

Causality as an emergent macroscopic phenomenon: The Lee-Wick $O(N)$ modelBenjamín Grinstein,^{1,*} Donal O’Connell,^{2,†} and Mark B. Wise^{3,‡}¹*Department of Physics, University of California at San Diego, La Jolla, California 92093, USA*²*School of Natural Sciences, Institute for Advanced Study, Princeton, New Jersey 08540, USA*³*California Institute of Technology, Pasadena, California 91125, USA*

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In quantum mechanics the deterministic property of classical physics is an emergent phenomenon appropriate only on macroscopic scales. Lee and Wick introduced Lorentz invariant quantum theories where causality is an emergent phenomenon appropriate for macroscopic time scales. In this paper we analyze a Lee-Wick version of the $O(N)$ model. We argue that in the large- N limit this theory has a unitary and Lorentz invariant S matrix and is therefore free of paradoxes in scattering experiments. We discuss some of its acausal properties.

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I. INTRODUCTION

It is interesting to try to understand if one or more of the pillars of modern physics may be violated by a theory that gives approximately the same experimental results as ordinary relativistic quantum field theory for experiments that are presently accessible. One such pillar is causality. Theories that are not causal appear, at first glance, to be fraught with paradoxical behavior that renders them inconsistent. In the late 1960’s Lee and Wick [1,2] proposed an extension of quantum electrodynamics where the Pauli-Villars regulator is treated as a finite mass scale. In this theory the Fourier transform of the gauge field two-point function has massive “Lee-Wick photon” poles with wrong-sign residues. It is easy to show that this theory is equivalent to a higher derivative theory. Naively, such theories are unstable and not unitary. Lee and Wick and Cutkosky, Landshoff, Olive, and Polkinghorne [3] gave rules (the “LW” and “CLOP” prescriptions) for calculating perturbative scattering amplitudes in this higher derivative theory that, for a wide class of Feynman diagrams, overcame these difficulties yielding Lorentz invariant and unitary scattering amplitudes. However, Lee-Wick electrodynamics is not causal for microscopic time scales. Their ideas provide a framework for studying acausal theories where the acausality is only detectable in experiments that can access very high energies and/or very short time scales [4].

In recent papers we extended the work of Lee and Wick to non-Abelian gauge theories and argued that they can solve the hierarchy puzzle [5,6]. Even if the ideas of Lee and Wick are not relevant for the hierarchy puzzle, it is worth exploring the physics of acausal theories and examining their consistency. Previous work has some limitations. While the LW and CLOP prescriptions have been

shown to give Lorentz invariant scattering amplitudes in a large class of Feynman graphs, it is not known whether this is true to all orders in perturbation theory. Moreover, there are serious obstacles to a nonperturbative path-integral formulation for the Lee-Wick theory with a CLOP prescription [7]. Other formulations with prescriptions different from CLOP’s can have a nonperturbative path-integral formulation [8,9], but these have yet to be shown to give a Lorentz invariant S matrix [10]. Tree-level LW-particle exchange is trivially unitary because the LW particles’ masses are complex and hence they cannot be on shell at tree level. However, two LW particles can be on shell, and unitarity beyond tree level is more difficult to establish. Perhaps there is some subtle obstacle that prevents the construction of nontrivial Lee-Wick theories that have a unitary and Lorentz invariant S matrix to all orders in perturbation theory. We will argue that this is not the case since at large N the Lee-Wick $O(N)$ model provides an example of an acausal theory that has a unitary and Lorentz invariant S matrix.

Lee and Wick gave an example of a soluble, but non-relativistic, theory with a unitary S matrix [1]. Tomboulis considered Einstein gravity coupled to N massless spinors in the large- N limit, holding M_{pl}^2/N fixed [11]. In this theory the graviton self-energy has, in addition to the usual pole and branch cut at $p^2 = 0$, pairs of complex poles, e.g., with $\text{Im}(p^2) \neq 0$. In the large- N limit LW graviton exchange only occurs at tree level, and so the theory is unitary. Antoniadis and Tomboulis [12] argued, using gauge invariance, that even beyond large N where the LW gravitons can occur in loops, this theory is unitary and furthermore does not need the CLOP prescription. However, these arguments were very formal and no explicit calculations have been done to support them.

At leading order in large N the scattering amplitudes in the $O(N)$ scalar model can be calculated [13]. In this paper we consider the Lee-Wick version of this theory and show that the Feynman graphs involved are all of the sort considered in the CLOP analysis. In fact, we go further: We

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show *by explicit calculation* that the scattering matrix is unitary and Lorentz invariant. Because the theory has unitary time evolution, there will be no paradoxes in experiments that involve normal scalars in the prepared initial state and the observed final state. After all, for any initial state there are various possible orthogonal final states, and the probability for each of them occurring sums to unity. This theory has unconventional acausal effects, some of which we illustrate with explicit calculations, but that does not make it inconsistent.

If a Lee-Wick extension of the standard model is relevant for the hierarchy puzzle, then the acausal effects can be studied indirectly in high energy accelerator experiments through unusual interference effects associated with a Lee-Wick resonance [14], like the wrong sign of the phase shift of the resonant scattering amplitude. However, there is no compelling reason that acausal effects should occur at the weak scale; perhaps low energy supersymmetry or a warped extra dimension provides the solution to the hierarchy puzzle. There are constraints on the masses of the Lee-Wick resonances from precision electroweak physics. These are quite strong because integrating out the Lee-Wick resonances gives tree-level contributions to the oblique parameters S and T [15].

It is possible that acausal effects, of the type we are studying in this paper, arise from the extension of the standard model to include a quantized theory of gravity. String theory is an extension of the standard model that includes a consistent quantum theory of gravity. At the present time it is widely accepted that string theory is realized in nature. This is because of a lack of alternatives and because, even though string theory is highly constrained, it has compactifications with enough light degrees of freedom to accommodate the known standard model physics as well as gravity. However, there is no experimental evidence to support the hypothesis that string theory is realized in nature. Therefore, even if string theory is incompatible with the type of acausality we are studying, it seems worth keeping an open mind on this issue and contemplating the possibility that acausal effects occurring on time scales of order the Planck time may arise from the extension of the standard model to include quantum gravity [16].

In the large- N theory, described above, of N spinors coupled to Einstein gravity, there is a Lee-Wick graviton with wrong-sign residue. Tomboulis argued this theory is renormalizable [17]. In a similar vein, in [18] it was argued that gravitational radiative corrections to the photon propagator in Maxwell-Einstein theory can induce higher derivative terms of the Lee-Wick type.

In the auxiliary field formulation of the $O(N)$ model at large N , the only loop diagram that enters the calculation of scattering amplitudes is the one-loop one-particle irreducible (1PI) auxiliary field self-energy. In the Lee-Wick extension of the $O(N)$ model, this Feynman diagram can

be treated using the LW and CLOP prescriptions. Hence this theory has a unitary and Lorentz invariant S matrix at large N , and this toy model provides a convenient laboratory to study the physics of acausal theories. In this paper we explore the Lee-Wick $O(N)$ model. We show by explicit calculation that the two-particle scattering amplitudes satisfy the optical theorem, argue that the S matrix is unitary, and calculate the acausal behavior that arises in some experiments.

Higher derivative versions of the $O(N)$ model at large N have been studied before [19], and the question of unitarity is taken up by Liu in Ref. [9]. Our investigations differ in several respects. The prescription that Liu uses differs from CLOP's prescription. In Ref. [9] terms of order ∂^6 are added to the Lagrangian and arranged so that *at tree level* there is a complex pair of poles in the propagators. The imaginary part of these poles is not associated with a decay width. In our model only terms of order ∂^4 are added to the Lagrangian so that at tree level the two poles in the propagator are at real and positive values of p^2 . It is the interactions that turn the wrong residue pole into a complete "di-pole," and the imaginary part is dictated by the physical width. Finally, Liu's proof of unitarity is indirect [20], while we demonstrate unitarity by direct calculation.

The work we present here is perhaps a minor extension of results in the literature. We are as interested in demonstrating that, indeed, there exist unitary, Lorentz invariant, higher derivative theories as we are in presenting a pedagogical account of the aspects of the theory that are unusual. So we begin in Sec. II by contrasting the time dependence in a scattering experiment in a normal versus a higher derivative theory. The results are not new; they can be found in the famous lectures by Coleman [4]. In Sec. III we review the $O(N)$ model, which allows us to compare with the discussion in Sec. IV of the Lee-Wick version of the model. We offer some concluding remarks in Sec. V.

II. TIME DEPENDENCE IN A SCATTERING EXPERIMENT

We start our discussion by entertaining the following question: How does one go about testing causality or looking for causality violation in a theory that gives only an S matrix? One may wonder if this is possible at all since formally the S matrix relates states of the infinite past to those in the infinite future. Intuitively it is clear that this is no impediment: We may prepare two localized wave packets in the infinite past to travel toward each other and set up detectors to look for the outcome of their collision. Moreover, if both the distances traveled by the prepared wave packets to the collision point and the distances from this point to the detectors are truly macroscopic, then there is an S matrix description of this process. Clearly, information on the position and timing of the first detection of collision products can then give information on the causal behavior of the interaction.

This section formalizes these statements mathematically. In theories with normal causal behavior, a resonant collision that takes place at some space-time point, z_0 , results in the production of outgoing stable particles that appear to arise from a second space-time point, z'_0 . This second point occurs later in time ($t'_0 > t_0$) and is typically separated from the collision point by a proper time of the order of $1/\Gamma$, the inverse of the width of the resonance. The distribution is a decaying exponential. This is encapsulated in Eq. (35), in which the separation between the two points is $w = z'_0 - z_0$.

The situation is quite different for resonant collisions through a Lee-Wick resonance. Here the detected particles appear to come from z'_0 , which occurs earlier than z_0 in time ($t'_0 < t_0$). The points are still separated by a proper time of the order of $1/\Gamma$, and still distributed as an exponential that decays away from z_0 . This is the content of the final equation in this section, Eq. (47).

We have been careful to state that the collision “appears to” take place at z'_0 . The measurement is made a long time after and a long distance from the collision region. Within the quantum mechanical S matrix formalism, there is no means by which we can investigate, nor is there meaning to, the question of precisely where or when the collision takes place. This observation is important in understanding the interpretation of the results in the case where the collision goes through a Lee-Wick resonance, where normal causal behavior is violated.

A. Kinematics

We prepare from stable particles of mass m an initial state consisting of two wave packets traveling towards each other from far away. They are initially localized about space-time points y_1 and y_2 , which we can assume are spacelike separated, $(y_2 - y_1)^2 < 0$, and have large negative time components (we imagine the interaction will take place at around zero time). So we take $y_i^0 < 0$ and $|y_i^0| \gg 1/m$, where m is the mass of the particles. Since they will have to travel a long distance to the interaction point, we also take $|\vec{y}_i| \gg 1/m$ [21].

We also want the wave packets to have specific momenta. That is, their Fourier transforms are localized about $p_1 = mv_1$ and $p_2 = mv_2$. Of course, the momenta have to point towards each other so that there is a collision. The collision occurs at a point z_0 such that

$$\frac{z_0 - y_1}{\tau_1} = v_1 \quad \text{and} \quad \frac{z_0 - y_2}{\tau_2} = v_2 \quad (1)$$

$$\begin{aligned} \langle 0|T[\phi(x'_2)\phi(x'_1)\phi(x_1)\phi(x_2)]|0\rangle &= \int \prod_i \frac{d^4 k_i}{(2\pi)^4} \frac{d^4 k'_i}{(2\pi)^4} e^{i(k_1 \cdot x_1 + k_2 \cdot x_2 - k'_1 \cdot x'_1 - k'_2 \cdot x'_2)} \\ &\times (2\pi)^4 \delta^{(4)}(k_1 + k_2 - k'_1 - k'_2) \prod_i \frac{i}{k_i^2 - m^2 + i\epsilon} \frac{i}{k_i'^2 - m^2 + i\epsilon} \Gamma^{(4)}(k_1, k_2, -k'_1, -k'_2). \end{aligned} \quad (7)$$

where $\tau_i = \sqrt{(z_0 - y_i)^2}$ is the proper time along the world line of the particle from the start point to the interaction point.

So we take for the initial state

$$|\psi_{\text{in}}\rangle = \int d^4 x_1 d^4 x_2 f_1(x_1 - y_1) f_2(x_2 - y_2) \phi(x_1) \phi(x_2) |0\rangle \quad (2)$$

with $f_i(x)$ concentrated about $x = 0$, and the Fourier transform

$$\tilde{f}_i(k_i) = \int d^4 x e^{ik_i \cdot x} f_i(x) \quad (3)$$

concentrated about $k_i = p_i$ with $p_i^2 = m^2$. Here $\phi(x)$ is a real scalar field that, when acting on the vacuum, creates a stable particle of mass m .

Similarly, we will set up two detectors for the outgoing particles that each record only at a particular point in space at a specific time, that is, at space-time points y'_i . These points can also be taken as spacelike separated and at late times and large distances, $y_i^0 \gg 1/m$ and $|\vec{y}'_i| \gg 1/m$. And we want to absorb specific momenta, $p'_i = mv'_i$. If the outgoing particles emerge from a point z'_0 , then

$$\frac{y'_1 - z'_0}{\tau'_1} = v'_1 \quad \text{and} \quad \frac{y'_2 - z'_0}{\tau'_2} = v'_2. \quad (4)$$

So we take for the final state

$$|\psi_{\text{out}}\rangle = \int d^4 x'_1 d^4 x'_2 g_1(x'_1 - y'_1) g_2(x'_2 - y'_2) \times \phi(x'_1) \phi(x'_2) |0\rangle \quad (5)$$

with $g_i(x)$ concentrated about $x = 0$, and their Fourier transforms concentrated at $p'_i = mv'_i$.

Consider now the amplitude for the state $|\psi_{\text{in}}\rangle$ to evolve into the state $|\psi_{\text{out}}\rangle$,

$$\begin{aligned} \langle \psi_{\text{out}} | \psi_{\text{in}} \rangle &= \int d^4 x_1 d^4 x_2 d^4 x'_1 d^4 x'_2 g_1^*(x'_1 - y'_1) g_2^*(x'_2 - y'_2) \\ &\times f_1(x_1 - y_1) f_2(x_2 - y_2) \\ &\times \langle 0 | \phi(x'_2) \phi(x'_1) \phi(x_1) \phi(x_2) | 0 \rangle. \end{aligned} \quad (6)$$

Since we have initial points that are spacelike separated, the order of the fields at x_1 and x_2 is irrelevant and the same goes for the fields at x'_1 and x'_2 . Also, the fields at x'_i have later times than those at x_i . So we can replace the product of fields above by the time ordered product, which is just the four-point Green function,

We have written this in Fourier space in terms of the amputated four-point function. We will consider cases where the amputated four-point function $\Gamma^{(4)}(k_1, k_2, -k'_1, -k'_2)$ is the sum of three terms which depend, respectively, on the Mandelstam variables, s , t , and u . At large separation $|z_0 - z'_0|$ the amplitude $\langle \psi_{\text{out}} | \psi_{\text{in}} \rangle$ will be negligible except when there is a narrow s -channel resonance in the four-point function. This is clear if we use customary causal intuition, that a narrow resonance can be thought of as a long-lived unstable particle produced at z_0 decaying *later* at z'_0 , but mathematically

it is true even when the resonance “decays” at z'_0 before z_0 . Therefore, to examine the leading dependence of $\langle \psi_{\text{out}} | \psi_{\text{in}} \rangle$ on $z_0 - z'_0$, for large $|z_0 - z'_0|$, only the term that depends on $s = (k_1 + k_2)^2$ is important and we drop the other pieces. We denote this term by $\Gamma_s^{(4)}(s)$.

Now, multiplying the above by

$$1 = \int \frac{d^4 q}{(2\pi)^4} (2\pi)^4 \delta(k_1 + k_2 - q) \quad (8)$$

we get

$$\begin{aligned} \langle \psi_{\text{out}} | \psi_{\text{in}} \rangle &= \int d^4 x_1 d^4 x_2 d^4 x'_1 d^4 x'_2 g_1^*(x'_1 - y'_1) g_2^*(x'_2 - y'_2) f_1(x_1 - y_1) f_2(x_2 - y_2) \int \prod_i \frac{d^4 k_i}{(2\pi)^4} \frac{d^4 k'_i}{(2\pi)^4} \frac{d^4 q}{(2\pi)^4} \\ &\quad \times (2\pi)^4 \delta^{(4)}(k_1 + k_2 - q) (2\pi)^4 \delta^{(4)}(k'_1 + k'_2 - q) e^{i(k_1 \cdot x_1 + k_2 \cdot x_2 - k'_1 \cdot x'_1 - k'_2 \cdot x'_2)} \prod_i \frac{i}{k_i^2 - m^2 + i\epsilon} \frac{i}{k_i'^2 - m^2 + i\epsilon} \Gamma_s^{(4)}(q^2) \\ &= \int \frac{d^4 q}{(2\pi)^4} \tilde{F}(q) \tilde{G}(q) \Gamma_s^{(4)}(q^2) \end{aligned} \quad (9)$$

where we have introduced

$$\begin{aligned} \tilde{F}(q) &= \int \prod_i \frac{d^4 k_i}{(2\pi)^4} (2\pi)^4 \delta(k_1 + k_2 - q) e^{i \sum k_i \cdot y_i} \\ &\quad \times \tilde{f}_1(k_1) \tilde{f}_2(k_2) \prod_i \frac{i}{k_i^2 - m^2 + i\epsilon} \end{aligned} \quad (10)$$

and

$$\begin{aligned} \tilde{G}(q) &= \int \prod_i \frac{d^4 k'_i}{(2\pi)^4} (2\pi)^4 \delta(k'_1 + k'_2 - q) e^{-i \sum k'_i \cdot y'_i} \\ &\quad \times \tilde{g}_1^*(k'_1) \tilde{g}_2^*(k'_2) \prod_i \frac{i}{k_i'^2 - m^2 + i\epsilon}. \end{aligned} \quad (11)$$

The integral in (10) can be broken into two single particle integrals by representing the delta function as an integral,

$$\begin{aligned} \tilde{F}(q) &= \int d^4 z \int \prod_i \frac{d^4 k_i}{(2\pi)^4} e^{iz \cdot (q - k_1 - k_2) + i \sum k_i \cdot y_i} \\ &\quad \times \tilde{f}_1(k_1) \tilde{f}_2(k_2) \prod_i \frac{i}{k_i^2 - m^2 + i\epsilon} \\ &= \int d^4 z e^{iz \cdot q} I_1(z) I_2(z) \end{aligned} \quad (12)$$

where

$$I_i(z) = \int \frac{d^4 k_i}{(2\pi)^4} \tilde{f}_i(k_i) e^{ik_i \cdot (y_i - z)} \frac{i}{k_i^2 - m^2 + i\epsilon}. \quad (13)$$

We now estimate I_i . We will repeatedly use the stationary phase approximation, justified by considering $\hbar \rightarrow 0$. For clarity, we temporarily reinstate the explicit \hbar dependence. We rewrite I_i by exponentiating the propagator,

using the $i\epsilon$ of the normal particle propagator,

$$I_i = \frac{1}{\hbar} \int_0^\infty ds \int \frac{d^4 k_i}{(2\pi)^4} \tilde{f}_i(k_i) e^{ik_i \cdot (y_i - z)/\hbar} e^{is(k_i^2 - m^2 + i\epsilon)/\hbar}. \quad (14)$$

We first do the k_i integration by the method of stationary phase as $\hbar \rightarrow 0$. The condition that the phase be stationary is

$$\frac{\partial}{\partial k_i^\mu} (k_i \cdot (y_i - z) + s k_i^2) = 0, \quad (15)$$

which implies that

$$k_i = \frac{z - y_i}{2s}. \quad (16)$$

So we obtain

$$\begin{aligned} I_i &\simeq \frac{\hbar}{(2\pi)^4} e^{-i(\pi/2)} \int_0^\infty ds \left(\frac{\sqrt{\pi}}{s} \right)^4 \tilde{f}_i \left(\frac{z - y_i}{2s} \right) \\ &\quad \times e^{-i((z - y_i)^2/4\hbar s) + is(-m^2 + i\epsilon)/\hbar}. \end{aligned} \quad (17)$$

Now we perform the s integration also using the method of stationary phase. The condition is

$$\frac{\partial}{\partial s} \left[\frac{(z - y_i)^2}{4s} - s(-m^2 + i\epsilon) \right]_{s=s_i} = 0 \quad (18)$$

which implies that

$$s_i = \frac{\sqrt{(z - y_i)^2}}{2m} \quad (19)$$

and leads to

$$I_i \simeq \frac{e^{-i(3\pi/4)}}{(2\pi)^4} \frac{\pi^{5/2} \hbar^{3/2}}{ms_i^{3/2}} \tilde{f}_i \left(\frac{z - y_i}{2s_i} \right) e^{-im\sqrt{(z-y_i)^2}/\hbar}. \quad (20)$$

We now put the pieces together. First, putting the result for I_i above into \tilde{F} as given in (12), we have

$$\tilde{F} \simeq \int d^4z e^{iz \cdot q/\hbar} \frac{-i\hbar^3}{2^8 \pi^3 m^2 (s_1 s_2)^{3/2}} \tilde{f}_1 \left(\frac{z - y_1}{2s_1} \right) \tilde{f}_2 \left(\frac{z - y_2}{2s_2} \right) \times e^{-im\sqrt{(z-y_1)^2}/\hbar - im\sqrt{(z-y_2)^2}/\hbar}. \quad (21)$$

Let us investigate this function. Using the method of stationary phase, which is justified because we can take $\hbar \rightarrow 0$, or equivalently, because we can think of q or m as large, we see that the stationary phase condition gives $z = z_*$, with z_* satisfying

$$q - m \frac{z_* - y_1}{\sqrt{(z_* - y_1)^2}} - m \frac{z_* - y_2}{\sqrt{(z_* - y_2)^2}} = 0. \quad (22)$$

The second and third terms are just the arguments of the functions \tilde{f}_i , which, we recall, have localized support. So the function \tilde{F} is nonzero only for

$$\frac{z_* - y_1}{\sqrt{(z_* - y_1)^2}} = v_1 \quad \text{and} \quad \frac{z_* - y_2}{\sqrt{(z_* - y_2)^2}} = v_2 \quad (23)$$

which implies the support is at

$$q = mv_1 + mv_2 = p_1 + p_2. \quad (24)$$

Moreover, we know how to solve for z_* in the region where the integral has support: By Eq. (1) we see that $z_* = z_0$, the point where the interaction takes place. We then conclude that

$$\tilde{F}(q) \simeq e^{iq \cdot z_0/\hbar} \hat{F}(q) \quad (25)$$

where \hat{F} is smooth [22] and has support localized at $q = p_1 + p_2$. Note that we are absorbing the phase factor, $\exp(-im\sqrt{(z_0 - y_1)^2}/\hbar) \exp(-im\sqrt{(z_0 - y_2)^2}/\hbar) = \exp(-im\tau_1/\hbar - im\tau_2/\hbar)$, into the definition of \hat{F} . Similarly, we conclude that

$$\tilde{G}(q) \simeq e^{-iq \cdot z'_0/\hbar} \hat{G}(q) \quad (26)$$

with \hat{G} smooth and with support localized at $q = p'_1 + p'_2$.

Using this information on the structure of \tilde{F} and \tilde{G} in (9), we have

$$\langle \psi_{\text{out}} | \psi_{\text{in}} \rangle \simeq \int \frac{d^4q}{(2\pi)^4} e^{-iq \cdot (z'_0 - z_0)} \hat{F}(q) \hat{G}(q) \Gamma_s^{(4)}(q^2). \quad (27)$$

Here and below we revert to $\hbar = 1$ units, since the role of the limit $\hbar \rightarrow 0$ in our approximations should remain clear. Eventually, we will calculate the $w = z'_0 - z_0$ dependence of $\langle \psi_{\text{out}} | \psi_{\text{in}} \rangle$ for large w^0 in the Lee-Wick $O(N)$ model. Before considering the $O(N)$ model, it is instructive to consider some simpler examples.

B. Standard resonant behavior

Here we recover the more familiar time dependence associated with resonant s -channel exchange in a toy model with two real scalar fields ϕ and χ and Lagrange density,

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 + \frac{1}{2} \partial_\mu \chi \partial^\mu \chi - \frac{1}{2} M^2 \chi^2 + \frac{g}{2} \phi^2 \chi. \quad (28)$$

For simplicity, we work at weak coupling $g/M \ll 1$ and also take $m/M \ll 1$. The Fourier transform of χ 's two-point function has the form

$$D_\chi(p^2) = \frac{i}{\pi} \int_{4m^2}^{\infty} ds \frac{\rho(s)}{p^2 - s + i\epsilon} \simeq \frac{i}{p^2 - M^2 + iM\Gamma} \quad (29)$$

where at order g^2 in perturbation theory the χ width is equal to $\Gamma = g^2/(32\pi M)$. In the narrow resonance approximation

$$\rho(s) \simeq \frac{M\Gamma}{(s - M^2)^2 + M^2\Gamma^2}. \quad (30)$$

Since $\rho(s)$ in Eq. (30) is strongly peaked near $s = M^2$, we can extend the s integration in Eq. (29) over the whole real s line. Performing this integration using contour methods with the value for $\rho(s)$ in Eq. (30) reproduces the usual resonance form of the propagator on the far right-hand side of Eq. (29).

We calculate the dependence of $\langle \psi_{\text{out}} | \psi_{\text{in}} \rangle$ on $w = z'_0 - z_0$ for large proper time $\sqrt{w^2}$, under the assumption that the functions $F(q)$ and $G(q)$ are slowly varying and have most of their support around $q = p_1 + p_2$ and $q = p'_1 + p'_2$, respectively. We also assume that the coupling g is small. The amputated four-point function is

$$\Gamma_s^{(4)}(q^2) = -g^2 D_\chi(q^2), \quad (31)$$

and so Eq. (27) becomes

$$\begin{aligned} \langle \psi_{\text{out}} | \psi_{\text{in}} \rangle &= -ig^2 \int \frac{d^4q}{(2\pi)^4} \hat{F}(q) \hat{G}(q) e^{-iqw} \frac{1}{q^2 - M^2 + iM\Gamma} \\ &= -g^2 \int_0^\infty ds \int \frac{d^4q}{(2\pi)^4} \hat{F}(q) \hat{G}(q) e^{-iqw} \\ &\quad \times e^{is(q^2 - M^2 + iM\Gamma)}. \end{aligned} \quad (32)$$

The integration over the components of the momentum q is done using the stationary phase approximation. The stationary point is at $q = w/(2s)$ and we find that (up to a constant phase)

$$\begin{aligned} \langle \psi_{\text{out}} | \psi_{\text{in}} \rangle &\simeq \frac{g^2}{(2\pi)^2} \int_0^\infty ds \left(\frac{1}{2s} \right)^2 \hat{F}(w/(2s)) \hat{G}(w/(2s)) \\ &\quad \times e^{-i(w^2/(4s) + sM^2)} e^{-\Gamma Ms}. \end{aligned} \quad (33)$$

Next we perform the s integration using the stationary phase approximation. The stationary point is at $s = \sqrt{w^2}/(2M)$ and we arrive at

$$\langle \psi_{\text{out}} | \psi_{\text{in}} \rangle \simeq \frac{g^2 \sqrt{M}}{2(2\pi\sqrt{w^2})^{3/2}} \hat{F}(Mw/\sqrt{w^2}) \hat{G}(Mw/\sqrt{w^2}) \times e^{-iM\sqrt{w^2}} e^{-\Gamma\sqrt{w^2}/2}. \quad (34)$$

Since the functions $\hat{F}(q)$ and $\hat{G}(q)$ are peaked around $q = p_1 + p_2$ and $q = p'_1 + p'_2$, respectively, the amplitude is appreciable only if $p_1 + p_2 \approx p'_1 + p'_2 \approx Mw/\sqrt{w^2}$. In the center of mass (CM) frame, the initial state must be prepared with total energy near M , and the detectors in the final state are designed to find ordinary particles that are back to back, with total energy near M . The amplitude is dominated by $w^0 > 0$ and $\vec{w} \simeq 0$, and we can write, in the CM frame,

$$\langle \psi_{\text{out}} | \psi_{\text{in}} \rangle \simeq \theta(w^0) \frac{g^2 \sqrt{M}}{2(2\pi w^0)^{3/2}} \hat{F}(Mw/\sqrt{w^2}) \times \hat{G}(Mw/\sqrt{w^2}) e^{-iMw^0} e^{-\Gamma w^0/2}. \quad (35)$$

Note that the theta function means that the outgoing ϕ wave packets appear to emerge from the χ decay at a time z'_0 that is after the time z_0 that the incoming ϕ wave packets collide. The factor of $\exp(-\Gamma w^0/2)$ gives the characteristic exponential decay associated with the χ resonance, and the factor of $(1/w^0)^{3/2}$ arises from the spreading of the χ wave packet.

C. Lee-Wick resonant behavior

Here we illustrate the acausal behavior of $\langle \psi_{\text{out}} | \psi_{\text{in}} \rangle$ in the simple Lee-Wick toy model introduced in [5]. The Lagrange density for this theory is

$$\mathcal{L} = \frac{1}{2} \partial_\mu \hat{\phi} \partial^\mu \hat{\phi} - \frac{1}{2M^2} (\partial^2 \hat{\phi})^2 - \frac{1}{2} m^2 \hat{\phi}^2 - \frac{1}{3!} g \hat{\phi}^3. \quad (36)$$

The higher derivative term can be removed by adding a field $\tilde{\phi}$ in terms of which the Lagrange density becomes

$$\mathcal{L} = \frac{1}{2} \partial_\mu \hat{\phi} \partial^\mu \hat{\phi} - \frac{1}{2} m^2 \hat{\phi}^2 - \tilde{\phi} \partial^2 \hat{\phi} + \frac{1}{2} M^2 \tilde{\phi}^2 - \frac{1}{3!} g \hat{\phi}^3. \quad (37)$$

Next we define $\phi = \hat{\phi} + \tilde{\phi}$ since in terms of ϕ and $\tilde{\phi}$ the two derivative terms are not coupled. The Lagrange density now takes the form

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} \partial_\mu \tilde{\phi} \partial^\mu \tilde{\phi} + \frac{1}{2} M^2 \tilde{\phi}^2 - \frac{1}{2} m^2 (\phi - \tilde{\phi})^2 - \frac{1}{3!} g (\phi - \tilde{\phi})^3. \quad (38)$$

Provided that $M > 2m$ the mass matrix can be diagonalized by a symplectic transformation

$$\phi = \cosh\theta \phi' + \sinh\theta \tilde{\phi}', \quad (39a)$$

$$\tilde{\phi} = \sinh\theta \phi' + \cosh\theta \tilde{\phi}' \quad (39b)$$

where

$$\tanh 2\theta = -2m^2/(M^2 - 2m^2). \quad (39c)$$

The Lagrange density then takes the form

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi' \partial^\mu \phi' - \frac{1}{2} m'^2 \phi'^2 - \frac{1}{2} \partial_\mu \tilde{\phi}' \partial^\mu \tilde{\phi}' + \frac{1}{2} M'^2 \tilde{\phi}'^2 - \frac{1}{3!} g (\cosh\theta - \sinh\theta)^3 (\phi' - \tilde{\phi}')^3. \quad (40)$$

Defining $g' = g(\cosh\theta - \sinh\theta)^3$ and then dropping all the primes gives the Lagrange density in a convenient form. For simplicity, we take $m \ll M$.

The free field propagator for the normal scalar takes the usual form $i/(p^2 - m^2)$; however, the free field propagator for the Lee-Wick field, $\tilde{\phi}$, is $-i/(p^2 - M^2)$ which differs by an overall minus sign from a conventional scalar of mass M . That minus sign means that the propagator that one gets from summing Lee-Wick self-energy insertions develops a complex pole at $p^2 = M_c^2 = M^2 + iM\Gamma$. Note that this has a positive imaginary part. Since the propagator remains real and regular on a segment of the real axis (below the two-normal-particle cut), the propagator satisfies Schwarz reflection principle, $(D_{\tilde{\phi}}(p^2))^* = D_{\tilde{\phi}}(p^{2*})$. There is therefore a second pole at $p^2 = M_c^{*2}$. The propagator can be written as the sum of terms with poles at $M_c^2 = M^2 + iM\Gamma$, M_c^{*2} , and the two-particle cut,

$$D_{\tilde{\phi}}(p^2) = \frac{-i}{p^2 - M_c^2} + \frac{-i}{p^2 - M_c^{*2}} + \frac{i}{\pi} \int_{4m^2}^{\infty} ds \frac{\rho(s)}{p^2 - s + i\epsilon}. \quad (41)$$

These poles must not give rise to additional imaginary parts in matrix elements since only the normal ϕ particle is in the spectrum of the theory.

In the narrow resonance approximation the spectral density $\rho(s)$ is again given by Eq. (30). The spectral representation for the propagator given in Eq. (29) contains no poles in p^2 but rather has a cut associated with the integral over s . However, for a Lee-Wick resonance there are poles at $p^2 = M_c^2$ and $p^2 = M_c^{*2}$. In the narrow resonance approximation the term in Eq. (41) that has a pole at $p^2 = M_c^{*2}$ cancels against the term that contains the integral over s (i.e. the cut piece),

$$D_{\tilde{\phi}}(p^2) \simeq \frac{-i}{p^2 - M^2 - iM\Gamma}, \quad (42)$$

where $\Gamma \simeq g^2/(32\pi M)$.

We want to calculate the large w^0 behavior of $\langle \psi_{\text{out}} | \psi_{\text{in}} \rangle$ that arises from s -channel exchange of the Lee-Wick resonance $\tilde{\phi}$ at tree level making the same assumptions that we did in the case where there was s -channel exchange of the ordinary resonance χ . In the case of Lee-Wick resonant exchange,

$$\Gamma_s^{(4)}(q^2) = -g^2 D_{\tilde{\phi}}(q^2), \quad (43)$$

with $D_{\tilde{\phi}}$ given in Eq. (42). We follow the same steps that were used for the χ case. Since the width term in the propagator has the opposite sign, the phase of the exponential proportional to s must be flipped to get convergence at infinity. Hence, we find that

$$\begin{aligned} \langle \psi_{\text{out}} | \psi_{\text{in}} \rangle &= ig^2 \int \frac{d^4 q}{(2\pi)^4} \hat{F}(q) \hat{G}(q) e^{-iqw} \frac{1}{q^2 - M^2 - iM\Gamma} \\ &= -g^2 \int_0^\infty ds \int \frac{d^4 q}{(2\pi)^4} \hat{F}(q) \hat{G}(q) \\ &\quad \times e^{-iqw} e^{-is(q^2 - M^2 - iM\Gamma)}. \end{aligned} \quad (44)$$

The stationary point for the q integration is now at $q = -w/(2s)$, and we find that (up to a constant phase)

$$\begin{aligned} \langle \psi_{\text{out}} | \psi_{\text{in}} \rangle &\simeq \frac{g^2}{(2\pi)^2} \int_0^\infty ds \left(\frac{1}{2s} \right)^2 \hat{F}(-w/(2s)) \\ &\quad \times \hat{G}(-w/(2s)) e^{i(w^2/(4s) + sM^2)} e^{-\Gamma Ms}. \end{aligned} \quad (45)$$

The stationary point for the s integration is in the same place as before, $s = \sqrt{w^2}/(2M)$, and we arrive at

$$\begin{aligned} \langle \psi_{\text{out}} | \psi_{\text{in}} \rangle &\simeq \frac{g^2 \sqrt{M}}{2(2\pi \sqrt{w^2})^{3/2}} \hat{F}(-Mw/\sqrt{w^2}) \hat{G}(-Mw/\sqrt{w^2}) \\ &\quad \times e^{iM\sqrt{w^2}} e^{-\Gamma \sqrt{w^2}/2}. \end{aligned} \quad (46)$$

Since in the CM frame the functions $\hat{F}(q)$ and $\hat{G}(q)$ are peaked around $q^0 = M$ and $\vec{q} = 0$, the amplitude is dominated by $w^0 < 0$ and $\vec{w} \simeq 0$, and we can write

$$\begin{aligned} \langle \psi_{\text{out}} | \psi_{\text{in}} \rangle &\simeq \theta(-w^0) \frac{g^2 \sqrt{M}}{2(2\pi |w^0|)^{3/2}} \hat{F}(-Mw/\sqrt{w^2}) \\ &\quad \times \hat{G}(-Mw/\sqrt{w^2}) e^{-iMw^0} e^{\Gamma w^0/2}. \end{aligned} \quad (47)$$

The theta function in Eq. (47) means that the outgoing ϕ wave packets appear to emerge from the $\tilde{\phi}$ decay at a time z'_0 that is before the time z_0 that the incoming ϕ wave packets collide. The factor of $\exp(\Gamma w^0/2)$ gives exponential decay of the $\tilde{\phi}$ resonance that is backwards in time, and the factor of $(1/|w^0|)^{3/2}$ arises from the spreading of the $\tilde{\phi}$ wave packet.

III. REVIEW OF THE $O(N)$ MODEL

The theory contains N scalar fields $\phi^a(x)$, $a = 1, \dots, N$, and is invariant under orthogonal transformations of them, $\phi^a(x) \rightarrow \phi'^a(x) = O^{ab} \phi^b$. The Lagrange density is

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi^a \partial^\mu \phi^a - \frac{1}{2} m_0^2 \phi^a \phi^a - \frac{1}{8} \lambda_0 (\phi^a \phi^a)^2. \quad (48)$$

To get a sensible large- N limit of this theory, we must take $\lambda_0 \sim \mathcal{O}(1/N)$. Then the $\phi\phi \rightarrow \phi\phi$ scattering amplitude is $\mathcal{O}(1/N)$, the $\phi\phi \rightarrow \phi\phi\phi\phi$ amplitude is $\mathcal{O}(1/N^2)$, etc. This is similar to the behavior of QCD in the large number of colors limit [23]. There, color singlet mesons M with interpolating fields of the form [24] $\bar{q}q/\sqrt{N_c}$ have $MM \rightarrow MM$ scattering amplitudes that are $\mathcal{O}(1/N_c)$ and $MM \rightarrow MMMM$ scattering amplitudes that are $\mathcal{O}(1/N_c^2)$. However, in the case of the $O(N)$ model the $\phi\phi \rightarrow \phi\phi$ cross section averaged over initial values of the $O(N)$ quantum number a and summed over final values is $\mathcal{O}(1/N)$, while in the large- N_c limit of QCD $MM \rightarrow MM$ scattering cross sections are $\mathcal{O}(1/N_c^2)$.

It is convenient to remove the quartic interaction term in Eq. (48) by introducing a nondynamical scalar σ , and make the N dependence explicit by introducing $\lambda = N\lambda_0$. The Lagrangian density takes the form

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi^a \partial^\mu \phi^a - \frac{1}{2} m_0^2 \phi^a \phi^a + \frac{N}{2\lambda} \sigma^2 - \frac{1}{2} \sigma \phi^a \phi^a. \quad (49)$$

One can show that the Lagrange density in Eq. (49) is equivalent to that in Eq. (48) by integrating out the scalar σ using its equations of motion, $\sigma = \lambda \phi^a \phi^a / (2N)$. In this formulation the only interaction vertex is between a sigma and two ϕ 's.

It is straightforward to argue that in this formulation of the theory, at large N , the only loop diagrams that must be computed are a one-loop σ self-energy, Σ_0 , and a σ tadpole. The physical effects of the tadpole can be absorbed into the ϕ mass by making the replacement $m_0^2 \rightarrow m^2$. Using dimensional regularization with \overline{MS} subtraction,

$$\Sigma_0(p^2) = -\frac{N}{32\pi^2} \int_0^1 dx \log \left(\frac{m^2 - p^2 x(1-x) - i\epsilon}{\mu^2} \right), \quad (50)$$

where μ is the subtraction point. The sigma propagator is

$$D_\sigma(p^2) = \frac{i}{\frac{1}{\lambda_0} + \frac{i}{\lambda_0} i \Sigma_0(p^2) \frac{i}{\lambda_0}} + \dots = \frac{i}{1/\lambda_0 + \Sigma_0(p^2)}. \quad (51)$$

Writing the scattering matrix as $S = 1 + iT$, we define the scattering amplitude \mathcal{M} by

$$\begin{aligned} \langle k'_1, c; k'_2, d | T | k_1, a; k_2, b \rangle \\ = (2\pi)^4 \delta^4(k_1 + k_2 - k'_1 - k'_2) \\ \times \mathcal{M}(k_1, a; k_2, b \rightarrow k'_1, c; k'_2, d). \end{aligned} \quad (52)$$

Using our expression for the sigma propagator, we find that the scattering amplitude is given by

$$\mathcal{M}(k_1, a; k_2, b \rightarrow k'_1, c; k'_2, d) = -\frac{\lambda}{N} \left(\frac{\delta_{ab}\delta_{cd}}{1 + \lambda\Sigma(s)} + \dots \right), \quad (53)$$

where s, t, u are the usual Mandelstam variables and the ellipsis denotes the two terms, similar to the one presented, that are functions of t and u . Note we have written $\Sigma_0 = N\Sigma$ to make all the N dependence explicit.

A. Unitarity of the two-particle scattering amplitudes

With these results in hand, we can explicitly check unitarity of two-particle scattering in the $O(N)$ model to leading order in $1/N$ and to all orders in λ . Unitarity of the S matrix, i.e., $S^\dagger S = 1$, is equivalent to $i(T^\dagger - T) = T^\dagger T$. Taking the two-particle matrix element of this equation gives

$$\begin{aligned} & i(\mathcal{M}(k'_1, c; k'_2, d \rightarrow k_1, a; k_2, b)^* \\ & - \mathcal{M}(k_1, a; k_2, b \rightarrow k'_1, c; k'_2, d)) \\ & = \sum_\psi \mathcal{M}(k_1, a; k_2, b \rightarrow \psi) \mathcal{M}^*(k'_1, c; k'_2, d \rightarrow \psi). \end{aligned} \quad (54)$$

To simplify this equation, note first that at leading order in N , we may restrict the summation above to two-particle states. Next, we use the fact that the theory is invariant under the combined time reversal times parity discrete symmetry, so that $\mathcal{M}(a \rightarrow b)^* = \mathcal{M}(b \rightarrow a)$. Thus, the requirement of unitarity becomes

$$\begin{aligned} & \sum_{e,f} \iota_{ef} \int \frac{d^3 q_1}{(2\pi)^3 2E_1} \frac{d^3 q_2}{(2\pi)^3 2E_2} (2\pi)^4 \delta^4(q_1 + q_2 - p_1 - p_2) \mathcal{M}(k_1, a; k_2, b \rightarrow q_1, e; q_2, f) \mathcal{M}^*(k'_1, c; k'_2, d \rightarrow q_1, e; q_2, f) \\ & = \frac{N}{2} \delta_{ab} \delta_{cd} \int \frac{d^3 k_1}{(2\pi)^3 2E_1} \frac{d^3 k_2}{(2\pi)^3 2E_2} (2\pi)^4 \delta^4(k_1 + k_2 - p) \left| \frac{\lambda}{N} \frac{1}{1 + \lambda\Sigma(s)} \right|^2 \\ & = \frac{\lambda^2}{16\pi N} \delta_{ab} \delta_{cd} \sqrt{1 - \frac{4m^2}{s}} \left| \frac{1}{1 + \lambda\Sigma(s)} \right|^2, \end{aligned} \quad (58)$$

where $s = p^2$. Consequently, we see that the S matrix of the theory is unitary to leading order in N on the two-particle subspace of the Hilbert space. Notice that this argument is sensitive only to the imaginary part of $\Sigma(p^2)$. We will see that in the Lee-Wick case, the real part of the one-loop correction is changed but the imaginary part (for two-particle final states) is the same. Since the only nontrivial imaginary part is associated with the σ propagator, it should be clear that unitarity also holds for the higher particle parts of the Hilbert space.

B. Time dependence of two-particle scattering processes

In preparation for our discussion of acausal processes in the Lee-Wick $O(N)$ model, we review some aspects of the

$$\begin{aligned} & 2 \operatorname{Im} \mathcal{M}(k_1, a; k_2, b \rightarrow k'_1, c; k'_2, d) \\ & = \sum_{e,f} \iota_{ef} \int \frac{d^3 q_1}{(2\pi)^3 2E_1} \frac{d^3 q_2}{(2\pi)^3 2E_2} \\ & \quad \times (2\pi)^4 \delta^4(q_1 + q_2 - p_1 - p_2) \\ & \quad \times \mathcal{M}(k_1, a; k_2, b \rightarrow q_1, e; q_2, f) \\ & \quad \times \mathcal{M}^*(k'_1, c; k'_2, d \rightarrow q_1, e; q_2, f), \end{aligned} \quad (55)$$

where the identical particle factor ι_{ef} is equal to $1/2$ if $e = f$, and is unity otherwise. It is now trivial to check unitarity. First, notice that we can express the one-loop correction $\Sigma(s)$ as

$$\begin{aligned} \Sigma(p^2) & = -\frac{1}{32\pi^2} \left[\int_0^1 dx \log|m^2 - p^2 x(1-x)| \right. \\ & \quad \left. - i\pi \sqrt{1 - \frac{4m^2}{p^2}} \theta(p^2 - 4m^2) \right]. \end{aligned} \quad (56)$$

Therefore, since both t and u are negative, the imaginary part on the left-hand side of Eq. (55) is given by

$$2 \operatorname{Im} \mathcal{M}(k_1 a; k_2 b \rightarrow k'_1 c; k'_2 d) = \frac{\lambda^2}{16\pi N} \frac{\sqrt{1 - \frac{4m^2}{s}} \delta_{ab} \delta_{cd}}{|1 + \lambda\Sigma(s)|^2}. \quad (57)$$

On the other hand, the sum over e, f on the right-hand side of Eq. (55) is enhanced by one power of N when the scattering is in the s channel. Thus, to leading order, the sum is given by

time dependence of the two-particle scattering amplitude in the normal $O(N)$ model. To simplify the discussion we will assume $\lambda \ll 1$ and work to one-loop order in perturbation theory. This approximation allows us to do explicit computations while retaining the salient features of the causal structure of the time-dependent amplitude. Using the results of Sec. II A, the transition amplitude $\langle \psi_{\text{out}} | \psi_{\text{in}} \rangle$ is given by Eq. (27):

$$\langle \psi_{\text{out}} | \psi_{\text{in}} \rangle = \int \frac{d^4 q}{(2\pi)^4} e^{-iqw} \hat{F}(q) \hat{G}(q) \Gamma_s^{(4)}(q^2), \quad (59)$$

where $w = z'_0 - z_0$. The four-point function $\Gamma_s^{(4)}(q^2)$ can be deduced from Eq. (53) by expanding in the small parameter λ . We ignore the tree-level amplitude as it leads

to trivial time dependence of the amplitude, $\exp(-i(p_1 + p_2) \cdot w)$ times a function localized about $w = 0$. The one-loop four-point function describing $(a, a) \rightarrow (b, b)$ scattering is given by

$$\Gamma_s^{(4)}(q^2) = \frac{-i\lambda^2}{32\pi^2 N} \int_0^1 dx \log\left(\frac{m^2 - q^2 x(1-x) - i\epsilon}{\mu^2}\right). \quad (60)$$

Thus, the transition amplitude is

$$\begin{aligned} \langle \psi_{\text{out}} | \psi_{\text{in}} \rangle &= \frac{-i\lambda^2}{32\pi^2 N} \int_0^1 dx \int \frac{d^4 q}{(2\pi)^4} \hat{F}(q) \hat{G}(q) \\ &\times e^{i\vec{q} \cdot \vec{w}} \left(\frac{1}{-iw^0} \frac{d}{dq^0} \right) (e^{-iq^0 w^0}) \\ &\times \log(m^2 - q^2 x(1-x) - i\epsilon), \end{aligned} \quad (61)$$

where we have introduced a derivative with respect to q^0 . Integrating by parts, this derivative acts on $\hat{F}(q)$, $\hat{G}(q)$ and the logarithm. Since the functions $\hat{F}(q)$ and $\hat{G}(q)$ are slowly varying, we will only keep the term where the derivative acts on the logarithm. Therefore, the amplitude can be written as

$$\begin{aligned} \langle \psi_{\text{out}} | \psi_{\text{in}} \rangle &\simeq \frac{-\lambda^2}{32\pi^2 N w^0} \int_0^1 dx \int \frac{d^4 q}{(2\pi)^4} \hat{F}(q) \hat{G}(q) \\ &\times e^{-iq \cdot w} \frac{2q^0}{q^2 - m^2(x) + i\epsilon}, \end{aligned} \quad (62)$$

where $m(x) = m/\sqrt{x(1-x)}$. Introducing an integration over a variable s to write the propagator as a phase gives

$$\begin{aligned} \langle \psi_{\text{out}} | \psi_{\text{in}} \rangle &\simeq \frac{\lambda^2}{32\pi^2 N w^0} \int_0^1 dx \int_0^\infty ds \int \frac{d^4 q}{(2\pi)^4} \\ &\times \hat{F}(q) \hat{G}(q) e^{-iq \cdot w} e^{is(q^2 - m^2(x) + i\epsilon)} 2q^0. \end{aligned} \quad (63)$$

As before, we use the stationary phase approximation to evaluate the various integrations. The stationary point for the integrations over the components of q is located at $q = w/(2s)$, and performing these integrations gives (up to a constant phase)

$$\begin{aligned} \langle \psi_{\text{out}} | \psi_{\text{in}} \rangle &\simeq \frac{\lambda^2}{16\pi^2 N} \frac{1}{(2\pi)^2} \int_0^1 dx \int_0^\infty ds \left(\frac{1}{2s}\right)^3 \\ &\times e^{-i(w^2/(4s) + m^2(x)s - i\epsilon s)} \hat{F}(w/(2s)) \hat{G}(w/(2s)). \end{aligned} \quad (64)$$

Next, the s integration is performed using the method of stationary phase. The stationary point is at $s = \sqrt{w^2}/(2m(x))$, and we find that

$$\begin{aligned} \langle \psi_{\text{out}} | \psi_{\text{in}} \rangle &\simeq \frac{\lambda^2}{32\pi^2 N} \frac{1}{(2\pi)^{3/2}} \left(\frac{1}{\sqrt{w^2}}\right)^{5/2} \int_0^1 dx (m(x))^{3/2} \\ &\times \hat{F}(m(x)w/\sqrt{w^2}) \hat{G}(m(x)w/\sqrt{w^2}) e^{-im(x)\sqrt{w^2}}. \end{aligned} \quad (65)$$

Finally, we have to do the x integral. We use the method of stationary phase once again. The stationary point is at $x =$

$1/2$ and the transition amplitude is

$$\begin{aligned} \langle \psi_{\text{out}} | \psi_{\text{in}} \rangle &= \frac{\lambda^2}{64\pi^3 N} \frac{m}{(\sqrt{w^2})^3} e^{-2im\sqrt{w^2}} \hat{F}(2mw/\sqrt{w^2}) \\ &\times \hat{G}(2mw/\sqrt{w^2}). \end{aligned} \quad (66)$$

Recall that the functions $\hat{F}(q)$ and $\hat{G}(q)$ have support at $q = p_1 + p_2$ and $q = p'_1 + p'_2$, respectively. The amplitude is appreciable only if $p_1 + p_2 \approx p'_1 + p'_2 \approx 2mw/\sqrt{w^2}$. Since the energy is positive, choosing the CM frame the above can be rewritten as

$$\begin{aligned} \langle \psi_{\text{out}} | \psi_{\text{in}} \rangle &= \theta(w^0) \frac{\lambda^2}{64\pi^3 N} \frac{m}{(w^0)^3} e^{-2imw^0} \hat{F}(2mw/\sqrt{w^2}) \\ &\times \hat{G}(2mw/\sqrt{w^2}). \end{aligned} \quad (67)$$

It is worth comparing this expression with the transition amplitude in the case where the scattering is mediated by a resonance, Eq. (35). In that case there is an exponential decay due to the width of the resonance, which is absent in Eq. (67) because the mediators are stable. In the tree-level case the scattering is mediated by one particle; its wave packet spreads out like $(w^0)^{3/2}$. In the loop case the presence of two wave packets leads to a power-law falloff of the amplitude as $(w^0)^3$. The θ function in Eq. (67) indicates that the decay particles appear at times after the collision.

IV. THE LEE-WICK $O(N)$ MODEL

Let us now move on to study the Lee-Wick $O(N)$ model. We begin by discussing the Lagrangian of the model before examining the loop structure. Once the loop structure is understood at leading order in the $1/N$ expansion, we will compute the two-particle scattering to leading order in $1/N$ and to all orders in λ . We will use these results to demonstrate unitarity of the theory. Finally, we will explicitly compute the time dependence of one-loop scattering processes, demonstrating aspects of the acausality of the model.

A. The Lagrangian

The theory is the usual $O(N)$ model, augmented with a higher derivative term. The Lagrangian is given by

$$\begin{aligned} \mathcal{L} &= \frac{1}{2} \partial_\mu \hat{\phi}^a \partial^\mu \hat{\phi}^a - \frac{1}{2M^2} \partial^2 \hat{\phi}^a \partial^2 \hat{\phi}^a - \frac{1}{2} m_0^2 \hat{\phi}^a \hat{\phi}^a \\ &- \frac{\lambda}{8N} (\hat{\phi}^a \hat{\phi}^a)^2, \end{aligned} \quad (68)$$

where, as in the normal $O(N)$ model, the fields $\hat{\phi}^a$ are scalar fields in the fundamental representation of the group $O(N)$. We can remove the higher derivative term from the Lagrangian by introducing N scalar fields $\tilde{\phi}^a$. Then an equivalent Lagrange density is

$$\begin{aligned} \mathcal{L} = & \frac{1}{2} \partial_\mu \hat{\phi}^a \partial^\mu \hat{\phi}^a - \frac{1}{2} m_0^2 \hat{\phi}^a \hat{\phi}^a - \tilde{\phi}^a \partial^2 \hat{\phi}^a \\ & + \frac{1}{2} M^2 \tilde{\phi}^a \tilde{\phi}^a - \frac{\lambda}{8N} (\hat{\phi}^a \hat{\phi}^a)^2. \end{aligned} \quad (69)$$

We may diagonalize the derivative terms by defining $\phi^a = \hat{\phi}^a + \tilde{\phi}^a$ and performing an integration by parts. The Lagrangian becomes

$$\begin{aligned} \mathcal{L} = & \frac{1}{2} \partial_\mu \phi^a \partial^\mu \phi^a - \frac{1}{2} m_0^2 (\phi^a - \tilde{\phi}^a) (\phi^a - \tilde{\phi}^a) \\ & - \frac{1}{2} \partial_\mu \tilde{\phi}^a \partial^\mu \tilde{\phi}^a + \frac{1}{2} M^2 \tilde{\phi}^a \tilde{\phi}^a \\ & - \frac{\lambda}{8N} [(\phi^a - \tilde{\phi}^a) (\phi^a - \tilde{\phi}^a)]^2. \end{aligned} \quad (70)$$

This Lagrangian has a simple interpretation. There are N normal scalar fields ϕ^a and N Lee-Wick scalar fields $\tilde{\phi}^a$; these fields have quartic interactions. Note that the Lee-Wick scalars $\tilde{\phi}^a$ can decay to three ϕ^a quanta provided that $\tilde{\phi}^a$ is heavy enough. We will assume this decay channel is open so that the width Γ of the Lee-Wick scalars is nonzero. There is mass mixing between the normal and Lee-Wick scalars which can be removed by a symplectic transformation on the fields or treated as a perturbation when the ordinary scalars are very light compared with the Lee-Wick scalars. Neglecting the mass of the ordinary scalars, for large N and small coupling λ ,

$$\Gamma \simeq \frac{\lambda^2 M}{2^{10} \pi^3 N}. \quad (71)$$

To obtain a unitary S matrix it is crucial that the $\tilde{\phi}$ propagator have poles at complex energy, as in (41). For this reason, even though the width is order $1/N$, it will be important to retain it in intermediate steps in our calculations.

As was the case in the normal $O(N)$ model, it is convenient to introduce a nondynamical scalar field σ so that the Lagrangian may be written as

$$\begin{aligned} \mathcal{L} = & \frac{1}{2} \partial_\mu \phi^a \partial^\mu \phi^a - \frac{1}{2} m_0^2 (\phi^a - \tilde{\phi}^a) (\phi^a - \tilde{\phi}^a) \\ & + \frac{N}{2\lambda} \sigma^2 - \frac{1}{2} \partial_\mu \tilde{\phi}^a \partial^\mu \tilde{\phi}^a + \frac{1}{2} M^2 \tilde{\phi}^a \tilde{\phi}^a \\ & - \frac{1}{2} \sigma (\phi^a - \tilde{\phi}^a) (\phi^a - \tilde{\phi}^a). \end{aligned} \quad (72)$$

In this formulation of the theory, it is straightforward to establish a power-counting argument which shows that at leading order in $1/N$, the only relevant graphs are the σ tadpole and the 1PI σ self-energy $\Sigma_0(q^2)$. We can absorb the effects of the σ tadpole by replacing m_0^2 by m^2 . In the following, we will treat m as a small parameter. It remains to compute the self-energy, $\Sigma_0(q^2)$, of σ to leading order in $1/N$.

B. Loops

Before we embark on the computation of the self-energy, let us pause for a moment to consider the properties of the Lee-Wick resonances. We are familiar with the properties of ordinary resonances in quantum field theory. One familiar fact is the importance of resumming the width of a resonance in order to avoid the appearance of spurious poles in Feynman graphs. This resummation changes the analytic structure of the theory in a manner consistent with the nonperturbative information of the Lehmann representation. Similarly, resummation of the width of Lee-Wick resonances changes the analytic structure of the theory in crucial ways.

At tree level, the Lee-Wick propagator is

$$\tilde{D}(p^2) = -\frac{i}{p^2 - M^2}. \quad (73)$$

At loop level, the particle develops a width. Just as in the example we discussed earlier, in Sec. II C, the loop-corrected Lee-Wick propagator is given by

$$\begin{aligned} \tilde{D}(p^2) = & -\frac{i}{p^2 - M_c^2} - \frac{i}{p^2 - M_c^{*2}} \\ & + \frac{i}{\pi} \int_{9m^2}^{\infty} ds \frac{\rho(s)}{p^2 - s + i\epsilon}, \end{aligned} \quad (74)$$

where $M_c^2 = M^2 + iM\Gamma$. We shall use this form of the propagator to compute the σ self-energy even though the corrections due to the width are formally subdominant in the $1/N$ expansion. The subdominant corrections modify the analytic structure, and it is this modification that allows the theory to be unitary.

The poles present in the Lee-Wick propagator are in unusual locations in the complex p^2 plane, so we must take care to define the contour of integration in Feynman graphs appropriately. We must understand how to define expressions such as

$$I = \int \frac{d^4 p}{(2\pi)^4} \frac{-i}{(p+q)^2 - M_1^2} \frac{-i}{p^2 - M_2^2}, \quad (75)$$

where M_1 and M_2 may be complex masses, either in the upper or lower half-plane of the Feynman integration. Let us consider the p^0 integral. The integrand has four poles. Two of the poles are located at

$$p^0 = \pm \sqrt{\vec{p}^2 + M_2^2}. \quad (76)$$

The location of the other two poles depends on the value of the external four-momentum q . For timelike q we can go to a frame where $\vec{q} = 0$, and these two poles are located at

$$p^0 = -q^0 \pm \sqrt{\vec{p}^2 + M_1^2}. \quad (77)$$

The contour Lee and Wick suggested is such that, once the Green's functions are computed by Fourier transform from momentum space to space-time, there is no exponen-

tial growth in time, and it can be described as follows. Consider the position of the poles as a function of the coupling λ present in the theory. At $\lambda = 0$ the widths vanish, so M_1 and M_2 are real masses. Then the contour is defined to be the usual Feynman contour. As λ increases away from zero, the Lee-Wick particles become unstable, and the poles on the real line become complex pairs of poles that move away from the real axis. The Lee-Wick prescription is to deform the contour, as λ increases from zero, so that the complex poles do not cross the contour; see Fig. 1. Hence, a pole which was initially below the contour remains below the contour, for example.

If the external momentum is in the unphysical region, this prescription is unambiguous. For example, for the integral in (75), one can start with $|q^0| < |M_1 + M_2|$. As the momentum is varied (for fixed, nonzero λ) poles may cross a contour. However, the integral can still be defined by deforming the contour so as to avoid the pole. This leads to a well-defined contour unless poles pinch it. The pinching occurs when a pole in Eq. (76) coincides with one in Eq. (77), and signals the presence of a singularity, usually a branch cut, in the integral I (as a function of q). An additional prescription is required to define the integral in this case.

We need a prescription only when the new singularity occurs for real-valued energy q^0 . This may occur if one propagator carries mass M_1^2 while the other one has mass M_1^{*2} —that is, we could have $M_2^2 = M_1^{*2}$ in Eq. (75). Then it is easy to see that, when q^0 satisfies the equation

$$(q^0)^2 = 2(\vec{p}^2 + \text{Re}M_1^2) + 2|\vec{p}^2 + M_1^2|, \quad (78)$$

two of the poles in Eqs. (76) and (77) overlap and the contour is pinched. The CLOP prescription is as follows: Define the Feynman integral by taking the masses M_1^2 and M_1^{*2} to be unrelated complex mass parameters so that the poles do not overlap. At the end of the calculation, impose the condition that M_2^2 is the complex conjugate of M_1^2 . With this prescription, the self-energy Σ_0 is unambiguously defined and Lorentz invariant, and may be computed using standard methods. In particular, the contour we have chosen allows us to Wick-rotate the integral in Eq. (75) and in all Feynman integrals that we will encounter. In the following, we will compute the integrals in dimensional regularization and discard the divergent pieces that are proportional to $1/(d-4)$ in addition to the finite pieces involving the logarithm of 4π and Euler's constant. In the Lee-Wick $O(N)$ model they cancel since the theory is finite by naive power counting in the higher derivative formulation [25].

C. Computation of the self-energy

We now embark on the explicit computation of the self-energy. We define $\Sigma = N\Sigma_0$ as before. There are various graphs contributing. The graph involving only the normal particles reproduces the self-energy in the normal $O(N)$

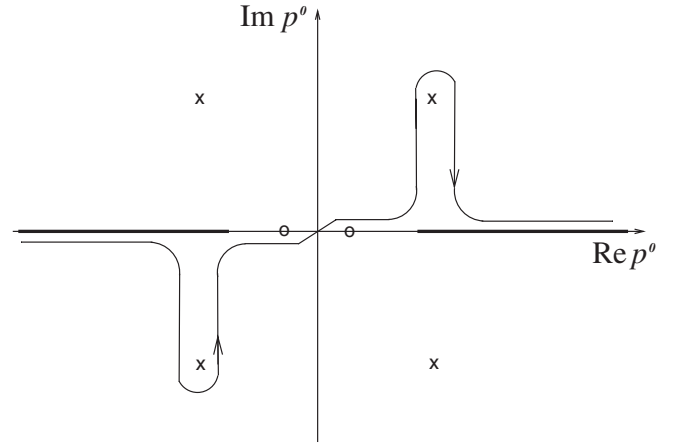


FIG. 1. Contour given by the Lee-Wick prescription for integration in the complex p^0 plane. The crosses denote the poles at $p^0 = \pm\sqrt{\vec{p}^2 + M_c^2}$ and at $p^0 = \pm\sqrt{\vec{p}^2 + M_c^{*2}}$ and the circles those at $p^0 = -q^0 \pm \sqrt{(\vec{p} + \vec{q})^2 + m^2}$. The heavy line denotes the cuts on the real axis starting at $\pm 3m$. The contour of integration is deformed as the interactions are turned on and the LW poles move into the complex plane so that the complex poles do not cross the contour.

model, Eq. (56). To simplify the notation, we define a function

$$F(M_1^2, M_2^2, q^2) = \frac{i}{2} \int \frac{d^d p}{(2\pi)^d} \frac{i}{(p+q)^2 - M_1^2} \frac{i}{p^2 - M_2^2}. \quad (79)$$

Discarding the divergent piece that is proportional to $1/(d-4)$ and the finite constant pieces involving the logarithm of 4π and Euler's constant, we find

$$F(M_1^2, M_2^2, q^2) = -\frac{1}{32\pi^2} \int_0^1 dx \log(xM_1^2 + (1-x)M_2^2 - x(1-x)q^2). \quad (80)$$

It is easy to verify that the only candidate singularity of $F(M_1^2, M_2^2, q^2)$, as a function of the complex variable q^2 , is a branch cut with branch point at $q^2 = (M_1 + M_2)^2$. The σ self-energy can then be expressed in terms of sums of these functions evaluated at various arguments. Thus, the contribution Σ_1 of the normal particles to the total self-energy Σ is

$$\Sigma_1(q^2) = F(m^2, m^2, q^2). \quad (81)$$

Since one of our main goals is to understand unitarity of the theory, we will focus on understanding any possible imaginary parts of Σ .

Next we consider graphs with both propagators being of the Lee-Wick field, (74). It is convenient to consider the contributions of the pole parts involving the Lee-Wick particles separately from contributions involving the spectral density $\rho(s)$. First, consider the terms involving only the Lee-Wick poles. Following the CLOP prescription we use different complex masses for the two propagators, with $M_1^2 = M_c^2 + i\delta$ and $M_2^2 = M_c^2$. The loop integral is

$$\begin{aligned}
\Sigma_2(q^2) &= -\frac{i}{2} \int \frac{d^d p}{(2\pi)^d} \left[\frac{1}{p^2 - M_c^2 - i\delta} \right. \\
&\quad \left. + \frac{1}{p^2 - M_c^{*2} + i\delta} \right] \left[\frac{1}{(p+q)^2 - M_c^2} \right. \\
&\quad \left. + \frac{1}{(p+q)^2 - M_c^{*2}} \right] \\
&= F(M_c^2 + i\delta, M_c^2, q^2) + F(M_c^{*2} - i\delta, M_c^{*2}, q^2) \\
&\quad + F(M_c^2 + i\delta, M_c^{*2}, q^2) + F(M_c^{*2} - i\delta, M_c^2, q^2). \tag{82}
\end{aligned}$$

It is easy to see that Σ_2 is continuous across the real line. The CLOP prescription has effectively moved the two branch points that would have occurred at $q^2 = (M_c + M_c^*)^2$ away from the real axis, by an amount of order δ , to $\sqrt{M_c^2 + i\delta} + M_c^*$ and $\sqrt{M_c^{*2} - i\delta} + M_c$. The remaining two terms appearing in the self-energy have complex branch points even for $\delta = 0$. Hence the discontinuity across the real line vanishes, and this persists in the limit that δ goes to zero. An explicit example may clarify this. The expression $F(M_c^2 + i\delta, M_c^{*2}) + F(M_c^{*2} - i\delta, M_c^2)$ contains the dangerous terms appearing in the Feynman integral in which poles on opposite sides of the contour may pinch when $\delta = 0$ (and q^2 is real). But this expression is explicitly real on the real axis and analytic in a band of width $\sim \delta$ containing the whole real axis:

$$\begin{aligned}
&F(M_c^2 + i\delta, M_c^{*2}, q^2) + F(M_c^{*2} - i\delta, M_c^2, q^2) \\
&= -\frac{1}{32\pi^2} \int_0^1 dx [\log(x(M_c^2 + i\delta) + (1-x)M_c^{*2} - x(1-x)q^2) \\
&\quad + \log(x(M_c^{*2} - i\delta) + (1-x)M_c^2 - x(1-x)q^2)]. \tag{83}
\end{aligned}$$

Since for real-valued q^2 the imaginary part of this vanishes (there is no need to define this as a discontinuity) independent of $\delta \neq 0$, the imaginary part remains zero in the limit as $\delta \rightarrow 0$. In the remainder of this section, we will omit the parameter δ to simplify the equations.

In the next section we will use the self-energy to verify the unitarity of the S matrix in this theory. It will be useful to write the result for the self-energy concisely. While the width is of order $1/N$, its presence is crucial in demonstrating that Σ_2 is real. But once we establish that the CLOP-defined Σ_2 is real, we can neglect the width and give a simple expression for the self-energy:

$$\Sigma_2(q^2) = -\frac{1}{16\pi^2} \int_0^1 dx \log|M^2 - x(1-x)q^2|. \tag{84}$$

Next, we compute the Feynman integrals involving the Lee-Wick pole and the normal particle. We find that the self-energy in this case is given by

$$\begin{aligned}
\Sigma_3(q^2) &= \frac{i}{2} \int \frac{d^d p}{(2\pi)^d} \left[\frac{1}{p^2 - m^2} \left(\frac{1}{(p+q)^2 - M_c^2} \right. \right. \\
&\quad \left. \left. + \frac{1}{(p+q)^2 - M_c^{*2}} \right) + \frac{1}{(p+q)^2 - m^2} \right. \\
&\quad \left. \times \left(\frac{1}{p^2 - M_c^2} + \frac{1}{p^2 - M_c^{*2}} \right) \right] \\
&= -2F(m^2, M_c^2, q^2) - 2F(m^2, M_c^{*2}, q^2). \tag{85}
\end{aligned}$$

The branch points are both off the real axis, at $q^2 = (M_c + m)^2$ and $(M_c^* + m)^2$. As above, since Σ_3 is real we can neglect the width and write, concisely,

$$\begin{aligned}
\Sigma_3(q^2) &= \frac{1}{16\pi^2} \int_0^1 dx \log|xM^2 + (1-x)m^2 \\
&\quad - x(1-x)q^2|^2. \tag{86}
\end{aligned}$$

Now we turn to terms involving the spectral density $\rho(s)$. Terms in the self-energy involving products of the Lee-Wick poles and the spectral density give

$$\begin{aligned}
\Sigma_4 &= \frac{i}{\pi} \int_{9m^2}^{\infty} ds \rho(s) \int \frac{d^d p}{(2\pi)^d} \frac{1}{p^2 - s + i\epsilon} \\
&\quad \times \left(\frac{1}{(p+q)^2 - M_c^{*2}} + \frac{1}{(p+q)^2 - M_c^2} \right) \\
&= -\frac{2}{\pi} \int_{9m^2}^{\infty} ds \rho(s) [F(s, M_c^2, q^2) + F(s, M_c^{*2}, q^2)]. \tag{87}
\end{aligned}$$

The sum of F functions is similar to that appearing in Eq. (85): Evidently, this is also real and so the integral against ρ is real. In the narrow resonance approximation, we find that the self-energy due to these terms is

$$\Sigma_4 = \frac{1}{16\pi^2} \int_0^1 dx \log|M^2 - x(1-x)q^2|^2. \tag{88}$$

Finally, there are terms involving the spectral density ρ and the normal pole, and involving a double integral over two powers of ρ . These terms do lead to an imaginary part, describing real scattering from two-particle states into four- or six-particle states. It is important to understand that the imaginary parts arising from these expressions involve final states containing only normal particles. In the narrow resonance approximation, we find that these terms lead to a contribution to the self-energy given by

$$\begin{aligned}
\Sigma_5 &= -\frac{1}{32\pi^2} \int_0^1 dx \log|M^2 - x(1-x)q^2| \\
&\quad + \frac{i\pi}{32\pi^2} \sqrt{1 - \frac{4M^2}{q^2}} \theta(q^2 - 4M^2) \\
&\quad - \frac{1}{16\pi^2} \int_0^1 dx \log|xM^2 + (1-x)m^2 - x(1-x)q^2| \\
&\quad + \frac{i\pi}{16\pi^2} \left(1 - \frac{(m+M)^2}{q^2} \right) \theta(q^2 - (m+M)^2). \tag{89}
\end{aligned}$$

In total, we find an explicit expression for the self-energy which is simple when we treat the width Γ to be negligible compared to the mass M :

$$\begin{aligned} \Sigma(q^2) = & -\frac{1}{32\pi^2} \left[\int_0^1 dx \log \frac{|x(1-x)q^2| |M^2 - x(1-x)q^2|}{|xM^2 - x(1-x)q^2|^2} \right. \\ & - i\pi\theta(q^2) - i\pi\sqrt{1 - \frac{4M^2}{q^2}}\theta(q^2 - 4M^2) \\ & \left. - 2i\pi\left(1 - \frac{M^2}{q^2}\right)\theta(q^2 - M^2) \right], \end{aligned} \quad (90)$$

where we have neglected the normal mass m . Notice that the width Γ of the Lee-Wick resonances does not appear in this result. It was important for defining the contour for the loop integration but not in the final form of the answer. It will also play a role in our understanding of the unitarity of the S matrix.

D. Unitarity

Unitarity of the S matrix is equivalent to requiring $i(T^\dagger - T) = T^\dagger T$. We consider two-particle matrix elements of the right- and left-hand sides of this equation and verify their equality to leading order in $1/N$. For convenience, we restate the requirement of unitarity for the amplitude describing scattering of a two-particle state into another two-particle state:

$$\begin{aligned} i(\mathcal{M}(k'_1, c; k'_2, d \rightarrow k_1, a; k_2, b)^* \\ - \mathcal{M}(k_1, a; k_2, b \rightarrow k'_1, c; k'_2, d)) \\ = \sum_\psi \mathcal{M}(k_1, a; k_2, b \rightarrow \psi) \mathcal{M}^*(k'_1, c; k'_2, d \rightarrow \psi), \end{aligned} \quad (91)$$

where $\mathcal{M}(k_1, a; k_2, b \rightarrow k'_1, c; k'_2, d)$ is the amplitude for the two-particle scattering. Previously, in our discussion of unitarity of scattering in the normal $O(N)$ model, we argued that the only allowed final state at leading order is the two-particle final state. However, the situation is different in the Lee-Wick $O(N)$ model. At leading order, two-, four-, and six-particle final states are accessible. Intuitively, this is because we can create Lee-Wick resonances which subsequently decay into three normal particles with unit probability. This is the reason it was necessary to retain the width of the Lee-Wick resonances, even if it is subleading in $1/N$; after all, for a nonzero width, however small, given enough time the unstable ‘‘particle’’ will decay. Therefore, we will have to include these additional final states in the sum of the right-hand side of Eq. (91). In Eq. (91) the initial and final states and the intermediate states ψ only involve the stable ordinary particles. Even though the propagator for the Lee-Wick resonances contains poles at $p^2 = M_c^2$ and $p^2 = M_c^{*2}$, these particles are not considered to be in the spectrum of the theory.

The two-particle scattering amplitude in the Lee-Wick theory is given in terms of the self-energy by the same

expression as in the normal $O(N)$ model:

$$\mathcal{M}(k_1, a; k_2, b \rightarrow k'_1, c; k'_2, d) = -\frac{\lambda}{N} \left(\frac{\delta_{ab}\delta_{cd}}{1 + \lambda\Sigma(s)} + \dots \right), \quad (92)$$

where $s = (k_1 + k_2)^2$ as usual, and the dots indicate the t and u channel terms in addition to higher order terms in $1/N$. Since t and u are negative quantities for physical scattering, we find that the left-hand side of the unitarity relation Eq. (91) is

$$\begin{aligned} i(\mathcal{M}(k'_1, c; k'_2, d \rightarrow k_1, a; k_2, b)^* \\ - \mathcal{M}(k_1, a; k_2, b \rightarrow k'_1, c; k'_2, d)) \\ = \frac{\lambda^2}{16\pi N} \frac{\delta_{ab}\delta_{cd}}{|1 + \lambda\Sigma(s)|^2} \left(1 + \sqrt{1 - \frac{4M^2}{s}}\theta(s - 4M^2) \right. \\ \left. + 2\left(1 - \frac{M^2}{s}\right)\theta(s - M^2) \right). \end{aligned} \quad (93)$$

For simplicity, and without loss of generality, here and below we neglect the normal mass m . Now we must compute the right-hand side of the unitarity relation. This is straightforward when the state $|\psi\rangle$ in Eq. (91) is a two-particle state; in that case the sum becomes an integral over the two-body phase space of the normal particles and the amplitude is simply the two-to-two scattering amplitude of Eq. (92). Since we are neglecting the mass of ϕ^a the sum becomes

$$\begin{aligned} \sum_{|\psi\rangle=|q_1, e; q_2, f\rangle} \mathcal{M}(k_1, a; k_2, b \rightarrow \psi) \mathcal{M}^*(k'_1, c; k'_2, d \rightarrow \psi) \\ = \frac{\lambda^2}{16\pi N} \delta_{ab}\delta_{cd} \left| \frac{1}{1 + \lambda\Sigma(s)} \right|^2. \end{aligned} \quad (94)$$

We see that this part of the complete sum over accessible final states $|\psi\rangle$ reproduces the first term in parentheses on the right-hand side of Eq. (93). The other two terms arise from four- and six-particle final states. Consider first the process with a four-particle final state. To leading order in N , the two-to-four particle scattering must contain an intermediate $\tilde{\phi}$ which then decays; the Feynman graph is shown in Fig. 2.

Since we are summing over all final states in the decay of the Lee-Wick resonance, we only need to compute the amplitude to make the intermediate state containing a Lee-Wick resonance and one normal particle, and then integrate over the two-particle phase space. This is familiar from the theory of ordinary resonances. In more detail, the four-particle phase space integral can be organized into an integral over the three-body phase space for the Lee-Wick decay into three normal particles and an integral over the two-particle phase space of the Lee-Wick particle and the fourth normal particle times an energy-momentum conserving delta function. The result of the three-particle phase space integral is the width of the Lee-Wick particle; however, in the region of the two-body phase space where the Lee-Wick particle is nearly on shell, a factor of $1/\Gamma$

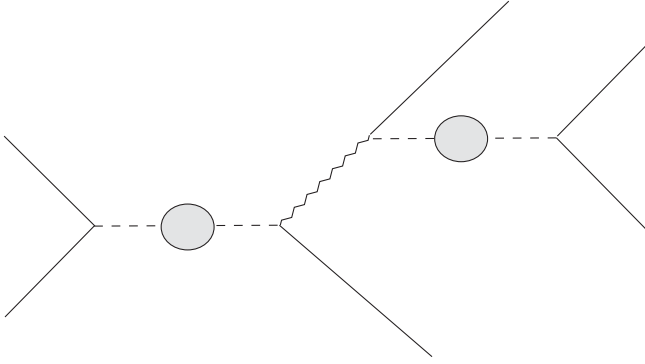


FIG. 2. Feynman diagram for the two-to-four scattering amplitude proceeding through the decay of a Lee-Wick particle. The solid line denotes a “normal” particle, the zigzag line a “Lee-Wick” particle, and the dashed line with the shaded blob denotes the dressed σ auxiliary field propagator.

appears from the Lee-Wick propagator. Thus, the whole computation reduces to a simple integral over the two-body phase space. The remaining parts of the integrand can then be interpreted as the amplitude to create the intermediate Lee-Wick particle plus one of the final state normal particles.

The amplitude to create the intermediate state is given, up to an overall irrelevant sign, by Eq. (92). Thus, we can explicitly perform the sum over four-body phase space to find

$$\begin{aligned} & \sum_{4 \text{ particle}} \mathcal{M}(k_1, a; k_2, b \rightarrow \psi) \mathcal{M}^*(k'_1, c; k'_2, d \rightarrow \psi) \\ &= \frac{2\lambda^2}{16\pi N} \delta_{ab} \delta_{cd} \left(1 - \frac{M^2}{s}\right) \theta(s - M^2) \left| \frac{1}{1 + \lambda \Sigma(s)} \right|^2. \end{aligned} \quad (95)$$

Similarly, the six-particle phase space integral becomes an integral over the two-particle phase space of two intermediate Lee-Wick particles. Near the region where the Lee-Wick particles are on shell, there is an enhancement by $1/\Gamma^2$. The result is

$$\begin{aligned} & \sum_{6 \text{ particle}} \mathcal{M}(k_1, a; k_2, b \rightarrow \psi) \mathcal{M}^*(k'_1, c; k'_2, d \rightarrow \psi) \\ &= \frac{\lambda^2}{16\pi N} \delta_{ab} \delta_{cd} \sqrt{1 - \frac{4M^2}{s}} \theta(s - 4M^2) \left| \frac{1}{1 + \lambda \Sigma(s)} \right|^2. \end{aligned} \quad (96)$$

Since the sum of Eqs. (94)–(96) equals Eq. (93), we have verified, for two-particle matrix elements, that $i(T^\dagger - T) = T^\dagger T$.

It is easy to extend this argument to show unitarity for any matrix element. To leading order in $1/N$ any amplitude is given by a sum of skeleton diagrams with the propagators, including the σ propagator, replaced by the full propagators. In the absence of a Kallen-Lehman decomposition, we cannot proceed with the usual cutting rules to show unitarity. For example, it is not obvious how to set up

a “largest time equation” [26]. However, one can still analyze individual graphs by cutting the diagrams. A cut through a σ propagator is handled using the results for the two-to-two amplitude demonstrated above. Cuts through normal particle propagators never produce an imaginary part: They are just as in the standard analysis and, since we only have skeleton graphs, these propagators are never on shell. Finally, there are “cuts” through the Lee-Wick propagators. These just correspond to taking the imaginary part of $\tilde{D}(p^2)$ in (74). The imaginary part of the sum of complex poles vanishes. We are left with the imaginary part of the integral over the spectral function $\rho(s)$. This has precisely the structure that a normal resonance in the standard unitarity analysis has, so it leads to the correct unitarity relation. In particular, it corresponds to a sum, in $T^\dagger T$, over intermediate three-normal-particle states.

Since the S matrix provides a one-to-one map from the past to the future in scattering experiments, the existence of a well-defined S matrix is enough to show that there can be no paradoxes in these scattering processes. Nevertheless, the theory is acausal as we shall now explore.

E. Time dependence: Acausality

To study the time dependence of scattering in the Lee-Wick theory, we will work to one-loop order in perturbation theory. The graph containing normal particles reproduces the transition amplitude of the normal $O(N)$ model, shown in Eq. (67). Our main focus is on the acausal behavior associated with poles in the upper half-plane. All the acausality decays exponentially with time except for the case where, in the loop, one of the poles is at M_c^2 and the other is at M_c^{*2} . Then for real incoming momentum one can create an on-shell configuration with two Lee-Wick resonances, and this leads to acausal behavior that falls off with a power of time. In this section we calculate this power-law acausal behavior. The part of the four-point function with Lee-Wick poles at M_c^2 and M_c^{*2} is

$$\begin{aligned} \Gamma_s^{(4)}(q^2) &= \frac{-i\lambda^2}{16\pi^2 N} \int_0^1 dx \log(M^2 - i(1 - 2x)M\Gamma \\ &\quad - x(1 - x)q^2). \end{aligned} \quad (97)$$

Since the sign of the $M\Gamma$ term in the logarithm changes over the region of integration, it is convenient to break the integral into two terms as

$$\begin{aligned} \Gamma_s^{(4)}(q^2) &= \frac{-i\lambda^2}{16\pi^2 N} \int_0^{1/2} dx [\log(M^2 - i(1 - 2x)M\Gamma \\ &\quad - x(1 - x)q^2) + \log(M^2 + i(1 - 2x)M\Gamma \\ &\quad - x(1 - x)q^2)]. \end{aligned} \quad (98)$$

We make the same assumptions as before. In particular, the functions $\hat{F}(q)$ and $\hat{G}(q)$ are taken to be slowly varying and to have support around $q^0 = 2M$ and $\vec{q} = 0$. We find it convenient to decompose the transition amplitude as

$$\langle \psi_{\text{out}} | \psi_{\text{in}} \rangle = \langle \psi_{\text{out}} | \psi_{\text{in}} \rangle_+ + \langle \psi_{\text{out}} | \psi_{\text{in}} \rangle_- \quad (99)$$

where

$$\begin{aligned} \langle \psi_{\text{out}} | \psi_{\text{in}} \rangle_{\pm} \simeq & -\frac{\lambda^2}{8\pi^2 N w^0} \int_0^{1/2} dx \int \frac{d^4 q}{(2\pi)^4} \hat{F}(q) \hat{G}(q) e^{-iq \cdot w} \\ & \times \left[\frac{q^0}{q^2 - M(x)^2 \pm i \frac{1-2x}{x(1-x)} M\Gamma} \right]. \end{aligned} \quad (100)$$

Here, $M(x) = M/\sqrt{x(1-x)}$. We now put the denominator of the propagator into an exponential by introducing an integration over a proper time variable s ,

$$\begin{aligned} \langle \psi_{\text{out}} | \psi_{\text{in}} \rangle_{\pm} \simeq & \frac{\pm i \lambda^2}{8\pi^2 N w^0} \int_0^{1/2} dx \int_0^{\infty} ds \int \frac{d^4 q}{(2\pi)^4} \\ & \times \hat{F}(q) \hat{G}(q) q^0 \exp(-iq \cdot w \\ & \pm is(q^2 - M(x)^2) \\ & - s(1-2x)M\Gamma/(x(1-x))). \end{aligned} \quad (101)$$

It is now straightforward to successively do the q , s , and x integrations using the stationary phase approximation. The stationary points are at $q = \pm w/(2s)$, $s = \sqrt{w^2}/(2M(x))$, and $x = 1/2$, and we find that (up to an overall constant phase)

$$\begin{aligned} \langle \psi_{\text{out}} | \psi_{\text{in}} \rangle_{\pm} \simeq & \frac{\lambda^2 M}{32(\sqrt{w^2})^3 \pi^3 N} e^{\mp i 2M\sqrt{w^2}} \hat{F}(\pm 2Mw/\sqrt{w^2}) \\ & \times \hat{G}(\pm 2Mw/\sqrt{w^2}). \end{aligned} \quad (102)$$

Given where the functions \hat{F} and \hat{G} have support, we can rewrite this as

$$\begin{aligned} \langle \psi_{\text{out}} | \psi_{\text{in}} \rangle_{\pm} \simeq & \frac{\theta(\pm w^0) \lambda^2 M}{32|w^0|^3 \pi^3 N} e^{-i 2Mw^0} \hat{F}(\pm 2Mw/\sqrt{w^2}) \\ & \times \hat{G}(\pm 2Mw/\sqrt{w^2}). \end{aligned} \quad (103)$$

We have checked by explicit calculation that the other one-loop contributions are exponentially suppressed in $|w^0|$, and so for very large $|w^0|$ the power-law term that falls off as $1/|w^0|^3$, displayed above, dominates the acausality in the one-loop contribution to $\langle \psi_{\text{out}} | \psi_{\text{in}} \rangle$. Note that Eq. (103) has a very different behavior than one would expect based on the example of single Lee-Wick resonant exchange that we discussed earlier. It is not exponentially suppressed for large times and contains both acausal and causal pieces.

V. CONCLUDING REMARKS

We have studied the Lee-Wick $O(N)$ model and argued that the prescription of Lee and Wick and Cutkosky *et al.* yields an S matrix for this theory that is unitary and Lorentz invariant in large N . This suggests that, even though the theory is not causal, there will not be paradoxical behavior in scattering experiments.

In this model we demonstrated, by explicit calculation, some of the acausal behavior in two-to-two scattering of the ordinary scalars that arises from virtual ‘‘Lee-Wick particles.’’ The Lee-Wick $O(N)$ model presents a playground to examine the consistency of theories where causality emerges only for long enough times and low enough energies. There are other theories that are worth exploring for this purpose. For example, there are two-dimensional models that can be solved exactly, and it would be interesting to see if Lee-Wick versions of some of these theories are also soluble and, if so, explore their properties.

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