Crossover between Abelian and non-Abelian confinement in $\mathcal{N} = 2$ supersymmetric QCD

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In this paper we investigate the nature of the transition from Abelian to non-Abelian confinement (i.e. crossover vs phase transition). To this end we consider the basic $\mathcal{N} = 2$ model where non-Abelian flux tubes (strings) were first found: supersymmetric QCD with the U(N) gauge group and $N_f = N$ flavors of fundamental matter (quarks). The Fayet-Iliopoulos term ξ triggers the squark condensation and leads to the formation of non-Abelian strings. There are two adjustable parameters in this model: ξ and the quark mass difference Δm. We obtain the phase diagram on the ($\xi, \Delta m$) plane. At large ξ and small Δm the world-sheet dynamics of the string orientational moduli is described by the $\mathcal{N} = 2$ two-dimensional $CP(N-1)$ model. We show that as we reduce ξ the theory exhibits a crossover to the Abelian (Seiberg-Witten) regime. Instead of N^2 degrees of freedom of non-Abelian theory, now only N degrees of freedom survive in the low-energy spectrum. Dyons with certain quantum numbers condense, leading to the formation of the Abelian Z_N strings whose fluxes are fixed inside the Cartan subalgebra of the gauge group. As we increase N this crossover grows exceedingly sharper, becoming a genuine phase transition at $N = \infty$.

I. INTRODUCTION

Transition from Abelian to non-Abelian confinement emerged as a central question in the current explorations of Yang-Mills theories. By Abelian confinement we mean that only those gauge bosons that lie in the Cartan sub-algebra are dynamically important in the infrared, i.e. at distances of the order of the flux tube (string) size. By non-Abelian confinement we mean such a dynamical regime in which at distances of the flux tube formation all gauge bosons are equally important. In supersymmetric $\mathcal{N} = 2$ Yang-Mills theories slightly deformed by a $\mu\text{Tr}\Phi^2$ term, linear confinement was discovered [1], explained by the dual Meissner effect. In the limit of small μ amenable to analytic studies [1], confinement is Abelian. It is believed that as μ gets large, $\mu \gtrsim \Lambda$, a smooth transition to non-Abelian confinement takes place in the Seiberg-Witten model.

In nonsupersymmetric theories a similar-purpose construction, with an adjustable parameter, was engineered in [2] (see also [3]), where Yang-Mills theories on $R_3 \times S_1$ were considered. The radius of the compact dimension $r(S_1)$ was treated as a free parameter. At small $r(S_1)$, after a center-symmetric stabilization, linear Abelian confinement sets in by virtue of the Polyakov mechanism [4]. Then it was argued that the transition from the small-$r(S_1)$ Abelian confinement regime to the decompactification limit of large r, $r(S_1) \gg \Lambda^{-1}$, where confinement is non-Abelian, is smooth. No obvious order parameter that could discontinuously change in passing from small to large $r(S_1)$ was detected.

This paper presents our new results on this issue. The nature of the transition from the Abelian to non-Abelian regime—i.e. phase transition vs crossover—appears to be nonuniversal. If there is a discrete symmetry on the string world sheet and the mode of realization of this symmetry changes in passing from the Abelian to non-Abelian regime, then these two domains are separated by a phase transition. A particular nonsupersymmetric example [5,6] of such a situation will be discussed in the bulk of the paper.

On the other hand, if the mode of realization of the discrete symmetry does not change, or there is no appropriate symmetry whatsoever, then the Abelian confinement is separated from the non-Abelian one by a crossover rather than a phase transition. In this paper we focus on $\mathcal{N} = 2$ Yang-Mills theory with the gauge group U(N) and $N_f = N$ flavors. To simplify our discussion we mostly consider the $N = 2$ case of U(2) theory with two flavors. We will show that this model belongs to the second class, with smooth transitions of the crossover type. Although there is no phase transition, we find that both perturbative and nonperturbative low-energy spectra of the theory are drastically changed when we pass from the Abelian to the non-Abelian regime.

The benchmark model we will deal with—the U(2) theory with $N_f = 2$ (s)quark multiplets—is described in detail in the review paper [7]. It is worth recalling that the model is characterized by two adjustable parameters: the coefficient of the Fayet-Iliopoulos (FI) term ξ and the difference Δm of the mass terms of the first and second flavors. If $\xi \gg \Lambda^2$, where Λ is the dynamical scale of the gauge theory at hand, the theory is at weak coupling and can be exhaustively analyzed using quasiclassical methods. The domain of large $|\Delta m|$ is that of the Abelian

confinement. At small $|\Delta m|$ confinement is non-Abelian. A discrete Z_2 symmetry inherent to the Lagrangian of the world-sheet theory is spontaneously broken in both limits, albeit the order parameters are different. Thus we expect (and, in fact, demonstrate) a crossover in $|\Delta m|$.

The domain of small ξ was not considered previously in the context of the problem we pose. Our task is to take it into consideration. Various regimes of the theory in the $(\xi, \Delta m)$ plane are schematically shown in Fig. 1. We choose Δm real, which is always possible to achieve by an appropriate U(1) rotation. The vertical axis in this figure denotes the values of the FI parameter ξ, while the horizontal axis represents the quark mass difference.

As was mentioned, domain I is that of non-Abelian confinement. In this domain the perturbative spectrum of the bulk theory has N^2 light states.[1] In the limit of degenerate quark masses the bulk theory has an unbroken global $SU(2)_{C+F}$ symmetry (the so-called color-flavor locking; see Sec. II), and the light states come in adjoint and singlet representations of this group. The nonperturbative spectrum contains mesons built from monopole-antimonopole pairs connected by two strings [7]. These strings are non-Abelian [8–11] (see also the reviews [7,12–14]). The non-Abelian SU(2) part of their fluxes is determined by moduli parameters, whose dynamics is described by $\mathcal{N} = 2$ supersymmetric $CP(1)$ on the string world sheet. Because of large quantum fluctuations in the $CP(1)$ model, the average non-Abelian flux of such a string vanishes.

Domain II is that of Abelian confinement at weak coupling. As we increase Δm, the W bosons and their superpartners become heavy and decouple from the low-energy spectrum. We are left with two photon states and their quark $\mathcal{N} = 2$ superpartners. Strings also become Abelian Z_2 strings. The moduli of the $CP(1)$ model at large Δm are fixed in two definite directions—the north and south poles of the S_2 sphere [the target space of the $CP(1)$ sigma model is S_2]. The non-Abelian parts of their fluxes no longer vanish.

As we reduce ξ and $|\Delta m|$ we enter domain III. It is nothing but the Abelian Seiberg-Witten confinement [1,15]. The W bosons and their superpartners decay on the curves of marginal stability (CMS). The set of light states includes photons and dyons with certain quantum numbers (the quarks we started with become dyons due to monodromies as we reduce $|\Delta m|$). Condensation of dyons leads to the formation of Abelian Z_2 strings. The nonperturbative spectrum still contains mesons built of monopole-antimonopole pairs connected by two distinct Z_2 strings. However, now these strings are different from those in domain I. They have nonvanishing fluxes directed in the Cartan subalgebra of SU(2).
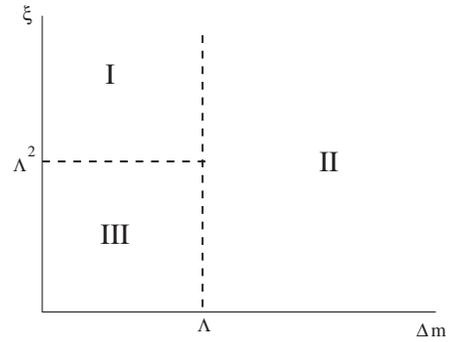

FIG. 1. Various regimes in the benchmark $\mathcal{N} = 2$ model are separated by crossovers.

These three regimes are separated by crossover transitions. Thus, non-Abelian strings can smoothly evolve into Abelian ones and vice versa. This is our main finding.

If, instead of the benchmark U(2) model we considered its U(N) generalization, we would see that, as we increase the number of colors N, these crossovers become exceedingly sharper and transform into genuine phase transitions in the limit $N \rightarrow \infty$. The width of the crossover domain scales as $1/N$. At finite N there is no discontinuity in physical observables (and no change in the breaking pattern of relevant global symmetries) along the dashed lines in Fig. 1. Still, both the perturbative spectrum and confining strings are dramatically different in regimes I, II, and III. In particular, if we keep the quark mass difference $\Delta m = 0$ and reduce the FI parameter ξ, we pass from the regime with non-Abelian strings and non-Abelian monopole confinement to the regime with Abelian strings and Abelian confinement. The low-energy spectrum of the theory in, say, domain I is not mapped onto that in domain III.

The paper is organized as follows. In Sec. II we describe our bulk theory at large ξ. In Sec. III we make some preliminary remarks on the behavior of the theory at small ξ. In Sec. IV we briefly review non-Abelian strings and discuss the order parameter which can separate a non-Abelian string from an Abelian one. In Sec. V we describe our theory in domain III, while in Sec. VI we consider the limit $N \rightarrow \infty$. Section VII summarizes our findings. In the Appendix we present in more detail the $CP(N-1)$ model with twisted masses and Z_{2N} global symmetry.

II. THE BULK THEORY: LARGE VALUES OF THE FI PARAMETER

In this section, for convenience of the reader, we will briefly outline some basic features of the $\mathcal{N} = 2$ bulk theory we will work with. Since these features are general, we will assume the gauge group to be U(N), and only later we will set $N = 2$.

We introduce N flavors (each of which is described by two $\mathcal{N} = 1$ superfields, Q and $\tilde{Q}$). The field content is as

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[1]By light we mean those states whose masses are less than or of the order of the inverse size of the string.

follows. The $\mathcal{N} = 2$ vector multiplet consists of the U(1) gauge field A_μ and the SU(N) gauge field A_μ^a, where $a = 1, \ldots, N^2 - 1$, and their Weyl fermion superpartners plus complex scalar fields a, and a^a. The latter are in the adjoint representation of SU(N).

The quark multiplets of the SU(N) × U(1) theory consist of the complex scalar fields q^{kA} and $\tilde{q}_{Ak}$ (squarks) and their fermion superpartners, all in the fundamental representation of the SU(N) gauge group. Here $k = 1, \ldots, N$ is the color index while A is the flavor index, $A = 1, \ldots, N$. Note that the scalars q^{kA} and $\bar{\tilde{q}}^{kA}$ form a doublet under the action of the global SU(2)$_R$ group.

The bosonic part of the bulk theory has the form [9] (see also the review paper [7])

$$S = \int d^4x \left[\frac{1}{4g_2^2}(F_{\mu\nu}^a)^2 + \frac{1}{4g_1^2}(F_{\mu\nu})^2 + \frac{1}{g_2^2}|D_\mu a^a|^2 \right.$$
$$+ \frac{1}{g_1^2}|\partial_\mu a|^2 + |\nabla_\mu q^A|^2 + |\nabla_\mu \bar{\tilde{q}}^A|^2$$
$$\left. + V(q^A, \tilde{q}_A, a^a, a) \right]. \tag{2.1}$$

Here D_μ is the covariant derivative in the adjoint representation of SU(N), while

$$\nabla_\mu = \partial_\mu - \frac{i}{2}A_\mu - iA_\mu^a T^a. \tag{2.2}$$

We suppress the color SU(N) indices. The normalization of the SU(N) generators T^a is as follows:

$$\text{Tr}\,(T^a T^b) = \frac{1}{2}\delta^{ab}.$$

The coupling constants g_1 and g_2 correspond to the U(1) and SU(N) sectors, respectively. With our conventions, the U(1) charges of the fundamental matter fields are ±1/2.

The potential $V(q^A, \tilde{q}_A, a^a, a)$ in the action (2.1) is the sum of D and F terms,

$$V(q^A, \tilde{q}_A, a^a, a) = \frac{g_2^2}{2}\left(\frac{1}{g_2^2}f^{abc}\bar{a}^b a^c + \bar{q}_A T^a q^A - \tilde{q}_A T^a \bar{\tilde{q}}^A\right)^2$$
$$+ \frac{g_1^2}{8}(\bar{q}_A q^A - \tilde{q}_A \bar{\tilde{q}}^A - N\xi)^2$$
$$+ 2g^2|\tilde{q}_A T^a q^A|^2 + \frac{g_1^2}{2}|\tilde{q}_A q^A|^2$$
$$+ \frac{1}{2}\sum_{A=1}^{N}\{|(a + \sqrt{2}m_A + 2T^a a^a)q^A|^2$$
$$+ |(a + \sqrt{2}m_A + 2T^a a^a)\bar{\tilde{q}}^A|^2\}. \tag{2.3}$$

Here f^{abc} denote the structure constants of the SU(N) group, m_A is the mass term of the Ath flavor, and the sum over the repeated flavor indices A is implied.

We introduced the FI D term for the U(1) gauge factor with the FI parameter ξ.

Now we briefly review the vacuum structure and the excitation spectrum in the bulk theory. The vacua of the

theory (2.1) are determined by the zeros of the potential (2.3). The adjoint fields develop the following vacuum expectation values (VEVs):

$$\left\langle\left(\frac{1}{2}a + T^a a^a\right)\right\rangle = -\frac{1}{\sqrt{2}}\begin{pmatrix} m_1 & \ldots & 0 \\ \ldots & \ldots & \ldots \\ 0 & \ldots & m_N \end{pmatrix}. \tag{2.4}$$

For generic values of the quark masses, the SU(N) subgroup of the gauge group is broken down to U(1)$^{N-1}$. However, in the special limit

$$m_1 = m_2 = \ldots = m_N, \tag{2.5}$$

the SU(N) × U(1) gauge group remains unbroken by the adjoint field. In this limit the theory acquires the global flavor SU(N) symmetry.

We can exploit gauge rotations to make all squark VEVs real. Then, in the case at hand, they take the color-flavor-locked form

$$\langle q^{kA}\rangle = \sqrt{\xi}\begin{pmatrix} 1 & \ldots & 0 \\ \ldots & \ldots & \ldots \\ 0 & \ldots & 1 \end{pmatrix}, \qquad \langle\bar{\tilde{q}}^{kA}\rangle = 0,$$
$$k = 1, \ldots, N \qquad A = 1, \ldots, N, \tag{2.6}$$

where we write down the quark fields as an $N \times N$ matrix in the color and flavor indices. This particular form of the squark condensates is dictated by the first and second lines in Eq. (2.3). Note that the squark fields stabilize at nonvanishing values entirely due to the U(1) factor represented by the second line.

The vacuum field (2.6) results in the spontaneous breaking of both gauge and flavor SU(N) symmetries. A diagonal global SU(N)$_{C+F}$ survives, however,

$$\text{U}(N)_{\text{gauge}} \times \text{SU}(N)_{\text{flavor}} \to \text{SU}(N)_{C+F}. \tag{2.7}$$

Thus, a color-flavor locking takes place in the vacuum. The presence of the global SU(N)$_{C+F}$ group is a key reason for the formation of non-Abelian strings. For generic quark masses the global symmetry (2.7) is broken down to U(1)$^{(N-1)}$.

Let us move on to the issue of the excitation spectrum in this vacuum [9,16]. The mass matrix for the gauge fields (A_μ^a, A_μ) can be read off from the quark kinetic terms in (2.1). It shows that all SU(N) gauge bosons become massive, with one and the same mass

$$m_W = g_2\sqrt{\xi}. \tag{2.8}$$

The equality of the masses is no accident. It is a consequence of the unbroken SU(N)$_{C+F}$ symmetry (2.7). The mass of the U(1) gauge boson is

$$m_\gamma = g_1\sqrt{\frac{N}{2}\xi}. \tag{2.9}$$

Thus, the bulk theory is fully Higgsed. The mass spectrum

of the adjoint scalar excitations is the same as the one for the gauge bosons. This is enforced by $\mathcal{N} = 2$.

The mass spectrum of the quark excitations can be read off from the potential (2.3). We have $4N^2$ real degrees of freedom of quark scalars q and $\tilde{q}$. Out of those, N^2 are eaten up by the Higgs mechanism. The remaining $3N^2$ states split into three plus $3(N^2 - 1)$ states with masses (2.9) and (2.8), respectively. Combining these states with the massive gauge bosons and the adjoint scalar states, we get [9,16] one long $\mathcal{N} = 2$ Bogomol'nyi-Prasad-Sommerfield (BPS) multiplet (eight real bosonic plus eight fermionic degrees of freedom) with mass (2.9) and $N^2 - 1$ long $\mathcal{N} = 2$ BPS multiplets with mass (2.8). Note that these supermultiplets come in representations of the unbroken $SU(N)_{C+F}$ group, namely, the singlet and adjoint representations.

Now let us have a closer look at quantum effects in the theory (2.1). The $SU(N)$ sector is asymptotically free. The running of the corresponding gauge coupling, if not interrupted, would drag the theory into the strong-coupling regime. This would invalidate our quasiclassical analysis. Moreover, strong-coupling effects on the Coulomb branch would break the $SU(N)$ gauge subgroup down to $U(1)^{N-1}$ by virtue of the Seiberg-Witten mechanism [1]. No non-Abelian strings would emerge.

The semiclassical analysis above is valid if the FI parameter ξ is large,

$$\xi \gg \Lambda, \qquad (2.10)$$

where Λ is the scale of the $SU(N)$ gauge theory. This condition ensures weak coupling in the $SU(N)$ sector because the $SU(N)$ gauge coupling does not run below the scale of the quark VEVs which is determined by ξ. More explicitly,

$$\frac{8\pi^2}{g_2^2(\xi)} = N \ln\frac{\sqrt{\xi}}{\Lambda} \gg 1. \qquad (2.11)$$

III. TOWARDS SMALLER ξ

Below we will see that if we pass to small ξ along the line $\Delta m = 0$, into the strong-coupling domain, where the condition (2.10) is not met, the theory undergoes a crossover transition into the Seiberg-Witten Abelian confinement regime. In this regime the low-energy perturbative sector contains no nontrivial representations of the unbroken $SU(N)_{C+F}$ group. Moreover, no non-Abelian strings develop.

The main tool which allows us to identify this crossover transition is the presence of the global unbroken $SU(N)_{C+F}$ symmetry in the theory at hand. First, we note that it is not spontaneously broken in the bulk. If it were broken this would imply the presence of massless Goldstone states. However, we showed above that the perturbative sector of the theory at large ξ has a large mass gap of the order of $g^2\sqrt{\xi}$, and masses of no states can be shifted to zero by

small quantum corrections of the order of Λ. Nor do we see the adjoint multiplet of massless Goldstones at small ξ.

The presence of the global unbroken $SU(N)_{C+F}$ symmetry means that all multiplets should come in representations of this group. We showed above that at large ξ this is the case indeed: all light states come in adjoint and singlet representations $[(N^2 - 1) + 1]$. We will see later that at small ξ (in the Seiberg-Witten regime) the low-energy spectrum is very different. It contains only N states which do not fill any nontrivial representations of $SU(N)_{C+F}$. They are all singlets.

To elucidate the point let us note the following. All $(N^2 - 1)$ states of the adjoint gauge-boson multiplet of $SU(N)_{C+F}$ have degenerate masses (2.8) at large ξ. The presence of the unbroken global $SU(N)_{C+F}$ ensures that they are not split. Imagine that these states were split with small splittings of the order of Λ. Then in the limit of small $\xi, \xi \ll \Lambda$, some of these states could, in principle, evolve into $(N - 1)$ light Abelian states while other members of the multiplet could acquire large masses, of the order of Λ (i.e. the photons could become light, while the W bosons could become heavy). We stress that this does not happen in the theory at hand. The adjoint multiplet is not split at large ξ and therefore can disappear from the low-energy spectrum at small ξ only as a whole. Hence, the light photons in the Seiberg-Witten regime at small ξ have nothing to do with the diagonal (Cartan) entries of the gauge adjoint $SU(N)_{C+F}$ multiplet at large ξ. Similarly, the light dyons in the Seiberg-Witten regime at small ξ have nothing to do with the light quarks q^{kA}, $\tilde{q}_{Ak}$ of the non-Abelian confinement regime at large ξ. The latter fill the adjoint representation of $SU(N)_{C+F}$ (see the discussion above), while the former are singlets.

In order to see what happens with the low-energy spectrum of the bulk theory as we reduce ξ, we use the following method. First we introduce quark mass differences $(m_A - m_B)$ and make them large, $|m_A - m_B| \gg \Lambda$. Then we can reduce the parameter ξ, keeping the theory at weak coupling and under control. Next, we reach the Coulomb branch at zero ξ and use the exact solution of the theory [1,15] to go back to the desired limit of degenerate quark masses (2.5). Thus, our routing is as follows: domain I → domain II → domain III. In domain II the global $SU(N)_{C+F}$ is lost, and a level crossing occurs.

The program outlined above will be carried out in full in Sec. V.

To conclude this section we briefly review the theory (2.1) at nonzero quark mass differences $(m_A - m_B) \neq 0$; see [7,10]. At nonvanishing $(m_A - m_B)$ the global $SU(N)_{C+F}$ is explicitly broken down to $U(1)^{(N-1)}$. The adjoint multiplet is split. The diagonal entries (photons and their $\mathcal{N} = 2$ quark superpartners) have masses given in (2.8), while the off-diagonal states (W bosons and the off-diagonal entries of the quark matrix q^{kA}) acquire additional contributions to their masses proportional to

($m_A - m_B$). As we make the mass differences larger, the W bosons become exceedingly heavier, decouple from the low-energy spectrum, and we are left with N photon states and N diagonal elements of the quark matrix. The low-energy spectrum becomes Abelian.

IV. NON-ABELIAN STRINGS AT LARGE ξ

Here we will study the passage from domain I $\rightarrow$ domain II. At first, we will briefly review non-Abelian strings [8–11] in the theory (2.1); see [7] for details. The Abelian Z_N-string solutions break the $SU(N)_{C+F}$ global group. Therefore strings have orientational zero modes, associated with rotations of their color flux inside the non-Abelian $SU(N)$. This makes these strings non-Abelian. The global group is broken by the Z_N string solution down to $SU(N-1) \times U(1)$. Therefore the moduli space of the non-Abelian string is described by the coset

$$\frac{SU(N)}{SU(N-1) \times U(1)} \sim CP(N-1). \qquad (4.1)$$

The $CP(N-1)$ space can be parametrized by a complex vector n^l in the fundamental representation of $SU(N)$ subject to the constraint

$$n_l^* n^l = 1, \qquad (4.2)$$

where $l = 1, \ldots, N$. As we will show below, one $U(1)$ phase will be gauged away in the effective sigma model. This gives the correct number of degrees of freedom, namely, $2(N-1)$.

With this parametrization the elementary string solution (with the lowest winding number) can be written as [5,10]

$$q = \frac{1}{N}[(N-1)\phi_2 + \phi_1] + (\phi_1 - \phi_2)\left(n \cdot n^* - \frac{1}{N}\right),$$

$$A_i^{SU(N)} = \left(n \cdot n^* - \frac{1}{N}\right)\varepsilon_{ij}\frac{x_i}{r^2}f_{NA}(r),$$

$$A_i^{U(1)} = \frac{1}{N}\varepsilon_{ij}\frac{x_i}{r^2}f(r), \qquad \bar{\bar{q}}^{kA} = 0, \qquad (4.3)$$

where $i = 1, 2$ labels coordinates in the plane orthogonal to the string axis and r and α are the polar coordinates in this plane. For brevity we suppress all $SU(N)$ indices. The profile functions $\phi_1(r)$ and $\phi_2(r)$ determine the profiles of the scalar fields, while $f_{NA}(r)$ and $f(r)$ determine the $SU(N)$ and $U(1)$ gauge fields of the string solution, respectively. These functions satisfy the first-order equations [9] which can be solved numerically.

The tension of the elementary string is given by

$$T = 2\pi\xi. \qquad (4.4)$$

Making the moduli vector n^l a slowly varying function of the string world-sheet coordinates x_k ($k = 0, 3$), we can derive the effective low-energy theory on the string world sheet [5,9,10]. From the topological reasoning above [see (4.1)] it is clear that we will get the two-dimensional

$CP(N-1)$ model. The $\mathcal{N} = 2$ supersymmetric $CP(N-1)$ model can be understood as a strong-coupling limit of a $U(1)$ gauge theory [17]. Then the bosonic part of the action takes the form

$$S_{CP(N-1)} = \int d^2x \left\{ 2\beta|\nabla_k n^\ell|^2 + \frac{1}{4e^2}F_{kl}^2 + \frac{1}{e^2}|\partial_k\sigma|^2 \right.$$
$$\left. + 4\beta|\sigma|^2|n^\ell|^2 + 2e^2\beta^2(|n^\ell|^2 - 1)^2 \right\}, \qquad (4.5)$$

where $\nabla_k = \partial_k - iA_k$ while σ is a complex scalar field. The condition (4.2) is implemented in the limit $e^2 \rightarrow \infty$. Moreover, in this limit the gauge field A_k and its $\mathcal{N} = 2$ bosonic superpartner σ become auxiliary and can be eliminated by virtue of the equations of motion,

$$A_k = -\frac{i}{2}n_\ell^* \overleftrightarrow{\partial}_k n^\ell, \qquad \sigma = 0. \qquad (4.6)$$

The two-dimensional coupling constant β here is determined by the four-dimensional non-Abelian coupling via the relation

$$\beta = \frac{2\pi}{g_2^2}. \qquad (4.7)$$

The above relation between the four-dimensional and two-dimensional coupling constants (4.7) is obtained at the classical level [9,10]. In quantum theory both couplings run. In particular, the $CP(N-1)$ model is asymptotically free [18] and develops its own scale Λ_σ. The ultraviolet cutoff of the sigma model on the string world sheet is determined by $g_2\sqrt{\xi}$. Equation (4.7) relating the two- and four-dimensional couplings is valid at this scale, implying

$$\Lambda_\sigma^N = g_2^N \xi^{N/2} e^{-(8\pi^2/g_2^2)} = \Lambda^N. \qquad (4.8)$$

Note that in the bulk theory *per se*, because of the VEVs of the squark fields, the coupling constant is frozen at $g_2\sqrt{\xi}$; there are no logarithms below this scale. The logarithms of the string world-sheet theory take over. Moreover, the dynamical scales of the bulk and world-sheet theories turn out to be the same [10].

The $CP(N-1)$ model was solved by Witten in the large-N limit [19]. We will briefly summarize Witten's results and translate them in terms of strings in four dimensions [10].

Classically, the field n^ℓ can have an arbitrary direction; therefore, one might naively expect a spontaneous breaking of $SU(N)$ and the occurrence of massless Goldstone modes on the string world sheet. However, this cannot happen in two dimensions. Quantum effects restore the symmetry. Moreover, the condition (4.2) gets, in effect, relaxed. Because of strong coupling we have more degrees of freedom than in the original Lagrangian, namely, all N fields n become dynamical and acquire masses Λ_σ.

Deep in the quantum non-Abelian regime the $CP(N-1)$-model strings carry no average $SU(N)$ magnetic flux. To see that this is indeed the case, note that the $SU(N)$ mag-

netic flux of the non-Abelian string (4.3) is given by

$$\int d^2x (F_3^*)_{SU(N)} = 2\pi \left(n \cdot n^* - \frac{1}{N} \right), \qquad (4.9)$$

where

$$F_i^* = \frac{1}{2} \varepsilon_{ijk} F_{jk} \qquad (i, j, k = 1, 2, 3). \qquad (4.10)$$

As was shown by Witten [19], in the $CP(N-1)$ model strong quantum fluctuations of n^ℓ result in

$$\langle n^\ell \rangle = 0, \qquad (4.11)$$

implying, in turn, that the average SU(N) magnetic flux of the non-Abelian string vanishes. We will use this circumstance later, to distinguish between large-ξ non-Abelian and small-ξ Abelian Z_N strings in the Seiberg-Witten regime below the crossover. The latter do carry the magnetic flux directed inside the Cartan subalgebra of SU(N). The $CP(N-1)$ model has N vacua [19]. They are interpreted in the problem at hand as N different elementary non-Abelian strings. These N vacua differ from each other by the expectation value of the chiral bifermion operator; see e.g. [20]. At strong coupling the chiral condensate is the order parameter for Z_N breaking (instead of the flux; see the Appendix). The U(1) chiral symmetry of the $CP(N-1)$ model is explicitly broken to a discrete Z_{2N} symmetry by the chiral anomaly (for a discussion of the global symmetry on the string world sheet, see the Appendix). The bifermion condensate breaks Z_{2N} down to Z_2. That is the origin of the N-fold degeneracy of the vacuum state.

Now, to make our consideration simpler we will focus on the simplest case $N = 2$. For arbitrary N the emerging dynamical pattern is similar. The solution for the non-Abelian string (4.3) in the $N = 2$ case takes the form

$$q = \frac{1}{2}(\phi_1 + \phi_2) + \frac{\tau^a}{2} S^a (\phi_1 - \phi_2), \qquad \tilde{q} = 0,$$
$$A_i^a(x) = S^a \varepsilon_{ij} \frac{x_j}{r^2} f_{NA}(r), \qquad A_i(x) = \varepsilon_{ij} \frac{x_j}{r^2} f(r), \qquad (4.12)$$

where S^a ($a = 1, 2, 3$) is a real moduli vector subject to the constraint

$$(S^a)^2 = 1. \qquad (4.13)$$

Its relation to the complex vector n^ℓ is as follows:

$$S^a = \bar{n} \tau^a n. \qquad (4.14)$$

We have $CP(1)$ as the effective world-sheet theory. It is equivalent to the O(3) sigma model. In terms of the real vector S^a the bosonic part of the world-sheet theory has the form

$$S = \beta \int d^2x \frac{1}{2} (\partial_k S^a)^2. \qquad (4.15)$$

Now let us introduce quark mass differences $(m_A - m_B)$. In the $N = 2$ case we have just one (generally speaking

complex) parameter

$$\Delta m = m_1 - m_2. \qquad (4.16)$$

The vacuum expectation values of the adjoint field reduce to

$$\langle a^3 \rangle = -\frac{\Delta m}{\sqrt{2}}, \qquad \langle a \rangle = -\sqrt{2} \frac{m_1 + m_2}{2}; \qquad (4.17)$$

see (2.4). The non-Abelian string (4.12) is no longer a solution of the first-order equations for arbitrary S^a. The global SU(2)$_{C+F}$ is explicitly broken down to U(1) by $\Delta m \neq 0$. Nevertheless, if we keep Δm small, we can consider S^a as quasimoduli, with a shallow potential on the $CP(1)$ moduli space. The string solution (4.12) in this case should be supplemented by a nontrivial profile for the adjoint field [7,10],

$$a^a = -\frac{\Delta m}{\sqrt{2}} \left[\delta^{a3} \frac{\phi_1}{\phi_2} + S^a S^3 \left(1 - \frac{\phi_1}{\phi_2} \right) \right]. \qquad (4.18)$$

Plugging this modified string solution in the action of the theory gives [7,10] the effective string world-sheet theory: an $\mathcal{N} = 2$ $CP(1)$ model with twisted mass [21]. The bosonic part of the action is

$$S_{CP(1)} = \beta \int d^2x \left\{ \frac{1}{2} (\partial_k S^a)^2 + \frac{|\Delta m|^2}{2} (1 - S_3^2) \right\}. \qquad (4.19)$$

This is the only functional form that allows $\mathcal{N} = 2$ completion. The mass-splitting parameter Δm of the bulk theory exactly coincides with the twisted mass of the world-sheet model.

The $CP(1)$ model (4.19) has two vacua located at $S^a = (0, 0, \pm 1)$. Clearly, these two vacua correspond to two elementary Z_2 strings.

With nonvanishing Δm we can introduce a gauge-invariant quantity which measures the SU(2) non-Abelian flux of the string. We define[2]

$$\Phi = \int d^2x a^a F_3^{*a}. \qquad (4.20)$$

This order parameter will be used below to distinguish between different regimes of the theory.

Substituting (4.18) and (4.19) into (4.20) we get

$$\Phi = -2\pi \frac{\Delta m}{\sqrt{2}} S^3. \qquad (4.21)$$

At small Δm, $\Delta m \ll \Lambda_\sigma$, the fields S^a strongly fluctuate and $\langle S^3 \rangle = 0$ [see (4.11)]. Therefore,

$$\langle \Phi \rangle_I \to 0 \quad \text{at } \Delta m \ll \Lambda_\sigma, \qquad (4.22)$$

where the subscript I refers to the non-Abelian domain at large ξ and small Δm, as it is indicated in Fig. 1.

---

[2]The subscript 3 indicates the direction along the string axis; cf. Eq. (4.9).

Instead, at large Δm ($\Delta m \gg \Lambda_\sigma$) the O(3) sigma model (4.19) is at weak coupling. Fluctuations are small, and the S^a acquires vacuum values at the north and south poles of the S_2 sphere, $\langle S^a \rangle = (0, 0, \pm 1)$. As a result

$$\langle \Phi \rangle_{\mathrm{II}} \rightarrow \mp 2\pi \frac{\Delta m}{\sqrt{2}}, \tag{4.23}$$

where the subscript II marks domain II in Fig. 1.

We see that the behavior of the string flux Φ drastically changes as we pass from the non-Abelian domain I of large ξ and small Δm to the Abelian domain II of large Δm; see Fig. 2. At large Δm this theory is in the weak-coupling regime and fluctuations are small, $\langle S^3 \rangle \approx \pm 1$, and the flux Φ is given by (4.23). At small Δm the world-sheet theory is in the strong coupled quantum regime, fluctuations are large, and the vector S^a is smeared over the whole sphere. Therefore, $\langle S^3 \rangle \approx 0$ and $\langle \Phi \rangle \approx 0$. The crossover between these two regimes is at $\Delta m \sim \Lambda_\sigma$. Note that the drastic change of behavior in the world-sheet $CP(1)$ model is correlated with the dynamics of the bulk theory.

In Sec. II we saw that the perturbative spectrum of the bulk theory is different in these two domains: it is essentially non-Abelian in domain I, while in domain II the W bosons become exceedingly heavier and decouple from the low-energy spectrum; we are left with N photons and their superpartners, the diagonal elements of the quark matrix. The pattern repeats itself at the nonperturbative level: the non-Abelian strings evolve into the Abelian strings as we increase $|\Delta m|$.

Later we will see that the crossover becomes exceedingly more pronounced as we increase the number of colors N. In the limit $N \rightarrow \infty$ the crossover evolves into a genuine phase transition. Note also that in nonsupersymmetric theories we do have a phase transition between the phase with non-Abelian strings at small $|m_A - m_B|$ and the phase with Abelian strings at large $|m_A - m_B|$ [6]. It is related to the restoration of the broken discrete Z_N symmetry at small $|m_A - m_B|$.[3] In the supersymmetric theory
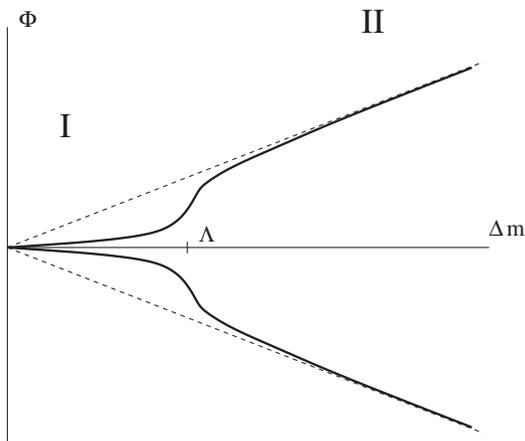

FIG. 2. Flux (4.20) as a function of Δm in domains I and II.

at hand the discrete Z_N symmetry is always broken (by VEVs of n^ℓ at large $|m_A - m_B|$ or by the two-dimensional bifermion condensates at small $|m_A - m_B|$; see the Appendix). Therefore, in the supersymmetric case we have a crossover rather than a phase transition.

To conclude this section, we briefly review the world-sheet theory on the non-Abelian string for generic N. It is described by the twisted-mass-deformed $CP(N-1)$ model. It can be nicely written [22] as a strong-coupling limit of a U(1) gauge theory. With twisted masses of the n^ℓ fields taken into account, the bosonic part of the action (4.5) takes the form

$$S = \int d^2x \left\{ 2\beta |\nabla_k n^\ell|^2 + \frac{1}{4e^2} F_{kl}^2 + \frac{1}{e^2} |\partial_k \sigma|^2 \right.$$
$$\left. + 4\beta \left| \sigma - \frac{\tilde{m}_\ell}{\sqrt{2}} \right|^2 |n^\ell|^2 + 2e^2 \beta^2 (|n^\ell|^2 - 1)^2 \right\}, \tag{4.24}$$

where

$$\tilde{m}_\ell = m_\ell - m, \qquad m \equiv \frac{1}{N}\sum_\ell m_\ell, \tag{4.25}$$

and the sum over ℓ in (4.24) is implied.

V. THE THEORY IN DOMAIN III

Now, we will consider the passage from domain II to domain III. In order to study the theory in regime III (see Fig. 1) we first assume the quark mass differences to be large. Then the theory stays at weak coupling and we can safely decrease the value of the FI parameter ξ. Next, we use the exact Seiberg-Witten solution of the theory on the Coulomb branch (at $\xi = 0$) to pass from regime II to regime III. To simplify our discussion, we will consider here only the case $N = 2$.

A. The $r = 2$ quark vacuum

Our first task is to identify the $r = 2$ quark vacuum (which we described semiclassically above) using the exact Seiberg-Witten solution [1,15].[4] The Seiberg-Witten curve for the U(2) gauge theory with $N_f = 2$ flavors has the form

$$y^2 = (x - \phi_1)^2 (x - \phi_2)^2 - 4\Lambda^2 \left(x + \frac{m_1}{\sqrt{2}} \right)\left(x + \frac{m_2}{\sqrt{2}} \right), \tag{5.1}$$

where ϕ_1 and ϕ_2 are gauge-invariant parameters on the Coulomb branch.

---

[3] At $N > 2$ the above discrete symmetry of the Lagrangian takes place under a special choice of the mass parameters; see the Appendix.

[4] This solution was obtained by Seiberg and Witten in the SU (2) gauge theories. Generalizations to SU(N) were obtained in [23–26].

Semiclassically,

$$\phi_1 \approx a_1 \equiv \tfrac{1}{2}(a + a^3), \qquad \phi_2 \approx a_2 \equiv \tfrac{1}{2}(a - a^3). \quad (5.2)$$

Let us make a shift in the variable x introducing a new variable z,

$$x = -\frac{m}{\sqrt{2}} + z, \qquad m = \frac{1}{2}(m_1 + m_2). \quad (5.3)$$

With $\Delta m \gg \Lambda$ we identify the $r = 2$ singularity, the point where both quarks q^{11} and q^{22} become massless. Upon switching on $\xi \neq 0$, this $r = 2$ singularity turns into the $r = 2$ vacuum we considered in the semiclassical approximation in the previous sections.

It turns out that in the $r = 2$ vacuum

$$\phi_1 + \phi_2 = -2\frac{m}{\sqrt{2}}. \quad (5.4)$$

We parametrize the deviations of ϕ_1 and ϕ_2 from their mean value $-\frac{m}{\sqrt{2}}$ by a new parameter ϕ,

$$\phi_1 = -\frac{m}{\sqrt{2}} + \phi, \qquad \phi_2 = -\frac{m}{\sqrt{2}} - \phi. \quad (5.5)$$

With this parametrization the curve (5.1) reduces to[5]

$$y^2 = (z - \phi)^2(z + \phi)^2 - 4\Lambda^2\left(z + \frac{\Delta m}{2\sqrt{2}}\right)\left(z - \frac{\Delta m}{2\sqrt{2}}\right)$$

$$= (z^2 - \phi^2)^2 - 4\Lambda^2\left(z^2 - \frac{\Delta m^2}{8}\right). \quad (5.6)$$

Next, we look for the values of the parameter ϕ which ensure that this curve has two double roots associated with two quarks being massless. This curve is a perfect square,

$$y^2 = \left[z^2 - \frac{1}{4}\left(\frac{\Delta m^2}{2} + 4\Lambda^2\right)\right]^2, \quad (5.7)$$

(see [22]) at

$$\phi = \frac{1}{2}\sqrt{\frac{\Delta m^2}{2} - 4\Lambda^2}. \quad (5.8)$$

In fact, there are two solutions with plus and minus signs in front of the square root above. They correspond to ϕ_1 and ϕ_2, namely,

$$\phi_1 = -\frac{m}{\sqrt{2}} - \frac{1}{2}\sqrt{\frac{\Delta m^2}{2} - 4\Lambda^2},$$

$$\phi_2 = -\frac{m}{\sqrt{2}} + \frac{1}{2}\sqrt{\frac{\Delta m^2}{2} - 4\Lambda^2}. \quad (5.9)$$

In the semiclassical limit $\Delta m \gg \Lambda$ these solutions reduce to

---

[5]Δm^2 is shorthand for $(\Delta m)^2$.

$$\phi_1 \approx -\frac{m_1}{\sqrt{2}}, \qquad \phi_2 \approx -\frac{m_2}{\sqrt{2}}, \quad (5.10)$$

which coincides with Eq. (2.4). This means that we correctly identified the $r = 2$ quark vacuum where two quarks, q^{11} and q^{22}, are massless; see Sec. II.

Two double roots of the curve in the quark vacuum are

$$e_1 = e_2 = -\frac{m}{\sqrt{2}} - \frac{1}{2}\sqrt{\frac{\Delta m^2}{2} + 4\Lambda^2},$$

$$e_3 = e_4 = -\frac{m}{\sqrt{2}} + \frac{1}{2}\sqrt{\frac{\Delta m^2}{2} + 4\Lambda^2}. \quad (5.11)$$

The Seiberg-Witten exact solution of the theory relates VEVs of the fields a, a^3 and a_D, a_D^3 (which, in turn, determine the spectrum of the BPS states on the Coulomb branch) to certain integrals along α and β contours in the x plane [1,15]. That is, the derivatives of the VEVs of a_1 and a_2 are given by the following integrals along the α contours:

$$\frac{\partial a_1}{\partial \phi_1} = \frac{1}{2\pi i}\int_{\alpha_1} dx \frac{x - \phi_2}{y},$$

$$\frac{\partial a_1}{\partial \phi_2} = \frac{1}{2\pi i}\int_{\alpha_1} dx \frac{x - \phi_1}{y},$$

$$\frac{\partial a_2}{\partial \phi_1} = \frac{1}{2\pi i}\int_{\alpha_2} dx \frac{x - \phi_2}{y}, \quad (5.12)$$

$$\frac{\partial a_2}{\partial \phi_2} = \frac{1}{2\pi i}\int_{\alpha_2} dx \frac{x - \phi_1}{y},$$

while the derivatives of the a_D's are given by similar integrals along the β contours.

The presence of two massless quarks q^{11} and q^{22} in the $r = 2$ vacuum at $\Delta m \gg \Lambda$ implies

$$a_1 + \frac{m_1}{\sqrt{2}} = 0, \qquad a_2 + \frac{m_2}{\sqrt{2}} = 0. \quad (5.13)$$

Thus, the fields a_1, a_2 are regular at the singularity while the fields a_D have logarithmic divergences related to the β functions of the low-energy $U(1) \times U(1)$ theory. This ensures that the α_1 contour should go around the roots e_1, e_2, while the α_2 contour should go around the roots e_3, e_4. In the $r = 2$ vacuum (5.11) both contours shrink and produce regular a's. The basis of the α and β contours is shown in Fig. 3. Here we consider our $U(2)$ theory as a two-flavor $SU(3)$ gauge theory broken down to $U(2)$ at a very high scale. In terms of the Seiberg-Witten curve this corresponds to two extra roots of the $SU(3)$ curve being far away from four roots of the $U(2)$ curve (5.1).

As a double-check of our identification of the quark vacuum, let us calculate the derivatives (5.12) in the semiclassical approximation $\Delta m \gg \Lambda$. Substituting (5.7) into (5.12) we get, at $\Delta m \gg \Lambda$,
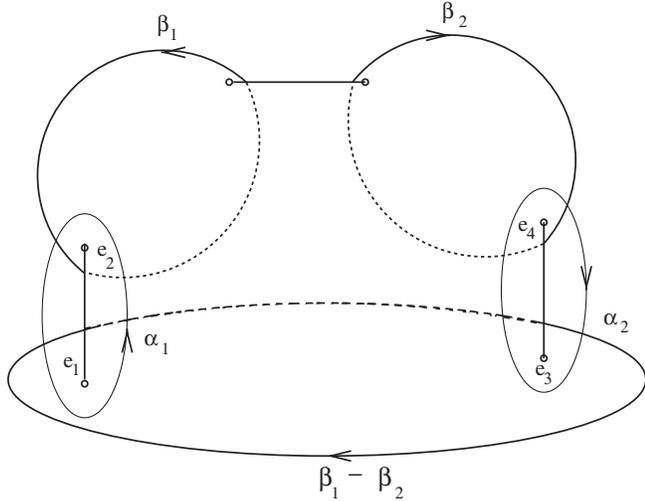
FIG. 3. Basis of the α and β contours of our U(2) gauge theory viewed as an SU(3) theory broken down to U(2). Two extra roots of the SU(3) theory are far away in the x plane.

$$\frac{\partial a_1}{\partial \phi_1} \approx 1, \qquad \frac{\partial a_1}{\partial \phi_2} \approx 0, \qquad \frac{\partial a_2}{\partial \phi_1} \approx 0, \qquad \frac{\partial a_2}{\partial \phi_2} \approx 1. \tag{5.14}$$

This is in accordance with (5.2) and confirms our choice of the α contours in Fig. 3.

To conclude this subsection we note that the monopole singularity [the point on the Coulomb branch where the SU(2) monopole becomes massless, $a_D^3 = 0$] corresponds to the shrinking of the $(\beta_1 - \beta_2)$ contour, i.e., to $e_1 = e_3$.

B. Monodromies

Let us study how the quantum numbers of massless quarks q^{11} and q^{22} change as we reduce $|\Delta m|$ and go from domain II into domain III where the theory is at strong coupling. The quantum numbers change due to monodromies with respect to Δm. The complex plane of Δm has cuts, and when we cross these cuts, the a and a_D fields acquire monodromies and the quantum numbers of states change accordingly. Monodromies with respect to quark masses were studied in [27] in the theory with the SU(2) gauge group using a monodromy matrix approach.

Here we will investigate the monodromies in the U(2) theory with two quark flavors using a slightly different approach, similar to that of Ref. [28]. If two roots of the Seiberg-Witten curve coincide, the contour which goes around these roots shrinks and produces a regular potential. That is, as was discussed above, at $\Delta m \gg \Lambda$ we have two double roots $e_1 = e_2$ and $e_3 = e_4$ in the $r = 2$ vacuum. Thus, two contours α_1 and α_2 shrink (see Fig. 3), and potentials a_1 and a_2 are regular. This is associated with the masslessness of the two quarks; see (5.13).

Instead, in the monopole singularity $e_1 = e_3$; thus the $(\beta_1 - \beta_2)$ contour shrinks producing a regular a_D^3. This is

associated with the masslessness of the SU(2) monopole, $a_D^3 = 0$ [1].

If we decrease $|\Delta m|$ and cross the cuts in the Δm plane, the root pairing in the given vacuum may change. This would mean that a different combination of a and a_D becomes regular, implying a change of the quantum numbers of the massless states in the given vacuum. To see how it works for our $r = 2$ vacuum, we go to the Argyres-Douglas (AD) point [29,30]. The AD point is a particular value of the quark mass parameters where more mutually nonlocal states become massless. In fact, we will study the collision of the $r = 2$ quark vacuum with the monopole singularity. We approach the AD point from domain II at large Δm. We will show below that as we pass through the AD point the root pairings change in the $r = 2$ vacuum, implying a change of the quantum numbers of the massless states. Two massless quarks transform into two massless dyons.

To be more precise, we collide the $r = 2$ vacuum with two massless quarks with the quantum numbers

$$(n_e, n_m; n_e^3, n_m^3) = (1/2, 0; 1/2, 0),$$
$$(n_e, n_m; n_e^3, n_m^3) = (1/2, 0; -1/2, 0) \tag{5.15}$$

with the monopole singularity

$$(n_e, n_m; n_e^3, n_m^3) = (0, 0; 0, 1) \tag{5.16}$$

where the monopole becomes massless. Here n_e and n_m denote electric and magnetic charges of a state with respect to the U(1) gauge group, while n_e^3 and n_m^3 stand for electric and magnetic charges with respect to the Cartan generator of the SU(2) gauge group [broken down to U(1) by Δm].

As was already mentioned, the $(0, 0; 0, 1)$ monopole is massless if $e_1 = e_3$. Equation (5.11) shows that this can happen in the $r = 2$ vacuum only if all four roots of the U(2) curve coincide at

$$\Delta m^2 = -8\Lambda^2, \qquad e_1 = e_2 = e_3 = e_4 = -\frac{m}{\sqrt{2}}. \tag{5.17}$$

This is the position of the AD point where both the quarks and the SU(2) monopole become simultaneously massless.

In order to see how the root pairings in the $r = 2$ vacuum change as we decrease $|\Delta m|$ and pass from domain II into domain III through the AD point (5.17), we have to slightly split the roots by shifting ϕ from its $r = 2$ solution (5.8). Let us take

$$\phi^2 = \frac{1}{4}\left(\frac{\Delta m^2}{2} - 4\Lambda^2\right) + \frac{1}{4\Lambda^2}\delta^2, \tag{5.18}$$

where δ is a small deviation. Then the curve (5.1) can be approximately (at small δ) written as

$$y^2 \approx \left[z^2 - \frac{1}{4}\left(\frac{\Delta m^2}{2} + 4\Lambda^2\right)\right]^2 - \delta^2. \tag{5.19}$$

Now all four roots split as follows:

$$e_1 = -\frac{m}{\sqrt{2}} + \sqrt{\mu^2 + \delta}, \qquad e_2 = -\frac{m}{\sqrt{2}} + \sqrt{\mu^2 - \delta},$$

$$e_3 = -\frac{m}{\sqrt{2}} - \sqrt{\mu^2 + \delta}, \qquad e_4 = -\frac{m}{\sqrt{2}} - \sqrt{\mu^2 - \delta},$$

(5.20)

where we introduced the shorthand notation

$$\mu \equiv \frac{1}{2}\sqrt{\frac{\Delta m^2}{2} + 4\Lambda^2}.$$

(5.21)

This parameter vanishes at the AD point.

In order to pass through the AD point from domain II into domain III, we decrease $|\Delta m|$, keeping Δm pure imaginary,

$$\Delta m = |\Delta m| e^{i(\pi/2)}.$$

(5.22)

Then μ goes along the imaginary axis towards the origin (which is the AD point), and below the AD point it increases along the positive axis. We also fix the parameter δ to be imaginary too, $\delta = |\delta| e^{i(\pi/2)}$. This is convenient as all four roots stay split at any $|\Delta m|$.

As we decrease $|\Delta m|$ the roots (5.20) move as shown in Fig. 4. We see that the root pairings in the $r = 2$ vacuum change. Namely, at large $|\Delta m|$ we have (at $\delta = 0$)

$$e_1 = e_2, \qquad e_3 = e_4,$$

(5.23)

which, as was explained above, corresponds to the shrinking of the α_1 and α_2 contours and the masslessness of two quarks (5.15). Below the AD point at small $|\Delta m|$, we have

$$e_2 = e_3, \qquad e_1 = e_4,$$

(5.24)

which corresponds to the shrinking of the contours

$$\beta_1 - \beta_2 + \alpha_1 \to 0, \qquad -\beta_1 + \beta_2 + \alpha_2 \to 0. \quad (5.25)$$
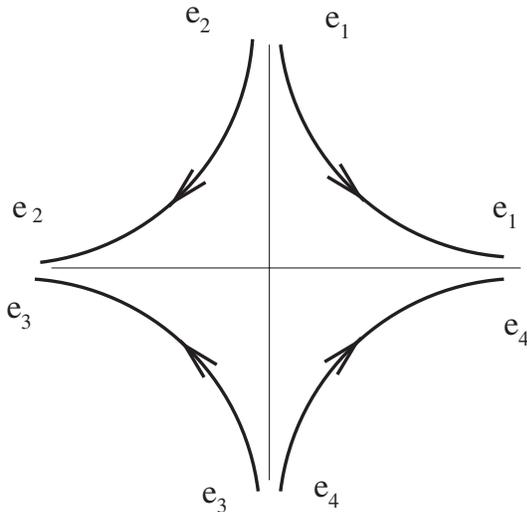

FIG. 4. As we decrease $|\Delta m|$ (keeping Δm imaginary) and pass through the AD point, the roots $e_{1,2,3,4}$ move in the x plane.

This means that massless quarks in the $r = 2$ vacuum transform into massless dyons D_1 and D_2 with the quantum numbers

$$D_1: (1/2, 0; 1/2, 1), \qquad D_2: (1/2, 0; -1/2, -1). \quad (5.26)$$

We see that the quantum numbers of the massless quarks in the $r = 2$ vacuum after the collision with the monopole singularity get shifted, the shift being equal to $\pm$ (monopole magnetic charge).

The monodromy discussed above implies

$$a_1 \to a_1 + a_D^3, \qquad a_2 \to a_2 - a_D^3, \qquad a^3 \to a^3 + 2a_D^3.$$

(5.27)

Therefore, the conditions (5.13) for the masslessness of the q^{11} and q^{22} quarks are replaced in domain III by the conditions of the masslessness of the dyons D_1 and D_2, namely,

$$a_1 + a_D^3 + \frac{m_1}{\sqrt{2}} = 0, \qquad a_2 - a_D^3 + \frac{m_2}{\sqrt{2}} = 0. \quad (5.28)$$

C. The low-energy theory

In this subsection we present the low-energy theory in the $r = 2$ vacuum in domain III at small ξ and small $|\Delta m|$ (below the AD point). It should be stressed that none of the fields in this low-energy theory belong to nontrivial representations of SU(2)$_{C+F}$.

As we already know, the massless quarks q^{11} and q^{22} transform into the massless dyons D_1 and D_2. The latter interact with two photons. According to the dyon quantum numbers (5.26), one of these photons is

$$A_\mu,$$

(5.29)

while the other photon is the following linear combination:

$$B_\mu = \frac{1}{\sqrt{5}}(A_\mu^3 + 2A_\mu^{3D}).$$

(5.30)

In fact, these are the only light states to be included in the low-energy effective theory in domain III. All other states are either heavy (with masses of the order of Λ) or decay on curves of marginal stability. In the case at hand, the CMS is located around the origin in the Δm complex plane and goes through the AD point [31]. In fact, the W bosons of the underlying non-Abelian gauge theory, as well as the off-diagonal states of the quark matrix q^{kA}, decay on the CMS. Let us illustrate this statement, say, for the W bosons. To this end we can go to the AD point. At this point we have for the W-boson mass

$$m_W = \sqrt{2}|a^3| = \sqrt{2}|(a^3 + a_D^3) - a_D^3|$$

$$= \sqrt{2}(|a^3 + a_D^3| + |a_D^3|) = m_M + m_D, \quad (5.31)$$

where m_M and m_D are the masses of the SU(2) monopole and SU(2) dyon with charges $(0, 0; 0, 1)$ and $(0, 0; 1, 1)$, respectively. This relation is valid at the AD point just

because the monopole becomes massless at this point, $a_D^3 = 0$. This means that the W boson decays into the SU(2) monopole and dyon at this point and is not present in domain III, in full accordance with the analysis of the SU(2) theory in [27].

Taking this into account we write the effective low-energy action of the theory in domain III as follows:

$$
S_{\text{III}} = \int d^4x \bigg[\frac{1}{4\tilde{g}_2^2}(F_{\mu\nu}^B)^2 + \frac{1}{4g_1^2}(F_{\mu\nu})^2 + \frac{1}{\tilde{g}_2^2}|\partial_\mu b|^2
$$
$$
+ \frac{1}{g_1^2}|\partial_\mu a|^2 + |\nabla_\mu^1 D_1|^2 + |\nabla_\mu^1 \tilde{D}_1|^2 + |\nabla_\mu^2 D_2|^2
$$
$$
+ |\nabla_\mu^2 \tilde{D}_2|^2 + V(D, \tilde{D}, b, a) \bigg], \qquad (5.32)
$$

where

$$
b = \frac{1}{\sqrt{5}}(a^3 + 2a_D^3) \qquad (5.33)
$$

is the scalar $\mathcal{N} = 2$ superpartner of the photon (5.30) while $F_{\mu\nu}^B$ is the field strength of the U(1) gauge field B_μ. Covariant derivatives are defined in accordance with the charges of the dyons D_1 and D_2. Namely,

$$
\nabla_\mu^1 = \partial_\mu - i\Big(\frac{1}{2}A_\mu + \frac{1}{2}A_\mu^3 + A_\mu^{3D}\Big)
$$
$$
= \partial_\mu - \frac{i}{2}(A_\mu + \sqrt{5}B_\mu),
$$
$$
\nabla_\mu^2 = \partial_\mu - i\Big(\frac{1}{2}A_\mu - \frac{1}{2}A_\mu^3 - A_\mu^{3D}\Big) \qquad (5.34)
$$
$$
= \partial_\mu - \frac{i}{2}(A_\mu - \sqrt{5}B_\mu).
$$

The coupling constants g_1 and $\tilde{g}_2$ correspond to two U(1) gauge groups. The potential $V(D, \tilde{D}, b, a)$ in the action (5.32) is

$$
V(D, \tilde{D}, b, a) = \frac{5\tilde{g}_2^2}{8}(|D_1|^2 - |\tilde{D}_1|^2 - |D_2|^2 + |\tilde{D}_2|^2)^2
$$
$$
+ \frac{g_1^2}{8}(|D_1|^2 - |\tilde{D}_1|^2 + |D_2|^2 - |\tilde{D}_2|^2 - 2\xi)^2
$$
$$
+ \frac{5\tilde{g}_2^2}{2}|\tilde{D}_1 D_1 - \tilde{D}_2 D_2|^2 + \frac{g_1^2}{2}|\tilde{D}_1 D_1
$$
$$
+ \tilde{D}_2 D_2|^2 + \frac{1}{2}\Big\{|a + \sqrt{5}b + \sqrt{2}m_1|^2
$$
$$
\times (|D_1|^2 + |\tilde{D}_1|^2) + |a - \sqrt{5}b + \sqrt{2}m_2|^2
$$
$$
\times (|D_2|^2 + |\tilde{D}_2|^2) \Big\}. \qquad (5.35)
$$

Now we are ready to move to the desired limit of equal quark masses, $\Delta m = 0$. In this limit the global SU(2)$_{C+F}$ symmetry is restored in the underlying theory. The vacuum of the theory (5.32) is located at the following values of scalars a and b:

$$
a = -\sqrt{2}m, \qquad \sqrt{5}b = -\frac{\Delta m}{\sqrt{2}}, \qquad (5.36)
$$

while the VEVs of the dyons are determined by the FI parameter ξ,

$$
D_1 = \sqrt{\xi}, \qquad D_2 = \sqrt{\xi}, \qquad \tilde{D}_1 = \tilde{D}_2 = 0. \qquad (5.37)
$$

Thus, the U(1) × U(1) gauge group is broken by dyon condensation. Both photons and dyons become massive, with masses proportional to $\sqrt{\xi}$. In particular, at $\Delta m = 0$ the vacuum value of b vanishes.

Note also that the theory (5.32) is the Abelian U(1) × U(1) gauge theory and hence is not asymptotically free. It stays at weak coupling at small ξ.

The low-energy theory (5.32) does not seem to have any global SU(2) symmetry. However, the underlying theory does have a global SU(2) symmetry in the limit $\Delta m = 0$. As was explained in Sec. II, this global SU(2) is not broken in domain I at large ξ. This symmetry is realized in a color-flavor-locked form in this domain [see Eq. (2.7)], and no Goldstone bosons are present. We showed that no massless states are present in domain III at nonzero ξ (and no light states are present other than two dyons and two photons discussed above); therefore, the global SU(2) cannot be spontaneously broken in this domain either. The only way out of this puzzle is to conclude that the SU(2) global symmetry is realized trivially in the low-energy description (5.32), i.e. that all states in (5.32) are singlets of the unbroken flavor SU(2).

This means, as was already mentioned in Sec. III, that the photon B_μ which appears in domain III has nothing to do with the third component of the SU(2) gauge field A_μ^a of domain I. At $\Delta m = 0$ the former is a singlet of the global SU(2), while the latter is a component of a triplet.[6]

Moreover, dyons D_1 and D_2 present in domain III have nothing to do with diagonal entries of the quark matrix q^{kA} of domain I. Dyons are singlets while the quarks q^{kA} form the singlet and triplet states.

Since we have a crossover between domains I and III rather than a phase transition, this means that in the full theory triplets become heavy and decouple as we pass from domain I into domain III along the line $\Delta m = 0$. Moreover, some composite singlets which are heavy and invisible in domain I become light in domain III and form dyons $D_{1,2}$ and photon B_μ (level crossing). Although this crossover is smooth in the full theory, from the standpoint of the low-energy description the passage from domain I into domain III means a dramatic change: the low-energy theories in these domains are completely different; in particular, the degrees of freedom in these theories are different (non-Abelian in domain I vs Abelian in domain III).

---

[6]We reveal the physical nature of these triplets in [32]. They are formed by monopole-antimonopole pairs connected by two confining strings.

D. Strings in domain III

It is obvious that the low-energy theory (5.32) has Z_2 string solutions in the vacuum (5.36) and (5.37). That is, the D_1 dyon can have a winding at infinity. In this case the string solution has the following behavior at $r \to \infty$:

$$D_1 \sim e^{i\alpha}\sqrt{\xi}, \qquad D_2 \sim \sqrt{\xi}, \qquad A_i \sim \partial_i \alpha, \qquad \sqrt{5}B_i \sim \partial_i \alpha,$$
(5.38)

where the indices $i = 1, 2$ denote the plane orthogonal to the string axis and r and α are polar coordinates in this plane. Another elementary string can be obtained from the one in (5.38) by the replacement $D_1 \to D_2$, $D_2 \to D_1$, and $B_i \to -B_i$.

These Z_2 elementary strings are BPS saturated. Their tensions are given by the formula (4.4) in the same way as the tensions of elementary strings in domains I and II.

The Z_2 strings are Abelian (of the Abrikosov-Nielsen-Olesen type [33]) in domain III. They do not have any orientational moduli, in contrast with non-Abelian strings in domain I.

Let us calculate the gauge-invariant non-Abelian flux (4.20) for these strings. In the Abelian domain II at large $|\Delta m|$,

$$a^a F_3^{*a} \to a^3 F_3^{*3}.$$
(5.39)

With $|\Delta m|$ decreasing, as we pass through monodromies, we get

$$a^3 \to a^3 + 2a_D^3 = \sqrt{5}b,$$
(5.40)

$$A_\mu^3 \to \sqrt{5}B_\mu.$$
(5.41)

Equation (5.41) follows from Eq. (5.40) by $\mathcal{N} = 2$ supersymmetry. Therefore

$$\Phi_{\mathrm{III}} = \int d^2x (\sqrt{5}b)(\sqrt{5}F_3^{*B}).$$
(5.42)

Equation (5.36) gives

$$\sqrt{5}b = -\Delta m/\sqrt{2}$$

in the $r = 2$ vacuum, while the flux of the field B_μ of the Z_2 strings can be read off from Eq. (5.38). In this way we arrive at

$$\langle \Phi \rangle_{\mathrm{III}} \to \mp 2\pi \frac{\Delta m}{\sqrt{2}}.$$
(5.43)

We see that the string flux in domain III is given by the same formula as in domain II. This is a flux of the Abelian string. The non-Abelian part of the flux is directed in the Cartan subalgebra of the gauge group. No orientational moduli appear. In contrast, in domain I the flux of the non-Abelian string is proportional to the orientational vector S^a. At small $|\Delta m|$ the expectation value $\langle S^a \rangle \to 0$, and the string flux is averaged to zero; see (4.22). Domains I and III are separated by a crossover at $\xi \sim \Lambda^2$.

Let us also mention one more dramatic distinction of nonperturbative spectra in domains I and III at $\Delta m = 0$. The confined SU(2) monopoles [with quantum numbers (5.16)] are the junctions of two different elementary strings in both domains. In the non-Abelian domain I the confined monopoles are seen as kinks of the world-sheet $CP(N-1)$ model [10,11,34]. As was shown by Witten [19], deep in the quantum regime at $(m_A - m_B) = 0$ the kink of the $CP(N-1)$ model is described by the field n^l and therefore acquires a global flavor quantum number with respect to the unbroken $SU(N)_{C+F}$. In fact, the kink/monopole is in the fundamental representation of this group (a doublet in the case $N = 2$) [19,35]. Therefore, a meson formed by a monopole connected to an antimonopole by two strings (see the review paper [7] for details) belongs to the singlet or adjoint representations of the global SU(N) [singlet or triplet of SU(2) for $N = 2$].

Clearly, in domain III the monopole confined by strings does not acquire global quantum numbers. It is in the singlet representation of SU(2). Hence, a meson formed by a monopole connected to an antimonopole by two strings is a singlet too. Thus, in the nonperturbative spectra of the theory we observe the same phenomenon which was seen in the perturbative spectra: triplets of global SU(2) present at low energies in domain I are lifted and do not appear in the low-energy description in domain III. Both perturbative and nonperturbative states in domain III are singlets of the unbroken global SU(2).

VI. THE PHASE TRANSITION AT $N \to \infty$

In this section we will consider the N dependence of the crossover transitions (see Fig. 1) in parameters ξ and $(m_A - m_B)$. We will show that in the large-N limit the crossovers become exceedingly sharper, and at $N = \infty$ they transform into a genuine phase transition. We will start from the crossover in $(m_A - m_B)$ at large ξ (i.e. the passage from domain I to domain II).

This crossover in the nonperturbative sector of the theory can be seen as a crossover in the effective $CP(N-1)$ model (4.24) on the world sheet of the non-Abelian string; see Sec. IV. As was already explained, at large quark mass differences the $CP(N-1)$ model is at weak coupling. The VEV of the vector n^l does not vanish. If we make a special choice for the mass parameters,

$$m_k = m_0 e^{(2\pi k/N)i}, \qquad k = 1, \ldots, N,$$
(6.1)

where m_0 is a single common parameter (which we will take to be real), our theory has a discrete Z_{2N} symmetry; see the Appendix for further details. In fact, $\langle n^l \rangle$ is an order parameter for the spontaneous breaking of this Z_{2N} symmetry down to Z_2.

At weak coupling, at large m_0, dynamics can be described as follows. The action (4.24) contains a term

$$\left| \sigma - \frac{m_l}{\sqrt{2}} \right|^2 |n^l|^2. \tag{6.2}$$

At weak coupling the field n can develop a VEV if σ reduces to a particular mass parameter,

$$\sigma = \frac{m_k}{\sqrt{2}}, \qquad n^l = \sqrt{2\beta}\delta^{lk}, \tag{6.3}$$

where $k = 1, \ldots, N$ labels N different vacua (i.e. the elementary Z_N strings of the bulk theory) and we rescaled the field n^l in (4.24) to make its kinetic term canonic, namely, $n^l \rightarrow n^l/\sqrt{2\beta}$.

As we reduce the value of m_0 the vacuum expectation value of the n^l field becomes smaller and tends to zero at the left boundary of domain II. Simultaneously, the VEV of σ is no longer given by the mass, as in Eq. (6.3). In fact, σ determines the bifermion condensate; $|\sigma|$ becomes of the order of Λ_σ at $m_0 \rightarrow 0$. In both limits the Z_N symmetry is broken. This is the reason why two domains, I and II, are separated by a crossover rather than a phase transition.[7]

In order to study the crossover at any N (rather than at $N = \infty$) we can use the description of the supersymmetric $CP(N - 1)$ model in terms of an exact superpotential [17,22]. Upon integrating out n^l fields the model can be described by an exact twisted superpotential of the Veneziano-Yankielowicz type [36],

$$\mathcal{W}_{\text{eff}} = \beta\Sigma + \frac{1}{4\pi}\sum_{l=1}^{N}\left(\Sigma - \frac{m_l}{\sqrt{2}}\right)\ln\left(\Sigma - \frac{m_l}{\sqrt{2}}\right), \tag{6.4}$$

where Σ is a twisted superfield [17] (with σ being its lowest scalar component) and we ignore here the θ dependence (θ stands for the vacuum angle). Minimizing this superpotential with respect to σ, we find

$$\prod_{l=1}^{N}(\sqrt{2}\sigma - m_l) = \Lambda_\sigma^N, \tag{6.5}$$

where Λ_σ is the scale parameter of the $CP(N - 1)$ sigma model under consideration.

Let us examine this equation, determining the VEV of the field σ at finite rather than infinite N. If N is fixed, it is readily seen that at large $|m_l|$ (i.e. $m_0 \gg \Lambda_\sigma$) the solution for σ coincides with one of the masses, in accordance with our semiclassical analysis; see Eq. (6.3). In the opposite limit of zero masses ($m_0 = 0$),

$$\sigma = \frac{1}{\sqrt{2}}\Lambda_\sigma e^{2\pi i k/N}, \tag{6.6}$$

where $k = 1, \ldots, N$ marks N distinct vacua. As was mentioned above, the Z_N symmetry is spontaneously broken at any m_0.

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[7]In the nonsupersymmetric case the VEV of σ vanishes in domain I, and the Z_N symmetry is restored [5,6]. In this case we do have a phase transition between domains I and II.

As we increase the value of m_0, the vacuum expectation value of σ smoothly interpolates between the regime (6.6), where the order parameter which distinguishes different vacua of the $CP(N - 1)$ model (i.e. different elementary strings of the bulk theory) is a bifermion condensate $\sim\sigma$, and the regime (6.3), where σ is determined by one of the masses m_l, while n^l develops a VEV. For finite N the solution for σ is a smooth function of m_0. Thus, this is a crossover that takes place between domains I and II.

If we increase N, this crossover becomes more pronounced. Let us study Eq. (6.5) at large N. To simplify our analysis let us consider $N = 2^p$, where p is an integer. Then Eq. (6.5) can be rewritten as

$$(\sqrt{2}\sigma)^N - m_0^N = \Lambda_\sigma^N. \tag{6.7}$$

This equation has the following perfectly smooth solution:

$$\sigma = \frac{1}{\sqrt{2}}e^{(2\pi k/N)i}(m_0^N + \Lambda_\sigma^N)^{1/N}. \tag{6.8}$$

However, at $N \rightarrow \infty$ the above function takes the form

$$\sigma = \frac{1}{\sqrt{2}}e^{(2\pi k/N)i} \times \begin{cases} m_0 & m_0 > \Lambda_\sigma \\ \Lambda_\sigma & m_0 < \Lambda_\sigma. \end{cases} \tag{6.9}$$

Corrections to this expression are exponential in $(-N)$.

We see that the solution for σ develops a discontinuity in the first derivative with respect to m_0. The crossover becomes a phase transition in the limit $N = \infty$. We stress that this phase transition is an artifact of the large-N approximation and is not related to a change in the pattern of realization of any symmetry.

The solution (6.9) for σ ensures the following behavior of the vector n^l in the $N \rightarrow \infty$ limit:

$$\langle n^l \rangle = \begin{cases} \sqrt{2\beta_{\text{ren}}}\delta^{kl} & m_0 > \Lambda_\sigma \\ 0 & m_0 < \Lambda_\sigma, \end{cases} \tag{6.10}$$

where $k = 1, \ldots, N$. The renormalized coupling β_{ren} tends to zero at $m_0 = \Lambda_\sigma$ [6]; thus, the VEV of n^l develops a discontinuity in the first derivative with respect to m_0.

This solution implies that the gauge-invariant non-Abelian flux of the non-Abelian string strictly vanishes in domain I,

$$\langle \Phi \rangle_{\text{I}} = 0 \tag{6.11}$$

at $N = \infty$, while in domains II and III it is given by a $U(N)$ generalization of Eq. (5.43). Namely,

$$\langle \Phi \rangle_{\text{II}} = \langle \Phi \rangle_{\text{III}} = -2\pi\sqrt{2}m_k \tag{6.12}$$

for the kth elementary Z_N string, $k = 1, \ldots, N$; see Eqs. (2.4) and (4.9).

The result (6.11) is exact at $N = \infty$. Thus, at $N = \infty$ the string flux (4.20) develops a discontinuity as we pass from

domain I to domain II or III. This implies that both cross-overs in ξ and m_0 transform into phase transitions.

VII. DISCUSSION

In this paper we considered the $r = N$ vacuum in $\mathcal{N} = 2$ supersymmetric QCD (SQCD) with the U(N) gauge group and $N_f = N$ flavors. We demonstrated that this theory exhibits a crossover transition in ξ; see Fig. 1. Namely, at large ξ in domain I the theory is in the non-Abelian confinement regime; it has N^2 degrees of freedom (gauge bosons and quarks) at low energies and supports non-Abelian strings. In contrast, at small ξ the theory passes into the Abelian Seiberg-Witten regime III. The low-energy effective description includes N degrees of freedom (dyons and dual photons) and supports Abelian strings. We have shown that non-Abelian gauge bosons and quarks in domain I have nothing to do with Abelian dyons and photons of domain III. These states belong to a different representation of the unbroken global flavor group SU(N).

Although in this paper we considered a particular vacuum in a specially chosen version of $\mathcal{N} = 2$ SQCD [where the global SU(N) symmetry remains unbroken due to the color-flavor locking], we believe that our results are quite general. It seems plausible that many Abelian vacua of the Seiberg-Witten type in $\mathcal{N} = 2$ SQCD exhibit crossover transitions into non-Abelian regimes as we increase the FI parameter ξ. Usually we do not have appropriate extra parameters (such as the quark mass differences in our example) which would allow us to study these crossovers.

The lesson is that, generally speaking, non-Abelian strings can smoothly evolve into Abelian strings and vice versa. At the same time, the corresponding dynamical patterns are drastically different.

What conclusions apply to theories with less supersymmetry? In the simplest version of the Seiberg-Witten solution [1], $\mathcal{N} = 2$ supersymmetric QCD can be deformed by adding a mass term μ for the adjoint field. In the limit of large μ the theory flows to $\mathcal{N} = 1$ SQCD. At small μ the mass term for the adjoint field induces a Fayet-Iliopoulos F term in $\mathcal{N} = 2$ theory [16,37], with ξ proportional to μ times some mass scale, such as Λ or the quark mass. Thus, the deformation parameter μ translates, roughly speaking, into the FI parameter ξ.

It is commonly believed that the behavior of supersymmetric QCD is smooth in μ: the Abelian degrees of freedom of $\mathcal{N} = 2$ theory smoothly evolve into non-Abelian degrees of freedom of $\mathcal{N} = 1$ theory as we increase $|\mu|$. While on the conceptual side our results provide unambiguous evidence in favor of the smooth transition, dynamics-wise the emerging pictures on the opposite sides of domain lines separating domains I, II, and III hardly look alike. In particular, light degrees of freedom are completely different.

In addition, we should note that there is at least one example of a nonsupersymmetric model where the evolution is proven to be discontinuous,[8] with a phase transition [5,6]. And even in our basic $\mathcal{N} = 2$ model the crossover becomes a full-blown phase transition at $N = \infty$.

To conclude, we would like to comment on the recent paper [39]. In this paper it is argued that non-Abelian vacua with $r > N_f/2$ which support non-Abelian strings "dynamically Abelianize" in quantum theory. We disagree with this statement. As we demonstrated above, both the Abelian and non-Abelian regimes can be present in $\mathcal{N} = 2$ QCD in quantum theory. They just occur in different domains of the parameter space and are separated by crossovers.

APPENDIX: GLOBAL SYMMETRIES OF THE $CP(N-1)$ MODEL WITH Z_{2N}-SYMMETRIC TWISTED MASSES

First, let us outline the $\mathcal{N} = 2$ $CP(N-1)$ model with twisted masses [21] in one of a few possible formulations, the so-called gauge formulation [40]. This formulation is built on an N-plet of complex scalar fields n^i where $i = 1, 2, \ldots, N$. We impose the constraint

$$n_i^\dagger n^i = 1. \tag{A1}$$

This leaves us with $2N - 1$ real bosonic degrees of freedom. To eliminate one extra degree of freedom, we impose a local U(1) invariance $n^i(x) \rightarrow e^{i\alpha(x)} n^i(x)$. To this end we introduce a gauge field A_μ which converts the partial derivative into the covariant one,

$$\partial_\mu \rightarrow \nabla_\mu \equiv \partial_\mu - iA_\mu. \tag{A2}$$

The field A_μ is auxiliary; it enters in the Lagrangian without derivatives. The kinetic term of the n fields is

$$L = \frac{2}{g_0^2} |\nabla_\mu n^i|^2. \tag{A3}$$

The superpartner to the field n^i is an N-plet of complex two-component spinor fields ξ^i,

$$\xi^i = (\xi_R^i, \xi_L^i). \tag{A4}$$

---

[8]On the other hand, analyses [2,3] carried out in a nonsupersymmetric setting different from that treated here and in [5,6] show no sign of the phase transition, while [38] exhibits a chiral phase transition on the way from Abelian to non-Abelian confinement.

The auxiliary field A_μ has a complex scalar superpartner σ and a two-component complex spinor superpartner λ; both enter without derivatives. The full $\mathcal{N} = 2$ symmetric Lagrangian is

$$L = \frac{2}{g_0^2}\left\{ |\nabla_\mu n^i|^2 + \xi_i^\dagger i\gamma^\mu \nabla_\mu \xi^i + 2\sum_i \left| \sigma - \frac{m_i}{\sqrt{2}} \right|^2 |n^i|^2 \right.$$
$$+ \left[i\sqrt{2}\sum_i \left(\sigma - \frac{m_i}{\sqrt{2}} \right) \xi_{iR}^\dagger \xi_L^i \right.$$
$$\left. \left. + i\sqrt{2} n_i^\dagger (\lambda_R \xi_L^i - \lambda_L \xi_R^i) + \text{H.c.} \right] \right\}, \qquad (A5)$$

where m_i are twisted mass parameters. Equation (A5) is valid in a special case when

$$\sum_{i=1}^N m_i = 0. \qquad (A6)$$

We will make a specific choice of the parameters m_i, namely,

$$m_i = m\{e^{2\pi i/N}, e^{4\pi i/N}, \ldots, e^{2(N-1)\pi i/N}, 1\}, \qquad (A7)$$

where m is a single common parameter. Then the constraint (A6) is automatically satisfied. Without loss of generality m can be assumed to be real and positive. The U(1) gauge symmetry is built in. This symmetry eliminates one bosonic degree of freedom, leaving us with $2N - 2$ dynamical bosonic degrees of freedom inherent to the $CP(N - 1)$ model.

Now let us discuss global symmetries of this model. In the absence of the twisted masses the model was SU(N) symmetric. The twisted masses (A7) explicitly break this symmetry down to U(1)$^{N-1}$,

$$n^\ell \to e^{i\alpha_\ell} n^\ell, \qquad \xi_R^\ell \to e^{i\alpha_\ell} \xi_R^\ell \qquad \xi_L^\ell \to e^{i\alpha_\ell} \xi_L^\ell,$$
$$\ell = 1, 2, \ldots, N, \qquad \sigma \to \sigma, \qquad \lambda_{R,L} \to \lambda_{R,L}, \qquad (A8)$$

where α_ℓ are N constant phases that are different for different ℓ.

Next, there is a global vectorial U(1) symmetry which rotates all fermions ξ^ℓ in one and the same way, leaving the boson fields intact,

$$\xi_R^\ell \to e^{i\beta} \xi_R^\ell, \qquad \xi_L^\ell \to e^{i\beta} \xi_L^\ell, \qquad \ell = 1, 2, \ldots, N,$$
$$\lambda_R \to e^{-i\beta} \lambda_R, \qquad \lambda_L \to e^{-i\beta} \lambda_L, \qquad n^\ell \to n^\ell, \qquad \sigma \to \sigma. \qquad (A9)$$

Finally, there is a discrete Z_{2N} symmetry which is of the most importance for our purposes. Indeed, let us start from the axial U(1)$_R$ transformation which would be a symmetry of the classical action at $m = 0$ (it is anomalous, though, under quantum corrections),

$$\xi_R^\ell \to e^{i\gamma} \xi_R^\ell, \qquad \xi_L^\ell \to e^{-i\gamma} \xi_L^\ell, \qquad \ell = 1, 2, \ldots, N,$$
$$\lambda_R \to e^{i\gamma} \lambda_R, \qquad \lambda_L \to e^{-i\gamma} \lambda_L, \qquad \sigma \to e^{2i\gamma}\sigma, \qquad n^\ell \to n^\ell. \qquad (A10)$$

With m switched on and the chiral anomaly included, this transformation is no longer the symmetry of the model. However, a discrete Z_{2N} subgroup survives both the inclusion of the anomaly and $m \neq 0$. This subgroup corresponds to

$$\gamma_k = \frac{2\pi i k}{2N}, \qquad k = 1, 2, \ldots, N, \qquad (A11)$$

with the simultaneous shift

$$\ell \to \ell - k. \qquad (A12)$$

In other words,

$$\xi_R^\ell \to e^{i\gamma_k} \xi_R^{\ell-k}, \qquad \xi_L^\ell \to e^{-i\gamma_k} \xi_L^{\ell-k}, \qquad \lambda_R \to e^{i\gamma_k} \lambda_R,$$
$$\lambda_L \to e^{-i\gamma_k} \lambda_L, \qquad \sigma \to e^{2i\gamma_k}\sigma, \qquad n^\ell \to n^{\ell-k}. \qquad (A13)$$

This Z_{2N} symmetry relies on the particular choice of masses given in (A7).

The order parameters for the Z_N symmetry are as follows: (i) the set of the vacuum expectation values $\{\langle n^1 \rangle, \langle n^2 \rangle, \ldots \langle n^N \rangle\}$ and (i) the bifermion condensate $\langle \xi_{R,\ell}^\dagger \xi_L^\ell \rangle$. That is, a nonvanishing value of $\langle n^1 \rangle$ or $\langle \xi_{R,\ell}^\dagger \xi_L^\ell \rangle$ implies that the Z_{2N} symmetry of the action is broken down to Z_2. The first set of order parameters is more convenient for detection at large m, while the second-order parameter is more easily detected at small m.

It is instructive to illustrate the above conclusions in a different formulation of the sigma model, namely, in the geometrical formulation [for simplicity, we will consider $CP(1)$; generalization to $CP(N - 1)$ is straightforward]. In components, the Lagrangian of the model is

$$\mathcal{L}_{CP(1)} = G\left\{ \partial_\mu \phi^\dagger \partial^\mu \phi - |m|^2 \phi^\dagger \phi + \frac{i}{2}(\psi_L^\dagger \overleftrightarrow{\partial}_R \psi_L \right.$$
$$+ \psi_R^\dagger \overleftrightarrow{\partial}_L \psi_R) - i\frac{1 - \phi^\dagger \phi}{\chi}(m\psi_L^\dagger \psi_R$$
$$+ \bar{m}\psi_R^\dagger \psi_L) - \frac{i}{\chi}[\psi_L^\dagger \psi_L (\phi^\dagger \overleftrightarrow{\partial}_R \phi)$$
$$+ \psi_R^\dagger \psi_R (\phi^\dagger \overleftrightarrow{\partial}_L \phi)] - \frac{2}{\chi^2} \psi_L^\dagger \psi_L \psi_R^\dagger \psi_R \right\}, \qquad (A14)$$

where

$$\chi = 1 + \phi^\dagger \phi, \qquad G = \frac{2}{g_0^2 \chi^2} \qquad (A15)$$

and

$$\partial_L = \frac{\partial}{\partial t} + \frac{\partial}{\partial z}, \qquad \partial_R = \frac{\partial}{\partial t} - \frac{\partial}{\partial z}. \qquad (A16)$$

The Z_2 transformation corresponding to (A13) is

$$\phi \to -\frac{1}{\phi^\dagger}, \qquad \psi_R^\dagger \psi_L \to -\psi_R^\dagger \psi_L. \qquad (A17)$$

The order parameter which can detect breaking/nonbreaking of the above symmetry is

$$\frac{m}{g_0^2}\left(1 - \frac{g_0^2}{2\pi}\right)\frac{\phi^\dagger \phi - 1}{\phi^\dagger \phi + 1} - iR\psi_R^\dagger \psi_L. \qquad (A18)$$

Under the transformation (A17) this order parameter

changes sign. In fact, this is the central charge of the $\mathcal{N} = 2$ sigma model, including the anomaly [31].

Now, what changes if instead of the $\mathcal{N} = 2$ model we consider a nonsupersymmetric $CP(N-1)$ model with twisted masses? Then the part of the Lagrangian (A5) containing fermions must be dropped. The same must be done in the Z_2-order parameter. As was shown in [5,6], now at $m > \Lambda$ the Z_2 symmetry is broken, while at $m < \Lambda$ it is unbroken. A phase transition takes place.

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