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Realistic radiative fermion mass hierarchy in nonsupersymmetric $SO(10)$

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A nonsupersymmetric grand unified theory can exhibit a ''radiative fermion mass hierarchy'', in which the heavier quarks and leptons get mass at tree level and the lighter ones get mass from loop diagrams. Recently the first predictive model of this type was proposed. Here it is analyzed numerically and it is shown to give an excellent fit to the quark and lepton masses and mixings, including the CP violating phase $\delta_{\rm CKM}$. A relation between the neutrino angle θ_{13} and the atmospheric neutrino angle is obtained.

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I. INTRODUCTION

The masses of the known quarks and leptons exhibit a large hierarchy. This has suggested to many theorists [1] that the light fermion mass hierarchy could be ''radiative'', i.e. that the lightest fermions get mass from loop diagrams, while the heaviest get mass at tree level. In the early 1980's several papers showed that such an idea can be implemented naturally in the context of nonsupersymmetric grand unified theories (GUTs) [2]. In models of this type, the radiative masses come from loop diagrams containing virtual GUT-scale particles. That is why such models must be nonsupersymmetric: otherwise, the loops would be suppressed by $O(M^2_{\rm SUSY}/M^2_{\rm GUT})$ due to the nonrenormalization theorems of supersymmetry.

In a recent paper [3] a very simple nonsupersymmetric $SO(10)$ model with a radiative hierarchy was proposed. One thing that allows this model to be so simple is precisely that its hierarchy is radiative. The point is that terms have to exist in the Lagrangian corresponding to the larger elements of the quark and lepton mass matrices, but not to the smallest elements, since they arise automatically from loops. The simplification can be seen by comparing the model of [3] to the supersymmetric $SO(10)$ model on which it was based, which had a nonradiative hierarchy [4]. That earlier model had a somewhat larger particle content and more Yukawa terms.

Models with radiative hierarchies are also simpler in another way: in them it is not necessary to introduce *ad hoc* very small dimensionless parameters to account for the fermion mass hierarchies, since they are automatically accounted for by the loop factors $1/16\pi^2$. Despite radiative hierarchy models being able to have a simpler structure, one might think they would be less predictive, since loop diagrams tend to depend on many parameters. However, the model proposed in [3] shows that this need not be the case. In that paper it was shown that the model gives a qualitatively realistic pattern of quark and lepton masses and mixings with only 9 parameters.

While the nonsupersymmetric model proposed in [3] is economical and qualitatively realistic, the analysis in that paper was not sufficient to establish that it is realistic quantitatively. In particular, several issues were not addressed. First, it was not specified what the sequence and scales of breaking were of $SO(10)$ down to the standard model group $G_{\rm SM}$ (which must, of course be consistent with proton decay bounds and unification of gauge couplings). Unless that is done, the renormalization-group running of the quark and lepton masses needed for a global fit of parameters cannot be performed. Second, the forms of the mass matrices given [3] were derived under the assumption that certain $SU(5)$-breaking effects could be ignored. However, as will be seen, this assumption is not necessarily consistent with the pattern of $SO(10)$ breaking that needs to be assumed in order to satisfy the constraints of gauge coupling unification and proton decay. $SU(5)$-breaking effects will turn out to modify significantly the forms of the quark and lepton mass matrices given in [3]. Third, the m_b/m_τ ratio is problematic in the version of the model discussed in [3]. In that model, because certain $SU(5)$-breaking effects were treated as negligible, the classic prediction $m^0_b \cong m^0_\tau$ was obtained. (A superscript ''0'' indicates throughout this paper quantities evaluated at the GUT scale.) This is well-known to give a fairly good fit in supersymmetric models for certain values of $\tan\beta$ [5]; but in nonsupersymmetric models it results in a prediction of m_b/m_τ at low energies that is typically too large by at least 30% [6]. Fourth, the ratio m_s/m_b is predicted in the version of the model given in to have the Georgi-Jarlskog value $\frac{1}{3} m_\mu/m_\tau$ at the GUT scale [7]. However, lattice calculations [8] have suggested that m_s is significantly smaller than previous estimates of it, and the best-fit value is now somewhat smaller than the Georgi-Jarlskog prediction.

In this paper, we address all these issues. The paper is organized as follows. In Sec. II, the $SO(10)$ model of [3] is reviewed, and it is explained how both the tree-level and radiative contributions to the mass matrices arise, and why the resulting forms give a good qualitative description of the pattern of quark and lepton masses and mixings. In Sec. III, a breaking of $SO(10)$ down to the standard model consistent with gauge coupling unification and proton decay bounds is specified. In Sec. IV, the effect of this pattern of symmetry breaking on the quark and lepton mass matrices is discussed and it is shown that forms somewhat different from those given in [3] result. In SEc. V, the results of a global numerical fit to the quark and lepton masses and mixings is given. An excellent fit is found to the quark and lepton masses and mixings, including the *CP* phase $\delta_{\rm CKM}$. A relation between the neutrino angle θ_{13} and the atmospheric neutrino angle is obtained.

II. THE MODEL

The model proposed in [3], whose predictions we analyze in detail in this paper, is a nonsupersymmetric one with unified group $SO(10)$. In it the tree-level mass matrices of quarks and charged leptons are generated by only three effective Yukawa operators

$$\begin{aligned} O_1 &= \mathbf{16}_3\mathbf{16}_3\mathbf{10}_H \qquad O_2 = \mathbf{16}_2\mathbf{16}_3\mathbf{10}_H\mathbf{45}_H/M_2 \\ O_3 &= (c_i\mathbf{16}_i\mathbf{16}_{iH})(\mathbf{16}_3\mathbf{16}'_H)/M_3, \qquad i = 1,2 \end{aligned} \tag{1}$$

where $\mathbf{16}_1$, $\mathbf{16}_2$, and $\mathbf{16}_3$ are three families of quarks and leptons, and the multiplets with subscript H are Higgs fields. In O_3, the factors in parentheses are contracted into $\mathbf{10}$'s of $SO(10)$. The loop-level elements in the mass matrices arise very simply from the tree-level elements, as will be seen later. The three operators given in Eq. [(1)] do the following things: O_1 gives the 33 elements of the mass matrices, i.e. the masses of the third family. O_2 and O_3 generate the masses of the second family and its mixing with the third family (i.e. V_{cb} and $\theta_{\rm atm}$), and also $\theta_{\rm sol}$. The masses of the first family and its remaining mixings come from loops.

The operators in Eq. (1) come from integrating out some "extra" vectorlike fermion multiplets, consisting of an $SO(10)$ vector ($\mathbf{10}$) and a spinor-antispinor pair ($\mathbf{16} + \overline{\mathbf{16}}$). Thus the complete fermionic content of the model comprises the following (left-handed) multiplets: $\mathbf{16}_{i=1,2,3} + (\mathbf{16} + \overline{\mathbf{16}} + \mathbf{10})$. The scales M_2 and M_3 in the denominators of the operators O_2 and O_3 in Eq. (1) are set by the masses of the fermions being integrated out; so that M_2 and M_3 are superlarge. However, since the vacuum expectation values (VEVs) of $\mathbf{45}_H$ and $\mathbf{16}_{iH}$ are also superlarge and of the same order, these operators are not highly suppressed.

The Higgs sector of the model contains the following multiplets: (a) A rank-4 tensor field, $\mathbf{210}_H$, the role of which is to break $SO(10)$ down to the Pati-Salam group $SU(4)_c \times SU(2)_L \times SU(2)_R$ at a scale M_G. (b) Three adjoint fields, $\mathbf{45}_H$, $\mathbf{45}'_H$, $\mathbf{45}''_H$. One of these, $\mathbf{45}_H$, obtains a VEV in the $B - L$ direction, thus participating in the breaking of the Pati-Salam group at a scale $M_{\rm PS}$. It also couples to the quarks and leptons and contributes to the superheavy masses of some of them. The other two adjoint Higgs fields play no essential role in symmetry breaking and do not couple to quarks and leptons. As will be seen in Sec. III, they are needed only for their effect on the running of gauge couplings. There may also be a symmetric tensor Higgs field $\mathbf{54}_H$, in order to introduce $SO(10)$-breaking and $SU(5)$-breaking into certain entries in the fermion mass matrices, as will be explained in the second paragraph of Sec. IV. It would have a VEV in the standard-model-singlet direction. (c) There are three spinor Higgs fields, $\mathbf{16}_{1H}$, $\mathbf{16}_{2H}$, and $\mathbf{16}'_H$. The first two of these obtain superlarge VEVs of order $M_{\rm PS}$ in the standard-model-singlet direction and break the Pati-Salam group down to the standard model group. They also couple to quarks and leptons and contribute to the superlarge masses of some of them. $\mathbf{16}'_H$ obtains only a weak-scale VEV in the color-singlet/weak-doublet component, participates in breaking the weak interaction group, and contributes to quark and lepton masses. (d) There are two vector Higgs multiplets, $\mathbf{10}_H$ and $\mathbf{10}'_H$. These get weak-scale VEVs, participate in breaking the weak interactions, and contribute to quark and lepton masses. More will be said about the Higgs sector later. The complete set of fields of the model are given in Table I.

The Dirac mass matrices of the up-type quarks, down-type quarks, charged leptons, and neutrinos (denoted by M_U, M_D, M_L, and M_N, respectively) arise from the following set of Yukawa terms in the Lagrangian:

$$\begin{aligned} \mathcal{L}_{\rm Yuk} = {} & M_{16}(\overline{\mathbf{16}}\mathbf{16}) + M_{10}(\mathbf{10}\mathbf{10}) + a(\overline{\mathbf{16}}\mathbf{16}_3)\mathbf{45}_H \\ & + \sum_{i=1,2} c_i(\mathbf{10}\mathbf{16}_i)\mathbf{16}_{iH} + h_{33}(\mathbf{16}_3\mathbf{16}_3)\mathbf{10}_H \\ & + h_2(\mathbf{16}\mathbf{16}_2)\mathbf{10}_H + h_3(\mathbf{10}\mathbf{16}_3)\mathbf{16}'_H \\ & + h(\mathbf{16}\mathbf{16})\mathbf{10}'_H. \end{aligned} \tag{2}$$

It is shown in [3] that this form of the Yukawa interactions is the most general allowed by a certain simple $U(1)_F$ flavor symmetry. The $U(1)_F$ charges of the fermion and scalar multilets of the model are given in Table I. (The numbers x, y, z, u, v in Table I can have virtually any values, except a set of measure zero that would allow more terms into the Yukawa Lagrangian than those shown in Eq. (2).)

The terms on the first line of Eq. (2) are the $O(M_{\rm PS})$ masses of the extra fermion multiplets; the terms on the second line contribute $O(M_{\rm PS})$ masses that mix those extra

TABLE I.

$SO(10)$	$\mathbf{16}_1$	$\mathbf{16}_2$	$\mathbf{16}_3$	$\mathbf{16}$	$\overline{\mathbf{16}}$	$\mathbf{10}$	$\mathbf{10}_H$	$\mathbf{10}'_H$	$\mathbf{16}'_H$	$\mathbf{16}_{1H}$	$\mathbf{16}_{2H}$	$\mathbf{45}_H$	$\mathbf{45}'_H$	$\mathbf{45}''_H$	$\mathbf{54}_H$	$\mathbf{210}_H$
$U(1)_F$	z	$2y-x$	y	x	$-x$	0	$-2y$	$-2x$	$-y$	$-z$	$x-2y$	$x-y$	u	v	0	0
$O(VEV)$	-	-	-	-	-	-	M_W	M_W	M_W	$M_{\rm PS}$	$M_{\rm PS}$	$M_{\rm PS}$	-	-	$M_{\rm PS}$	$M_{\rm PS}$

fermions with the three chiral families $\mathbf{16}_i$; the terms on the third line generate the weak-scale $SU(2)_L \times U(1)_Y$-breaking masses; and the last term is needed to give radiative masses to the first family.

We will use the notation that $\mathbf{p}(\mathbf{q})$ stands for a $\mathbf{p}$ multiplet of $SU(5)$ contained in a $\mathbf{q}$ multiplet of $SO(10)$. For example, the vacuum expectation values of the Higgs fields $\mathbf{16}_{iH}$ lie in the $\mathbf{1}(\mathbf{16})$ direction. The electroweak gauge symmetry $SU(2)_L \otimes U(1)_Y$ is spontaneously broken by the Higgs multiplets denoted $\mathbf{10}_H$, $\mathbf{10}'_H$, and $\mathbf{16}'_H$ in Eq. (1) and, more specifically, by the neutral components of the $Y/2 = -1/2$ doublets contained in $\bar{\mathbf{5}}(\mathbf{10}_H)$, $\bar{\mathbf{5}}(\mathbf{10}'_H)$, and $\bar{\mathbf{5}}(\mathbf{16}'_H)$, and the neutral components of the $Y/2 = +1/2$ doublets contained in $\mathbf{5}(\mathbf{10}_H)$ and $\mathbf{5}(\mathbf{10}'_H)$. Of course, in the low-energy effective theory, which is just the standard model, there is only one Higgs doublet, which is some linear combination of these doublets (and their Hermitian conjugates).

According to [3], the mass matrices that result from the terms in Eq. (2) have the form

$$
\begin{aligned}
M_U &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{\epsilon}{3} \\ 0 & -\frac{\epsilon}{3} & 1 \end{pmatrix} m_U, \\
M_D &= \begin{pmatrix} 0 & 0 & \delta_{g1} \\ 0 & \delta_H & \frac{\epsilon}{3} + \delta_{g2} \\ C_1 & C_2 - \frac{\epsilon}{3} & 1 \end{pmatrix} m_D, \\
M_N &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\epsilon \\ 0 & \epsilon & 1 \end{pmatrix} m_U, \\
M_L &= \begin{pmatrix} 0 & 0 & C_1 \\ 0 & \delta_H & C_2 - \epsilon \\ \delta_{g1} & \epsilon + \delta_{g2} & 1 \end{pmatrix} m_D,
\end{aligned} \tag{3}
$$

where $m_U \equiv h_{33}\langle \mathbf{5}(\mathbf{10}_H)\rangle$ and $m_D \equiv h_{33}\langle \bar{\mathbf{5}}(\mathbf{10}_H)\rangle$. (It will be seen in Sec. IV that GUT-symmetry-breaking effects modify these forms somewhat.) The convention here is that the mass matrices are multiplied from the left by the left-handed fermions and from the right by the right-handed fermions.

The 33 elements of the mass matrices in Eq. (3) come simply from the term $h_{33}(\mathbf{16}_3\mathbf{16}_3)\mathbf{10}_H$, as is usually the case in $SO(10)$ models [9]. (This is just the operator O_1 in Eq. (1).)

The contributions to the 23 and 32 elements denoted by ϵ come from integrating out the family-antifamily pair $\overline{\mathbf{16}} + \mathbf{16}$. The antifamily $\overline{\mathbf{16}}$ appears in two mass terms from Eq. (2), which can be combined as follows: $\overline{\mathbf{16}}(M_{16}\mathbf{16} + a\langle \mathbf{45}_H\rangle \mathbf{16}_3)$. These terms have the effect of mixing the $\mathbf{16}$ with the $\mathbf{16}_3$. One linear combination of $\mathbf{16}$ and $\mathbf{16}_3$ obtains a superlarge mass $\sqrt{M_{16}^2 + (a\langle \mathbf{45}_H\rangle)^2} \simeq M_{16}$, while the orthogonal combination (denoted by the index $3'$) remains light. (From now on, primed indices will be used to denote the light families that remain after the superheavy fermions have been integrated out.) Thus, the $\mathbf{16}$ with no index has some of the third light family mixed in with it; and the amount of this mixing is proportional to the VEV $\langle \mathbf{45}_H\rangle$. As a result, the term $h_2(\mathbf{16}\mathbf{16}_2)\mathbf{10}_H$ from Eq. (2) leads to an effective operator of the form $(\mathbf{16}_{3'}\mathbf{16}_{2'})\mathbf{10}_H\mathbf{45}_H/M_2$, which is just the operator O_2 of Eq. (1). And it can easily be seen that $1/M_2 \cong h_2 a/\sqrt{M_{16}^2 + (a\langle \mathbf{45}_H\rangle)^2} \simeq 1/M_{16}$. Note that, since $\langle \mathbf{45}_H\rangle \sim M_{\rm PS}$, it must be that $M_{16} \sim M_{\rm PS}$, or else the operator O_2 would be suppressed. This operator produces the contributions denoted in Eq. (3) by ϵ. Since $\langle \mathbf{45}_H\rangle \propto B - L$, the ϵ contributions are $1/3$ times as large for the quarks as for the leptons.

The elements denoted by C_1 and C_2 arise in a similar fashion by integrating out the $SO(10)$-vector multiplet of quarks and leptons, $\mathbf{10}$. This multiplet contains a $\bar{\mathbf{5}} + \mathbf{5}$ of $SU(5)$. The $\mathbf{5}(\mathbf{10})$ appears in several mass terms from Eq. (2), which can be combined as $\mathbf{5}(\mathbf{10})[M_{10}\bar{\mathbf{5}}(\mathbf{10}) + \sum_{i=1,2} c_i\langle \mathbf{1}(\mathbf{16}_{iH})\rangle \bar{\mathbf{5}}(\mathbf{16}_i)]$. These terms have the effect of mixing the $\bar{\mathbf{5}}(\mathbf{10})$ with the $\bar{\mathbf{5}}(\mathbf{16}_1)$ and $\bar{\mathbf{5}}(\mathbf{16}_2)$. One linear combination of these $\bar{\mathbf{5}}$'s obtains an $O(M_{\rm PS})$ mass, while the two orthogonal linear combinations are in the light families and denoted $\bar{\mathbf{5}}_{1'}$ and $\bar{\mathbf{5}}_{2'}$. Consequently, the $\bar{\mathbf{5}}(\mathbf{10})$ has mixed in with it some of $\bar{\mathbf{5}}_{1'}$ and $\bar{\mathbf{5}}_{2'}$. That means that the term $h_3(\mathbf{10}\mathbf{16}_3)\mathbf{16}'_H$ in Eq. (2) leads to effective mass terms of the form $(C_1\bar{\mathbf{5}}_{1'} + C_2\bar{\mathbf{5}}_{2'})\mathbf{10}_{3'}m_D$. This is just the operator O_3 of Eq. (1), with $M_3 \cong h_3/\sqrt{M_{10}^2 + (\Sigma_i c_i\langle \mathbf{16}_{iH}\rangle)^2}$. This gives the terms denoted by the C_i in Eq. (3). These contributions appear only in M_L and M_D, because $\bar{\mathbf{5}}$'s of $SU(5)$ contain only charged leptons and down-type quarks. In both [3,4] the M_{10} was assumed to be an explicit (and therefore $SU(5)$-invariant) mass, and therefore the same C_i appear in both M_L and M_D. Note that, since $\langle \mathbf{16}_{iH}\rangle \sim M_{\rm PS}$, it must also be that $M_{10} \sim M_{\rm PS}$, as otherwise the operator O_3 and the elements C_i would be suppressed.

An aspect of the operators O_2 and O_3 that may at first be very puzzling, is that (as noted earlier) each operator contains in the numerator a VEV that is of the same order as the mass appearing in the denominator. This might appear inconsistent with an effective field theory approach; but actually there is nothing improper in this, and operators of this type very commonly appear when heavy fermions are integrated out to give effective tree-level Yukawa operators for the remaining light fermions. This happens, for example, in Froggatt-Nielson models [10] and in many grand unified models [11–16].

The point is that in models that seek to explain the pattern of quark and lepton masses by means of some symmetry (flavor symmetry or grand unified symmetry or a combination of the two) there is always an original "symmetry basis" of the fermions and a "mass basis". The heavy fermions (in grand unified theories they would be superheavy fermions) that are integrated out are linear combinations of certain fermions in the symmetry basis. Those linear combinations are expressable in terms of mixing angles (not CKM mixing angles, but angles describing the mixing of superheavy and light fermions). The tangents of those angles are typically given by dimensionless ratios of VEVs and masses, such as $a\langle \mathbf{45}_H\rangle/M_{16}$ and $c_i\langle \mathbf{16}_{iH}\rangle/M_{10}$ here. The effective operators obtained by integrating out the heavy fermions, will contain factors of sines of those angles, for example, in deriving O_2 there is a factor of $a\langle \mathbf{45}_H\rangle/\sqrt{M_{16}^2 + (a\langle \mathbf{45}_H\rangle)^2}$. When these sine factors are expanded out, they give infinite series of operators of higher and higher dimension (e.g. here containing higher and higher powers of $\mathbf{45}_H$) suppressed by higher powers of the heavy mass scale. These series simply converge to sine factors of order 1. In many model-building papers, these mixing angles are assumed to be somewhat smaller than 1. In that case, the sines can be replaced by tangents, and that is equivalent to replacing the infinite series by its first term, e.g. replacing $\sqrt{M_{16}^2 + (a\langle \mathbf{45}_H\rangle)^2}$ by M_{16}. In many papers, no explicit mention is made of the fact that such an approximation is being made (for example, it is not stated in [14–16]), but such an approximation is always being made when it is stated that integrating out some heavy fermions gives a $d = 5$ operator.

Thus, the expressions for the quark and lepton mass matrices given in Eq. (3) are approximate. The exact expressions involve factors, such as $1/\sqrt{1 + (a\langle \mathbf{45}_H\rangle/M_{16})^2}$ and $1/\sqrt{1 + (\sum_i c_i\langle \mathbf{16}_{iH}\rangle/M_{10})^2}$, which are just the cosines of angles describing the mixing between the extra fermions $\mathbf{16} + \overline{\mathbf{16}} + \mathbf{10}$ and the three chiral families $\mathbf{16}_i$. If these mixing angles are small, their cosines are very close to 1, and the mass matrices become insensitive to their values. This is an assumption that we make here (as in [3,14–16]), as it reduces the number of parameters. However, there is no *a priori* reason to assume that these angles are extremely small. (Indeed, if they vanished, so would ϵ.) If one of these angles were of order 0.25 radians, say, it would give 3% corrections to some of the elements of the mass matrices.

The elements denoted by δ_{gi} and δ_H in Eq. (3) are necessary to make the mass matrices M_L and M_D be of rank 3 rather than rank 2, and so generate masses and mixings for the first family. As will be seen, in order to fit the first family masses and mixings these δ are must be of order 10^{-2}, whereas the other parameters appearing inside in the mass matrices in Eq. (3) turn out to be of order 1 (or, in the case of ϵ, about 0.19). In [4], additional vectorlike quark and lepton fields besides those in Eq. (2) had to be introduced in order to generate these small δ's. In [3], however, it was noted that the terms in Eq. (2) are enough to generate the δ terms automatically by one-loop diagrams and also to explain why they are of order 10^{-2}.

The δ_{gi} terms are given by the one-gauge-boson-loop diagram shown in Fig. 1(a). The gauge boson in this diagram is in a $\mathbf{10}$ of $SU(5)$ (of course, it is in the adjoint $\mathbf{45}$ of $SO(10)$), so that it turns $\mathbf{10}$'s of $SU(5)$ into $\bar{\mathbf{5}}$'s and *vice versa*. That means that the small δ_{gi} elements that couple $\mathbf{10}_i$ to $\bar{\mathbf{5}}_3$ (namely $(M_D)_{i3}$ and $(M_L)_{3i}$) come from the large C_i elements that couple $\bar{\mathbf{5}}_i$ to $\mathbf{10}_3$ (namely $(M_L)_{i3}$ and $(M_D)_{3i}$) So $\delta_{gi} \propto C_i$. These diagrams were evaluated in [3] neglecting certain $SU(5)$-breaking effects, giving the result that the same δ_{gi} appear in M_L and M_D, as given in Eq. (3).

The diagram in Fig. 1(a) superficially looks divergent. However, the accidental symmetry $U(1)_F$ of the terms in Eq. (1) makes $(M_D)_{13}$ and $(M_L)_{31}$ vanish at tree level and guarantees that the loop is finite, as an exact calculation indeed shows. The finiteness of this diagram is more obvious if we write it in the form shown in Fig. 1(b). The calculation of these loops will be discussed in Sec. IV.

The 22 elements of M_L and M_D (denoted δ_H) arise from the one-Higgs-boson-loop diagram shown in Fig. 2. Whereas the one-gauge-boson-loop shown in Fig. 1 *must* exist if the tree-level masses in Eq. (3) exist, the diagram in Fig. 2 only exists if certain couplings not needed for the tree-level masses are present: namely, the last term in Eq. (2) ($h(\mathbf{16}\mathbf{16})\mathbf{10}'_H$) and a Higgs-mass term of the form $\mathbf{10}_H\mathbf{10}_H$. A diagram related by $SO(10)$ to the one in Fig. 2 gives a 22 element for the up-quark mass matrix M_U. However, if one supposes the contributions to $(M_U)_{22}$ and $(M_D)_{22}$ from these diagrams to be roughly comparable, then $(M_U)_{22}/(M_U)_{33}$ would be of order 10^{-4} and thus at most a few percent correction to m_c.

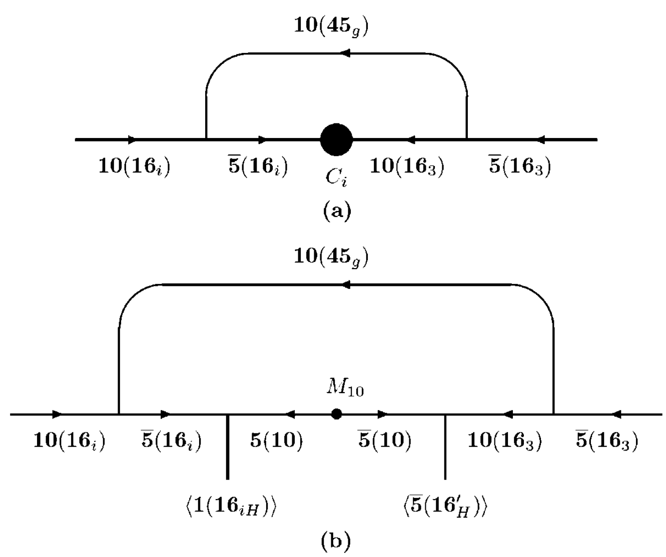

FIG. 1. The diagram in (a) shows how a tree-level mass for $\mathbf{10}_3\bar{\mathbf{5}}_i$ (shown as a blob in the center) leads to a one-loop mass for $\mathbf{10}_i\bar{\mathbf{5}}_3$: i.e. the δ_{gi} elements arise radiatively from the C_i elements. The $\mathbf{10}(\mathbf{45}_g)$ in the loop is a superheavy gauge boson. The diagram in (b) is more detailed and shows why the loop is finite.

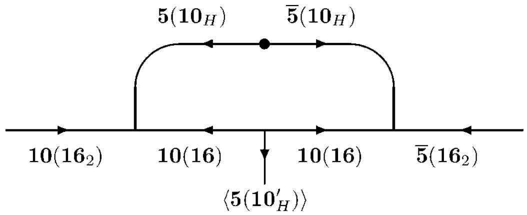

FIG. 2. A diagram showing how the 22 elements of the mass matrices can arise radiatively through Higgs-boson loops.

Before getting into a more detailed discussion of the model, it is useful to explain how the structure of the matrices given in Eq. (3) explains qualitatively many of the features of the observed pattern of masses and mixings of the quarks and leptons.

First, neglecting the δ parameters (which are of order 10^{-2} because they come from one-loop diagrams) and the parameter ϵ (which, though a tree-level effect, is somewhat smaller than 1), one has that all the mass matrices in Eq. (3) are of rank 1. In this approximation, $m_b^0 \cong m_\tau^0 \cong \sqrt{1+|C_1|^2+|C_2|^2}$, where the superscript "0" denotes quantities evaluated at the GUT scale. The relation $m_b^0 \cong m_\tau^0$ is known to fit fairly well in supersymmetric grand unified models with certain values of $\tan\beta$. It works less well in nonsupersymmetric grand unified models; however, this relation will be substantially modified when realistic symmetry breaking is taken into account in Sec. IV.

The large (i.e. $O(1)$) off-diagonal elements C_i produce large mixing angles in the left-handed lepton sector and the right-handed quark sector. This is because they result from mixings of $\bar{\mathbf{5}}$'s of $SU(5)$, which contain, of course, left-handed charged leptons and right-handed down-type quarks. Consequently, these elements produce large MNS neutrino mixing angles, but they do not produce large CKM mixing, since CKM mixing is of the left-handed not right-handed quarks. This is one of the basic ideas of so-called "lopsided" models [17].

Moreover, the present model is "doubly lopsided" in the sense that both C_1 and C_2 are large [18]. (In singly lopsided models C_2 is large but not C_1.) This doubly lopsided structure can give a very simple explanation of the "bi-large" pattern of neutrino mixing angles in the following way. The MNS matrix is given by $U_{\text{MNS}} = U_L U_\nu^\dagger$, where U_L and U_ν are the unitary transformations of the left-handed charged leptons and neutrinos needed to diagonalize M_L and M_ν, respectively. The transformation U_L is the product of three successive rotations. First, there is a rotation in the 1-2 plane by an angle $\theta_a = \tan^{-1}\frac{C_1}{C_2-\epsilon}$, which eliminates the 13 element of M_L (See Eq. (3) bringing the 23 element to $\sqrt{|C_1|^2+|C_2-\epsilon|^2}$. Second, there is a rotation in the 2-3 plane by an angle $\theta_s = \tan^{-1}\sqrt{|C_1|^2+|C_2-\epsilon|^2}$, which eliminates the 23 element of M_L, bringing the 33 element to $\sqrt{|C_1|^2+|C_2-\epsilon|^2+1}$. The product of these two rotations is

$$U_0 = \begin{pmatrix} c_s & s_s & 0 \\ -c_a s_s & c_a c_s & s_a \\ s_a s_s & -s_a c_s & c_a \end{pmatrix}, \qquad (4)$$

where $s_a \equiv \sin\theta_a$, $c_a \equiv \cos\theta_a$, $s_s \equiv \sin\theta_s$, and $c_s \equiv \cos\theta_s$. The result of these two rotations is to bring M_L to the form

$$M_L' = \begin{pmatrix} 0 & O(\delta_H) & 0 \\ O(\delta_{g1}) & O(\epsilon) & 0 \\ O(\delta_{g1}) & O(\epsilon) & \sqrt{|C_1|^2+|C_2-\epsilon|^2+1} \end{pmatrix}.$$

The third rotation (call it V) is in the 1-2 plane by an angle of $O(\delta_H/\epsilon)$, and is needed to eliminate the 12 element M_L'. (There are also small rotations of the right-handed leptons needed to eliminate the elements below the diagonal, but these have a negligible effect on the MNS matrix.) Altogether, then, $U_{\text{MNS}} = U_L U_\nu^\dagger = (VU_0)U_\nu^\dagger$. Since V and U_ν involve small rotations (the rotations in U_ν are small because the neutrino mass matrix is hierarchical, as we shall see), it follows that U_{MNS} is approximately given by the matrix U_0 given in Eq. (4), and therefore has the "bi-large" form, with the solar angle being approximately $\theta_s \sim 1$, the atmospheric angle being approximately $\theta_a \sim 1$, and the angle θ_{13} being approximately zero. (One thing that ought to be emphasized is that in the limit that ϵ and the δ's vanish, the mass matrix M_L is rank 1, and therefore one leptonic mixing angle is undefined. The small elements lift this degeneracy, however. In particular, it is the hierarchy $\delta_H \ll \epsilon$ that determines the angle in V to be small, and ensures a bi-large form of U_{MNS}.)

One of the great successes of lopsided and doubly lopsided models is that they elegantly explain the very widely differing magnitudes of the quantities related to the hierarchy between the second and third families. V_{cb}, m_s/m_b, and m_μ/m_τ are all of similar magnitude (0.04, 0.02, and 0.06, respectively), $U_{\mu 3}(\equiv \sin\theta_{\text{atm}})$ is much larger (0.7); and m_c/m_t is much smaller (0.003). This comes about from the structure of the mass matrices in lopsided models, which cause V_{cb}, m_s/m_b, and m_μ/m_τ to be all of order ϵ^1; $U_{\mu 3}$ to be of order ϵ^0; and m_c/m_t to be of order ϵ^2. This can be seen as follows. Looking at M_U, one sees that *both* its off-diagonal elements are $O(\epsilon)$, so that the mass of the second-family fermion is $O(\epsilon^2)$. More precisely, $m_c/m_t \cong \epsilon^2/9$. For M_L and M_D, on the other hand, only the off-diagonal elements on *one* side of the long diagonal are $O(\epsilon)$ (we neglect the $O(\delta)$ elements at this point), whereas the off-diagonal elements of the other side of the diagonal are dominated by the large lopsided elements C_i. Thus the mass of the second-family fermions in these cases are $O(\epsilon^1)$. More precisely, $m_\mu/m_\tau \simeq 3m_s/m_b \simeq \epsilon\frac{\sqrt{|C_1|^2+|C_2|^2}}{(|C_1|^2+|C_2|^2+1)}$. (Note that the Georgi-Jarlskog ratio emerges naturally.) The quark 2-3 mixing V_{cb} is controlled by the 23 elements of M_U and M_D, which are both $O(\epsilon)$, whereas the leptonic 2-3 mixing $U_{\mu 3}$ is controlled by the 23 (and in doubly lopsided models also the 13) element of M_L, which is $O(1)$.

The u quark is left massless by the matrices in Eq. (3). However, that is not a bad thing. Experimentally, m_u/m_t is of order 10^{-5}, which is far smaller than the corresponding ratios $m_d/m_b \sim 10^{-3}$ and $m_e/m_\tau \sim 0.3 \times 10^{-3}$. Thus, if m_d and m_e arise at one-loop level, one would expect m_u to vanish at one-loop level.

Before analyzing the structure of the quark and lepton mass matrices further, it is necessary to deal with the question of the pattern of breaking of $SO(10)$ down to the standard model group $G_{\rm SM} = SU(3)_c \otimes SU(2)_L \otimes U(1)_Y$.

III. THE BREAKING OF $SO(10)$

If $SO(10)$ broke at a single scale all the way to the standard model group $G_{\rm SM}$, it would give the same prediction for the low-energy gauge couplings as nonsupersymmetric $SU(5)$, which are known to be unsatisfactory. Moreover, as in nonsupersymmetric $SU(5)$, the proton lifetime would be too short. However, it is possible to get satisfactory gauge coupling unification and proton lifetime by assuming a two-stage breaking with the Pati-Salam group [19] as the intermediate symmetry:

$$SO(10) \stackrel{M_G}{\rightarrow} SU(4)_C \otimes SU(2)_L \otimes SU(2)_R \stackrel{M_{\rm PS}}{\rightarrow} SU(3)_C \otimes SU(2)_L \otimes U(1)_Y.$$

The breaking of $SO(10)$ to the Pati-Salam group at the higher scale M_G can be done by a $\mathbf{210}_H$. The breaking of the Pati-Salam group at $M_{\rm PS}$ is done by the VEVs of the adjoint and spinor Higgs fields, $\mathbf{45}_H$ and $\mathbf{16}_{iH}$, which also contribute to the superheavy quark and lepton masses through couplings that appear in Eq. (2).

In running the gauge couplings between M_G and $M_{\rm PS}$, the following matter multiplets contribute to the beta functions: (1) The quark and lepton multiplets, $\mathbf{16}_1$, $\mathbf{16}_2$, $\mathbf{16}_3$, $\mathbf{16}$, $\overline{\mathbf{16}}$, and $\mathbf{10}$. Since the masses of these multiplets are produced by coupling to the adjoint and spinor Higgs fields and not the $\mathbf{210}_H$, their splittings are of order $M_{\rm PS}$ and can be ignored in running between M_G and $M_{\rm PS}$. (2) The Higgs multiplets $\mathbf{16}_{1H}$, $\mathbf{16}_{2H}$, $\mathbf{16}'_H$, $\mathbf{10}_H$, $\mathbf{10}'_H$, and three adjoint Higgs multiplets. For the Higgs multiplets too, except for the adjoints, it is assumed that the splittings are of order $M_{\rm PS}$ and can be neglected in running between M_G and $M_{\rm PS}$. For the adjoints, however, we assume splittings of order M_G. Under the Pati-Salam group a $\mathbf{45}_H$ decomposes to $(15, 1, 1)_H + (6, 2, 2)_H + (1, 3, 1)_H + (1, 1, 3)_H$. We assume that the color-singlet pieces of the adjoints get mass of order M_G and the color-non-singlet pieces get mass of order $M_{\rm PS}$. This is not unreasonable, since the renormalizable couplings of the adjoints to the $\mathbf{210}_H$ produce splittings of order M_G between the color-singlet and color-non-singlet pieces. (One such term is $\langle H^{[IJKL]}\rangle H^{[IJ]} H^{[KL]}$, where the fundamental indices I, J, K, L take $SU(2)_L \otimes SU(2)_R$ values.) Of course, the whole Higgs potential must be tuned to give the hierarchy between M_G and $M_{\rm PS}$, so different patterns of splittings are possible. It is in order to get a value of M_G large enough to be consistent with proton-decay limits, that we assume there are three split adjoint Higgs multiplets. More such adjoints would push M_G higher. Below the scale $M_{\rm PS}$, we assume just the standard model field content with one-Higgs-doublet.

The results of the running are shown in Fig. 3. In the running, the input values used are $\alpha_1^{-1}(M_Z) = 58.97$, $\alpha_2^{-1}(M_Z) = 29.61$, and $\alpha_3^{-1}(M_Z) = 8.47$ [20]. The result of the running is that $M_{\rm PS} = 4.79 \times 10^{13}$ GeV. At $M_{\rm PS}$ the standard model couplings have the values $\alpha_1^{-1}(M_{\rm PS}) = 41.35$, $\alpha_2^{-1}(M_{\rm PS}) = 43.21$, and $\alpha_3^{-1}(M_{\rm PS}) = 38.55$. The unification scale comes out to be $M_G = 1.17 \times 10^{16}$ GeV. and $\alpha_U^{-1}(M_G) = 35.65$. These values are consistent with present bounds on proton decay. (In this model, the dominant contribution to proton decay comes from dimension-6 operators produced by the exchange of gauge bosons of mass M_G. The Pati-Salam gauge bosons do not give proton decay.) The values of the gauge couplings plotted in Fig. 3 are used in the running of the quark and lepton masses and mixing angles in Sec. V.

One consequence of the fact that $SO(10)$ is broken down to the Pati-Salam group at a high scale is that it makes more natural the assumption being made in this model that $\langle \mathbf{45}_H \rangle \propto B - L$. In the original supersymmetric version of the model [4] this assumption was justified by the fact that a $\mathbf{45}_H$ whose VEV is proportional to $B - L$ is needed to implement to Dimopoulos-Wilczek mechanism (or "missing VEV mechanism") [21] for doublet-triplet splitting. However, in a nonsupersymmetric $SO(10)$ model, that mechanism does not work—and, in fact, doublet-triplet splitting must be achieved through fine-tuning [22]. The justification for the assumption made in [3] that

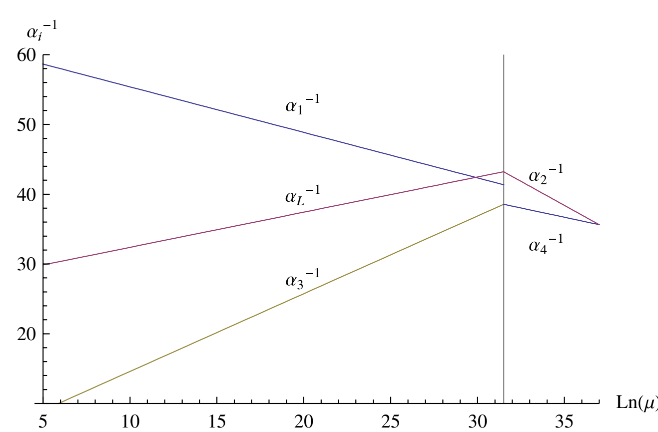

FIG. 3 (color online). Gauge couplings unification.

$\langle \mathbf{45}_H \rangle \propto B - L$ is therefore less clear. However, if $SO(10)$ breaks to the Pati-Salam group at the high scale M_G in such a way that only the $(1, 3, 1)_H + (1, 1, 3)_H$ in the adjoints have the large mass M_G, as assumed, then the residual $(15, 1, 1)_H$ naturally obtains a VEV in the $B - L$ direction (and, in fact can point in no other direction without breaking the standard model group at a superheavy scale).

Perhaps a word should be said about the Higgs potential and the possibility of obtaining the desired pattern of VEVs and Higgs splittings by minimizing it. Since there are two scales of symmetry breaking, M_G and $M_{\rm PS}$, and this is an ordinary nonsupersymmetric grand unified theory, there is inevitably some tuning in the Higgs potential to achieve a large ratio of these scales. However, the tuning is not so severe—only about a factor of 0.005. Moreover, one could imagine that some of the breaking is done dynamically, which might avoid any tuning of parameters, though we shall not pursue that idea here. This kind of tuning is characteristic of multiscale models without supersymmetry. On the other hand, in nonsupersymmetric models the Higgs potential is much less constrained, since it does not come from a superpotential. It contains all quadratic, cubic, and quartic terms that can be constructed out of both the Higgs fields and their Hermitian conjugates consistent with symmetry, whereas a superpotential is at most cubic and is made of the chiral superfields alone, without their Hermitian conjugates. As a result, any realistic nonsupersymmetric model has a very large number of free parameters in the Higgs potential. In the present case there are about 20 *types* of terms and of order a hundred free parameters. This makes it obvious that the simple pattern of VEVs that we assume can be obtained by a suitable choice of parameters, though for equally obvious reasons we shall not attempt a minimization. The $U(1)_F$ does constrain the Higgs potential, but still allows a very large number of terms, enough for realistic breaking. (In the Higgs potential the $U(1)_F$ constraint can be satisfied in many types of terms by having fields multiplied by their Hermitian conjugates, whereas this is not an option in the Yukawa sector, which is therefore much more constrained, and indeed must have the form in Eq. (2).)

IV. MODIFICATIONS TO THE MASS MATRICES

In this section the implications for the mass matrices of the breaking of $SO(10)$ down to the Pati-Salam group will be analyzed.

In Eq. (3) the term $M_{10}(\mathbf{10}\mathbf{10})$, as written, involves an explicit mass. This was the assumption made in [3]. It is also possible, however, and just as simple, to suppose that this mass arises from the VEV of some Higgs field(s) that break $SO(10)$. There are several ways that this might happen. Perhaps the simplest is through a coupling $(\mathbf{10}\mathbf{10})\langle \mathbf{54}_H \rangle$. This is allowed by the $U(1)_F$ symmetry. [Alternatively, one could dispense with the $\mathbf{54}_H$ field and have instead the effective coupling $(\mathbf{10}\mathbf{10})\langle \mathbf{45}_H \rangle \times \langle \mathbf{45}_H \rangle / M_{\rm PS}$. This operator would arise from integrating out fields with mass $O(M_{\rm PS})$, which would have to be introduced for this purpose. The simplest possibility would be to introduce a $\mathbf{10}'$ of fermions with $U(1)_F$ charge $(y - x)$, and a singlet Higgs $\mathbf{1}_H$ with $U(1)_F$ charge $(2x - 2y)$ and VEV of order $M_{\rm PS}$. The $\mathbf{10}'$ would get mass from the coupling $\mathbf{10}'\mathbf{10}'\mathbf{1}_H$, and it would also have coupling $\mathbf{10}\mathbf{10}'\mathbf{45}_H$.]

If M_{10} reflects the breaking of $SU(5)$ then it is a matrix that gives different values when acting on the down quarks and on the charged leptons in the $SO(10)$ **10** of fermions. Call its value for the leptons M_{10} and for the down quarks M'_{10}. One result of this splitting is that the entries C_i are no longer the same in the mass matrices M_L and M_D. If one assumes, as before, that the quantities $\sqrt{1 + (\Sigma_i c_i \langle \mathbf{16}_{iH} \rangle / M_{10})^2}$ are well approximated by 1, then one can write the C_i contributions to the mass matrices as C_i for M_L and fC_i for M_D, where $f = M_{10}/M'_{10}$. This $SO(10)$-breaking effect is, as it were, optional. However, it is quite important for fitting m_b^0/m_τ^0, which otherwise would be predicted to be 1. The effects that will now be described are unavoidable consequences of the breaking of $SO(10)$.

$SO(10)$-breaking and $SU(5)$-breaking effects come into the loop contributions δ_{gi} and δ_H in several ways. Consider δ_H first. If one examines the diagram in Fig. 2 closely, one finds that its contribution to M_D involves loops with scalars that can be either color triplets or color singlets, whereas its contributions to M_L involve only color-triplet scalars. When $SO(10)$ breaks, the degeneracy between these two types of scalars is badly broken, which means that one cannot assume that Fig. 2 gives the same contribution to the two matrices. We therefore introduce a factor f_H into the 22 element of M_L to reflect this fact.

The case of the elements δ_{gi} requires a more involved discussion. First, one must recall that the vacuum expectation values $\langle \mathbf{1}(\mathbf{16}_{iH}) \rangle$ and $\langle \mathbf{45}_H \rangle$ break the Pati-Salam group, and therefore must be no larger than $M_{\rm PS}$, and that the masses M_{10} and M_{16} cannot be too much larger than these VEVs, since otherwise the entries C_i and ϵ would be too small. Consequently, one can assume that all the superheavy fermion masses are much lighter than the scale M_G. This has implications for the loop diagrams in Fig. 1. Some of those diagrams contain gauge bosons whose mass is of order $M_{\rm PS}$ and others contain gauge bosons whose mass is of order M_G. Because the fermions in those loops are much lighter than M_G, as just argued, the loops with $O(M_G)$ gauge bosons are suppressed relative to the loops with $O(M_{\rm PS})$ gauge bosons by a large factor and are therefore negligible. To see what this implies, one must look in more detail at the diagrams in Fig. 1 to see how they contribute to M_L and M_D.

In Fig. 4(a), is shown the contribution to M_L. In this diagram there are three possible values of the color index a, (or, equivalently, of the pair of color indices bc on the

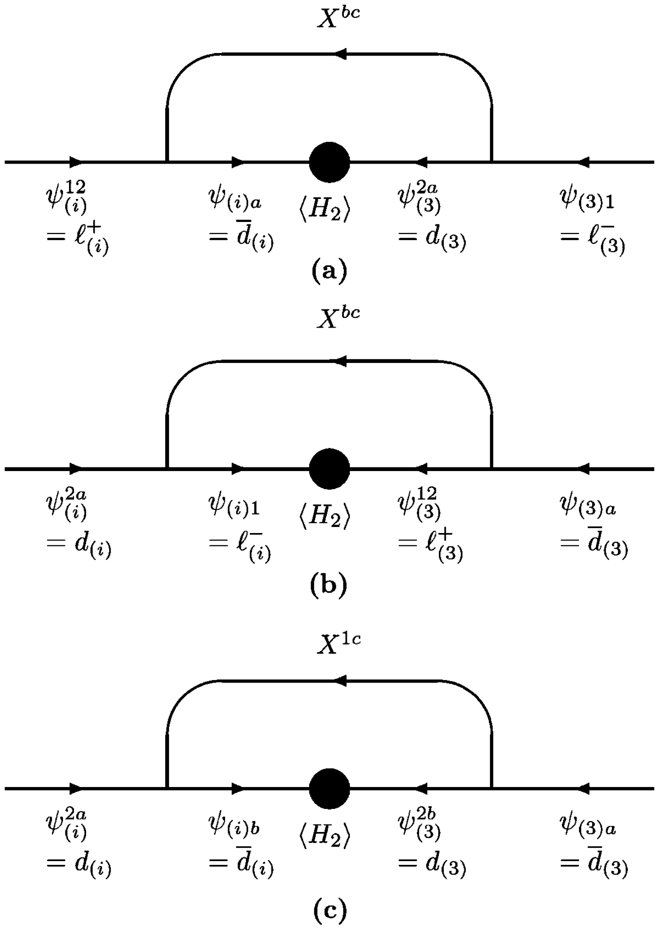

FIG. 4. Gauge-boson-loop contribution to mass matrices. Subscripts in parentheses are family labels. a, b, c are $SU(3)_c$ indices. 1, 2 are $SU(2)_L$ indices.

superheavy gauge boson X^{bc}). For any of these values, the gauge boson converts a left-handed charged lepton into a left-handed down-type quark at one vertex, and a left-handed charged antilepton into a left-handed down-type antiquark at the other vertex. That shows that the gauge boson is one of those of the Pati-Salam group $SU(4)_c$, which make such transitions. (The Pati-Salam multiplet $(\mathbf{4}, \mathbf{2}, \mathbf{1})$ unifies left-handed leptons and quarks, while the multiplet $(\bar{\mathbf{4}}, \mathbf{1}, \mathbf{2})$ unifies left-handed antileptons and antiquarks.) For all three values of a, therefore, the loop in Fig. 3(a) contains only gauge bosons whose mass is of order $M_{\rm PS}$.

In Figs. 4(b) and 4(c) are shown the diagrams that contribute to M_D. In Fig. 4(b), the external quark has a color index of fixed value a, which determines uniquely the values of the color indices on the virtual gauge boson X^{bc}. This gauge boson is [as in Fig. 4(a)] one of those of the Pati-Salam group $SU(4)_c$, as can be seen from the fact that it converts left-handed down quarks into left-handed charged leptons. In Fig. 4(c) there are two choices for the color index c on the gauge boson X^{1c}, since it is only required to be different from a. This gauge boson, however, is obviously *not* one of those of the Pati-Salam group, since it converts a left-handed quark into a left-handed antiquark (and a left-handed lepton into a left-handed antilepton), which are not unified together in the multiplets of the Pati-Salam group. Thus the gauge boson in Fig. 4(c) has mass of order $M_G \gg M_{\rm PS}$. The diagram in Fig. 4(c) is thus highly suppressed. From these considerations, one sees that the contribution to M_L has a factor 3 relative to the contribution to M_D because of the color degeneracy in the loop.

The gauge-boson-loop integrals can be written in a simple form if one makes the same approximation as before, namely $\sqrt{1 + (\Sigma_i c_i \langle \mathbf{16}_{iH} \rangle / M_{10})^2} \cong 1$. In that case the gauge-boson-loop contributions to M_L and M_D are given by

$$(M_L)^{gb\ell}_{3i} = 3\left(\frac{3\alpha_U}{16\pi^2}\right) C_i \frac{\ln x}{x-1}, \qquad (M_D)^{gb\ell}_{i3} = \left(\frac{3\alpha_U}{16\pi^2}\right)(f C_i) \frac{\ln x'}{x'-1}. \tag{5}$$

Here $x \equiv (M_g/M_{10})^2$ and $x' \equiv (M_g/M'_{10})^2$, where M_g is the mass of the colored "Pati-Salam" gauge bosons in Fig. 3(a) 3(b). Recalling that $f = M_{10}/M'_{10}$, one can rewrite these as

$$(M_L)^{gb\ell}_{3i} = 3\left(\frac{3\alpha_U}{16\pi^2}\right) C_i \left(\frac{M_{10}}{M_g}\right) F(x), \qquad (M_D)^{gb\ell}_{i3} = \left(\frac{3\alpha_U}{16\pi^2}\right) C_i \left(\frac{M_{10}}{M_g}\right) F(x'), \tag{6}$$

where $F(x) \equiv x^{1/2} \ln x/(x-1)$. It happens that the function $F(x)$ is very slowly varying for arguments of order 1. For example, $F(1+y) = 1 - \frac{1}{24}y^2 + \frac{1}{24}y^3 + \ldots$, and $F(2) = F(\frac{1}{2}) = 0.98$. Thus to a very good approximation one may write

$$(M_L)^{gb\ell}_{3i} = 3C_i\delta \qquad (M_D)^{gb\ell}_{i3} = C_i\delta. \tag{7}$$

The mass matrices that result are

$$M_U = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{\epsilon}{3} \\ 0 & -\frac{\epsilon}{3} & 1 \end{pmatrix} m_U, \qquad M_D = \begin{pmatrix} 0 & 0 & C_1\delta \\ 0 & \delta_H & \frac{\epsilon}{3} + C_2\delta \\ fC_1 & fC_2 - \frac{\epsilon}{3} & 1 \end{pmatrix} m_D,$$
$$M_N = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\epsilon \\ 0 & \epsilon & 1 \end{pmatrix} m_U, \qquad M_L = \begin{pmatrix} 0 & 0 & C_1 \\ 0 & f_H\delta_H & C_2 - \epsilon \\ 3C_1\delta & \epsilon + 3C_2\delta & 1 \end{pmatrix} m_D. \tag{8}$$

TABLE II.

m_u	1.27 ± 0.50 MeV
m_s/m_d	19.9 ± 0.8
m_s	55 ± 16 MeV
m_c	0.619 ± 0.084 GeV
m_b	2.89 ± 0.09 GeV
m_t	171.7 ± 3 GeV
m_e	$0.486570161 \pm 0.000000042$ MeV
m_μ	$102.7181359 \pm 0.0000092$ MeV
m_τ	1746.24 ± 0.20 MeV

V. FITTING THE FERMION MASSES AND MIXINGS

The forms of the mass matrices in Eq. (8) are those given by the model at the scale $M_{\rm PS}$, since that is the mass of the superheavy fields that are integrated out to give these matrices. Below $M_{\rm PS}$, the model reduces to the standard model. One can therefore use the renormalization-group equations (RGEs) of the one-Higgs-doublet standard model to run the measured quark and lepton masses and mixings from low scales up to $M_{\rm PS}$ and then fit the results using the forms in Eq. (8).

Running from M_Z to $M_{\rm PS}$ is done using the one-loop standard model RGEs given in [23]. The input values of the quark and lepton masses at M_Z, shown in Table II, are taken from [24], and were computed using updated Particle Data Group values. The input values of the CKM angles are taken from [25]: $s_{12} = 0.2243 \pm 0.0016$, $s_{23} = 0.0413 \pm 0.0015$, and $s_{13} = 0.0037 \pm 0.0005$. The leptonic angles are taken from [26]: $\theta_{\rm sol} = 33.9° \pm 2.4°$, $\theta_{\rm atm} = 45° \pm 10°$. The quark and lepton masses at $M_{\rm PS}$ that result from the running are shown in Table III. The quark mixing angles at the Pati-Salam scale are, $s_{12} = 0.2243 \pm 0.0016$, $s_{23} = 0.0464 \pm 0.0015$, and $s_{13} = 0.0041 \pm 0.0005$.

Note that we fit m_d/m_s, which is relatively well-known from current algebra, rather than fitting m_d itself. For several quantities, namely, the charged lepton masses, the mass of the u quark, and the atmospheric neutrino angle, we will add a "theoretical error bar" to the experimental error bar in doing the χ^2 fit. In the case of the charged leptons, we add a 1% fractional error to their masses, simply because we do not expect the forms in Eq. (8) to be more accurate than that. (We have made approximations of that order in deriving them.)

TABLE III.

	mass with error bar	fractional error (%)
m_u	0.571 ± 0.24 MeV	42
m_s/m_d	18.9 ± 0.8	4.23
m_s	25.387 ± 8.0 MeV	31
m_c	0.278 ± 0.042 GeV	15
m_b	1.186 ± 0.05 GeV	4.2
m_t	86.926 ± 4 GeV	4.6
m_e	$0.488848231 \pm 0.000000042$ MeV	10^{-5}
m_μ	$103.1990611 \pm 0.0000092$ MeV	10^{-5}
m_τ	1754.46 ± 0.20 MeV	10^{-4}

In the case of m_u, we add a theoretical error because the mass matrices we are using to do the fit include only tree-level and one-loop effects. A two-loop contribution to the 11 element of M_U would be expected to be roughly of order $(\frac{1}{16\pi^2})^2 m_t^0 \sim 3.5$ MeV. Thus we take the prediction of the model to be that $m_u^0 = 0 \pm 3.5$ MeV. In other words, the theoretical error for m_u is about 600% of the actual value of m_u, so it has no effect on the fitting.

In the case of the neutrino angles, there is an inherent uncertainty in the prediction of this model, due to the Majorana mass matrix of the right-handed neutrinos M_R being unknown and not predicted by the model. Because the Dirac neutrino mass matrix M_N is (to one-loop order) given by the form in Eq. (8), which has vanishing first row and column, it follows that the light neutrino mass matrix, given by the usual type-I seesaw formula $M_\nu = -M_N M_R^{-1} M_N^T$, also has vanishing first row and column at this order. Thus, the unitary matrix U_ν required to diagonalize M_ν is simply a rotation by some angle θ_ν in the 23 plane. Thus the mixing matrix of the neutrinos is given by

$$U_{\rm MNS} = U_L U_\nu^\dagger = U_L \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_\nu & -\sin\theta_\nu \\ 0 & \sin\theta_\nu & \cos\theta_\nu \end{pmatrix}, \qquad (9)$$

where U_L is the unitary rotation of the left-handed charged leptons required to diagonalize M_L. Consequently,

$$\begin{aligned} \sin\theta_{\rm atm} &= \cos\theta_\nu \sin\theta^L_{\rm atm} - \sin\theta_\nu \cos\theta^L_{\rm sol} \cos\theta^L_{\rm atm}, \\ &= \sin\theta^L_{\rm atm} + O(\theta_\nu) \\ \sin\theta_{13} &= \cos\theta_\nu \sin\theta^L_{13} - \sin\theta_\nu \sin\theta^L_{\rm sol}, \\ \sin\theta_{\rm sol} &= \cos\theta_\nu \sin\theta^L_{\rm sol} = \sin\theta^L_{\rm sol} + O(\theta_\nu^2), \end{aligned} \qquad (10)$$

where $\sin\theta^L_{\rm atm}$, $\sin\theta^L_{\rm sol}$, and $\sin\theta^L_{13}$ are the 23, 12, and 13 elements of U_L respectively. In performing the fit to the data, we take as the neutrino mixings predicted by the model $\sin\theta^L_{\rm sol}$, $\sin\theta^L_{\rm atm}$, and $\sin\theta^L_{13}$ (these are what are given under "model" for these quantities in Table IV), and we treat the effect of θ_ν as a "theoretical error". The question is how big to assume this error is in doing the fit.

Naively, one would expect that typically $\theta_\nu = O(\epsilon)$, since the 23 element of the Dirac neutrino mass matrix M_N is $-\epsilon$, so that θ_ν vanishes in the limit ϵ goes to zero. Though this conclusion is correct, a more careful argument is needed to establish it, because the form of M_R may be special, and in fact must be. The matrix M_N is hierarchical, and thus, for a "generic" M_R, one would expect $M_\nu = M_N M_R^{-1} M_N^T$ to be very strongly hierarchical. And indeed the ratio of eigenvalues m_2 and m_3 of M_ν comes out to be

TABLE IV.

	model (at $M_{\rm PS}$)	expt. (at $M_{\rm PS}$)	off (%)	expt. error* (%)
m_e	0.0004900	0.0004888	0.027	1.0*
m_μ	0.1031	0.1032	−0.13	1.0*
m_τ	1.756	1.754	0.07	1.0*
m_u	0	0.000571	100	600*
m_c	0.342	0.278	23.0	15.1
m_t	87.24	86.93	0.36	4.6
m_s/m_d	18.68	18.90	−1.14	4.23
m_s	0.0358	0.0254	40.8	31.5
m_b	1.17	1.186	−1.29	4.22
$\frac{m_e}{m_\mu}/\frac{m_d}{m_s}$	0.0886	0.0895	−0.99	
m_b/m_τ	0.667	0.676	−1.35	
V_{us}	0.2243	0.2243	0.002	0.71
V_{cb}	0.0456	0.0463	−1.51	3.24
$\|V_{ub}\|$	0.00368	0.00432	−14.8	11.6
δ_{13}	0.887	0.995	−10.8	24.12
$\sin\theta_{\rm sol}$	0.518	0.559	−7.33	7.51
$\sin\theta_{\rm atm}$	0.891	0.707	26.1	28*
$\sin\theta_{13}$	0.014	<0.178		
$\chi 2$	7.2			

$O(\epsilon^4)$ for generic M_R, which is much smaller than the observed ratio of about 1/5. This suggests that there is a hierarchy in M_R^{-1} that partially compensates in the seesaw formula for the hierarchy in M_N. It is therefore useful to parametrize the 23 block of the matrix M_R^{-1} with powers of $1/\epsilon$ as follows: $(M_R^{-1})_{33} = M$, $(M_R^{-1})_{23} = (M_R^{-1})_{32} = (a/\epsilon)M$, $(M_R^{-1})_{22} = (b/\epsilon)^2 M$. Then one finds that $tr\bar{M}_\nu = m_2 + m_3 = (1 + 2a + b^2 + \epsilon^2)(m_D^2/M)$ and $\det\bar{M}_\nu = m_2 m_3 = \epsilon^2(b^2 - a^2)(m_D^2/M)^2$, where $\bar{M}_\nu$ is the 23 block of M_ν. Defining $r \equiv \det\bar{M}_\nu/(tr\bar{M}_\nu)^2 \cong m_2/m_3 \simeq \frac{1}{5}$, and $K \equiv r/\epsilon^2 \simeq 5$, one obtains the following equation for a and b (which for real parameters is just that of an ellipse): $(K-1)b^2 + (a+K)^2 = [K^2 - K - r]$. It is easy to see that typically $|a| \sim |K|$ and $|b| \sim |K|^{1/2}$. Since the angle θ_ν is given by $\tan 2\theta_\nu = -\epsilon[\frac{2(1+a)}{1+2a+b^2-\epsilon^2}]$, one finds that except for special values of a and b, θ_ν is of order ϵ, in agreement with the naive estimate.

In view of the above considerations one expects θ_ν to be of order ϵ. While there are choices of a and b that fit the neutrino mass ratios and make $\theta_\nu \sim 1$, these are not typical. And so, while we could be generous to ourselves and use a "theoretical error bar" of order 1 for the atmospheric angle in the numerical fit (and obtain a better χ^2), it seems more justified to use a tighter theoretical error bar of ϵ. (As we shall see, this is indeed about the amount by which the value of $\theta^L_{\rm atm}$ given by the model misses the experimental value of the atmospheric angle.) There should also be a "theoretical error bar" for the solar angle coming from the factor $\cos\theta_\nu$ in Eq. (10), but for $\theta_\nu \sim \epsilon$ this is small compared to the experimental error bar, and so we ignore it.

In fitting, one has to take into account that the parameters appearing in Eq. (8) are, in general, complex. Because of the freedom to redefine the phases of the quark and lepton fields, most of the phases can be "rotated away" from the low-energy theory. We will consider two cases. If the group theoretical factors denoted f and f_H are real, then there are two physical phases in the mass matrices of Eq. (8), which one can take to be phases of the parameters ϵ and δ_H. We will denote these by θ_ϵ and θ_H. If f and f_H are complex, then there are two additional phases, which we will denote θ' and θ_{fH}. The phase θ' comes into the subleading terms of the 23 and 32 elements of M_D and M_L. The phase θ_{fH} comes only into the 22 element of M_L. The phase θ' has only a small effect on the fit, and θ_{fH} has almost none.

Altogether, then, we fit using 9 real parameters (M_U, M_D, C_1, C_2, ϵ, δ, δ_H, f, f_H) and two or four phases (θ_ϵ, θ_H, and if f and f_H are complex then also θ' and θ_{fH}). With these we fit 16 quantities: the 9 masses of the quarks and leptons (excluding the neutrino masses, which depend on the unknown M_R), the 3 CKM angles, the 1 CKM phase, and the 3 neutrino mixing angles.

The results of the fit assuming f and f_H are real are given in Table IV. The asterisks in the "experimental error" column are reminders that for certain entries a "theoretical error" is included, as explained above. The masses are all in GeV. The CKM phase δ_{13} is in radians. One notes that most quantities are fit excellently. The least good fits are to m_c, m_s, and $|V_{ub}|$. Considering that 11 parameters are fitting 16 quantities, the χ^2 of 7.2 is quite reasonable.

The parameter values for this fit are $\epsilon = 0.189$, $C_1 = 1.03$, $C_2 = -1.51$, $f = 0.566$, $f_H = 0.208$, $16\pi^2\delta = 2.22$, $16\pi^2\delta_H = 2.66$, $\theta_\epsilon = 1.52$ rad, $\theta_H = 0.514$ rad. Note that all these quantities are of order one. In other words, no small dimensionless parameters are needed to fit the quark and lepton mass hierarchies in this model. The scales called m_U and m_D in Eq. (8) are given, respectively, by 86.9 GeV and 0.79 GeV. The large ratio of these scales is not explained by the structure of the model or by symmetry, and presumably comes from the details of the sector that breaks the weak interactions.

The results of the fit assuming f and f_H are complex are given in Table V. The parameter values for the fit in Table IV are $\epsilon = 0.182$, $C_1 = 0.997$, $C_2 = -1.60$, $f = 0.573$, $f_H = 0.224$, $16\pi^2\delta = 2.16$, $16\pi^2\delta_H = 3.22$, $\theta_\epsilon = -0.554$ rad, $\theta_H = -1.56$ rad.

Comparison of Tables IV and V shows that the inclusion of the phases θ' and θ_{fH} makes very little difference to the fits. This is not surprising, since θ' appears only on subleading terms in the mass matrices, and θ_{fH} appears on the very small entry f_H. For the two fits, the values of the real parameters hardly changes. The phase angles θ_ϵ and θ_H both change by -2.07 rad, but that is essentially due to a rephasing: a shift in these two phases by a certain amount

TABLE V.

	model (at M_{PS})	expt. (at M_{PS})	off (%)	expt. error* (%)
m_e	0.0004888	0.0004888	−0.012	1.0*
m_μ	0.1032	0.1032	−0.01	1.0*
m_τ	1.755	1.754	0.02	1.0*
m_u	0	0.000571	100	600*
m_c	0.317	0.278	14.14	15.1
m_t	87.24	86.93	0.33	4.6
m_s/m_d	18.44	18.90	-2.39	4.23
m_s	0.0346	0.0254	36.17	31.5
m_b	1.17	1.186	−1.67	4.22
$\frac{m_e}{m_\mu}/\frac{m_d}{m_s}$	0.0874	0.0895	−2.39	
m_b/m_τ	0.665	0.676	−1.7	
V_{us}	0.2243	0.2243	−0.018	0.71
V_{cb}	0.0458	0.0463	−1.13	3.24
$\|V_{ub}\|$	0.00382	0.00432	−11.5	11.6
δ_{13}	0.889	0.995	−10.6	24.12
$\sin\theta_{sol}$	0.499	0.559	−10.7	7.51
$\sin\theta_{atm}$	0.895	0.707	26.1	28*
$\sin\theta_{13}$	0.015	<0.178		
$\chi 2$	6.0			

can be compensated by a shift in θ' together with a change in the phase of two small subleading terms. In other words, what is really changing a lot between the fits in Tables IV and V is the phase θ' (which is, of course, zero for the fit in Table IV). What this shows is that the fit is hardly affected by large changes in θ'.

In this model there is a relation between the atmospheric angle and θ_{13}, which is given in Eq. (10). Using the best-fit values given in Table IV, Eq. (10) yields

$$\begin{aligned} |\sin\theta_{atm}| &= |\cos\theta_\nu(0.891) - \sin\theta_\nu(0.396)|, \\ |\sin\theta_{13}| &= |\cos\theta_\nu(-0.014) + \sin\theta_\nu(0.518)|. \end{aligned} \tag{11}$$

If the parameter $\sin\theta_\nu$ were a real number, these equations would give a prediction for θ_{13} in terms of θ_{atm}. For values of the atmospheric angle near maximal mixing, i.e. $\theta_{atm} \cong \pi/4$, the prediction for the 13 angle would be approximately given by

$$|\sin\theta_{13}| \cong 0.160 - 0.72(\sin\theta_{atm} - 1/\sqrt{2}). \tag{12}$$

However, in fact, the parameter $\sin\theta_\nu$ can be complex. Therefore, the smaller of the two values that is obtained for $|\sin\theta_{13}|$ by solving Eq. (11) with real $\sin\theta_\nu$ is a *lower bound*. So, if the atmospheric mixing angle is near maximum, there is a lower bound on $\sin\theta_{13}$ given by Eq. (12).

VI. CONCLUSIONS

The model that we have studied here is the first predictive grand unified model with a radiative fermion mass hierarchy. In a number of ways, it is as economical as a model of quark and lepton masses can be. The masses and mixings of the second and third families come from only three effective Yukawa operators, shown in Eq. (1). These operators account for many features of the light fermion spectrum: (1) the fact that V_{cb} is of the same order as m_s/m_b and m_μ/m_τ; (2) the fact that m_c/m_t is much smaller than those ratios; (3) the largeness of the atmospheric and solar neutrino angles; (4) the smallness of the 13 angle; (5) the rough equality of m_b^0 and m_τ^0; and (6) the Georgi-Jarlskog factor of about 1/3 between m_s/m_b and m_μ/m_τ. The masses and mixings of the first family (except for the solar neutrino angle) come from loop diagrams. It is remarkable that one of these loop diagrams (Fig. 1) is present automatically, whereas the other (Fig. 2) requires only a single additional Yukawa term to be postulated.

It is striking that no small parameters are needed in this model to account for the dramatic hierarchies in the quark and lepton masses. The model yields a definite relation between the atmospheric angle and the angle θ_{13}.

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