about $p = 0$. We conclude that there is no deformed chiral symmetry operator $M_{\rm def}$ for Wilson fermions.

We summarize that the lattice blocking induces the continuum regularization. In turn, if we have chosen a specific continuum regularization, this restricts the lattice blocking compatible in the continuum limit. We conclude this analysis with the remark that an analysis of chiral transformations $\psi \to (1 \pm \gamma_5)/2\psi$ completely fixes the relations, as the related integrated anomaly is independent of the regularization. This is at the heart of the lattice observations made in [32,33].

F. Explicit solution for a quadratic action

For a quadratic action it is also possible to solve (5) for the effective action S explicitly. Assuming $S_{\rm cl}[\varphi] = \frac{1}{2}\varphi_x^i \tilde{K}_{xy}^{ij}\varphi_y^j$, the lattice action $S[\phi] = \frac{1}{2}\phi K \phi$ can be obtained via performing the Gaussian integration. It leads to

$$K = \alpha - \alpha f (f^T \alpha f + \tilde{K})^{-1} f^T \alpha. \tag{50}$$

After some manipulations that can be found in Appendix A the resulting fixed point operator reads in momentum space

$$K(p_k) = \left(\sum_{l\in\mathbb{Z}} \frac{f^*(p_k + l\frac{2\pi}{a}) f(p_k + l\frac{2\pi}{a})}{\tilde{K}(p_k + l\frac{2\pi}{a})} + \alpha^{-1}(p_k)\right)^{-1}. \tag{51}$$

Note that such a solution of the Ginsparg-Wilson relation was already mentioned in [13]. It is often called perfect lattice action.

In most cases $f(x)$ is considered to be the averaging over one lattice spacing, e.g. in one dimension

$$f(x) = \begin{cases} 1/a & \text{if } |x| < a/2, \\ 0 & \text{otherwise,} \end{cases} \tag{52}$$

which means $f(p_k) = \frac{2}{La}\frac{\sin(p_k a/2)}{p_k}$. Such an averaging was applied in [25] to construct a free supersymmetric (perfect) lattice theory. However, since the constraint (13) was not considered there the symmetry properties of the resulting effective action cannot be expressed in terms of a lattice symmetry involving only lattice fields: Eq. (13) demands for the derivative operator appearing in the supersymmetry transformations

$$\sum_m \nabla_{nm}\phi(am) = \frac{1}{a}(\varphi(an + a/2) - \varphi(an - a/2)) \tag{53}$$

and this cannot be fulfilled for any ∇_{nm} since the transformation involves the continuum fields.

To interpret the rhs of Eq. (53) a new field was introduced in [25], which is defined to be $\frac{1}{a}\varphi(an + a/2)$ at the lattice point an. Then the lattice fields are transformed into such fields under the supersymmetry transformations. They are, however, rather a continuum than a blocked lattice quantity. The correct SUSY continuum limit is therefore ensured in this approach because the lattice action is a direct solution of the blocking. But this property cannot be expressed in terms of a lattice symmetry that contains only lattice fields. A well-defined lattice symmetry is, however, desirable as a guiding principle for the construction of a more general lattice action.

III. ADDITIONAL CONSTRAINT

A. Discussion

In the derivation of the relation of the effective action there has emerged a novel constraint, Eq. (13),

$$Mf = f\tilde{M}, \tag{54}$$

on the symmetries M, $\tilde{M}$ and the averaging function f. It is trivially fulfilled if the symmetry transformation merely acts on the multiplet indices, e.g. with γ_5 in the chiral case.

However, whenever the symmetry transformation $\tilde{M}$ contains a derivative—as in the case of supersymmetry—the constraint becomes nontrivial. The problem can be considered in each space-time direction separately. It states that the derivative ∂ (in $\tilde{M}$) is "pulled through" the averaging function f to become a lattice derivative operator ∇ (in M) that acts among the averaged fields:

$$\nabla_{nm}\int dx f(am - x)\varphi(x) = \int dx f(an - x)\partial_x\varphi(x) \tag{55}$$

for all continuum fields $\varphi(x)$ (neglecting internal indices i, j) and for all lattice points n. [The meaning of (55) is illustrated in Fig. 1. It represents a commuting diagram.] This constraint will restrict the possible lattice derivatives ∇ to be used in the lattice symmetry transformations M as we show now.

In order to satisfy Hermiticity and translational invariance, ∇ should be an antisymmetric circulant matrix

$$\nabla_{nm} = \frac{1}{2a}\sum_{l=-(N-1)/2}^{(N-1)/2} c_l \delta_{n-m,-l}, \tag{56}$$

with real coefficients c_l fulfilling $c_{-l} = -c_l$. For simplicity we have specialized to an odd number N of lattice points. The Kronecker symbol δ on the rhs is periodic with periodicity N.

We use a partial integration in (55) and a Fourier transform with a discrete momentum $p_q = 2\pi q/L$ with $q \in \mathbb{Z}$ and a lattice volume $L = Na$ (for details see Appendix B) to arrive at

$$f(p_q)[\nabla(p_q) - ip_q] = 0 \quad \forall\ q \in \mathbb{Z}. \tag{57}$$

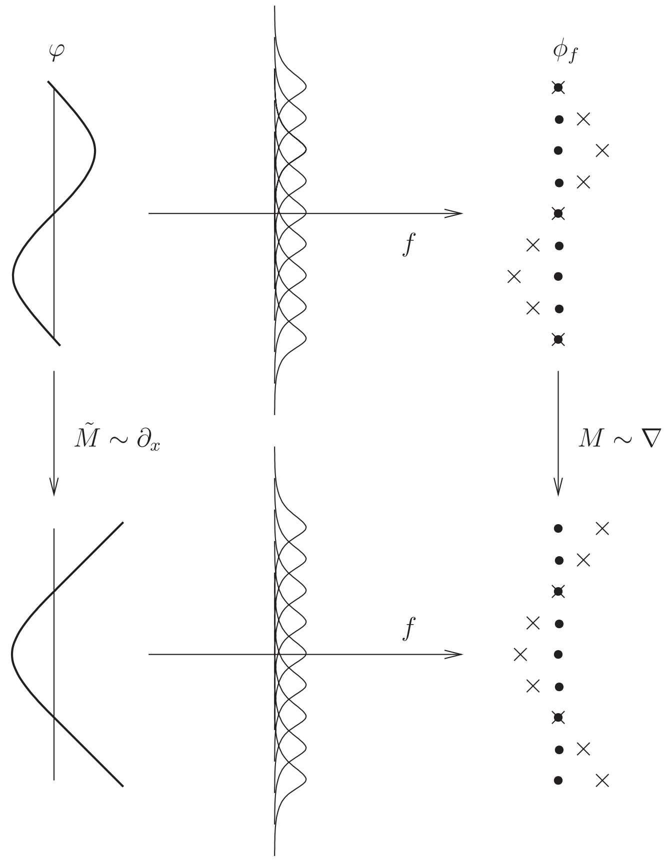

FIG. 1. A sketch of the blocking procedure: The averaging function f maps from continuum fields φ to averaged fields ϕ_f (that are connected to the lattice fields ϕ via α). The additional constraint Eq. (13) comes about because the diagram of f with the continuum symmetry $\tilde{M} \sim \partial_x$ and the lattice symmetry $M \sim \nabla$ has to commute.

Hence, the constraint states that the averaging function can have nonvanishing Fourier components $f(p_q)$ for each wave number p_q for which the difference operator has the "ideal" continuum dispersion relation $\nabla(p_q) = ip_q$. The latter condition means that the naive translation

$$(\partial_x e^{ip_q x})|_{x=an} = \sum_m \nabla_{nm} e^{ip_q am} \tag{58}$$

holds for all lattice points n for this wave number.

At this point let us stress that because of the periodicity $\nabla(p_{q+N}) = \nabla(p_q)$ the square brackets in Eq. (57) can vanish only once for every $q \bmod N$, for all other q $f(p_q)$ has to vanish. With regard to the correct continuum limit of ∇ we demand this to happen—if at all—in the first Brillouin zone, thus

$$f(p_q) = 0 \quad \text{for } |p_q| > \frac{\pi}{a}\left(1 - \frac{1}{N}\right). \tag{59}$$

This limits the spatial resolution of f to $|\Delta x| \sim a$ which, however, is a natural scale in the blocking approach to a lattice [see (52) for comparison].

B. Solutions

Inside the first Brillouin zone the constraint (57) introduces a kind of uncertainty relation between the averaging function f and the lattice difference operator ∇. For instance, if one demands an ultralocal operator ranging over only one neighboring point, ∇ will be proportional to the symmetric difference,

$$\nabla = c_1 \nabla^{\text{symm}}, \qquad \nabla^{\text{symm}}_{nm} = \frac{1}{2a}(\delta_{n+1,m} - \delta_{n-1,m}). \tag{60}$$

The dispersion relation is in this case

$$\nabla(p_q) = c_1 \frac{i}{a} \sin(ap_q), \tag{61}$$

and the bracket in (57) vanishes for $p = 1$ if and only if

$$c_1 = \frac{2\pi/N}{\sin(2\pi/N)}. \tag{62}$$

The proportionality factor c_1 indeed approaches 1 in the continuum limit and hence ∇ approaches the continuum derivative as ∇^{symm} does. As a consequence, f has only the lowest (and zeroth) Fourier components ($p_k = \{-2\pi/L, 0, 2\pi/L\}$):

$$f(x) = f_0 + f_1 \cos(2\pi x/L). \tag{63}$$

Hence for this ultralocal difference operator the averaging function f is very broad as it probes the whole space $x \in [-L/2, L/2]$.

The solution allowing for the next ($p = 2$) Fourier component in f demands the difference operator to spread over at least nearest and next-to-nearest neighbors with coefficients

$$\nabla_{nm} = \frac{1}{2a}(c_1(\delta_{n+1,m} - \delta_{n-1,m}) + c_2(\delta_{n+2,m} - \delta_{n-2,m})) \tag{64}$$

with the solutions

$$c_1 = \frac{2\pi}{N} \frac{2\sin(4\pi/N) - \sin(8\pi/N)}{\sin^2(4\pi/N) - \sin(2\pi/N)\sin(8\pi/N)}, \tag{65}$$

$$c_2 = -\frac{2\pi}{N} \frac{2\sin(2\pi/N) - \sin(4\pi/N)}{\sin^2(4\pi/N) - \sin(2\pi/N)\sin(8\pi/N)}, \tag{66}$$

which correctly approach $4/3$ and $-1/6$ in the limit $N \to \infty$.

One can proceed in this way. The more Fourier coefficients are included in the averaging function, the less localized is the derivative operator. In general, ∇ needs to spread to the nth neighbors to enable n nonvanishing Fourier coefficients in $f(p)$.[2]

[2]In this way derivatives interpolating between the symmetric derivative and the SLAC derivative are constructed.

At the extreme, in order to make f as narrow as it can get, all Fourier components $f(p_q)$ (in the first Brillouin zone) are needed. The constraint (57) then leads to the nonlocal SLAC operator by definition [2]. The coefficients in this case are

$$c_l^{\mathrm{SLAC}} = (-1)^l \frac{2\pi/N}{\sin(\pi l/N)}. \tag{67}$$

All of these operators can in principle be used inside M to generate the lattice supersymmetry transformations. But there are additional restrictions on f further reducing these possibilities.

IV. CONTINUUM LIMIT

The results of Sec. III necessitate a careful investigation of the continuum limit. We have argued in Sec. II A that the averaging function f needs to approach the delta distribution in the continuum limit. Hence, approaching this limit, more and more Fourier components $f(p_q)$ are needed. As a consequence of the additional constraint (57), the lattice derivative operator ∇ agrees in Fourier space with the SLAC derivative for the increasing number of modes p_q with nonvanishing $f(p_q)$. In other words, the difference operator becomes more and more extended over neighboring lattice sites, while f gets narrower. The more neighbors are included in the lattice derivative ∇, the more demanding numerical simulations will become. One is therefore tempted to use the most localized solution from this constraint.

Moreover, if combined with an appropriate blocking kernel α, the numerical effort could be reduced further. Thus it is advantageous to determine the general setting giving access to the full set of allowed blockings f's and, in particular, α's.

For that purpose we reconsider the full generating functional in the presence of external sources. The physics of the blocked fields ϕ_f with blocking f, (2), is carried by general correlation functions of this field, $\langle\phi_f(an_1)\cdots\phi_f(an_r)\rangle$. These correlation functions are generated by

$$Z_f[J] = \frac{1}{\mathcal{N}}\int d\varphi\, e^{-S_{\mathrm{cl}}[\varphi]+J\phi_f[\varphi]}. \tag{68}$$

The correlation functions of ϕ_f naturally live on a lattice defined by $n \in N_1 \times N_2 \times \cdots \times N_d$ resulting from the blocking f.

A. Continuum limit of the blocking f

In the limit $\phi_f[\varphi] \to \varphi$ the correlation functions $\langle\phi_f(an_1)\cdots\phi_f(an_r)\rangle$ tend toward the continuum correlation functions. Accordingly, the most restrictive constraint coming from the comparison of the lattice observables with their continuum counterparts is that f must approach the delta function in the continuum limit. Then the lattice theory resembles the continuum up to minor modifications. As stated in the previous section this means that Eq. (57) must be fulfilled for an increasing number of modes.

Let us work on a lattice with N points and let ∇ be more localized than the SLAC derivative, i.e. have an ideal dispersion relation $\nabla(p_q) = ip_q$ up to some momentum $p_{\max} < \pi(1-1/N)/a$. The corresponding f has nonvanishing Fourier components up to this momentum. Doublers will appear in the spectrum of such operators; they can be removed as shown in Appendix C within our solution for supersymmetric quantum mechanics. Analyzing the consequences for f, however, we will argue against these solutions in the following.

The momentum of the lattice fields $\phi(p_k)$ is restricted by this momentum cutoff of f [for an explicit formula see (B5)]. That means one introduces an additional momentum cutoff smaller than the usual lattice cutoff. In other words the number of degrees of freedom (Fourier modes) induced from the continuum via f is smaller than the actual number N of lattice degrees of freedom. This contradicts the blocking philosophy where all the lattice degrees of freedom should come from blocked continuum degrees of freedom. To be more precise, in the defining equation (5), the averaged fields ϕ_f span a vector space smaller than the one of the lattice fields ϕ. Therefore some lattice fields ϕ have no counterpart ϕ_f. Rather, their contribution to the lattice action $S[\phi]$ is a simple quadratic one with kernel α. This mismatch is depicted in Fig. 2.

Another way of stating the problem is that the blocking f gives rise to a resolution (an "effective lattice spacing") of $\pi(1-1/N)/p_{\max} > a$. On this coarser lattice the derivative is actually again SLAC. It is very unlikely that it

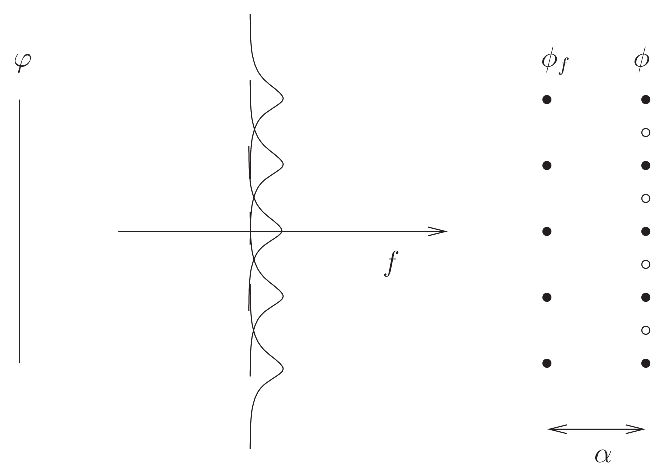

FIG. 2. A sketch of the mismatch in the blocking procedure, if f—in order to generate a local lattice derivative in the constraint (57)—has a limited number of Fourier components: the averaged fields ϕ_f have fewer degrees of freedom than the number of lattice points N used for ϕ. In other words, the ϕ_f transfer information from the continuum only to a coarser lattice (see text).

yields any improvement to work on the finer lattice with lattice spacing a, where not all of the degrees of freedom are induced by a blocking from continuum fields.

We conclude that the lattice derivative ∇ entering the lattice symmetry relation as M has to be the SLAC operator or some degrees of freedom on the lattice will have no continuum counterparts. In any case relation (57) must hold for an increasing number of lattice modes in the continuum limit if f should approach the delta distribution.

B. Continuum limit of the generating functional

It is left to discuss the consequences of a general choice for α. This is best done in terms of the generating functional $Z_f[J]$ defined in (68). The path integral in (68) can be conveniently rewritten in terms of a path integral over lattice fields ϕ. In Sec. II A we have done this already for vanishing external currents J, and a quadratic blocking kernel $\frac{1}{2}(\phi - \phi_f)\alpha(\phi - \phi_f)$, see (5), with symmetry properties (8) and (9). This procedure is readily extended to the general case with nonvanishing currents by rewriting the source term $\exp\int J\phi_f$ via the quadratic blocking kernel,

$$e^{J\phi_f} = \frac{e^{-(1/2)J\alpha^{-1}J}}{\int d\phi e^{-(1/2)\phi\alpha\phi}} \int d\phi e^{-(1/2)(\phi-\phi_f)\alpha(\phi-\phi_f)+J\phi}. \tag{69}$$

Inserting (69) into (68) the generating functional Z_f can be rewritten as a lattice generating functional

$$Z_f[J] = \frac{1}{\mathcal{N}[J]} \int d\phi e^{-S[\phi]+J\phi}, \tag{70}$$

with Wilsonian action S as defined in (5),

$$e^{-S[\phi]} = \mathrm{SDet}^{1/2}\alpha \int d\varphi e^{-(1/2)(\phi-\phi_f)\alpha(\phi-\phi_f)-S_{\mathrm{cl}}[\varphi]}. \tag{71}$$

The normalization $\mathcal{N}[J]$ carries a trivial quadratic dependence on the current J and reads

$$\mathcal{N}[J] = \mathcal{N} e^{(1/2)J\alpha^{-1}J}. \tag{72}$$

It reduces to $\mathcal{N}$ for vanishing current. We emphasize again that $Z_f[J]$ in (70) has no dependence on α, and reduces to (4) for vanishing current $J = 0$. Note also that the generating functional in (70) with $\mathcal{N}[0]$ is the standard lattice generating functional. As $\mathcal{N}[J]$ is a trivial Gaussian, lattice simulations for correlation functions straightforwardly relate to those from (70).

The above construction allows us to evaluate general choices of α. The blocking function f will be discussed in the next section; here we will assume f to approach the delta distribution such that $\phi_f \to \varphi$. We have already argued in Sec. II A that a diverging α ensures that the lattice field ϕ agrees with the blocked field $\phi_f[\varphi]$. A simple α is a diagonal one, $\alpha^{ij}_{nm} = c\delta^{ij}\delta_{mn}$ with $c \to \infty$; see also Sec. II A. Then we are led to

$$\mathrm{SDet}^{1/2}\alpha e^{-(1/2)(\phi-\phi_f)\alpha(\phi-\phi_f)} = \delta(\phi - \phi_f), \tag{73}$$

see (7), and the integral in (69) is trivially done. Moreover, the normalization loses its J dependence, $\mathcal{N}[J] \to \mathcal{N}$, as α^{-1} vanishes for $c \to \infty$. Finally, the Wilsonian action is given by

$$e^{-S[\phi]} = \int d\varphi \delta(\phi - \phi_f[\varphi]) e^{-S_{\mathrm{cl}}[\varphi]}. \tag{74}$$

This can be viewed as the canonical form of the Wilsonian action. Note also that in this case the symmetry relation (18) simplifies to the standard one, as the rhs vanishes with $\alpha^{-1} = 0$.

In consequence the limit $\alpha^{-1} \to 0$ is the natural choice for the continuum limit. In order to see how a general α scales with the lattice spacing a, we rewrite the blocking term as

$$(\phi - \phi_f)\alpha(\phi - \phi_f) = a^d \sum_{i,n} a^d \sum_{j,m} (\phi - \phi_f)^i_n \times \frac{\alpha^{ij}_{nm}}{a^{2d}} (\phi - \phi_f)^i_m. \tag{75}$$

First we remark that $a^d\sum_n \to \int d^d x$ in the continuum limit $a \to 0$. Second, for symmetric smearings α we can always diagonalize the matrix α. Third, a factor a^d appears because $\delta_{nm}/a^d \to \delta(x - y)$. We conclude that all eigenvalues α_n have to satisfy

$$\alpha_n/a^d \to \infty, \tag{76}$$

for guaranteeing that the Wilsonian action tends toward the classical action in the continuum limit.

In other words, the inverse α^{-1} has to vanish for $S \to S_{\mathrm{cl}}$. For practical purposes it might be advantageous to work with some vanishing eigenvalues of α^{-1} already at finite lattice spacing. Then the related eigenvalues α_n diverge already for finite lattice spacing, and the right-hand side of (73) will be proportional to delta functions for the related eigenfunctions ψ_n, leading to $(\psi_n, \phi) = (\psi_n, \phi_f[\varphi])$. We also note that the symmetry relation (18) contains only α^{-1}, which shows no divergencies for $\alpha_n \to \infty$ but zeros. Indeed we show in Appendix D that (18) can be derived without using α explicitly, and hence noninvertible α^{-1} are not problematic. On the contrary, vanishing eigenvalues of α^{-1} mean that the factor $N[J]$ in (72) is actually a J-independent constant in that subspace of fields.

Likewise, the eigenvalues of α^{-1} can have any sign whereas (71) would require positive eigenvalues of α.

What happens if some eigenvalues of α^{-1} do not vanish in the continuum limit? As the generating functionals $Z_f[J]$, and in particular $Z[J]$, do not depend on α, it is not mandatory that the Wilsonian action S approaches the classical action $S_{\rm cl}$. Assuming $\phi_f[\varphi] \to \varphi$, a finite α in (71) amounts to equivalence classes of actions with measures

$$d\phi \exp{-S[\phi]} \tag{77}$$

related by Gaussian integrations in the continuum. On the level of classical actions this is nothing but the introduction of auxiliary fields ϕ via the equations of motion.

The case of vanishing eigenvalues α_n (diverging α^{-1}), however, is different. From the path integral in (71) one reads off that the corresponding subspace of fields φ is simply integrated out and the Wilsonian action S does not depend on the corresponding eigenfunctions ψ_n. The corresponding singularities of α^{-1} in the normalization (72) can be avoided by considering currents J only in the orthogonal subspace, that is $(\psi_n, J) = 0$. Then the above derivations are unaltered, but with this procedure we have removed the φ modes in the singular subspace from our theory. A simple example for vanishing α is given by the Wilson mass term in a fermionic theory. Assume that we start with the naive lattice Dirac action with doublers. The blocking kernel α can be chosen such that it vanishes at the doublers. In turn α^{-1} provides a diverging mass for the doublers. More details on this and the general case with vanishing α is deferred to Appendix E.

We summarize our findings as follows: in order to recover the original action $S_{\rm cl} = S$ in the continuum limit $f \to \delta$, the blocking kernel α has to lead to a delta function in field space; see (73). It might, however, be advantageous to rely on a nontrivial classical Wilsonian action in the continuum limit, generated by other choices of α, in order to optimize the locality and hence to minimize the numerical effort.

V. FREE SUPERSYMMETRY ON THE LATTICE

We will now apply the blocking formalism to SUSYQM which is a supersymmetric theory in one dimension, i.e. all "fields" depend only on a time $x_1 = t$. It serves as a toy model for supersymmetric theories. After fixing the notation, only a quadratic theory is considered in this section; interacting SUSY theories will be considered in the next section.

A. Brief review of SUSYQM in the continuum

The field content of SUSYQM is the multiplet

$$\varphi_x = \{\chi(t), F(t), \psi(t), \bar{\psi}(t)\}, \tag{78}$$

where χ and F are real bosons, ψ is a complex fermion (Grassmannian), and $\bar{\psi}$ its complex conjugate. (The length dimensions of these fields are $\sqrt{L}$, $1/\sqrt{L}$, and L^0, respectively.)

The Euclidean action in the continuum has the following form:

$$S_{\rm cl}[\varphi] = \int dt \bigg[\frac{1}{2}(\partial_t \chi)^2 + \bar{\psi}\partial_t \psi - \frac{1}{2}F^2 + \bar{\psi}\frac{\partial W}{\partial \chi}\psi - FW(\chi)\bigg], \tag{79}$$

where the first three terms are kinetic ones (F is an auxiliary nondynamical field) and the last two terms represent a potential term for χ and Yukawa interactions.

This action is invariant under the supersymmetry transformations

$$\begin{aligned} \delta\chi &= -\bar{\epsilon}\psi + \epsilon\bar{\psi}, \qquad & \delta F &= -\bar{\epsilon}\partial_t\psi - \epsilon\partial_t\bar{\psi}, \\ \delta\psi &= -\epsilon\partial_t\chi - \epsilon F, \qquad & \delta\bar{\psi} &= \bar{\epsilon}\partial_t\chi - \bar{\epsilon}F, \end{aligned} \tag{80}$$

up to the following surface term:

$$\delta S_{\rm cl} = \int dt \partial_t(\epsilon\bar{\psi}(\partial_t\chi + F) + \epsilon\bar{\psi}W(\chi) + \bar{\epsilon}\psi W(\chi)). \tag{81}$$

According to our general notation we write

$$\delta\varphi = (\epsilon\tilde{M} + \bar{\epsilon}\bar{\tilde{M}})\varphi, \tag{82}$$

where

$$\tilde{M} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -\partial_t \\ -\partial_t & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \bar{\tilde{M}} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & -\partial_t & 0 \\ 0 & 0 & 0 & 0 \\ \partial_t & -1 & 0 & 0 \end{pmatrix}. \tag{83}$$

B. Transformations on the lattice

On the lattice it is very natural to take the same field multiplet, now evaluated at discrete lattice points

$$\phi_n = \{\chi_n, F_n, \psi_n, \bar{\psi}_n\}. \tag{84}$$

In the corresponding lattice transformations [as defined by (12)]

$$\delta\phi^i_n = (\epsilon M^{ij}_{nm} + \bar{\epsilon}\bar{M}^{ij}_{nm})\phi^j_m, \tag{85}$$

the matrices M and $\bar{M}$ will be of the same form as in the continuum,

$$M^{ij}_{nm} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -\nabla \\ -\nabla & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{nm}, \tag{86}$$

$$\bar{M}^{ij}_{nm} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & -\nabla & 0 \\ 0 & 0 & 0 & 0 \\ \nabla & -1 & 0 & 0 \end{pmatrix}_{nm}. \tag{87}$$

∇ is subject to the discussion in previous sections. We will come back to its locality properties in Sec. V E.

C. Ansatz for the quadratic lattice action

For the rest of this section we restrict ourselves to a quadratic theory. Even in this simple case one obtains a nontrivial action with nontrivial lattice derivative operators. These operators solve the relation (18) for SUSY just as the overlap operator solves it for the chiral symmetry.

We choose the following matrix K for the quadratic action (cf. Sec. II D):

$$\frac{K^{ij}_{mn}}{a} = \begin{pmatrix} -\Box & -m_b & 0 & 0 \\ -m_b & -\mathbb{1} & 0 & 0 \\ 0 & 0 & 0 & \hat{\nabla} - m_f \\ 0 & 0 & \hat{\nabla} + m_f & 0 \end{pmatrix}_{mn}, \quad (88)$$

which means

$$\frac{S[\phi]}{a} = -\frac{1}{2}\chi\Box\chi + \bar{\psi}(\hat{\nabla} + m_f)\psi - \frac{1}{2}F\mathbb{1}F - Fm_b\chi \quad (89)$$

(again, a sum over the lattice indices is understood). Though the symbols of the matrices K suggest them to be similar to the objects in a quadratic continuum action with $W = m\chi$, they are so far undetermined. (In particular the lattice masses and derivatives can be different for fermions and bosons.) $\Box$, m_b, and m_f must be symmetric and $\hat{\nabla}$ antisymmetric circulant matrices to guarantee Hermiticity and translational invariance. In the continuum limit we expect the behavior $\Box \to \partial^2$, $\hat{\nabla} \to \partial$, and $m_{b,f} \to m\mathbb{1}$, while at finite lattice spacing these matrices are chosen according to relation (18). This also means that the matrix $\hat{\nabla}$ can be different from the derivative operator ∇ in the naive generators M and $\bar{M}$. Generalizations of the mentioned ansatz for the quadratic action are possible. One could, for example, introduce an additional undetermined matrix in the F^2 term, but this is not necessary for the solution of the relation in the next two sections. In the last section of this chapter we will consider the most general ansatz to derive some statements for the general locality of the solutions.

D. First solution for an ultralocal blocking

To start with the simplest form of the blocking matrix α we take its inverse to be

$$a(\alpha^{-1})^{ij}_{mn} = \begin{pmatrix} a_2 & 0 & 0 & 0 \\ 0 & a_0 & 0 & 0 \\ 0 & 0 & 0 & a_1 \\ 0 & 0 & -a_1 & 0 \end{pmatrix}_{mn}, \quad (90)$$

where all a_i are symmetric circulant matrices of length dimension i (and allowed to be zero, cf. Sec. IV B). Note that these considerations also include a much more general ansatz. This happens because (90) is the same as a general blocking ansatz up to a symmetric part α_S as shown in Appendix F.

Circulant matrices commute. Using this property, a solution of the symmetry relation (18) for the quadratic SUSY action (89) is straightforward. In Sec. II D we have already specified the symmetry relation for quadratic theories. From the first line of (27) one can read the following equations:

$$-\Box + \nabla\hat{\nabla} + \nabla(m_f - m_b) = -[(a_2\nabla + a_1)\Box + (a_1\nabla + a_0)m_b](\hat{\nabla} + m_f), \quad (91)$$

$$\hat{\nabla} - \nabla + m_f - m_b = -[(a_2\nabla + a_1)m_b + (a_1\nabla + a_0)](\hat{\nabla} + m_f). \quad (92)$$

Two additional equations can be identified with the transposed of these if one reconsiders the symmetric or antisymmetric form of the matrices. The field-independent second line of (27) vanishes as M and hence $M\alpha^{-1}K^T$ always connect bosons with fermions. The corresponding relation for the generator $\bar{M}$ induces the same set of equations.

At first we proceed in the same manner as in the derivation of the Ginsparg-Wilson relation: we use an ultralocal blocking with a_i's diagonal in lattice sites and derive a solution for the lattice action in terms of these matrices. This solution corresponds to the overlap operator that also is a function of the blocking (appearing in the chiral case).

The second equation (92) can easily be solved for $\hat{\nabla} + m_f$ in terms of m_b and ∇. One gets $\hat{\nabla}$ and m_f as the antisymmetric respectively symmetric part of

$$\frac{[(1 + a_1m_b + a_0) - (a_2m_b + a_1)\nabla](\nabla + m_b)}{X}, \quad (93)$$

where

$$X = (1 + a_1m_b + a_0)^2 - (a_2m_b + a_1)^2\nabla^2. \quad (94)$$

The first equation (91) then gives $\Box$ and the complete solution reads

$$\hat{\nabla} = \frac{(1 + a_0 - a_2m_b^2)\nabla}{X}, \quad (95)$$

$$m_f = \frac{(1 + a_1m_b + a_0)m_b - (a_2m_b + a_1)\nabla^2}{X} \quad (96)$$

$$-\Box + m_b^2 = \frac{-\nabla^2 + m_b^2}{1 + a_0 - a_2\nabla^2}. \quad (97)$$

The last part is presented in terms of $-\Box + m_b^2$ because this operator appears in the bosonic sector after integrating out F, i.e. in the on-shell action.

As expected, in the limit $a_i \longrightarrow 0$ one has $\Box \longrightarrow \nabla^2$, $\hat{\nabla} \longrightarrow \nabla$, and $m_f \longrightarrow m_b$. At finite lattice spacing, however, these operators are nontrivial because each of them contains both the derivative operator ∇ and the mass term m_b. $\hat{\nabla}$ and m_f are nonsingular because X is positive since ∇ is anti-Hermitian; $\Box$ is nonsingular, if a_0 and a_2 are not largely negative and the denominator in (97) vanishes.

Now we have to check the locality of our resulting lattice action. As explained in Sec. IV, the solution of the additional constraint, (54), leads to the nonlocal SLAC derivative $\nabla(p) \sim p$. As the operators of the lattice action (95)–(97) are given in terms of ∇, one might expect that they inherit this locality problem. Therefore, the locality properties of the lattice action need to be examined carefully.

Since α is similar to a mass term (diagonal in lattice sites) the form of the denominators in (95)–(97) renders the appearing operators very similar to a massive propagator. In the continuum similar expressions lead to an exponential decay for large distances. On the lattice, however, the corresponding behavior is spoiled by terms decaying only algebraically. This is shown in Appendix G using methods of complex analysis.

E. Solution with a local action

Since we insist on the locality of the lattice action, more general blocking kernels α must be considered. With these it is possible to enforce locality in the SUSY lattice action. In our point of view this is a crucial feature of the modified symmetry including the blocking kernel compared to the naive symmetry without it.

Allowing now for an arbitrary momentum dependence of α one can solve the Eqs. (91) and (92) for the circulant matrices a_0, a_1, and a_2. Consequently these matrices are dependent on $\hat{\nabla}$, $\Box$, m_f, and m_b. Given $\nabla(p_k) = ip_k$ the solution in terms of the matrices of the lattice action is

$$a_0(p_k) = \frac{\Box}{m_b^2 - \Box} - \frac{i\hat{\nabla} p_k}{m_f^2 - \hat{\nabla}^2}, \tag{98}$$

$$a_1(p_k) = \frac{-m_b}{m_b^2 - \Box} + \frac{m_f}{m_f^2 - \hat{\nabla}^2}, \tag{99}$$

$$a_2(p_k) = \frac{1}{m_b^2 - \Box} + \frac{i\hat{\nabla}/p_k}{m_f^2 - \hat{\nabla}^2}. \tag{100}$$

Now one could use the simplest ultralocal operators (without doublers) in the action and read off the corresponding α^{-1} for a solution of the relation. Additional restrictions for these operators appear since singularities in the a_i must be excluded. A possible singularity appears at $p_k = 0$ if the mass is zero in the theory. To avoid this singularity one can use $m_b = m_f$ and $\Box = \hat{\nabla}\hat{\nabla}$. Note that in this way the fermionic and bosonic sectors are treated in a similar manner. So possible doublers of the fermionic sector also appear in the bosonic one, but can be removed with the same mass term. The result then simplifies to

$$a_0(p_k) = \frac{\hat{\nabla}(\hat{\nabla} - ip_k)}{m_f^2 - \hat{\nabla}^2}, \qquad a_2(p_k) = \frac{1 + i\hat{\nabla}/p_k}{m_f^2 - \hat{\nabla}^2}, \tag{101}$$

and $a_1(p_k) = 0$. Just as expected all a_i vanish if the derivative operator $\hat{\nabla}$ is the SLAC derivative and the deformed symmetry is reduced to the naive one.

The simplest solution for the fermionic operators is the standard Wilson fermion. With the corresponding bosonic operators one finally arrives at

$$\hat{\nabla}(p_k) = \frac{i}{a}\sin(ap_k), \qquad \Box = \hat{\nabla}\hat{\nabla}, \tag{102}$$

$$m_b(p_k) = m_f(p_k) = m + \frac{1}{a}(1 - \cos(ap_k)). \tag{103}$$

In this realization all possible doublers are removed by the Wilson mass terms in the bosonic and fermionic sectors.

Such a form for the quadratic lattice action was chosen in [34] and other lattice simulations. It has been shown that the choice of the same mass and derivative operators in the fermionic and bosonic sectors leads to a major improvement with respect to the lattice supersymmetry.

Note, however, that a major requirement is still not considered. One should not only insist on the locality of the action. To get a well-defined representation of the symmetry on the lattice also the deformed symmetry transformations must be local.

F. Local supersymmetry

In order to get a well-defined lattice supersymmetry, the deformed symmetry operator M_{def} has to be local and approach the continuum supersymmetry, as explained in Sec. II D. The solutions we have obtained so far contain the SLAC operator, either in the action, Eqs. (95)–(97), or in the inverse blocking kernel, Eqs. (98)–(101). This leads to a nonlocal behavior in $\alpha^{-1}K$, which in $M_{\mathrm{def}} = M - M\alpha^{-1}K$ can enhance or reduce the nonlocality of M (whereas in the chiral case a nonlocal $\alpha^{-1}K$ immediately induces a nonlocal M_{def}). Here we investigate the conditions under which the locality of the action and the deformed symmetry generator M_{def} can be achieved. We first derive a special solution which fulfills the locality condition (24) for the action and M_{def}. At the end of the section we will argue that this condition [and not (23)] is the best one can achieve in the present setup.

For the special solution we consider now a slightly generalized form of the deformed supersymmetry and the lattice action. Since the corresponding inverse blocking kernel has no direct physical implication it is adjusted accordingly.

In the ansatz for the quadratic lattice SUSY action, Eq. (88), a general symmetric circulant matrix $-I$ is used in the F^2 term instead of the diagonal matrix $-\mathbb{1}$. As a deformed supersymmetry generator we take

$$M_{\text{def}} = \begin{pmatrix} 0 & 0 & 0 & I_b \\ 0 & 0 & 0 & -\nabla_b \\ -\nabla_f & -I_f & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{104}$$

If α^{-1} vanishes in the continuum limit, it follows that $M_{\text{def}} \to \tilde{M}$, and hence $I_{b,f} \to 1$ and $\nabla_{b,f} \to \partial_t$.

Demanding that M_{def} is a symmetry of the lattice action K, Eq. (32), one arrives at the following sets of equations:

$$\frac{I_b}{I_f} = \frac{m_b}{m_f} = \frac{\nabla_b}{\nabla_f}, \tag{105}$$

$$\frac{\nabla_f}{I_b} = \frac{\Box}{\hat{\nabla}}, \tag{106}$$

$$\frac{\nabla_b}{I_b} = \frac{\hat{\nabla}}{I}. \tag{107}$$

This can be solved easily, e.g. for the choice (102) and (103) of straightforward operators in K, one obtains $I = 1$. In that case a natural solution is

$$I_b = I_f = 1, \qquad \nabla_b = \nabla_f = \hat{\nabla}. \tag{108}$$

With $\hat{\nabla}$ as defined in (102), for example, the deformed symmetry is ultralocal and obviously approaches the continuum supersymmetry. So it seems that there exists a local deformed symmetry of the considered local lattice action. The reason for the absence of locality problems is that the SLAC operator from the blocked lattice symmetry M has not appeared yet.

The deformed symmetry must, however, not only be a local symmetry of the lattice action. A further condition, Eq. (31), implies a relation between M and M_{def}. This additional condition is a direct consequence of the symmetry relation (18). Since α^{-1} is so far undetermined it is apparently not difficult to satisfy this condition. But restrictions of the lattice action, due to e.g. Hermiticity, impose further constraints on the blocking kernel. In the case of supersymmetry these constraints are of great importance since α^{-1} can only connect fermions with fermions, whereas M_{def} connects fermions and bosons with each other. So one has to investigate whether or not there exists an α^{-1} to ensure that this deformed symmetry M_{def} is indeed the result of a blocking procedure and hence fulfills $M_{\text{def}} = M(\mathbb{1} - \alpha^{-1}K)$. As M is not invertible we use a general ansatz for α^{-1}, namely

$$a(\alpha^{-1})^{ij}_{mn} = \begin{pmatrix} b_2 & b_1' & 0 & 0 \\ b_1' & b_0 & 0 & 0 \\ 0 & 0 & 0 & b_1 + b_1'' \\ 0 & 0 & -b_1 + b_1'' & 0 \end{pmatrix}_{mn}, \tag{109}$$

and compare the two sides of $M\alpha^{-1} = (M - M_{\text{def}})K^{-1}$ yielding

$$\begin{pmatrix} 0 & 0 & (-1)(b_1 - b_1'') & 0 \\ 0 & 0 & \nabla_b(b_1 - b_1'') & 0 \\ -\nabla b_2 - b_1' & -\nabla b_1' - b_0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & (1 - I_b)/(\hat{\nabla} - m_f) & 0 \\ 0 & 0 & (\nabla_b - \nabla)/(\hat{\nabla} - m_f) & 0 \\ \cdots & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{110}$$

The lower left block of this matrix equation can be satisfied by an appropriate choice of b_0, b_1', and b_2. The upper right block fixes $b_1 - b_1''$ and in addition implies

$$\frac{\nabla_b}{I_b} = \nabla. \tag{111}$$

Together with Eq. (107) this yields

$$I\nabla = \hat{\nabla}. \tag{112}$$

We remind the reader that ∇ is the SLAC derivative appearing in the lattice symmetry M due to the additional constraint and I and $\hat{\nabla}$ are the matrices for the F^2 and the kinetic fermion term in the lattice action, in which both should be local.

The SLAC operator is nonlocal because of its discontinuity at the boundary of the Brillouin zone. To ensure a local $\hat{\nabla} = I\nabla$, i.e. to make it periodic in momentum space, I and all its derivatives must vanish at the boundaries of the Brillouin zone. Then the behavior is not analytic in momentum space, but a stronger than polynomial decay is guaranteed (see Appendix H). With such a result for the matrix I all symmetry conditions for the action can be fulfilled with

$$m_f = m_b, \qquad I_f = I_b = I, \tag{113}$$

$$\nabla_f = \nabla_b = \hat{\nabla} = I\nabla, \qquad \Box = I\nabla^2. \tag{114}$$

All of these operators of the action satisfy locality stronger than polynomial. Note that this solution amounts to $M_{\text{def}} = IM$.

The only remaining problem arises since I vanishes at the boundaries of the Brillouin zone. This means that the on-shell bosonic mass term inherits a divergence $m_b(p)^2/2I(p)$ at the edge of the Brillouin zone. Hence an on-shell problem is expected although the off-shell action is local in the sense of condition (24).

It is instructive to recall that these results rely on the specific structure of the transformations in the case of supersymmetry. They transform fermions into bosons and vice versa. By contrast the blocking matrix and the action relates fermions with fermions and bosons with bosons. In the case of other symmetries the transformations α^{-1}, and K would have the same block diagonal structure, which makes it easier to find a local solution for K and M_{def}.

One may still wonder whether the current approach can be generalized to get a result that satisfies the more severe condition (23) for locality. To investigate this problem we look at the relation $M_{\mathrm{def}} = -M\alpha_S^{-1}K$ [see (31)], where only the symmetric part of α_S^{-1} of the blocking appears. To get an exponential decay for both M_{def} and the action K, also $M\alpha_S^{-1}$ has to fulfill this condition. The SLAC derivative in the generator M, (86), violates the locality of this matrix. Taking the most general α_S^{-1}, (F2), the only entries of $M\alpha_S^{-1}$ are $d_1 + \nabla d_1''$ ($d_1 - \nabla d_1''$ for the second supersymmetry, $\bar{M}\alpha_S^{-1}$) and the same matrix multiplied by the SLAC derivative. So both $d_1 + \nabla d_1''$ and $\nabla(d_1 + \nabla d_1'')$ must be local. In analogy to I and $I\nabla$ we conclude that it is impossible to get an exponential decay on the lattice for both matrices. So the condition (24) and not (23) can be fulfilled for both M_{def} and K.

VI. TOWARD INTERACTING SUPERSYMMETRY ON THE LATTICE

The final goal of our investigations of lattice SUSY is to write down a lattice action for an interacting theory. According to the usual argument, that is not blocking inspired, a supersymmetric lattice action for the quadratic case can be found. The lattice supersymmetry transformations are in that case defined as the continuum transformations with the derivative operator replaced by a local lattice derivative.

For the case of a quadratic action we have given a solution above that preserves the modified symmetry and is hence guided by the blocking of the symmetry. In this case it is also possible to derive a direct solution of the blocking transformations, cf. Equation (50). Therefore it is desirable to extend the above results to interacting theories.

For actions beyond second order we go back to the original equation (18) which provides a systematic relation to be fulfilled in order to keep the considered symmetry. The solutions are, however, much more complicated than in the quadratic case. This can already be observed by an analysis of a specific solvable example, namely, constant fields in SUSYQM, which gives nontrivial results and displays a new issue: the polynomial nature of the action.

A. Solution for constant fields in SUSYQM

In this section we work with the same parametrization for M, $\bar{M}$, and α^{-1}, see (86) and (90), but study only constant fields $\chi_n = \chi$ and so on. This amounts to the zero mode sector of the lattice derivative ∇, i.e. we can replace $\nabla \to 0$. Note that in this approximation F is invariant under the naive lattice transformations. Of course, locality will not be an issue for this toy model.

For the action we use the following ansatz:

$$\frac{S}{a} = N[\bar{\psi}\psi g(\chi) - h(\chi, F)], \tag{115}$$

where N, the number of lattice sites, is a remnant of the summation. In view of the continuum limit we have restricted ourselves to an F-independent function g coupling the boson and the fermion, like $\partial W/\partial\chi$ in the continuum action (79). Likewise, we expect the $Fg = \partial h/\partial\chi$ to hold in the continuum limit.

For such an action the relation (18) becomes a partial differential equation in g and h:

$$Fg - \frac{\partial h}{\partial \chi} = -Na_1 g\frac{\partial h}{\partial \chi} - Na_0 g\frac{\partial h}{\partial F} - \frac{a_1}{a}\frac{\partial g}{\partial \chi}. \tag{116}$$

This indeed approaches $Fg = \partial h/\partial\chi$ in the limit where the a_i vanish.

For finite a_0 and a_1 this equation can be solved for h for different choices of g. Among these solutions we consider those that consist of a term $F^2/2$ plus terms linear in F with an arbitrary χ dependence, such that the auxiliary field F can be integrated out easily.

The simplest case is $g(\chi) = 0$, which should include the "kinetic term," that in the zero mode sector degenerates to $-F^2/2$. Indeed, any function $h(F)$ is a solution of Eq. (116) in this case.

The case $g(\chi) = m_f$ resembles an additional mass term. The corresponding solution

$$h(\chi, F) = \frac{1}{2}F^2 + \frac{1 + a_0N}{1 - a_1Nm_f}m_fF\chi + \frac{a_0}{2}\frac{(1 + a_0N)Nm_f^2}{(1 - a_1Nm_f)^2}\chi^2 \tag{117}$$

becomes $F^2/2 + m_fF\chi$ in the limit $a_i \to 0$ as required. For finite a_i we obtain a bosonic mass different from m_f and an additional term proportional to χ^2. This action could also be gotten from Sec. V C, Eqs. (89) and (95)–(97), since putting $\nabla \to 0$ there also yields different bosonic and fermionic masses and a χ^2 term surviving from $\Box$.

The most interesting case is an interacting theory with a truly χ-dependent term $g(\chi)$. For the lowest possible power

$$g(\chi) = \lambda\chi, \tag{118}$$

one obtains the common Yukawa interaction. The general solution of (116) is again restricted by the requirement that for vanishing constants a_i the term h should resemble the continuum result $F^2/2 + \lambda F\chi^2/2$. One obtains the nonpolynomial solution

$$\begin{aligned} h(\chi, F) = &\frac{1}{2}F^2 - \frac{1 + a_0N}{a_1N}\chi F + \frac{a_0(1 + a_0N)}{2a_1^2N}\chi^2 \\ &- \left(\frac{1}{aN} + \frac{1 + a_0N}{a_1^2\lambda N^2}F - \frac{a_0(1 + a_0N)}{a_1^3\lambda N^2}\chi\right) \\ &\times \log(1 - a_1\lambda N\chi) + \frac{a_0(1 + a_0N)}{2a_1^4\lambda^2N^3} \\ &\times (\log(1 - a_1\lambda N\chi))^2, \end{aligned} \tag{119}$$

which upon expanding in a_1 becomes polynomial

$$\begin{aligned} h(\chi, F) = &\frac{1}{2}F^2 + \frac{\lambda}{2}(1 + a_0N)F\chi^2 \\ &+ \frac{a_0}{8}\lambda^2N(1 + a_0N)\chi^4 + O(a_1). \end{aligned} \tag{120}$$

Now it is obvious that the correct continuum limit is approached and that this interaction term does not diverge in the limit $a_1 \to 0$ as one might have expected from (119).

Note that in the zero mode sector there is no failure of the Leibniz rule. Hence the direct translations of the continuum action are naively supersymmetric fulfilling (19) which amounts to setting $a_{1,2} = 0$ (in other words using a SUSY preserving blocking α_S). The presented solutions are deformations thereof. Already in the simple case of constant fields these deformations lead to a nonpolynomial solution if an interacting theory is considered.

One might expect that the nonpolynomial form of the action is a generic consequence of the symmetry relation (18), not only for supersymmetry. Indeed, similar results have been obtained in the context of chiral symmetry [17]. We will discuss this in the next sections as well as the circumstances under which a polynomial action like (120) is possible.

B. Polynomial solutions of the symmetry relation

As shown in the previous section it seems difficult to find a polynomial solution of the lattice symmetry relation (18) beyond second order. In the following we argue that a nonpolynomial action is indeed the generic solution of this relation for an arbitrary linear symmetry. Only if special conditions are fulfilled, can the series in the fields be truncated. To show this general behavior we generalize the considerations of Sec. II D to interacting systems.

We consider a lattice action consisting of polynomials up to order R in the fields represented as

$$S[\phi] = \sum_{r=1}^{R} s^{(r)}[\phi], \qquad s^{(r)}[\phi] = K^{i_1\cdots i_r}_{n_1\cdots n_r}\phi^{i_1}_{n_1}\cdots\phi^{i_r}_{n_r}, \tag{121}$$

where $s^{(r)}$ contains the rth order in the fields. The purely quadratic case $R = 2$ (and $s^{(1)} = 0$) has been discussed in Sec. V. The coefficients K are so far not further specified; they can imply a simple multiplication of fields at the same lattice point, but are also allowed to contain lattice derivatives or to smear the powers of the fields over several lattice sites, as long as they obey the correct continuum limit.

The relation (18) is in general a complicated nonlinear differential equation coupling derivatives with respect to the fields at different lattice points. An expansion in the order of the fields using the ansatz (121) yields

$$\begin{aligned} O(\phi^0)\colon 0 = &M\alpha^{-1}\left(\frac{\delta s^{(1)}}{\delta\phi}\frac{\delta s^{(1)}}{\delta\phi} - \frac{\delta^2 s^{(2)}}{\delta\phi\delta\phi}\right) \\ &+ (\mathrm{STr}M - \mathrm{STr}\tilde{M}), \end{aligned} \tag{122}$$

$$\begin{aligned} O(\phi^{r=1\ldots R-2})\colon &M\phi\frac{\delta s^{(r)}}{\delta\phi} \\ &= M\alpha^{-1}\sum_{s+t=r+2}\left(\frac{\delta s^{(s)}}{\delta\phi}\frac{\delta s^{(t)}}{\delta\phi} - \frac{\delta^2 s^{(r+2)}}{\delta\phi\delta\phi}\right), \end{aligned} \tag{123}$$

$$O(\phi^{r=R-1,R})\colon M\phi\frac{\delta s^{(r)}}{\delta\phi} = M\alpha^{-1}\sum_{s+t=r+2}\frac{\delta s^{(s)}}{\delta\phi}\frac{\delta s^{(t)}}{\delta\phi}, \tag{124}$$

$$O(\phi^{r=R+1\cdots 2R-2})\colon 0 = M\alpha^{-1}\sum_{s+t=r+2}\frac{\delta s^{(s)}}{\delta\phi}\frac{\delta s^{(t)}}{\delta\phi}, \tag{125}$$

where we used a shorthand notation without indices. These coupled equations can be read as restrictions for the $K^{i_1\cdots i_r}_{n_1\cdots n_r}$ parametrizing $s^{(r)}$ imposed by the symmetry. In the case of $R = 2$ only the conditions (122) and (123) are relevant giving (27).

For interacting theories, $R > 2$, a set of equations with vanishing left-hand sides, Eq. (125), appears. The very difficulty to obtain a polynomial solution is to fulfill these equations with a finite number of interacting terms $s^{(r)}[\phi]$ (which in addition shall give the desired theories in the continuum). If this turns out to be impossible for some order R, this order has to be increased and finally one might be forced to nonpolynomial interactions.

As an example we consider the relation of the highest order $O(\phi^{2R-2})$, which reads

$$0 = (M\alpha^{-1})^{ij}_{nm}\left(\frac{\delta s^{(R)}}{\delta\phi^j_m}\frac{\delta s^{(R)}}{\delta\phi^i_n}\right), \tag{126}$$

and can be rewritten in matrix-vector notation as

$$0 = v^T(M\alpha^{-1})v \quad \text{with} \quad v^i_n = \frac{\partial s^{(R)}}{\partial\phi^i_n}. \tag{127}$$

This relation is a severe constraint, because it implies that

$$M\alpha^{-1} \pm (M\alpha^{-1})^T = 0 \tag{128}$$

within the subspace of lattice fields spanned by the v^i_n. If the v^i_n span the whole space of ϕ^i_n, the relation is immediately reduced to the naive symmetry.

If the v_n^i do not span the whole space of the fields they must be linearly dependent. Then some linear combinations of the v_n^i vanish and from the definition (127) it is clear that the highest part of the action $s^{(R)}$ does not depend on some particular combinations of fields. On this subspace there is no constraint like (128).

So after all it is only possible to get a truncation of the action, if (128) is fulfilled on that subspace of ϕ's, on which the highest term of the action, $s^{(R)}$, depends. Keeping translational invariance, it is impossible to have $s^{(R)}$ independent of fields at particular lattice points n, but $s^{(R)}$ may be independent of a whole field component (ϕ^i) of the multiplet. Such a case appears in the case of constant fields [see Eq. (120)] when a_1 is set to zero. Then the highest term of the action, χ^4, depends only on χ, and $M\alpha^{-1} + (M\alpha^{-1})^T$ has no matrix entries for this field component. In this way a polynomial solution can be achieved.

We stress that so far we have only discussed one necessary condition for a truncation of the action down to a polynomial. The remaining Eqs. (122)–(125) need to be solved as well.

The continuum limit of the resulting actions needs in general a careful investigation since additional interaction terms must be introduced to solve the relation. These terms have no corresponding continuum counterparts and should vanish in the continuum limit. It is easy to ensure this in a naive way, where one just introduces the appropriate power of the lattice spacing, a, in front of the terms to let them vanish when a goes to zero. Notice, however, that the additional interaction terms introduce new vertices in a perturbative expansion. Because of divergences (for $a \to 0$) the corresponding diagrams are not necessarily vanishing even though the vertices themselves vanish. So it seems nontrivial to find the correct form of the action in the continuum limit.

The situation gets even more difficult if the truncation constraint is not fulfilled. Then one has to accept a nonpolynomial action. Beside the feasibility of such actions for numerical simulations, one has to analyze fundamental aspects of them like renormalizability. Nevertheless, these solutions obey the deformed lattice symmetry that approaches the continuum one in the continuum limit. This limits the possible actions, e.g. by relating the different couplings and might even determine the entire form of the action.

How the discussion specializes to supersymmetry will be discussed in the next section.

C. Interacting SUSY

To illustrate the above rather formal general statements, we specify them to SUSY theories. For the sake of argument we consider now a higher order action of SUSYQM; other SUSY theories will lead to a similar situation.

We mimic the interaction term of the continuum theory, $\bar{\psi}\psi\chi + F\chi^2$ [see Eq. (79) with $W = \chi^2/2$], by the following lattice action:

$$s^{(3)} = K_{n_1,n_2,n_3}\bar{\psi}_{n_1}\psi_{n_2}\chi_{n_3} + K'_{n_1,n_2,n_3}F_{n_1}\chi_{n_2}\chi_{n_3} \tag{129}$$

with parameters K and K' as in (121).

The (super)vector v following from this action reads

$$v_n^i = \begin{pmatrix} K_{n_1,n_2,n}\bar{\psi}_{n_1}\psi_{n_2} + 2K'_{n_1,n_2,n}F_{n_1}\chi_{n_2} \\ K'_{n,n_1,n_2}F_{n_1}\chi_{n_2} \\ -K_{n_1,n,n_2}\bar{\psi}_{n_1}\chi_{n_2} \\ K_{n,n_1,n_2}\psi_{n_1}\chi_{n_2} \end{pmatrix}. \tag{130}$$

It is not hard to see that v takes all values in the considered space of lattice fields upon varying the fields $(\chi_n, F_n, \bar{\psi}_n, \psi_n)$ at all lattice sites. The rationale for this is, as explained above, the "completeness" of the highest term in the action $s^{(3)}$, Eq. (129), if it contains all fields of the multiplet. As a consequence, (128) has to be fulfilled in general, but this reduces the symmetry relation (18) to the naive symmetry. One can easily convince oneself that the same argument applies to all continuum-inspired actions of the form $(R-1)\bar{\psi}\psi\chi^{R-2} + F\chi^{R-1}$.

There are two options to consider at this point. The first one is to accept nonpolynomial actions like (119), which induces many new complications.

The second and rather intricate possibility is to fulfill (128) only on a subspace of the fields. For SUSY this can be achieved, e.g., by building the highest order term (in this case $s^{(3)}$) out of purely bosonic or purely fermionic fields. As $M\alpha^{-1}$ has only entries mixing bosons and fermions, (127) is fulfilled easily in this way. We repeat the constant field result, where setting $a_1 = 0$ and $a_0 \neq 0$ in Eq. (120) the highest term is of the form χ^4. In the present case with "smearing" in K, such a term could be represented as $K_{n_1,n_2,n_3,n_4}\chi_{n_1}\chi_{n_2}\chi_{n_3}\chi_{n_4}$.

Of course, such incomplete parts of the actions on their own are not giving the desired continuum limit since in the continuum SUSY actions all fields are present at any given order (i.e. in every $s^{(r)}$). The only meaning such terms could have is to be lattice artifacts needed to solve the symmetry relation (18) at finite lattice spacing and to vanish in the continuum limit, such that the continuum action contains only terms of lower order in the fields.

Again, in this approach we so far have discussed only the highest relation. To solve the remaining relations is a nontrivial task to be done in further investigations.

With the help of such additional terms it might be possible to fulfill the symmetry relation which then guarantees the (super)symmetry in the continuum limit.

VII. SUMMARY

In this work we have systematically approached lattice supersymmetry and have extended the Ginsparg-Wilson relation to general linear symmetries and interacting theories.

For a general blocking procedure we have derived a lattice relation, Eq. (18), from the continuum symmetry of the theory at hand. This relation can be viewed as the ''remnant'' of the continuum symmetry on the lattice and has to be satisfied by the lattice action. The relation also includes potential anomalies and reduces to the well-known Ginsparg-Wilson relation for chiral theories.

As this relation is derived for all possible blockings it does in general not comprise a lattice symmetry, but rather describes the breaking of the continuum symmetry by the blocking procedure. It can be interpreted as a deformed symmetry only for those blockings for which the symmetry operator in (21) is local, as defined in (24), and tends toward the continuum symmetry operator in the continuum limit. These important properties are discussed in Sec. II C. These requirements exclude for instance the Wilson fermion action as a solution for a chiral lattice theory.

Interestingly enough, the averaging function f defines the lattice formulation, but does not appear in the relation of the lattice action. Instead, f is involved in an additional constraint encountered in the derivation of the relation. This constraint is of particular importance when the symmetry transformations include derivatives, which is one of the characteristic features of supersymmetry (as opposed to e.g. the algebraic γ_5 transformations in the chiral case). In order to keep the resolution of the averaging function as demanded by the lattice cutoff, the derivative operator in the lattice transformations must be the SLAC derivative or approach the SLAC derivative in the continuum limit. Such a nonlocal operator is of course problematic in view of the locality of lattice action and symmetry.

The other ingredient of the blocking transformation, the blocking kernel α, appears explicitly in the symmetry relation of the lattice action. An α that respects the symmetry leads to a vanishing rhs of this relation and thus to a naive symmetry. For chiral theories this is forbidden because of vector symmetry. In case of supersymmetry a symmetric α can in principle be chosen, but the resulting deformed symmetry would then contain the SLAC derivative and hence be nonlocal. The corresponding invariant lattice actions inherit this problem. Thus we are led to nonsymmetric α's and a nonvanishing rhs of our relation to have a chance to obtain local lattice actions and local symmetry transformations for supersymmetry.

In the concrete example of quadratic SUSYQM one can find how α improves the locality of the action: An ultralocal α^{-1} introduces propagatorlike denominators in the lattice operators, which however are not enough to compensate for the nonlocality of the emerging SLAC operator. Considering α^{-1} as a momentum-dependent quantity allows (ultra-) local lattice actions as a solution of the symmetry relation. Beside the lattice action also the deformed lattice symmetry must be local. To get a connection to the blocking procedure, only a relaxed version of locality can be guaranteed (cf. Sec. V F). Hence this rather simple example already reveals a lot about the interplay of the blocking ingredients, the locality of actions, and symmetries on the lattice.

One of the main differences between SUSY and chiral theories is that the symmetry of the former acts on nonquadratic terms. Correspondingly, the relation for lattice SUSY extends beyond second order and couples different powers of fields. We have discussed that this generically results in nonpolynomial actions. This finding is not completely surprising as our construction is halfway toward the full quantum effective action: We integrate out the original fields and represent the remaining path integral in terms of the blocked lattice fields. The nonpolynomial actions are derived from the continuum theory in a prescribed way. One might for instance speculate that certain cancellations between bosons and fermions are still present and help to renormalize the system. We have given a necessary condition for a truncation in the action. Indeed, in the very special example of constant fields in SUSYQM this criterion could be satisfied and the action is local.

We conclude with a short outlook. To date, the main problem for performing SUSY lattice simulations is the implementation of the symmetry. We hope to use the blocking formalism to obtain local interacting theories, which are invariant under a local deformed symmetry. The corresponding solutions to the symmetry relation should either be obtained through reasonable approximations or by a resummation. This is work in progress.

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APPENDIX A: SOLUTION FOR A QUADRATIC ACTION

The starting point to get the fixed point operator in its usual form is the result of the Gaussian integration,

$$\begin{aligned} K &= \alpha - \alpha f (f^T \alpha f + \tilde{K})^{-1} f^T \alpha \\ &= \alpha [1 - f (f^T \alpha f + \tilde{K})^{-1} f^T \alpha]. \end{aligned} \tag{A1}$$

A direct inversion of $f^T \alpha f + \tilde{K}$ is difficult since $f^T \alpha f$ is, unlike $\tilde{K}$, nondiagonal in frequency space. This property comes from the f that maps a larger space onto a smaller one. So $f^T \alpha f$ acts on a larger space than the diagonal α and is in general nondiagonal. Therefore, one first gets a closed expression of the inverse of K. Assuming $\tilde{K}$ and α to be invertible an expansion in terms of a Neumann series yields

$$
\begin{aligned}
K^{-1} &= \sum_{n=0}^{\infty}[f(f^T\alpha f+\tilde{K})^{-1}f^T\alpha]^n\alpha^{-1}\\
&= \alpha^{-1}+f\sum_{n=0}^{\infty}[(f^T\alpha f+\tilde{K})^{-1}(f^T\alpha f+\tilde{K}-\tilde{K})]^n\\
&\quad\times(f^T\alpha f+\tilde{K})^{-1}f^T\\
&= \alpha^{-1}+f\sum_{n=0}^{\infty}[1-(f^T\alpha f+\tilde{K})^{-1}\tilde{K}]^n\\
&\quad\times(f^T\alpha f+\tilde{K})^{-1}f^T. \qquad \text{(A2)}
\end{aligned}
$$

After a resummation the result becomes

$$
\begin{aligned}
K^{-1} &= \alpha^{-1}+f[(f^T\alpha f+\tilde{K})^{-1}\tilde{K}]^{-1}(f^T\alpha f+\tilde{K})^{-1}f^T\\
&= \alpha^{-1}+f\tilde{K}^{-1}f^T. \qquad \text{(A3)}
\end{aligned}
$$

An inversion of this expression is the known result mentioned in Eq. (51).

Another way to derive it is done via the introduction of auxiliary fields σ_n^i,

$$
\begin{aligned}
e^{-S[\phi]} &= \int d\varphi e^{-(1/2)(\phi-f\varphi)\alpha(\phi-f\varphi)-(1/2)\varphi\tilde{K}\varphi}\\
&= \mathcal{N}'\int d\varphi\int d\sigma e^{-(1/2)\varphi\tilde{K}\varphi+i\sigma(\phi-f\varphi)-(1/2)\sigma\alpha^{-1}\sigma}.
\end{aligned}
$$

After a Gaussian integration (first of the φ then of the σ) one gets again the desired expression for K

$$
\begin{aligned}
e^{-S[\phi]} &= \mathcal{N}''e^{-(1/2)\phi(f\tilde{K}^{-1}f^T+\alpha^{-1})^{-1}\phi}\\
&= \mathcal{N}''e^{-(1/2)\phi_n^iK_{nm}^{ij}\phi_m}. \qquad \text{(A4)}
\end{aligned}
$$

With the same conventions as in Appendix B one can easily derive the Fourier representation of K.

APPENDIX B: FOURIER ANALYSIS OF THE ADDITIONAL CONSTRAINT

For simplicity we work in a one-dimensional finite volume of size L. A general continuum field has the Fourier series representation

$$\varphi(x)=\sum_{q=-\infty}^{\infty}\varphi(p_q)e^{ip_qx}, \qquad \text{(B1)}$$

$$\varphi(p_q)=\frac{1}{L}\int_0^L dx\varphi(x)e^{-ip_qx}, \qquad \text{(B2)}$$

with dimensionless wave numbers q ($\in\mathbb{Z}$) and $p_q=\frac{2\pi q}{L}$. The same representation can be applied to the averaging function $f(an-x)$.

We discretize $L=Na$ with an odd number N of lattice points and lattice spacing a. Functions on the lattice can be parametrized by N independent waves which we choose to be in the first Brillouin zone,

$$\phi(an)=\sum_{k=-(N-1)/2}^{(N-1)/2}\phi(p_k)e^{ip_kan}, \qquad \text{(B3)}$$

$$\phi(p_k)=\frac{1}{N}\sum_{n=0}^{(N-1)}\phi(an)e^{-ip_kan}. \qquad \text{(B4)}$$

From this relation it is clear that $\phi(p_k)$ is periodic in p_k, $\phi(p_k)=\phi(p_k+l2\pi/a)=\phi(p_{k+lN})\ \forall\ l\in\mathbb{Z}$. The same transformation is used for the $\phi_f(an)$. [For $\phi_f(p_k)$ k runs from $-(N-1)/2$ to $(N-1)/2$.]

In Fourier space the convolution in the averaged field of Eq. (2) becomes a product,

$$
\begin{aligned}
\phi_f(p_k) &= \frac{L}{N}\sum_{n=-\infty}^{\infty}\sum_{m=0}^{(N-1)}e^{i(p_q-p_k)an}f(p_q)\varphi(p_q)\\
&= L\sum_{l=-\infty}^{\infty}f\left(p_k+l\frac{2\pi}{a}\right)\varphi\left(p_k+l\frac{2\pi}{a}\right). \qquad \text{(B5)}
\end{aligned}
$$

This shows how the averaging projects the Fourier components of φ onto the first Brillouin zone. In addition one easily observes that the Fourier components of ϕ_f and the lattice fields are determined by f, which means that f introduces a cutoff for the lattice momentum if $f(p_q)$ vanishes for all p_q greater than the cutoff.

Because of the constraint (59), the sum on the rhs actually has at most one term.

The additional constraint (55) reads after partial integration

$$\sum_m\nabla_{nm}f(am-x)+\partial_xf(an-x)=0\quad\forall\ n,x. \qquad \text{(B6)}$$

Because of the circulant form of ∇ [cf. Eq. (56)] we define its Fourier transform as

$$\nabla_{nm}=\frac{1}{N}\sum_{k=-(N-1)/2}^{(N-1)/2}\nabla(p_k)e^{ip_ka(n-m)}, \qquad \text{(B7)}$$

with

$$\nabla(p_k)\delta_{kl}=\frac{1}{N}\sum_{m=0,n=0}^{(N-1)}\nabla_{nm}e^{ip_kan}e^{ip_lam}, \qquad \text{(B8)}$$

where k and l are integer numbers between $-(N-1)/2$ and $(N-1)/2$. For greater values of k the result of the Fourier transformation again fulfills $\nabla(p_{k+lN})=\nabla(p_k)$ and δ_{kl} has the same periodicity (which means it is 1 only for $k=l \bmod N$ and otherwise zero).

With these Fourier transforms the constraint becomes

$$\sum_{q=-\infty}^{\infty}f(p_q)[\nabla(p_q)-ip_q]e^{ip_q(an-x)}=0, \qquad \text{(B9)}$$

which for every individual component p_q gives the constraint (57).

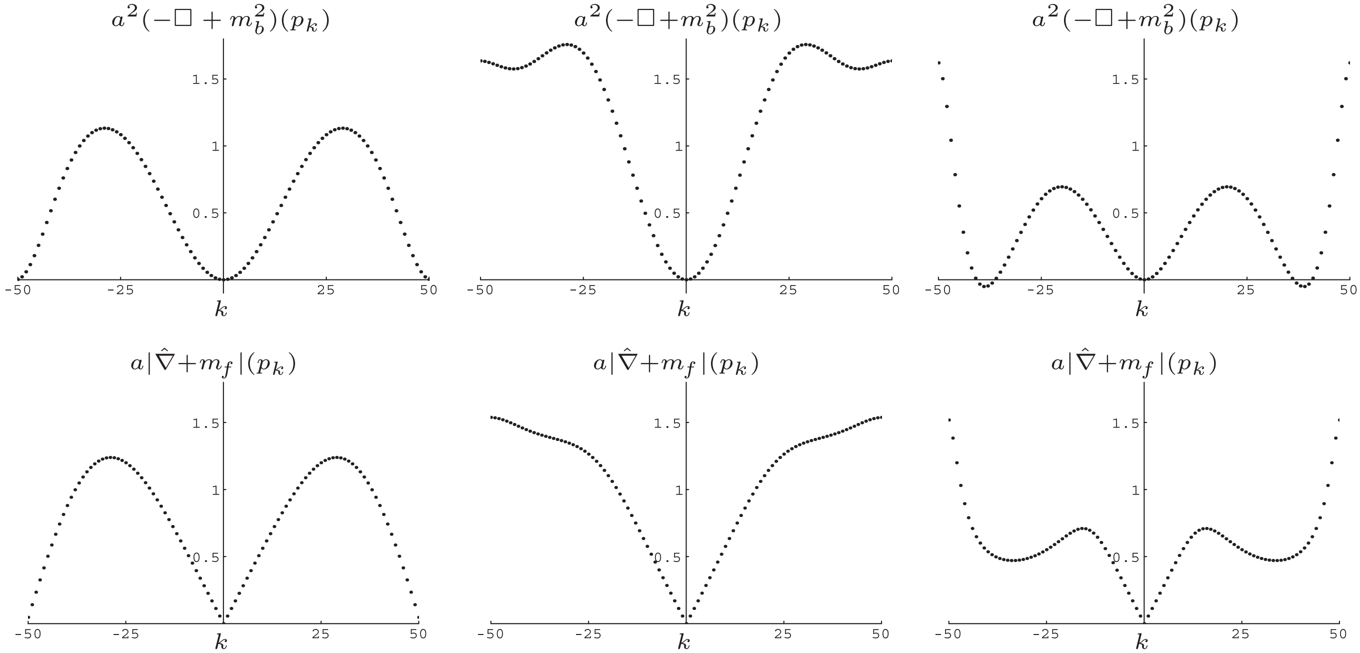

FIG. 3. Dispersion relations for the bosonic and fermionic bilinears with ultralocal ∇ and Wilson mass in m_b, cf. Eqs. (C2) and (C3), on an $N = 101$ lattice. In the top row we plot the bosonic $-\Box + m_b^2$ and in the bottom row the fermionic $|\hat{\nabla} + m_f|$, according to the solutions (95)–(97), as a function of the momentum p in the first Brillouin zone. On the left: $r = 0$ with doublers visible at the edges. In the middle: the doublers are removed by $r = 1$ and $(a_0, a_1, a_2) = (0.1, 0.1, 0.3)$. On the right: additional zeros in the bosonic sector occur for the choice $r = 1$ and $(a_0, a_1, a_2) = (0.1, 0.1, 2.0)$.

APPENDIX C: DOUBLER PROBLEM

In this Appendix we discuss solutions ∇ of the additional constraint which are ultralocal. These include the (slightly modified) symmetric difference, (60) and (62), and extensions thereof spreading over neighbors at larger distances, e.g. (64). They come with blocking functions f extended over the whole volume, (63), or subvolumes, depending on how many Fourier components f can have. In Sec. IV A we have argued that only the f with the highest resolution $O(a)$ leads to a reasonable, namely, nonredundant lattice theory. This amounts to ∇ being the SLAC operator.

Although the use of these ultralocal ∇'s is questionable, we nevertheless want to point out that doublers in them can be removed. For supersymmetric theories it is natural that also bosons have a doubling problem [34], so we investigate the kinetic operator in the bosonic sector, too.

Consider the solutions (95)–(97). The operator ∇ in them has doublers at the edge of the Brillouin zone, unless ∇ is the SLAC operator, and $\hat{\nabla}$ inherits them via the solution (95).

To resolve the doubler problem we make use of the fact that the matrix m_b in our ansatz (89) and in (95)–(97) is only restricted to be symmetric and circulant. In the spirit of the Wilson term, we will now replace m_b by

$$(m_b)_{nm} = m\delta_{nm} + \frac{r}{2a}(2\delta_{nm} - \delta_{n+1,m} - \delta_{n-1,m}) \quad \text{(C1)}$$

with r a dimensionless parameter. Such a momentum-dependent correction will then also occur in m_f through our solution[3] and all—bosonic and fermionic—doublers are removed as long as the parameter a_2 remains small.

The dispersion relations of m_b and ∇ are

$$m_b(p_k) = m + \frac{r}{a}(1 - \cos(ap_k)), \quad \text{(C2)}$$

$$\nabla(p_k) = \frac{i}{a}\sum_{l=1}^{(N-1)/2} c_l \sin(lap_k). \quad \text{(C3)}$$

We consider massless fields $m = 0$. Then because of $m_b(0) = 0$ and $\nabla(0) = 0$ it is clear that the bilinears

$$\tfrac{1}{2}\chi(-\Box + m_b^2)\chi, \qquad \bar{\psi}(\hat{\nabla} + m_f)\psi, \quad \text{(C4)}$$

according to (95)–(97) have zeros at vanishing p as they should. At the edge of the Brillouin zone, $p = \pi/a$, they would vanish as well because of $\nabla(\pi/a) = 0$, if there were no r corrections. The finite value $m_b(\pi/a) = 2r/a$, however, removes the doublers since

$$(-\Box + m_b^2)\left(p = \frac{\pi}{a}\right) = \left(\frac{2r}{a}\right)^2 \frac{1}{1 + a_0}, \quad \text{(C5)}$$

[3] In our setting, where $m_f(m_b)$, it is the simplest to change m_b. One could also have solved $m_b(m_f)$ and then change m_f.

$$(\hat{\nabla} + m_f)\left(p = \frac{\pi}{a}\right) = \frac{2r}{a}\frac{1}{1 + 2ra_1/a + a_0}. \tag{C6}$$

These values are nonzero and scale depending on the behavior of the a_i.

What remains to be shown is that there are no other zeros at intermediate values of p. For that we specialize to the solution of ∇ ranging to next-to-nearest neighbors with $c_{1,2}$ from Eqs. (65) and (66), and plot in Fig. 3 the operators $-\Box + m_b^2$ and $\hat{\nabla} + m_f$ in momentum space. As $-\Box + m_b^2$ and m_f are real symmetric (also after including r), their Fourier transforms are real, whereas that of the real antisymmetric $\hat{\nabla}$ is purely imaginary. In order to seek for zeros, we therefore plot $(-\Box + m_b^2)(p)$ and $|(\hat{\nabla} + m_f) \times (p)| = \sqrt{\hat{\nabla}^2(p) + m_f^2(p)}$, both as functions of p. As one can see from that figure, the doublers are removed and no additional zeros appear, unless a_2 is too big.

APPENDIX D: ALTERNATIVE DERIVATION OF THE SYMMETRY RELATION

Let us discuss how the symmetry relation emerges in the setting of Sec. IV B with currents. Here we take the point of view that the Wilsonian action S is defined to give the same generating functional $Z_f[J]$ in (70),

$$Z_f[J] = \frac{1}{\mathcal{N} e^{(1/2)J\alpha^{-1}J}} \int d\phi e^{-S[\phi]+J\phi}, \tag{D1}$$

for all currents J as the classical action in (68),

$$Z_f[J] = \frac{1}{\mathcal{N}} \int d\varphi e^{-S_{\mathrm{cl}}[\varphi]+J\phi_f[\varphi]}. \tag{D2}$$

This yields the same expectation values for all observables composed out of the blocked fields ϕ_f.

Now performing a symmetry variation $\varphi \to \varphi + \epsilon\tilde{M}\varphi$ on the classical fields φ and changing the integration variable in (D2) accordingly, one gets to $O(\epsilon)$:

$$\epsilon \mathrm{STr}\tilde{M} Z_f[J] = \frac{1}{\mathcal{N}} \int d\varphi e^{-S_{\mathrm{cl}}[\varphi]+J\phi_f[\varphi]} \epsilon J\phi_f[\tilde{M}\varphi]. \tag{D3}$$

The second term can be rewritten as $\epsilon JM\phi_f[\varphi]$ using the additional constraint. Then it is just the change of $Z_f[J]$ under $J \to J + \epsilon JM$,

$$\epsilon \mathrm{STr}\tilde{M} Z_f[J] = Z_f[J + \epsilon JM] - Z_f[J]. \tag{D4}$$

This can be computed in the representation (D1), where again we change the integration variable $\phi \to \phi + \epsilon M\phi$. This yields a STr$M$ part, the variation of S, and a bilinear term in J from the variation of the prefactor:

$$0 = \epsilon\{\mathrm{STr}\tilde{M} - \mathrm{STr}M + \delta S - JM\alpha^{-1}J\} Z_f[J]. \tag{D5}$$

The currents in the last term can be rewritten as ϕ derivatives of $\exp(J\phi)$ in $Z_f[J]$ and, by partial integration, of $\exp(-S[\phi])$:

$$0 = \epsilon \int d\phi e^{-S[\phi]+J\phi}\Big\{\mathrm{STr}\tilde{M} - \mathrm{STr}M + \delta S - e^{S[\phi]}\frac{\partial}{\partial\phi} M\alpha^{-1}\frac{\partial}{\partial\phi} e^{-S[\phi]}\Big\} Z_f[J]. \tag{D6}$$

Demanding this equality for all currents J, the curly bracket in the integrand has to vanish giving (17) and thus our symmetry relation (18).

Notice that we have used the same ingredients as in the original derivation of the relation, including the additional constraint and the inverse blocking kernel α^{-1}, but we have not relied on the definition (71) of the Wilsonian action based on α itself.

APPENDIX E: SINGULAR α^{-1}

The results of Sec. IV B extend to α's with some diverging matrix elements α_{mn}. The corresponding eigenmodes $\alpha\phi_{\mathrm{sing}} = \lambda_{\mathrm{sing}}\phi_{\mathrm{sing}}$ with $\lambda_{\mathrm{sing}} \to \infty$ are fixed: $\phi_{\mathrm{sing}} = \phi_{\mathrm{sing,f}}[\varphi]$. The remaining modes in ϕ still undergo the smearing procedure.

The other extremal case is $P_{\mathrm{sing}}\alpha^{-1}P_{\mathrm{sing}} \to \infty$, where P_{sing} is the projection operator on the singular part of α^{-1}. Still (69) is valid but the definitions of the Wilsonian action (71) and the normalization (72) develop singularities. If we want to keep these definition we could simply change the generating functional $Z_f[J]$ to $Z_f[(1 - P_{\mathrm{sing}})J]$ which generates correlation functions $\langle\phi_{f,\mathrm{reg}} \cdots \phi_{f,\mathrm{reg}}\rangle$. Here $\phi_{f,\mathrm{reg}} = P_{\mathrm{reg}}\phi_f$ with $P_{\mathrm{reg}} = (1 - P_{\mathrm{sing}})$. Then the above derivations are unaltered with $\alpha \to \alpha_{\mathrm{reg}} = P_{\mathrm{reg}}\alpha P_{\mathrm{reg}}$ and $\alpha_{\mathrm{reg}}^{-1} = P_{\mathrm{reg}}\alpha^{-1}P_{\mathrm{reg}}$. With this procedure we have removed the ϕ_{sing} modes from our theory. In particular the definition of the Wilsonian action (71) with α_{reg} leads to $S[\phi] = S[\phi_{\mathrm{reg}}]$.

APPENDIX F: GENERAL BLOCKING MATRICES α_G

The restrictions of the blocking matrix originate from the Hermiticity of the lattice action. In addition, the most general blocking matrix α_G should connect fermionic and bosonic fields only among each other. Therefore, the form of its inverse is restricted to be a

$$(\alpha_G^{-1})^{ij}_{mn} = \begin{pmatrix} b_2 & b_1' & 0 & 0 \\ b_1' & b_0 & 0 & 0 \\ 0 & 0 & 0 & b_1 + b_1'' \\ 0 & 0 & -b_1 + b_1'' & 0 \end{pmatrix}_{mn}, \tag{F1}$$

where b_1'' has to be antisymmetric, whereas all other matrices are symmetric. In order to get translation invariance all matrices must be circulant.

The most general symmetric matrix, fulfilling (20) with the symmetry operators M and $\bar{M}$ defined by (86) and (87), is

$$a(\alpha_S^{-1})^{ij}_{mn} = \begin{pmatrix} d_2 & d_1 & 0 & 0 \\ d_1 & \nabla^2 d_2 & 0 & 0 \\ 0 & 0 & 0 & -d_1 - \nabla d_2 \\ 0 & 0 & d_1 - \nabla d_2 & 0 \end{pmatrix}_{mn}, \tag{F2}$$

where d_2 and d_1 are symmetric circulant matrices. Such a symmetric part, α_S^{-1}, can always be added to α^{-1} without changing the relation (27). So the matrix elements of

$$\alpha^{-1} = \alpha_G^{-1} + \alpha_S^{-1} \tag{F3}$$

can be reduced without any loss of generality concerning the modified symmetry using the freedom of choosing d_2 and d_1. We choose them to be

$$d_1 = -b_1 \quad \text{and} \quad \nabla d_2 = -b_1'' \tag{F4}$$

and define

$$a_0 := b_0 - \nabla b_1'', \qquad a_1 := b_1 + b_1, \qquad a_2 := b_2 - \nabla^{-1} b_1'', \tag{F5}$$

where all a_i are now symmetric circulant matrices. The resulting matrix with reduced matrix elements compared to α_G^{-1} is just the one given in Eq. (90). Since α_S^{-1} is not important for the deformed symmetry of the action the result is not changed compared to α_G^{-1}. However, the deformed symmetry $M_{\rm def}$ depends on this part. So if one wants to investigate the locality of $M_{\rm def}$ the most general blocking matrix α_G^{-1} instead of the α^{-1} must be considered.

From these considerations it is clear that all entries of α^{-1} can in principle be zero after splitting off α_S^{-1}. Then the blocking matrix would consist only of a symmetric part α_S. This was done in [24] where $\alpha^{-1} = \alpha_S^{-1}$ with $d_2 = 0$ was used. The lattice symmetry is in that case reduced to the naive one.

APPENDIX G: DECAY OF SUSY LATTICE OPERATORS

In this Appendix we discuss the locality properties of the operators (95)–(97) with momentum-independent a_i, which due to the denominators resemble massive propagators. Nonetheless, we will demonstrate that the position space representations of these operators do not decay exponentially on the lattice.

Let us, as is common in the discussion of locality, consider the limit of large lattices. That is, we replace ∇ in Eq. (95) by p_k for $k = -(N-1)/2 \cdots (N-1)/2$ and take N to infinity. Then for the position space representation of the (circulant) fermionic operator $\hat{\nabla}$

$$\hat{\nabla}_{nm} = \frac{1}{2a} \sum_{l=-(N-1)/2}^{(N-1)/2} \hat{c}_l \delta_{n-m,-l}, \tag{G1}$$

cf. (56), the discrete Fourier transformation turns into a Fourier integral

$$\hat{c}_l = \lim_{N\to\infty} \frac{1}{N} \sum_{k=-(N-1)/2}^{(N-1)/2} \hat{\nabla}(p_k) e^{ip_k l} = \int_{-\pi/a}^{\pi/a} dp \hat{\nabla}(p) e^{ipla}, \tag{G2}$$

where p is a continuous momentum and

$$\hat{\nabla}(p) = i \frac{(1 + a_0 - a_2 m_b^2) p}{(1 + a_1 m_b + a_0)^2 + (a_2 m_b + a_1)^2 p^2} \tag{G3}$$

from the solution (95).

In the limit $a \to 0$ and with p-independent a_i, the Fourier transform gives an exponential decay because

$$\int_{-\infty}^{\infty} dp \frac{p}{\eta_1^2 + \eta_2^2 p^2} e^{ipla} \sim \exp\left(-\text{sign}(l) \frac{la|\eta_1|}{|\eta_2|}\right). \tag{G4}$$

At finite lattices the Fourier transform (G2) can be computed in the complexified p space. The Fourier integral consists of contributions from poles where the denominator of $\hat{\nabla}(p)$ vanishes and an additional part from the corresponding complex contours that encloses the poles.

Because of the regularity of $\hat{\nabla}(p)$ on the real axis, the poles appear away from it. For positive l we enclose the poles with positive imaginary parts by adding paths $p = \pm\pi/a + i\zeta$, $\zeta \in [0, \infty)$. The residue at each pole is some number coming from $\hat{\nabla}$, which we assume to be nonzero, times a term decaying exponentially in l from the Fourier factor in (G2). For the same reason there is no contribution from the asymptotic contour at large Imp parallel to the real interval $[-\pi/a, \pi/a]$.

The two new contours contribute

$$\pm \int_0^{\infty} d\zeta \hat{\nabla}(p) e^{ipla}|_{p=\pm\pi/a+i\zeta}, \tag{G5}$$

which turn into Laplace transforms

$$\int_0^{\infty} d\zeta [\hat{\nabla}(\pi/a + i\zeta) - \hat{\nabla}(-\pi/a + i\zeta)] e^{-\zeta la}. \tag{G6}$$

For large spatial distances l only the value of $\hat{\nabla}$ at $p = \pi/a$ remains, namely

$$[\hat{\nabla}(\pi/a) - \hat{\nabla}(-\pi/a)] \int_0^{1/la} d\zeta (1 - \zeta la) = \hat{\nabla}(\pi/a) \frac{1}{la}, \tag{G7}$$

where we used that $\hat{\nabla}$ is odd. Hence the additional contours give *algebraic corrections to* $\hat{\nabla}$ *in position space*, unless $\hat{\nabla}$ vanishes at the boundary of the Brillouin zone $p = \pi/a$. Of course the problem of algebraic tails is absent, if ∇ is ultralocal. Here, however, we have considered $\hat{\nabla}$ as a function of the SLAC operator ∇ according to the solution (95). The way out, which is followed in the body of the paper, Sec. V E, is to consider general momentum-dependent blocking kernels.

We remark that the discussion is similar for negative l (where one encloses the poles with negative imaginary parts). Furthermore, the algebraic corrections do not appear in m_f and $-\Box + m_b^2$, Eqs. (96) and (97), because these are even functions in p.

APPENDIX H: LOCALITY CONDITION

All of the following one-dimensional considerations can be extended easily to higher dimensions. The considered lattice operators O are, because of translational invariance, circulant matrices,

$$O_{mn} = O_{n-m} = F(a(n-m)). \tag{H1}$$

The slightly modified condition for locality demands that F decays faster than any polynomial. That means

$$|x^r F(x)| < \infty \quad \forall\ r \in \mathbb{N}, \qquad x, y \in a\mathbb{N}. \tag{H2}$$

If the Fourier transform of $F(x)$, $f(p)$, and its derivatives have no singularities and fulfill periodic boundary conditions at the edge of the BZ the following estimation can be made:

$$\begin{aligned} |x^r F(x)| &= \left| \int_{\mathrm{BZ}} (\partial_p^r f(p)) e^{ipx} \right| \le \int_{\mathrm{BZ}} |\partial_p^r f(p)| \\ &\le C_r < \infty. \end{aligned} \tag{H3}$$

Consider now a nonlocal operator similar to the SLAC derivative. This nonlocal operator should have no singularities within the BZ for all of its derivatives. The boundary conditions are, however, not periodic. According to the discussion of the locality of K and M_{def} it should support the modified locality after a multiplication with a local operator. In view of the above argument the boundary conditions must hence be enforced by this local operator without spoiling the differentiability of f. Its representation in Fourier space, $I(p)$, must therefore be a function that vanishes together with all its derivatives at the edge of the BZ. In addition no singularities should appear within the BZ for any of its derivatives. One function that fulfills these requirements is

$$I(p) = \begin{cases} \exp(-\frac{\epsilon^2}{\epsilon^2 - p^2}) & |p| < \epsilon, \\ 0 & |p| \ge \epsilon, \end{cases} \quad \text{with} \quad \epsilon \le \frac{\pi}{a}. \tag{H4}$$

It is clear that $I(p)$ cannot be analytic since any analytic function that vanishes with all its derivatives at a specific point must be identical to zero. So the common definition of locality in terms of analyticity in momentum space cannot be satisfied.

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