Dark energy, inflation, and extra dimensions

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We consider how accelerated expansion, whether due to inflation or dark energy, imposes strong constraints on fundamental theories obtained by compactification from higher dimensions. For theories that obey the null energy condition (NEC), we find that inflationary cosmology is impossible for a wide range of compactifications; and a dark energy phase consistent with observations is only possible if both Newton's gravitational constant and the dark energy equation of state vary with time. If the theory violates the NEC, inflation and dark energy are only possible if the NEC-violating elements are inhomogeneously distributed in the compact dimensions and vary with time in precise synchrony with the matter and energy density in the noncompact dimensions. Although our proofs are derived assuming general relativity applies in both four and higher dimensions and certain forms of metrics, we argue that similar constraints must apply for more general compactifications.

I. INTRODUCTION

Compelling evidence exists that the present Universe is dominated by some form of dark energy and undergoing a period of accelerated expansion. Also, a widely accepted hypothesis is that the early Universe underwent inflation, a period of accelerated expansion shortly after the big bang that smoothed and flattened the Universe and generated a nearly scale-invariant spectrum of density perturbations.

The purpose of this paper is to explore the implications of cosmic acceleration for fundamental theories obtained by compactification from a higher-dimensional theory, a feature common to Kaluza-Klein theory, Randall-Sundrum models, string theory, and M-theory, for example. A general property of compactified theories is that the expansion of the noncompact directions required for any realistic big bang cosmology has the tendency to cause the extra dimensions to contract unless some interaction prevents it. The contraction of the extra dimensions has undesirable physical effects, such as the time variation of Newton's constant or other fundamental constants and a deviation from standard Friedmann-Robertson-Walker (FRW) evolution. For decelerating universes, these problems can be avoided, in principle, by introducing ordinary interactions.

In this paper, though, combining techniques developed in Refs. [1,2] with new approaches, we shall derive a series of no-go theorems showing how one is forced to consider more exotic solutions in order to obtain accelerated expansion in compactified theories. The power of these theorems may surprise some readers, yet they emerge from fairly simple considerations. The key constraint is that the models are described by Einstein gravity both in the 4d effective theory and in the higher-dimensional theory. What seems relatively innocuous in the 4d effective theory— e.g., accelerated expansion of the noncompact directions—

can require something extraordinary when lifted into the higher-dimensional Einstein gravity. As a simple example, consider the original Kaluza-Klein model with a single static extra dimension whose size, we will assume, has been frozen by some interaction. Accelerated expansion of the 4d effective theory means that the 5d theory is described by a metric $ds^2 = -dt^2 + a^2(dx_1^2 + dx_2^2 + dx_3^2) + dx_4^2$, where the FRW scale factor satisfies $\dot{a} > 0$ and $\ddot{a} > 0$. By substituting the metric into the 5d Einstein equations, it is possible to show [1] that the equation of state in the compact dimension (the ratio of the 4-4 to the 0-0 components of the energy-momentum tensor) is $w_5 < -1$; for example, for an expanding Universe with $a(t) \sim t^{p>1}$, its value is $w_5 = (1 - 2p)/p < -1$. The fact that this ratio is less than -1 means the higher-dimensional theory necessarily violates the null energy condition (NEC), an extraordinary constraint. The NEC is not violated by any observed matter fields or by unitary two-derivative quantum field theories; and violating the NEC can produce problems of its own. Under many conditions it leads to unacceptable consequences, such as superluminal propagation, instabilities, or violations of unitarity [3–7].

We begin in Sec. III by considering compactified theories that do not violate the NEC, including the original Kaluza-Klein model, the Randall-Sundrum II model [8], and many string theory models, and see how difficult it is to accommodate accelerated expansion. For a wide class of models, we derive a no-go theorem that rules out inflationary cosmology altogether and additional no-go theorems that rule out the simplest dark energy models, including ΛCDM. We further show that a dark energy phase with accelerated expansion consistent with current observations is only possible if both Newton's gravitational constant and the dark energy equation of state vary with time.

Then, we turn our attention in Sec. IV to models that do violate the NEC. In spite of the potential dangers cited above and in Refs. [3–7], models of this type have been suggested that may safely violate NEC, such as the Randall-Sundrum I model [9] and recently proposed flux compactifications on the string landscape. Examples of NEC-violating components invoked in string constructions include orientifold planes, which have negative tension, and quantum effects analogous to Casimir energy. Here, though, we find another set of new no-go theorems that rule out some forms of NEC violation and impose precise conditions on how any NEC-violating components must vary with time as the Universe evolves. Although the discussion here is confined to certain common types of metrics and assumes Einstein's general theory of relativity applies in higher dimensions, we argue in Sec. V that similar no-go theorems must apply in more general cases.

Our approach complements but is quite different from previous no-go theorems based on supersymmetry or supergravity [10]; supersymmetry is not assumed in our analysis, so our conclusions apply to more general compactified theories. Our results are also different from inflationary no-go theorems based on requiring small values of the slow-roll parameters ϵ and η in the case of inflation; or constructions leading to the long-lived metastable de Sitter minima in the string landscape [11–15]. Previous theorems are based on what might be called "micro-to-macro" approaches where the microphysics is specified first and then the constraints on the macroscopic pressure, energy density and equation of state are derived.

Ours is a more "macro-to-micro" approach in which we assume a certain equation of state on macroscopic scales (based on observations) and derive constraints on the microphysics. This method is more closely related to the one used by various authors [16–20] to constrain compactified theories with *purely static de Sitter minima*. A purely static de Sitter universe has equation of state $w = w_{DE} = -1$, where we use throughout this paper the symbol w (without subscript) to represent the ratio of *total* pressure to *total* energy density and w_{DE} to represent the pressure-to-density ratio for the dark energy component alone. References [1,2] and this paper, though, use somewhat different technical approaches resulting in several advances over previous work, including:

(i) extension to cases where w is greater than -1;

(ii) extension to time-varying extra-dimensional metrics;

(iii) extension to practical cases where w is time varying (e.g., the present Universe has $w \approx -0.74$ today and varying with time) [21–23];

(iv) extension from $\mathcal{R}$ flat (RF) to $\mathcal{R}$ flat up to a conformal factor (CRF) metrics;

(v) in cases where NEC is satisfied, demonstration that substantial variations in G and w are required;

(vi) in cases where NEC violated, demonstration that many additional conditions on the NEC-violating

component must be satisfied to match observed cosmology (e.g., the introduction of negative tension branes in inflation or dark energy models is not sufficient).

II. COMPACTIFIED MODELS AND NEC VIOLATION

The NEC is commonly assumed in fundamental theories to avoid the classical and quantum instabilities (closed timelike curves, big rips, ghosts, and unitarity violation) normally associated with its violation [3–7]. Nevertheless, we will show that, for a wide range of compactified models, inflationary cosmology and the NEC are completely incompatible and that dark energy is compatible only if Newton's gravitational constant G_N and the dark energy equation of state w_{DE} vary with time.

A. Assumptions

Our conclusions rest on rigorous theorems that apply to compactified satisfying certain conditions in addition to NEC:

(i) *GR condition:* both the higher-dimensional theory and the 4d theory are described by Einstein's theory of general relativity (GR), either exactly or with small corrections;

(ii) *Flatness condition:* the 4d theory is spatially flat;

(iii) *Boundedness condition:* the extra dimensions are bounded;

(iv) *Metric condition:* the metric of the higher-dimensional theory is RF or CRF:

$$ds^2 = e^{2\Omega}(-dt^2 + \bar{a}^2(t)d\mathbf{x}^2) + g_{mn}dy^m dy^n, \quad (1)$$

where the $\mathbf{x}$ are the noncompact spatial dimensions; $y \equiv \{y^m\}$ are the extra dimensions; $\bar{a}(t)$ is the usual FRW scale factor; and

$$g_{mn}(t, y) = e^{-2\bar{\Omega}}\bar{g}_{mn}, \quad (2)$$

where $\bar{g}_{mn}$ has Ricci (scalar) curvature $\mathcal{R} = 0$, as evaluated in the compact dimensions. We do not require that $\bar{g}_{mn}$ have zero Ricci tensor. We call the metric RF if $\bar{\Omega} = $ const and CRF if $\bar{\Omega}(t, y) = \Omega(t, y)$. We will use indices $\{M, N\}$ to represent all $4 + k$ dimensions, $\{\mu, \nu\}$ to represent the noncompact dimensions, and $\{m, n\}$ to represent the extra dimensions.

These conditions are common to many models published in the literature. The *GR condition* dates back to the original Kaluza-Klein theory and underlies the idea of unified theories based on compactifying extra dimensions. It is reasonable to expect corrections, such as higher derivative terms, in the higher and 4d effective theory. So long as those are small, the theorems will apply with obvious caveats (as discussed in Sec. VI). The *spatial flatness condition* for the 4d theory is motivated by cos-

mological observations, e.g., from WMAP [23]. The *boundedness condition* on the extra dimensions is needed because the theorems rely on integrating fields and warp factors over the compact direction. In particular, the boundedness condition insures that, if Ω is nontrivial and has continuous first derivative, then the Laplacian $\Delta\Omega$ must be nonzero for some y; this fact is useful in some of the proofs.

The *metric condition* is motivated by common constructions in the literature, especially string theory. The original Kaluza-Klein model, the Randall-Sundrum models, and Calabi-Yau based models are all RF; some useful theorems for this case were developed in Refs. [1,2]. Metrics of CRF type appear in warped Calabi-Yau [11] and warped conifold [24] constructions (where they are sometimes referred to as conformally Calabi-Yau metrics). Here, we derive no-go theorems for both RF and CRF models. Our constraints for RF and CRF are slightly different in terms of the number of extra dimensions and the moduli fields to which they apply. However, the differences do not affect our conclusions for practical cases relevant to string theory, M-theory, the Kaluza-Klein model, etc., so we will only present the details for CRF models and ignore the fine distinctions.

B. Detecting NEC violation

In this subsection, we develop some basic relations that make it possible to detect easily when a higher-dimensional theory is forced to violate the NEC.

To describe a spatially flat FRW spacetime after dimensional reduction, the metric $g_{mn}(t, y)$ and warp function $\Omega(t, y)$ must be functions of time t and the extra-dimensional coordinates y^m only. Following the convention in Ref. [1], we parameterize the rate of change of g_{mn} using quantities ξ and σ_{mn} defined by

$$\frac{1}{2}\frac{dg_{mn}}{dt} = \frac{1}{k}\xi g_{mn} + \sigma_{mn}, \qquad (3)$$

where $g^{mn}\sigma_{mn} = 0$ and where ξ and σ are functions of time and the extra dimensions; this relation assumes the gauge choice discussed in Ref. [1].

It is important to note that all discussions of the equation of state, the NEC, accelerated expansion, the energy-momentum tensor T_{MN}, and the pressure and density of any components always refer to Einstein frame quantities in either the higher-dimensional or 4d effective theory. The space-space components of the energy-momentum tensor are block diagonal with a 3×3 block describing the energy momentum in the three noncompact dimensions and a $k \times k$ block for the k compact directions. The 0-0 component is the higher-dimensional energy density ρ. The 0-m components are generally nonzero but will be of no special interest for our theorems.

Associated with the two blocks of space-space components of T_{IJ} are two trace averages:

$$p_3 \equiv \frac{1}{3}\gamma_3^{\mu\nu}T_{\mu\nu} \quad \text{and} \quad p_k \equiv \frac{1}{k}\gamma_k^{mn}T_{mn}, \qquad (4)$$

where $\gamma_{3,k}$ are, respectively, the 3×3 and $k \times k$ blocks of the higher-dimensional space-time metric. Violating the NEC means that $T_{MN}n^M n^N < 0$ for at least one null vector n^M and at least one space-time point.

Our approach in this paper is not to identify *all* cases where the NEC is violated, which can be complicated; rather we find simple methods for identifying a subset of cases where it must be violated. For this purpose, the following two lemmas, proven in Ref. [1], are very useful:

Lemma 1: If $\rho + p_3$ or $\rho + p_k$ is less than zero for any space-time point, then the NEC is violated. (Note that the converse is not true, $\rho + p_3 \geq 0$ and $\rho + p_k \geq 0$ does not guarantee that the NEC is satisfied.)

The second lemma utilizes the concept of *A*-averaged quantities introduced in Ref. [1]:

$$\langle Q\rangle_A = \left(\int Q e^{A\Omega}\sqrt{g}\,d^k y\right)\Big/\left(\int e^{A\Omega}\sqrt{g}\,d^k y\right); \qquad (5)$$

that is, quantities averaged over the extra dimensions with weight factor $e^{A\Omega}$ where, for simplicity, we restrict ourselves to constant A. Using the fact that the weight function in the A average is positive definite, a straightforward consequence is:

Lemma 2: If $\langle \rho + p_3\rangle_A < 0$ or $\langle \rho + p_k\rangle_A < 0$ for any A and any $\{t, \mathbf{x}\}$, then the NEC must be violated.

As with the case of Lemma 1, this test is asymmetrical: finding an A average less than zero proves NEC is violated, but finding a positive average is not sufficient to conclude NEC is satisfied.

To illustrate the utility of A averaging, we introduce the CRF metric into the higher-dimensional Einstein equations, and then try to express terms dependent on $\bar{a}$ in terms of the 4d effective scale factor using the relation $a(t) \equiv e^{\phi/2}\bar{a}(t)$, where [1]

$$e^\phi \equiv \ell^{-k}\int e^{2\Omega}\sqrt{g}\,d^k y \qquad (6)$$

and ℓ is the $4 + k$-dimensional Planck length. The 4d effective scale factor $a(t)$ obeys the usual 4d Friedmann equations:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{3}\rho_{4d}, \qquad (7)$$

$$\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{\ddot{a}}{a} = -p_{4d} \qquad (8)$$

(henceforth, we use reduced Planck units, $8\pi G_N = 1$ in 4d; also, except where displayed explicitly, we choose $\ell = 1$ in the $4 + k$-dimensional theory). Note that the 4d effective energy density ρ_{4d} and pressure p_{4d} are generally different from ρ and p_3 in the higher-dimensional theory if the warp factor Ω is nontrivial. Then, using the Einstein equations, we obtain

$$e^{-\phi}\langle e^{2\Omega}(\rho + p_3)\rangle_A = (\rho_{4d} + p_{4d}) - \frac{k+2}{2k}\langle\xi\rangle_A^2$$
$$- \frac{k+2}{2k}\langle(\xi - \langle\xi\rangle_A)^2\rangle_A - \langle\sigma^2\rangle_A,$$

$$(9)$$

$$e^{-\phi}\langle e^{2\Omega}(\rho + p_k)\rangle_A = \frac{1}{2}(\rho_{4d} + 3p_{4d})$$
$$+ 2\left(\frac{A}{4} - 1\right)\frac{k+2}{2k}\langle(\xi - \langle\xi\rangle_A)^2\rangle_A$$
$$- \frac{k+2}{2k}\langle\xi\rangle_A^2 - \langle\sigma^2\rangle_A + \left[-5 + \frac{10}{k}\right.$$
$$+ k + A\left(-3 + \frac{6}{k}\right)\bigg]\langle e^{2\Omega}(\partial\Omega)^2\rangle_A$$
$$+ \frac{k+2}{2k}\frac{1}{a^3}\frac{d}{dt}(a^3\langle\xi\rangle_A).$$

$$(10)$$

A averaging is a powerful tool because, with a judicious choice, one can insure that certain coefficients on the right-hand side, the ones that depend explicitly on A, are non-positive. This opens a path for proving some of the no-go theorems below.

This freedom is possible provided there is a range where

$$4 \geq A \geq \frac{10 - 5k + k^2}{3k - 6} \equiv A_*,$$

$$(11)$$

which is the case for $13 \geq k \geq 3$ (for CRF). Some theorems below rely on choosing $A = 2$; for this value to be within the range given in Eq. (11), it is necessary that $8 \geq k \geq 3$. (The corresponding ranges of k in the RF case are given the Appendix.) Since this includes the relevant string and M-theory models, we will implicitly assume this range of k for CRF models for the remainder of this paper. (For $k = 1$, the metric reduces to RF and similar theorems in Ref. [1] apply.)

The two relations in Eq. (9) can be rewritten

$$e^{-\phi}\langle e^{2\Omega}(\rho + p_3)\rangle_A = \rho_{4d}(1 + w) - \frac{k+2}{2k}\langle\xi\rangle_A^2$$
$$+ \text{non-positive terms for all } A,$$

$$(12)$$

$$e^{-\phi}\langle e^{2\Omega}(\rho + p_k)\rangle_A = \frac{1}{2}\rho_{4d}(1 + 3w)$$
$$+ \frac{k+2}{2k}\frac{1}{a^3}\frac{d}{dt}(a^3\langle\xi\rangle_A)$$
$$+ \text{non-positive terms for some } A,$$

$$(13)$$

where the values of A that make the last term nonpositive are those that are in the range in Eq. (11). Henceforth, unless stated otherwise, we always choose A to be in that range. Recall that w represents the ratio of the total 4d

effective pressure p_{4d} to the total 4d effective energy density ρ_{4d}. In the Appendix, we provide the coefficients of the last term in Eqs. (12) and (13) relations for the case where the moduli are frozen $\xi = 0$.

On the left-hand side of Eqs. (12) and (13), both ϕ and $\langle\ldots\rangle_A$ depend on the warp factor, Ω, but the combination is invariant under shifts $\Omega \to \Omega + C$, where C is a constant. Furthermore, the combination tends to have a weak dependence on Ω. For example, if $\rho + p_k$ is homogeneous in $\{y^m\}$, the left-hand side reduces to $K(\rho + p_k)$, where the dimensionless coefficient K is not very sensitive to Ω or A; in particular, $K = \ell^k I(A + 2)/I(A)I(2)$, where

$$I(\bar{A}) \equiv \int e^{\bar{A}\Omega}\sqrt{g}d^k y.$$

$$(14)$$

In this notation, the k-dimensional volume of the compact space is $V_k = I(0)$; then, K is equal to ℓ^k/V_k, a coefficient which is strictly less than unity. Similarly, if $\rho + p_k$ is smooth and Ω has a sharp maximum on some subspace of dimension s and volume v_s, then the left-hand side of Eq. (13) is $O(1)(\ell^s/v_s)(\rho + P_k)$, where $(\rho + p_k)_{max}$ is the value of $\rho + p_k$ evaluated on the subspace, where Ω is maximal. We will use this example in Sec. V.

III. NO-GO THEOREMS FOR MODELS THAT SATISFY NEC

The lemmas of the previous subsection can be used to prove that compactified theories satisfying NEC and meeting the other assumptions given at the beginning of Sec. (II) are incompatible with inflation and the simplest dark energy models consistent with observations. The theories include the original Kaluza-Klein model and many string theories. The Randall-Sundrum II model [8], with a single brane, also satisfies NEC; formally, it does not satisfy the boundedness condition, but, because the warp factor is well behaved at infinite distances from the brane, we conjecture that the same theorems apply.

As a first step, we show that w must be strictly greater than -1. The argument is simple. If $w = -1$, the first term in Eq. (12) is precisely zero, and the second two are non-positive. Consequently, NEC can only be satisfied if the last two terms are precisely zero as well. However, in Eq. (13), the first term is strictly negative, and the last term is nonpositive. Hence, the middle term must be positive for this equation to satisfy the NEC; but this requires ξ and/or its time derivative to be nonzero. But this is incompatible with having the middle term in Eq. (13) be zero. Hence, one or both equations must violate the NEC if $w = -1$.

An immediate consequence is a first dark energy no-go theorem. (Theorems labeled IA, IB, etc. refer to models obeying NEC, and models labeled IIA, IIB, etc. refer to models that violate NEC.)

Dark Energy No-go Theorem IA: ΛCDM (the current concordance model in cosmology) is incompatible with compactified models [25] satisfying the NEC.

A pure de Sitter universe is obviously ruled out by the argument above. Also, $w < -1$ is forbidden by the assumption that the 4d effective theory obeys the NEC. It is further apparent that $w > -1$ but close to -1 is subject to the same problems. Consequently, a ΛCDM universe, with a mixture of matter and positive cosmological constant that approaches $w = -1$ in the future, is also ruled out.

This point can be made more forcefully and precisely. Depending on the number of extra dimensions and whether the metric is RF or CRF, there exists a $w_{\text{transient}}$ between $-1/3$ and -1 such that w can only remain in the interval $(-1, w_{\text{transient}})$ for a few e-folds. The condition $w < w_{\text{transient}}$ cannot be maintained indefinitely because it requires either NEC violation of Eq. (13), if the average ξ and $d\xi/dt$ are kept small or negative, or NEC violation of Eq. (12), if ξ is made large and positive enough to avoid NEC violation in Eq. (13). In principle, it is possible to satisfy NEC for both relations if ξ is near zero and $d\xi/dt$ is large and positive, but this can only be maintained for a brief period.

How brief is brief? In order for the right-hand side of Eq. (13) to remain positive, it is necessary that

$$\frac{k+2}{2k}\frac{1}{a^3}\frac{d}{dt}(a^3\langle\xi\rangle_A) > -\frac{1}{2}\rho_{4d}(1 + 3w). \quad (15)$$

The right-hand side is positive for $w < w_{\text{transient}}$ and has magnitude $\mathcal{O}(\rho_{4d}) = \mathcal{O}(H^2)$, where $H \equiv \dot{a}/a$ is the Hubble parameter. Hence, $H^{-2}d\langle\xi\rangle_A/dt = \mathcal{O}(1)$. Now suppose $\langle\xi\rangle_A$ begins small so that Eq. (12) is satisfied initially. Integrating over a Hubble time, we find that $\langle\xi\rangle_A$, grows until

$$\langle\xi\rangle_A/H = \mathcal{O}(1) \quad (16)$$

at which point Eq. (12) violates NEC. In other words, the brief period during which ξ remains small cannot last more than a few Hubble times.

To reach $w < w_{\text{transient}}$ in the first place, it must be that w_{DE} is less than $w_{\text{transient}}$. But, then, the only way to avoid violating NEC is for w to increase above $w_{\text{transient}}$ after a few e-folds, which is only possible if w_{DE} itself increases above $w_{\text{transient}}$ after a few e-folds (which we will take to be three e-folds, for the purposes of this paper).

A plot of $w_{\text{transient}}$ as a function of the number of extra dimensions is given in Fig. 1; note that $w_{\text{transient}}$ is substantially greater than -1 for the cases of greatest interest, such as string theory ($k = 6$) or M-theory ($k = 7$). [See Ref. [26] for a more detailed quantitative discussion].

Two additional no-go theorems follow from this analysis:

Dark Energy No-go Theorem IB: Dark energy models with constant w_{DE} less than $w_{\text{transient}}$ or time-varying w_{DE}
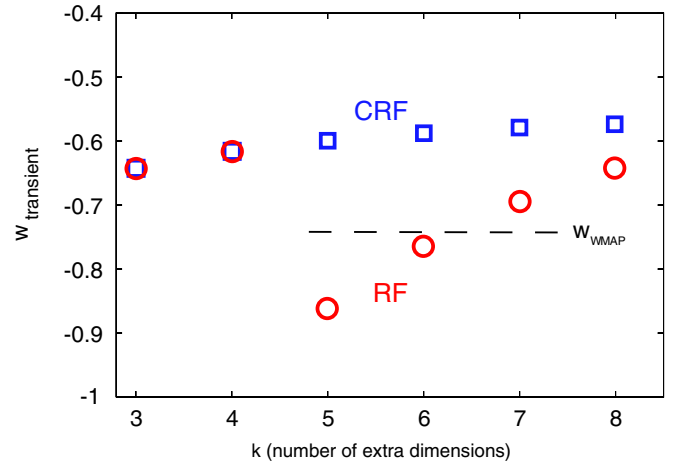

FIG. 1 (color online). Plot of $w_{\text{transient}}$ vs the number of extra dimensions k for extra-dimensional models based on RF (circle) and CRF (square) metrics. The dashed horizontal segment represents the current value of w according to WMAP.

whose value remains less than $w_{\text{transient}}$ for a continuous period lasting more than a few Hubble times are incompatible with compactified models [25] satisfying the NEC.

This theorem rules out a wide spectrum of dark energy models, including a range that is currently allowed observationally and that the JDEM mission is designed to explore [27]. Conversely, if JDEM indicates $w_{\text{DE}} < w_{\text{transient}}$ and constant, this would rule out this entire class of compactified models.

Inflationary No-go Theorem IA: Inflationary models consistent with observations are incompatible with compactified models [25] satisfying the NEC.

Inflationary cosmology requires a period of 40 or more e-folds of accelerated expansion with $w \approx -1$ to within a few percent in order to smooth and flatten the Universe and to obtain a scalar spectral index within current observational bounds [23]. For the compactified models considered here, this value of w is far below $w_{\text{transient}}$ and cannot be maintained for more than a few e-folds—certainly not for 40 e-folds.

Inflationary Corollary: Compactified models [25] satisfying the NEC are counterexamples to the common assertion that inflation with nearly scale-invariant spectra are an inevitable consequence given chaotic or generic initial conditions after the big bang.

The common lore is that, after the big bang, the Universe is chaotic but there are always rare patches of space that are smooth enough and have the right conditions to initiate inflation (assuming an inflaton with a sufficiently flat potential); and these patches soon dominate the volume of the Universe. For the entire class of theories considered here, though, no patches of space undergo inflation that is slow and long lasting enough to produce a spectral tilt anywhere near the observational bounds. The problem is not finding a scalar field with a sufficiently flat potential, because none

of the compactified models explicitly forbids that. Rather, the problem is that accelerated expansion induces a rapid variation of ξ which, in the 4d effective theory in Einstein frame, appears as a time-varying field whose kinetic energy increases the overall equation of state w and prevents inflation from continuing long enough.

Additionally, for all models (RF or CRF, and for any k), $\langle \xi \rangle_A = \dot{G}_N / G_N$ for $A = 2$, if the theory is expressed in the Jordan-Brans-Dicke (JBD) frame. Although we keep to the Einstein frame in this paper generally, it useful to express ξ in terms of $\dot{G}_N$ for the purpose of comparison to observational constraints on $\dot{G}_N$, which implicitly assume the JBD frame. For RF models, $A = 2$ lies outside the corresponding range of A (see the Appendix) and so theorems below about changing G_N should be re-expressed as conditions on changing ξ or, equivalently, changing size of the extra dimensions. For CRF models, $A = 2$ lies in the range in Eq. (11) when $8 \geq k \geq 3$. Converted to the JBD frame, we could restate our conclusion for CRF models as follows: accelerated expansion induces a rapid variation of the gravitational constant $\dot{G}_N / G_N H = \mathcal{O}(1)$.

Returning now to Eq. (15), we note that it requires that, regardless of the value of A, $d\langle \xi \rangle_A / dt = \mathcal{O}(H^2)$ be positive not only for $w < w_{\text{transient}}$, but also for $w < -1/3$. From this emerges:

Dark Energy No-go Theorem IC: All dark energy models are incompatible with compactified models [25] satisfying the NEC if the moduli fields are frozen (or, specifically, G_N is constant, in the case of CRF models).

This follows trivially because any form of dark energy requires w reach a value less than $-1/3$, and, as we just argued, ξ must vary with time whenever $w < -1/3$ if NEC is satisfied.

As a practical matter, the current value of $w \approx -0.74$ is already less than $w_{\text{transient}}$ for $k = 7$ dimensions (e.g., M theory). In this case, both dark energy no-go theorem IC and IB apply, but Theorem IB is more stringent. Theorem IB says that both w_{DE} and G_N must vary with time and at high rates. One might wonder: Is it already possible to rule out all RF and CRF compactified models [25] satisfying the NEC based on current observations? In Ref. [26], we show that the answer is no; there remains a small window in the parameter space $\{w_{\text{DE}}, dw_{\text{DE}}/dt, \dot{G}_N/G_N\}$ consistent with all current observations. However, anticipated observations will be able to check this remaining window to determine if this class of theories is empirically ruled out or not.

IV. NO-GO THEOREMS FOR MODELS THAT VIOLATE NEC

In this section, we continue to consider compactified models satisfying the GR, flatness, boundedness, and metric conditions assumed in Sec. II. The difference is that, before, we only considered models that satisfy the NEC, in which case we showed that moduli fields ξ (and G_N) must

vary with time at a fast rate barely compatible with current observational constraints and potentially ruled out by near-future observations. So now we consider models that violate NEC but with fixed (or very slowly varying) moduli. Theories of this type include the Randall-Sundrum I model [9], because it includes a negative tension brane, and some models that arise in flux compactifications of Calabi-Yau manifolds when NEC-violating components, such as orientifold planes or Casimir energy, are introduced.

If the only requirement for incorporating accelerated expansion were NEC violation, then it would suffice if $\rho + p_3 < 0$ or $\rho + p_k < 0$ at any one space-time point. However, we now will present a set of no-go theorems that show that cosmic acceleration imposes a host of stringent conditions on the spatial distribution and temporal variation of the NEC-violating elements. The no-go theorems in this section are qualitatively the same for dark energy and inflation because the theorems only rely on the fact that the universe must evolve from $w > -1/3$ to $w < -1/3$ or vice versa, which is required both for inflation and dark energy cosmology. Recall that w refers to the ratio of the total pressure (p_{4d}) to the energy density (ρ_{4d}) in the 4d effective theory.

Inflationary/Dark Energy No-go Theorem IIA: Inflation and dark energy are incompatible with compactified models [25] (with fixed moduli) if the NEC is satisfied in the compact dimensions (i.e., $\rho + p_k \geq 0$ for all t and y^m)— whether or not NEC is violated in the noncompact directions.

The first step in the proof is to note that, since G_N (and other moduli) are assumed to be fixed, the middle term in the expression for $e^{-\phi} \langle e^{2\Omega}(\rho + p_k) \rangle_A$ in Eq. (13) is zero. In this case, the relations in the Appendix apply. We can use the freedom to choose A in our A averaging so that the third term in Eq. (13) is zero; this corresponds to $A = A_*$ in Eq. (11). That leaves only the first term, proportional to $1 + 3w$, which is positive for $w > -1/3$ and negative for $w < -1/3$. Hence, whenever the Universe is accelerating ($w < -1/3$), NEC violation must occur in the compact dimensions. (It may or may not occur in the noncompact dimensions as well.)

Inflationary/Dark Energy No-go Theorem IIB: Inflation and dark energy are incompatible with compactified models [25] (with fixed moduli) for which the net NEC violation ($\rho + p_k$) is time independent.

This theorem relies on the fact that both inflation and dark energy models have a transition from phases with $w > -1/3$ to phases with $w < -1/3$. (This proof does not apply to a pure de Sitter phase where w is always equal to -1.) Since $e^{-\phi} \langle e^{2\Omega}(\rho + p_k) \rangle_{A_*}$ is proportional to $3w + 1$, which switches sign as w evolves past $w = -1/3$, the NEC violation (summing over all energy density and pressure contributions) in the compact direction must be time dependent. In the case of inflation, there is also a transition in which w changes from less than $-1/3$ to greater than $-1/3$. This leads to an important corollary:

Inflationary Corollary: Inflationary cosmology is only compatible with compactified theories [25] that include an NEC-violating component in the compact dimensions whose magnitude is of order the vacuum density (that is, $e^{-\phi}\langle e^{2\Omega}(\rho + p_k)\rangle_{A_*} \sim \rho_{4d}$); such that $\langle \rho + p_k\rangle_{A_*}$ switches from positive to negative when inflation begins *and* switches back when inflation is complete.

The corollary means that the requirements usually associated with inflation—a scalar field with a flat potential, stringent conditions on slow-roll parameters, a reheating mechanism, etc.—are not sufficient to have inflation in compactified theories since they do not produce or annihilate NEC violations. Furthermore, the magnitude of the NEC violation is 100 orders of magnitude larger than what is required to support a dark energy phase; so the source of NEC violation must be different from whatever is used to produce the current vacuum state. Finally, after inflation is over, $e^{-\phi}\langle e^{2\Omega}\rho + p_k\rangle_{A_*}$ must switch sign again, so the reheating in the noncompact dimensions must somehow have backreaction that changes the NEC violation in the compact directions by 100 orders of magnitude.

There is more to be said. Theorem IIA imposes the condition that NEC is violated in the compact dimensions. The next no-go theorems constrain the spatial distribution of the NEC-violating elements within those compact directions.

Inflationary/Dark Energy No-go Theorem IIC: Inflation and dark energy are incompatible with compactified models [25] with fixed moduli if the warp factor $\Omega(t, y)$ is nontrivial and has continuous first derivative and if any of the following quantities is homogeneous in y:

   (1) $\rho + p_3$;
   (2) $x\rho + p_k$ for RF metric, for any $(1/2) \times (1 - 3w) > x > 4(k - 1)/3k$;
   (3) ρ for CRF metric for $k > 4$;
   (4) $2\rho + p_k$ for CRF metric for $k > 3$ and $w > -1$;

The first condition follows straightforwardly from Eqs. (A1) and (A9), which show that $\rho + p_3 = e^{\phi - 2\Omega}(\rho_{4d} + p_{4d})$. This expression must be inhomogeneous because Ω is y *dependent* (by assumption) and the 4d effective energy density ρ_{4d} and pressure p_{4d} are y *independent* (by definition).

The remaining conditions are proven by using Eq. (A9) in the Appendix to express each of the linear combinations of ρ and p_k in the list above as

$$C\Delta\Omega + D(\partial\Omega)^2 + Ee^{-2\Omega}\rho_{4d}, \qquad (17)$$

where C and E have the same sign. For example, consider the case where C and E are positive. If Ω is nontrivial and has continuous first derivative and if the compact dimensions are bounded, then Ω must have a nonzero maximum and minimum on the compact manifold. At the maximum, we have that $\partial\Omega = 0$ (so the middle term is zero), $\Delta\Omega < 0$ and $e^{-2\Omega}$ is minimal; similarly, at the minimum, the middle term is also zero but $\Delta\Omega > 0$ and $e^{-2\Omega}$ is maximal.

Hence, for positive C and E, both terms in Eq. (17) are smaller for maximal Ω compared to their values for minimal Ω; the sum cannot be a homogeneous function of y. (A similar argument applies if C and E are both negative.)

For the RF case, a similar argument can be used to show that $x\rho + p_k$ must be inhomogeneous for a continuum of set of choices $(1/2)(1 - 3w) > x > 4(k - 1)/3k$. Note that there exists a nonzero range of x provided $w < -5/9 + (8/9k)$, which includes all $w < -5/9$. Since all observationally acceptable dark energy and inflation models must pass through phases where $w < -5/9$, these models require $x\rho + p_k$ be inhomogeneous for a finite range of x. A similar argument holds for the CRF case, but here we have, for simplicity, limited ourselves to two linear combinations: ρ alone and $2\rho + p_k$, which must both be inhomogeneous for all $w > -1$.

We have made no attempt to be exhaustive here because these examples suffice to make the point that the energy density and pressure must have nontrivial distributions across the extra dimension to satisfy the higher-dimensional Einstein equations. Further constraints are given by the following no-go theorems that rely on somewhat different methods of proof.

Inflationary/Dark Energy No-go Theorem IID: Inflation and dark energy are incompatible with compactified models [25] with fixed moduli if the warp factor $\Omega(t, y)$ is nontrivial if $\rho + p_k$ is homogeneous.

Note that this linear combination is the indicator of NEC violation, so this no-go theorem says that the degree of NEC violation must itself be inhomogeneously distributed in the compact dimensions. To prove this result, it suffices to restrict ourselves to showing that $\rho + p_k$ is inhomogeneous for $w = -1/3$ since both dark energy and inflation models must pass through this value of w. For $w = -1/3$, the last term in $\rho + p_k$ in Eq. (A1) ([for RF] and Eq. (A9) [for CRF] in the Appendix is zero. Using Lemma A1 in the Appendix, the remaining terms can be rewritten as $\Gamma e^{-\gamma\Omega}\Delta e^{\gamma\Omega}$ where γ and Γ are positive. For nontrivial Ω, $\Delta\Omega$ must be nonzero and have different signs at the maximum and minimum of Ω on the compact manifold. Hence, $\rho + p_k$ must be inhomogeneous.

Inflationary/Dark Energy No-go Theorem IIE: Inflation and dark energy are incompatible with compactified models [25] with fixed moduli if $w_k(A_*) \equiv \langle p_k\rangle_{A_*}/\langle \rho_k\rangle_{A_*} > -1$ for $\langle\rho\rangle_{A_*} > 0$ or if $w_k(A_*) \equiv \langle p_k\rangle_{A_*}/\langle\rho_k\rangle_{A_*} < -1$ for $\langle\rho\rangle_{A_*} < 0$.

We will present the proof before explaining its significance: Let us first consider the case where $\langle\rho\rangle_{A_*} > 0$. Based on Eqs. (A7) and (A16) in the Appendix, we can express $w_k(A)$ as

$$w_k(A) = \frac{g(A)\langle(\partial\Omega)^2\rangle_A + \frac{3w-1}{2}X}{f(A)\langle(\partial\Omega)^2\rangle_A + X}, \qquad (18)$$

where $X = \langle e^{\phi - 2\Omega}\rho_{4d}\rangle_A > 0$. (Recall that $\rho_{4d} > 0$ in in-

flation and dark energy models.) Recall that the denominator is $<\rho_A$, the A-averaged energy density. For $A = A_*$ [as given in Eq. (11)], $f(A_*)/g(A_*) = -1$. For $w < -1/3$, as required for inflation or dark energy models, the coefficient of X in the numerator is less than -1. Straightforward algebra then shows that $w_k(A_*)$ is strictly less than -1. [A similar argument can be used to show $w_k(A_*)$ is strictly greater than -1 if $\langle \rho_A \rangle < 0$.]

The quantity w_k is the ratio of the volume-averaged pressure to volume-averaged energy density with positive definite weight $e^{-2\Omega}$. To have NEC violation in the compact dimensions, as required by Theorem IIA, it suffices that $p_k/\rho < -1$ for $\rho > 0$ at a single point; or $p_k/\rho > -1$ for $\rho < 0$ at a single point. Here, we have shown that the ratio of the volume weighted averages must satisfy these inequalities, generally a much stronger condition.

This no-go theorem is useful because it shows that simply violating the NEC is not enough; one must be deeply within the NEC-violating regime. For example, for constant warp factor Ω, $w_k(A) = (3w - 1)/2$ (independent of A), which approaches -2 as $w \to -1$. This value is far below the minimal value needed to violate the NEC; e.g., inconsistent with simply Casimir energy or a single orientifold plane as the source of NEC violation.

There are some other curiosities. For example, for constant warp factor Ω, radiation alone exerts positive pressure in the noncompact dimensions, but must exert zero pressure in the compact dimensions; and matter exerts no pressure in the noncompact dimensions, but must exert negative pressure in the compact dimensions.

V. CONSTRAINTS ON MODELS VIOLATING THE GR OR METRIC CONDITIONS

Formally, the theorems derived here apply strictly to models in which the higher-dimensional theory satisfies Einstein's equations and is described by an RF or CRF metric. However, the theorems provide useful insights for some models that violate one or both conditions. For example, some string inflation models satisfy the GR conditions perturbatively but violate them nonperturbatively [11,13,28]. One might inquire whether these models evade the no-go theorems derived in this paper. Absent an explicit expression for the nonperturbative interactions, a quantitatively precise answer cannot be reached. Nevertheless, qualitatively, it is clear that the no-go theorems may only be evaded if the violations are large and time-dependent.

For example, if the violations can be expressed as additions to the right-hand side of Eqs. (12) and (13), then, these modifications have to balance the equations by satisfying similar time-variation conditions as required for the NEC-violating components in the proofs of the no-go theorems. That is, there must be some sort of backreaction in the compact directions in either case. By the argument given below Eq. (13), the modifications to $e^{-\phi}\langle e^{2\Omega}(\rho + p_3)\rangle_A$ and $e^{-\phi}\langle e^{2\Omega}(\rho + p_k)\rangle_A$ must be of

order ρ_{4d}, and they must change by an amount $\mathcal{O}(1)\rho_{4d}$ whenever the Universe switches from accelerating to decelerating (or vice versa) in order to change the sign of $e^{-\phi}\langle e^{2\Omega}\rho + p_k\rangle_{A_*}$ (as required by the kind of argument presented for Theorem IIB).

What makes the backreaction problematic is that, phenomenologically, the change from acceleration to deceleration (or vice versa) in the 4d effective theory is supposed to be due entirely to the production of matter and radiation (in the case of inflation) or redshifting of matter energy density (at the onset of dark energy domination) that acts in the noncompact dimensions; so it would seem that any backreaction in the compact dimensions required to satisfy Eq. (13) had better turn out to be quantitatively small enough to have a negligible effect on the 4d effective theory. If the effect of backreaction on the 4d effective theory is not negligible, it will alter the course of accelerated expansion in undesirable ways, such as shortening or eliminating the acceleration phase, as was shown to be the case for models that satisfy the GR and metric conditions. In the case of inflation, even if the backreaction does not prevent inflation, it may change the transition from inflation to reheating and, thereby, the predictions.

In fact, in certain flux compactifications in string theory, there is an argument to suggest that the backreaction will have a very large effect. These models invoke orientifold planes (extended objects with negative tension) that serve as sources of the NEC violation necessary to stabilize a true vacuum with positive cosmological constant [11,13]. Averaged over the bulk volume, the large negative tension of the orientifold planes is nearly canceled by large positive density contributions, such as branes. There can also be positive density contributions in the throat. However, several of the no-go theorems entail the A_* average of $\rho + p_k$, where $A_* \geq 1$. For example, Theorem IIB requires that this average switch sign and change by an amount of order ρ_{4d} when the Universe transitions from acceleration to deceleration (or vice versa). Because the A_* average over the compact volume weights contributions to $\rho + p_k$ by a factor of $e^{A_*\Omega}$, contributions from regions in the compact volume where Ω is maximal will be strongly weighted compared to regions where Ω is small. In the case of orientifold planes, singular surfaces near which $G_{00} < 0$ and $(\partial\Omega)^2$ approaches zero, Eq. (A9) implies $\Delta\Omega < 0$; hence, orientifold planes are (local or global) maxima of Ω and tend to be strongly weighted in the A_* average.

Consider, for example, a setup where Ω is maximal along the orientifold planes which have some constant $(\rho + p_k)_{\text{neg}} < 0$ in a volume of dimension m and volume v_s; further suppose that Ω_{pos} is somewhat smaller but nearly uniform over the rest of the bulk where there is some average stress energy $(\rho + p_k)_{\text{pos}} > 0$ that nearly balances the orientifold-plane component; finally, as in

the case of d-brane inflation, suppose there is some positive $(\rho + p_k)_{\text{throat}} > 0$ contribution in the throat. The A_*-weighted combination of these components is then

$$\frac{\ell^s}{v_s}(\rho + p_k)_{\text{neg}} + \frac{\ell^{2s-k}V_k}{v_s^2}e^{-(A_*+2)\Delta\Omega_{\text{bulk}}}(\rho + p_k)_{\text{pos}}$$

$$+ \frac{\ell^{2s-k}V_k}{v_s^2}e^{-(A_*+2)\Delta\Omega_{\text{throat}}}(\rho + p_k)_{\text{throat}}, \qquad (19)$$

where $\Delta\Omega_{\text{bulk}} = \Omega_{\text{neg}} - \Omega_{\text{pos}} > 0$ and $\Delta\Omega_{\text{throat}} = \Omega_{\text{neg}} - \Omega_{\text{throat}} > 0$. This sum is supposed to switch from an amount of order $-\rho_{4d}$ to $+\rho_{4d}$ at the end of inflation (or the reverse at the onset of dark energy domination). Because of the A_* weights, the exponentially dominant contribution to Eq. (19) is the due to the orientifold plane, which contributes an amount $(l^s/v_s)(\rho + p_k)_{\text{neg}}$, that is exponentially enhanced compared to the positive energy density contributions in the bulk or in the throat because, by assumption, the warp factor Ω is much larger near orientifold planes. In order for $e^{-\phi}\langle e^{2\Omega}\rho + p_k\rangle_{A_*}$ to switch sign when the Universe changes from accelerating to decelerating (or vice versa), the backreaction in the bulk must either change the contribution of the orientifold planes by an amount of order ρ_{4d}, which seems unlikely given their topological character; or the backreaction must change the positive energy density components by an exponentially larger amount. In the latter case especially, the effect of the backreaction on the effective 4d theory is likely to be overwhelmingly large. [One could switch the scenario so that Ω is maximal in the bulk positive $(\rho + p_k)$ regions and smaller on the orientifolds; even so, the only way to change the sign on the left-hand side of Eq. (13) is to have a backreaction in which some energy components change by an amount of at least ρ_{4d}; and, in most cases, by an amount exponentially greater amount.] It is, therefore, essential to track the effect of this back-reaction on the 4d effective theory (where the leading contribution is supposed to be of order ρ_{4d}) to be sure the cosmological scenario is not spoiled. As of this writing, though, the backreactions during the transition from inflation to reheating and from matter domination to dark energy domination are not well understood: In particular, they have not been included in string inflation calculations and predictions or in discussions of stringy dark energy models.

We note that our analysis has been restricted to the case of RF or CRF metrics that are Ricci flat or conformally Ricci flat and that we have ignored nonperturbative corrections to GR. However, a similar argument applies if they are included. They can be viewed as amendments to the right-hand side of Eq. (13); then, by the same reasoning, they must change by an amount of order ρ_{4d} to balance the equation. So, as in the case above, one must be concerned about the effect of their backreaction in the 4d effective theory.

VI. CONCLUSIONS

The essence of this paper is that cosmic acceleration is surprisingly difficult to incorporate in compactified models. The problem arises in trying to satisfy simultaneously the 4d and higher-dimensional Einstein equations. Both must be satisfied for any equation of state, but we have shown that, for the metrics assumed in this paper, this requires increasingly exotic conditions as the Universe goes from decelerated to accelerated expansion or, equivalently, as w decreases below $-1/3$. For dark energy models, either moduli fields (including G_N) have to change with time at a rate that is nearly ruled out (and may soon be excluded observationally altogether [26]) or NEC must be violated. For inflation, only the second option remains viable.

If the NEC is violated, it must be violated in the compact dimensions; it must be violated strongly (w_k significantly below the minimally requisite value for NEC violation); and the violation in the compact dimensions must vary with time in a manner that precisely tracks the equation of state in the 4d effective theory. For example, in realistic cosmological models, there are known matter and radiation components (baryons and photons, for example) that contribute to the energy and density of the 4d effective theory but are not normally related to NEC violation. Nevertheless, the no-go theorems say that the magnitude of NEC violation must vary with time in sync with how the conventional matter and radiation energy density and pressure evolve.

Satisfying these equations for ΛCDM is difficult, but satisfying them for inflation is even harder. A period of inflation with w within a few percent of -1 (as required to meet the observational constraints on the spectral tilt) must be sustained for at least 40 e-folds to resolve the flatness and homogeneity problems; this requirement restricts us to the case that the NEC is violated, according to Inflationary Theorem IA. The magnitude of the NEC violation is proportional to ρ_{4d} according to Eq. (13), which is roughly 10^{100} times greater during the inflationary epoch than during the present dark energy dominated epoch. Hence, the source of NEC violation for inflation must be different and 10^{100} stronger. Also, identifying a scalar inflaton field with a flat potential or branes and antibranes approaching one another in some warped throat does not suffice because they do not violate NEC, either. For example, as a hypothetical, imagine that a D3 brane-antibrane pair collide and annihilate into ordinary radiation; they do not change $\rho + p_k$ at all since neither branes nor radiation exert pressure in the compact directions and the energy density remains the same. Yet, after inflation is over and the equation of state increases to $w = +1/3$ (the radiation epoch), the NEC violation must be reduced or eliminated to continue to satisfy the Einstein equations. This suggests some backreaction effect must be built into the higher-dimensional theory

that creates and later eliminates exponentially large NEC-violating contributions at the beginning and end of inflation, leaving behind exponentially small NEC-violating effects needed for the current dark energy dominated epoch. This needs to be incorporated into any realistic theory of reheating [29].

The added complexity is disappointing. Inflation and dark energy in 4d have always had the problem that they require special degrees of freedom and fine-tuning. One would have hoped that extra dimensions, which are introduced to simplify the unification of fundamental forces, would also alleviate the conditions needed for inflation. The no-go theorems say the opposite: the number and complexity of conditions needed to have inflation or dark increase significantly.

The fact that NEC violation is required to have inflation in theories with extra dimensions is unexpected since this was not a requirement in the original inflationary models based on four dimensions only. Curiously, a criticism raised at times about models with bounces from a contracting phase to an expanding phase, such as the ekpyrotic [30,31] and cyclic [32] alternatives to inflationary cosmology, is that the bounce requires a violation of the NEC [or quantum gravity corrections to GR as the FRW scale factor $a(t) \to 0$ that serve the same function]. Now we see that, although the details are different, all of these cosmologies require NEC violation when incorporated into theories with extra dimensions.

In general, the no-go theorems are powerful because they span a broad sweep of theories. They say that one should be wary of focusing on one localized region of the extra dimensions, such as a warped throat, since there are nontrivial global constraints. Second, just because some elements appear to add to the vacuum energy or provide an inflaton potential in the 4d effective theory does not mean the theory is viable; they may force unacceptable conditions in the higher-dimensional theory. Thirdly, the NEC violation must be time varying, at least for the class of metrics considered here. This power of the no-go theorems derives from the fact that they arise from a macro-to-micro approach in which the analysis only relies on known macroscopic properties, although this also means that they tell us nothing directly about the detailed microphysics needed to satisfy or evade them.

We note that, thus far, we have restricted the analysis to no-go theorems that are simple to express and simple to prove. There are numerous other relations that must be satisfied to have cosmic acceleration that will be considered in future work. However, we hope the examples shown here and in Ref. [1] suffice to show how these no-go theorems can be remarkably informative, complementing other ways of thinking about how to construct higher-dimensional models.

APPENDIX: SOME USEFUL RELATIONS

For RF models, we have the following useful relations in the case of fixed ξ (breathing mode) and metric g_{mn}:

$$G_{00} = -3\Delta\Omega - 6(\partial\Omega)^2 + e^{\phi-2\Omega}\rho_{4d}, \qquad (A1)$$

$$p_3 = 3\Delta\Omega + 6(\partial\Omega)^2 + e^{\phi-2\Omega}p_{4d}, \qquad (A2)$$

$$p_k = \left(4 - \frac{4}{k}\right)\Delta\Omega + \left(10 - \frac{4}{k}\right)(\partial\Omega)^2, \qquad (A3)$$

$$+ e^{\phi-2\Omega}(\tfrac{1}{2}\rho_{4d}(3w - 1)), \qquad (A4)$$

$$\rho + p_3 = e^{\phi-2\Omega}(\rho_{4d} + p_{4d}), \qquad (A5)$$

$$\rho + p_k = \left(1 - \frac{4}{k}\right)\Delta\Omega + \left(4 - \frac{4}{k}\right)(\partial\Omega)^2$$
$$+ e^{\phi-2\Omega}\left(\frac{1}{2}\rho_{4d}(1 + 3w)\right). \qquad (A6)$$

Taking A averages and using $\langle\Delta\Omega\rangle_A = -A(\partial\Omega)^2$, we can obtain an expression for the equation of state of the compact directions

$$w_k^{RF}(A) = \frac{[(10 - 4A) + \frac{4A-4}{k}](\partial\Omega)^2 + (\frac{3w-1}{2})e^{\phi}\langle e^{-2\Omega}\rho_{4d}\rangle_A}{(3A - 6)(\partial\Omega)^2 + e^{\phi}\langle e^{-2\Omega}\rho_{4d}\rangle_A}.$$
$$(A7)$$

Following Ref. [1], we can obtain the RF analogue of (11) by multiplying both sides of (A6) by $e^{2\Omega-\phi}$ and taking the A average. As shown in [1], for all $k \geq 1$ an A can be found such that A-dependent coefficients are nonpositive. The RF version of A_*, for which the warp term contribution to $e^{-\phi}\langle e^{2\Omega}(\rho + p_k)\rangle_A$ vanishes, is given by

$$A_* = \frac{2(k + 2)}{k - 4}, \qquad (A8)$$

and A can be chosen equal to A_* for all $k \geq 1$. The choice $A = 2$ is inconsistent with keeping the A-dependent coefficients nonpositive in the RF case.

For CRF models, the analogous relations to (A1)–(A6) are

$$G_{00} = (k - 4)\Delta\Omega + \tfrac{1}{2}(k^2 - 3k - 10)(\partial\Omega)^2 + e^{\phi-2\Omega}\rho_{4d},$$
$$(A9)$$

$$p_3 = -(k-4)\Delta\Omega - \tfrac{1}{2}(k^2 - 3k - 10)(\partial\Omega)^2 + e^{\phi - 2\Omega}p_{4\mathrm{d}}, \tag{A10}$$

$$p_k = \left(7 - \frac{6}{k} - k\right)\Delta\Omega + \left(6 - \frac{2}{k} + \frac{5k}{2} - \frac{k^2}{2}\right)(\partial\Omega)^2, \tag{A11}$$

$$+ e^{\phi - 2\Omega}(\tfrac{1}{2}\rho_{4\mathrm{d}}(3w - 1)), \tag{A12}$$

$$\rho + p_3 = e^{\phi - 2\Omega}(\rho_{4\mathrm{d}} + p_{4\mathrm{d}}), \tag{A13}$$

$$\rho + p_k = \left(3 - \frac{6}{k}\right)\Delta\Omega + \left(k + 1 - \frac{2}{k}\right)(\partial\Omega)^2 + e^{\phi - 2\Omega}\left(\frac{1}{2}\rho_{4\mathrm{d}}(1 + 3w)\right), \tag{A14}$$

$$\mathring{R} = 2(k-1)\Delta\Omega + (k-1)(k-2)(\partial\Omega)^2, \tag{A15}$$

where $\mathring{R}$ is the Ricci curvature of the compact manifold. Then, the effective equation of state is

$$w_k^{\mathrm{CRF}}(A) = \frac{[-(7 - \frac{6}{k} - k)A + (6 - \frac{2}{k} + \frac{5k}{2} - \frac{k^2}{2})](\partial\Omega)^2 + (\frac{3w-1}{2})e^{\phi}\langle e^{-2\Omega}\rho_{4\mathrm{d}}\rangle_A}{[-(k-4)A + \frac{1}{2}(k^2 - 3k - 10)](\partial\Omega)^2 + e^{\phi}\langle e^{-2\Omega}\rho_{4\mathrm{d}}\rangle_A}. \tag{A16}$$

In addition, the following Lemma proven in Ref. [1] is useful in deriving dark energy theorems:

Lemma A1: For real and nonzero α and β,

$$\alpha\Delta\Omega + \beta(\partial\Omega)^2 = \Gamma e^{-\gamma\Omega}\Delta e^{\gamma\Omega}, \tag{A17}$$

where $\alpha = \Gamma\gamma$ and $\beta = \Gamma\gamma^2$.

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