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Cylindrical wormholes

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It is shown that the existence of static, cylindrically symmetric wormholes does not require violation of the weak or null energy conditions near the throat, and cylindrically symmetric wormhole geometries can appear with less exotic sources than wormholes whose throats have a spherical topology. Examples of exact wormhole solutions are given with scalar, spinor and electromagnetic fields as sources, and these fields are not necessarily phantom. In particular, there are wormhole solutions for a massless, minimally coupled scalar field in the presence of a negative cosmological constant, and for an azimuthal Maxwell electromagnetic field. All these solutions are not asymptotically flat. A no-go theorem is proved, according to which a flat (or string) asymptotic behavior on both sides of a cylindrical wormhole throat is impossible if the energy density of matter is everywhere nonnegative.

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I. INTRODUCTION

Lorentzian traversable wormholes as smooth bridges between different universes, or smooth shortcuts between remote parts of a single universe, have been widely discussed from different theoretical standpoints for many years; see [1–4] for reviews. It is well known that a wormhole geometry can only appear as a solution to the Einstein equations if the stress-energy tensor (SET) of matter violates the null energy condition (NEC) at least in a neighborhood of the wormhole throat [2]. This conclusion, however, has been proved under the assumption that the throat is a compact 2D surface, having a finite (minimum) area (at least in the static case, since for dynamic wormholes it has proved to be necessary to generalize the notion of a throat [5]; see, however [6] for other definitions of a throat). In other words, it was implied that, as seen from outside, a wormhole entrance is a local object like a star or a black hole.

But, in addition to such objects, the Universe may contain structures which are infinitely extended along a certain direction, like cosmic strings. And, while starlike structures are, in the simplest case, described in the framework of spherical symmetry, the simplest stringlike configurations are cylindrically symmetric. Their possible wormhole properties will be the subject of the present paper. For simplicity, we here consider only static configurations. It should be stressed that we will deal with genuine cylindrical symmetry, unlike Kuhfittig [7] who considered cylindrical configurations of finite length, which are actually axially symmetric due to z dependence. A special case of wormholes related to the present setting (with a cosmic string metric and thin shells of negative density) has been considered in [8].

For static, spherically symmetric space-times with the metric

$$ds^2 = A(u)dt^2 - B(u)du^2 - r^2(u)(d\theta^2 + \sin^2\theta d\phi^2) \quad (1)$$

(where u is an arbitrary admissible spherical radial coordinate), we say that there is a wormhole geometry if at some $u = u_0$ the function $r(u)$ has a regular minimum $r(u_0) > 0$ (which is then called a throat) and, on both sides of this minimum, $r(u)$ grows to much larger values than $r(u_0)$. It is supposed that, at least in some range of u containing u_0, the functions $A(u)$ and $B(u)$ are also smooth, finite and positive, which guarantees regularity and absence of horizons.[1]

Likewise, consider static, cylindrically symmetric space-times with the general metric taken in the form

$$ds^2 = e^{2\gamma(u)}dt^2 - e^{2\alpha(u)}du^2 - e^{2\xi(u)}dz^2 - e^{2\beta(u)}d\phi^2, \quad (2)$$

where u is an arbitrary admissible cylindrical radial coordinate, $z \in \mathbb{R}$ is the longitudinal coordinate, and $\phi \in [0, 2\pi]$ is the angular one. The main global features of such space-times are defined in terms of the behavior of the circular radius $r(u) = e^{\beta(u)}$: namely, a spatial asymptotic (if any) corresponds to $r(u) \to \infty$, and a symmetry axis (if

[1] We thus do not restrict ourselves to asymptotically flat wormholes and, moreover, admit that a horizon may occur somewhere far from the throat, as it happens, e.g., if a wormhole is asymptotically de Sitter due to a small cosmological constant.

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any) is defined by vanishing $r(u)$, which means that the coordinate circles shrink to points. It therefore seems reasonable to accept the following definition of a cylindrical wormhole:

Definition 1. We say that the metric (2) describes a wormhole geometry if the circular radius $r(u)$ has a minimum $r(u_0) > 0$ at some $u = u_0$, if, on both sides of this minimum, $r(u)$ grows to much larger values than $r(u_0)$, and, in some range of u containing u_0, all four metric functions in (2) are smooth and finite (which guarantees regularity and absence of horizons). The cylinder $u = u_0$ is then called a throat.

The notion of a wormhole is, as in other similar cases, not rigorous because of the words "much larger," but the notion of a throat as a minimum of $r(u)$ is exact. Asymptotically regular wormholes, to be discussed below, are also defined exactly.

We will give some examples of wormhole solutions whose sources are scalar, nonlinear spinor and electromagnetic fields. It is important that these fields need not be phantom, and, in particular, there are Einstein-Maxwell wormhole solutions with azimuthal electric or magnetic fields.

It is also possible to define a cylindrical wormhole by analogy with spherically symmetric or other starlike configurations, using, instead of $r(u)$, the area function $a(u) = e^{\beta+\xi}$ of 2D cylindrical surfaces $t = \text{const}$, $u = \text{const}$. (Their area is certainly infinite but becomes finite if we identify some points on the z axis, thus converting cylindrical symmetry to toroidal.) We still believe that Definition 1 is more appropriate for cylindrical symmetry, although it is useful to compare the properties of different notions of a throat.

The paper is organized as follows. In Sec. II, we present necessary conditions for the existence of a cylindrically symmetric wormhole throat and formulate some further observations. We also discuss an alternative definition of a throat in terms of the area function $a(u)$. Comparing the consequences of different definitions, we arrive at an important no-go theorem for wormholes with flat or string asymptotic behavior at both sides of the throat: it turns out that, in order to make such a wormhole, it is necessary to have matter with negative energy density. Section III considers a few examples of matter sources of cylindrically symmetric geometries and the corresponding wormhole solutions. Two no-go theorems of more specific nature are presented there. Section IV contains some concluding remarks.

II. CYLINDRICAL WORMHOLES: GEOMETRY AND MATTER CONTENT

A. Basic equations

Let us begin with presenting the nonzero components of the Ricci tensor for the metric (2) in its general form, without specifying the choice of the radial coordinate u:

$$\begin{aligned}
R^0_0 &= -e^{-2\alpha}[\gamma'' + \gamma'(\gamma' - \alpha' + \beta' + \xi')],\\
R^1_1 &= -e^{-2\alpha}[\gamma'' + \xi'' + \beta'' + \gamma'^2 + \xi'^2\\
&\quad + \beta'^2 - \alpha'(\gamma' + \xi' + \beta')],\\
R^2_2 &= -e^{-2\alpha}[\xi'' + \xi'(\gamma' - \alpha' + \beta' + \xi')],\\
R^3_3 &= -e^{-2\alpha}[\beta'' + \beta'(\gamma' - \alpha' + \beta' + \xi')],
\end{aligned} \tag{3}$$

where the prime denotes d/du and the coordinates are numbered according to the scheme $(0, 1, 2, 3) = (t, u, z, \phi)$. It is also helpful to present the component G^1_1 of the Einstein tensor $G^\nu_\mu = R^\nu_\mu - \frac{1}{2}\delta^\nu_\mu R$ which does not contain any second-order derivatives:

$$G^1_1 = e^{-2\alpha}(\gamma'\xi' + \beta'\gamma' + \beta'\xi'). \tag{4}$$

The Einstein equations are written as

$$G^\nu_\mu = -\varkappa T^\nu_\mu, \qquad \varkappa = 8\pi G, \tag{5}$$

where G is Newton's constant of gravity, or equivalently,

$$R^\nu_\mu = -\varkappa \tilde{T}^\nu_\mu, \qquad \tilde{T}^\nu_\mu = T^\nu_\mu - \frac{1}{2}\delta^\nu_\mu T^\alpha_\alpha, \tag{6}$$

The above relations were written with an arbitrary u coordinate. In many cases it is helpful to use this coordinate freedom and to choose u as a harmonic radial coordinate, which is defined by the condition [9]

$$\alpha = \beta + \gamma + \xi. \tag{7}$$

In particular, with this choice, the expressions for R^0_0, R^2_2 and R^3_3 do not contain first-order derivatives.

B. Conditions on the throat

Now, let us take the SET T^ν_μ in the most general form admitted by the space-time symmetry:

$$T^\nu_\mu = \text{diag}(\rho, -p_r, -p_z, -p_\phi), \tag{8}$$

where ρ is the density and p_i are pressures of any physical origin in the respective directions.

It is straightforward to find out how the SET components should behave on a wormhole throat. At a minimum of $r(u)$, due to $\beta' = 0$ and $\beta'' > 0$,[2] we have $R^3_3 < 0$, and from the corresponding component of (6) it follows that

$$T^* := T^0_0 + T^1_1 + T^2_2 - T^3_3 = \rho - p_r - p_z + p_\phi < 0. \tag{9}$$

If $T^2_2 = T^3_3$, which means $p_z = p_\phi$, the condition (9) leads to $\rho - p_r < 0$, or $p_r > \rho$, which violates the *dominant energy condition* if we assume, as usual, $\rho \geq 0$. (This is true, in particular, for Pascal isotropic fluids, in which all

[2]Here and henceforth we restrict ourselves for convenience to generic minima, at which $\beta'' > 0$. If there is a special minimum at which $\beta'' = 0$, we still have $\beta'' > 0$ in its vicinity, along with all consequences of this inequality. The same concerns minima of $a(u)$ discussed below.

p_i are equal to each other.) In the general case of anisotropic pressures, (9) does not necessarily violate any of the standard energy conditions.

Let us discuss what changes if we define a throat using the area function $a(u)$ instead of $r(u)$. We will call it an *a-throat* for clarity.

Definition 2. In a space-time with the metric (2), an a-throat is a cylinder $u = u_1$ where the function $a(u) = e^{\beta+\xi}$ has a regular minimum.

By Definition 2, at $u = u_1$ we have $\beta' + \xi' = 0$ and $\beta'' + \xi'' > 0$. The minimum occurs in terms of *any* admissible coordinate u, in particular, in terms of the harmonic coordinate (7). Using it in Eqs. (3) and (6), we find that the condition $\beta'' + \xi'' > 0''$ implies

$$R_2^2 + R_3^3 < 0 \Rightarrow T_0^0 + T_1^1 = \rho - p_r < 0. \tag{10}$$

In addition, substituting $\beta' + \xi' = 0$ into the Einstein equation $G_1^1 = -\varkappa T_1^1$, we find

$$G_1^1 = e^{-2\alpha}\beta'\xi' = -e^{-2\alpha}\beta'^2 \le 0 \Rightarrow -T_1^1 = p_r \le 0. \tag{11}$$

Combining (10) and (11), we obtain

$$\rho < p_r \le 0 \quad \text{at } u = u_1. \tag{12}$$

Thus at and near an a-throat there is necessarily a region with negative energy density ρ.

Let us recall for comparison the wormhole throat conditions in static spherical symmetry. There, the Einstein equation $\binom{0}{0} - \binom{1}{1}$ leads to the well-known inequality $\rho + p_r < 0$, which violates the *null energy condition*. The equation $G_1^1 = -\varkappa T_1^1$ leads to $p_r < 0$ on the throat. Meanwhile, the density ρ can have any sign.

We conclude that both conditions (9) and (12) radically differ from their counterpart in spherical symmetry. Moreover, these two conditions themselves are drastically different: while (9) admits quite usual kinds of matter (as will be seen from the examples below), (12) definitely requires $\rho < 0$, i.e., even more exotic matter than in spherical symmetry.

C. Asymptotic conditions; a no-go theorem

So far we discussed the local conditions that must hold on the throat. To describe a wormhole as a global entity, it is mandatory to consider the geometry far from the throat, on both sides from it. We will consider a situation that seems the most natural, in which the wormhole is observed as a stringlike source of gravity from an otherwise very weakly curved or even flat environment.

So we require the existence of a spatial infinity, i.e., a value $u = u_\infty$ such that $r = e^\beta \to \infty$, where the metric is either flat or corresponds to the gravitational field of a cosmic string.

First, as $u \to u_\infty$, a correct behavior of clocks and rulers requires $|\gamma| < \infty$ and $|\xi| < \infty$, i.e.,

$$\gamma \to \text{const}, \qquad \xi \to \text{const} \quad \text{as } u \to u_\infty. \tag{13}$$

Choosing proper scales along the t and z axes, one can turn these constants to zero; but if a wormhole has two such asymptotics, this operation, in general, can be done only at one of them.

Second, at large r we require

$$|\beta'|e^{\beta-\alpha} \to 1 - \mu, \qquad \mu = \text{const} < 1 \quad \text{as } u \to u_\infty, \tag{14}$$

so that the circumference-to-radius ratio for the circles $u = \text{const}$, $z = \text{const}$ tends to $2\pi(1-\mu)$ instead of 2π which should be the case if the space-time is asymptotically flat. So the parameter μ is an angular defect. Under the asymptotic conditions (13) and (14), $\mu > 0$, the solution can simulate a cosmic string. A flat spatial asymptotic takes place if $\mu = 0$. Negative values of μ are also not *a priori* excluded, they correspond to an angular excess. In what follows we will use the words "*regular asymptotic*" in the sense "*flat or string asymptotic*."

Third, the Riemann curvature tensor, hence the Ricci tensor, should vanish at large r, and, due to the Einstein equations, all SET components must decay quickly enough.[3]

It is easy to verify that in the coordinates (7) a regular asymptotic can only occur as $u \to \pm\infty$. Indeed, due to (13) at such an asymptotic we have

$$\alpha \sim \beta \to \infty \Rightarrow |\beta'| \to \text{const}. \tag{15}$$

So (14) and the two conditions (15) are compatible with each other only if $u \to \pm\infty$. If we deal with a wormhole with two regular asymptotics, one of them occurs at $u = +\infty$, the other at $u = -\infty$.

Evidently, at a regular asymptotic, both $r(u)$ and $a(u)$ tend to infinity. If there are two such asymptotics, both functions have minima at some finite u, i.e., there occur both a throat as a minimum of $r(u)$ and an a-throat (they do not necessarily coincide if there is no symmetry with respect to $u = u_1$). This leads to the following result:

Proposition 1. In general relativity, any static, cylindrically symmetric wormhole with two regular asymptotics contains a region where the energy density is negative.

This conclusion can be equivalently formulated as a no-go theorem:

Proposition 1a. In general relativity, a static, cylindrically symmetric, twice asymptotically regular wormhole cannot exist if the energy density T_0^0 is everywhere nonnegative.

[3] In general, this requirement should be formulated in terms of Lorentz tetrad components of all tensors. However, for our diagonal metric (2) these tetrad components coincide with coordinate components written with mixed indices, such as $R^{\mu\nu}{}_{\rho\sigma}$ for the Riemann tensor, R_μ^ν for the Ricci tensor and T_μ^ν for the SET. They behave as scalars at reparametrizations of the radial coordinate u, which makes a transition to tetrad components redundant.

It should be stressed that this conclusion holds with any of the two definitions of a throat.

III. EXAMPLES OF CYLINDRICAL WORMHOLES

In what follows, we will everywhere adhere to Definition 1 of a cylindrical wormhole. A few examples of wormhole space-times will be presented with different matter sources. None of them have two regular asymptotics, even though in some cases the energy density is (partly) negative.

A. Vacuum

Vacuum space-time ($T_\mu^\nu = 0$) cannot contain a wormhole since the condition (9) does not hold anywhere. Let us still consider it for comparison. The easiest way to solve the equations $R_\mu^\nu = 0$ is to choose u as a harmonic radial coordinate; see (7). Then the equations $R_\mu^\nu = 0$ lead to $\beta'' = \gamma'' = \xi'' = 0$, so that

$$\gamma(u) = au + a_0, \qquad \beta(u) = bu + b_0, \qquad \xi(u) = cu + c_0, \tag{16}$$

with six integration constants, among which a_0, b_0, c_0 may be turned to zero by changing scales along the t and z axes and choosing the zero point of the u coordinate. Moreover, the first-order equation $G_1^1 = 0$ leads to a relation between the remaining constants a, b, c:

$$ab + ac + bc = 0, \tag{17}$$

hence there are two essential constants. The metric takes the form

$$ds^2 = \mathrm{e}^{2au} dt^2 - \mathrm{e}^{2(a+b+c)u} du^2 - \mathrm{e}^{2cu} dz^2 - \mathrm{e}^{2bu} d\phi^2. \tag{18}$$

It is the well-known Levi-Civita solution whose more usual form (see, e.g., [10])

$$ds^2 = \bar{r}^{2m} dt^2 - \bar{r}^{2m(m-1)}(d\bar{r}^2 + dz^2) - \mathcal{C}\bar{r}^{2(1-m)} d\phi^2 \tag{19}$$

is obtained from (18) using the relations and notations

$$\mathrm{e}^{(a+b)u} = k\bar{r}, \qquad k = (a+b)^{-(a+b)/c}, \qquad m = \frac{a}{a+b}, \qquad \mathcal{C} = (a+b)^{2b/c} \tag{20}$$

valid for $a + b \neq 0$, $c \neq 0$. [Note that $a + b = 0$ leads to $a = b = 0$ while $c = 0$ leads to $ab = 0$ due to (17).] The two parameters in (19) are m called the mass parameter and $\mathcal{C}$ called the conicity parameter.

In the special case $m = 0$ in (19), or $a = c = 0$ in (17) [in this case the two metrics are not related by (20) but the result is the same] we obtain the flat metric in which $1 - \mathcal{C}$ or $1 - b^2$ is the angular defect.

For our treatment it is important that even the vacuum solution is in general not asymptotically flat, and only for $m = 0$ or $a = c = 0$ one obtains a flat (for $b^2 = 1$) or string (for $b^2 \neq 1$) asymptotic behavior. This means that in the general case, when T_μ^ν vanishes asymptotically and the metric approaches the Levi-Civita solution, the conditions (13) and (14) make a very strong restriction. Recall for comparison that in spherically symmetric systems the vacuum (Schwarzschild) solution is asymptotically flat, and the same is true for a wealth of nonvacuum solutions, certainly, in the absence of a cosmological constant.

B. Scalar fields; two more no-go theorems

Consider a scalar field with the Lagrangian

$$L_s = \frac{1}{2}\varepsilon\varphi^{,\alpha}\varphi_{,\alpha} - V(\varphi), \tag{21}$$

where $V(\varphi)$ is an arbitrary function, and $\varepsilon = \pm 1$ distinguishes normal ($\varepsilon = +1$) and phantom ($\varepsilon = -1$) scalar fields. For the metric (2) and $\varphi = \varphi(u)$, the SET has the form

$$T_\mu^\nu = \varepsilon\varphi_{,\mu}\varphi^{,\nu} - \delta_\mu^\nu L_s = \frac{1}{2}\varepsilon\varphi'^2 \mathrm{e}^{-2\alpha}\,\mathrm{diag}(1, -1, 1, 1) + V(\varphi)\delta_\mu^\nu, \tag{22}$$

so that

$$T_0^0 = T_2^2 = T_3^3. \tag{23}$$

Therefore, in the coordinates (7), three second-order equations (6) combine to give

$$\beta'' = \gamma'' = \xi'' = \frac{1}{3}\alpha'', \tag{24}$$

where the last equality is due to (7), whence

$$\xi = \frac{1}{3}(\alpha - Au), \qquad \gamma = \frac{1}{3}(\alpha - Bu), \qquad \beta = \frac{1}{3}(\alpha + Au + Bu), \tag{25}$$

where A and B are integration constants and two more constants are ruled out by moving the origin of u and rescaling the z axis. The remaining unknowns α and φ obey the equations [the scalar field equation and combinations of (5)]

$$\alpha'' + 3\varkappa V(\varphi)\mathrm{e}^{2\alpha} = 0, \tag{26}$$

$$\varepsilon\varphi'' - (dV/d\varphi)\mathrm{e}^{2\alpha} = 0, \tag{27}$$

$$\alpha'^2 - N^2 = \frac{3}{2}\varepsilon\varkappa\varphi'^2 - 3\varkappa V \mathrm{e}^{2\alpha}, \qquad N^2 := \frac{1}{3}(A^2 + AB + B^2), \tag{28}$$

where (28), following from the $\binom{1}{1}$ component of (5), is a first integral of (26) and (27).

For a SET satisfying (23), the condition (9) leads to $V < 0$ for both normal and phantom fields. Thus wormholes solutions are not excluded but require a negative potential, at least on and near the throat.

Suppose now that there is a *regular spatial asymptotic*. Without loss of generality it can be placed at $u \to +\infty$, where due to (15) $\alpha \approx \beta$ and due to (25) we have

$$A = B = N > 0, \qquad \alpha \approx \beta \approx Nu.$$

Another regular asymptotic might occur at $u \to -\infty$; however, since the relation for the integration constants $A = B = N$ still holds, if we assume that γ and ξ are finite there, we arrive again at $\alpha \sim \beta \sim Nu$, but now it means that $\beta \to -\infty$, that is, an axis (which can in principle be regular); another regular spatial infinity cannot exist. We arrive at the following result (see also [11]):
Proposition 2. Static, cylindrically symmetric wormholes with two regular asymptotics do not exist in general relativity with matter whose SET satisfies Eq. (23).

If we deny the asymptotic regularity condition but require symmetry of the wormhole with respect to its throat, then at such a throat ($u = u_0$) $\beta' = \gamma' = \xi' = 0$, and Eqs. (5) may be combined to give

$$2\beta'' = -\varkappa e^{2\alpha}(T^0_0 + T^2_2 - T^3_3) \tag{29}$$

at $u = u_0$. Since it is a minimum of β, we have there $\beta'' > 0$. Assuming $T^2_2 = T^3_3$ (which is true for scalar fields), Eq. (29) means $T^0_0 < 0$. The result is (see also [11]):
Proposition 3. A static, cylindrically symmetric wormhole, symmetric with respect to its throat, cannot exist in general relativity with matter whose SET satisfies the conditions $T^2_2 = T^3_3$ and $T^0_0 \geq 0$.

Propositions 2 and 3 not only apply to scalar fields with the Lagrangian (21) but to any matter whose SET satisfies the corresponding condition, for instance, scalar fields with more general Lagrangians like $F(\phi, X)$, $X = g^{\mu\nu}\phi_{,\mu}\phi_{,\nu}$ frequently used to model dark energy (k-essence, generalized Chaplygin gas models etc.) as well as spinor fields to be briefly discussed further in this section. Proposition 3 also applies to any Pascal (not necessarily perfect) fluids with isotropic pressure.

C. Scalar fields: Some wormhole solutions

Consider a solution to Eqs. (26)–(28) in a special case of negative potential, putting

$$3\varkappa V = 3\Lambda = -\lambda^2 < 0, \qquad \lambda > 0, \tag{30}$$

where $\Lambda < 0$ is a cosmological constant. Thus we are dealing with a self-gravitating massless scalar field in the presence of a cosmological constant. We can expect wormhole solutions but certainly without regular asymptotics, not only due to Proposition 2 but simply because the curvature must be nonzero at an asymptotic if $\Lambda \neq 0$.

From Eqs. (26) and (27) we get

$$\varphi' = C = \text{const}, \qquad \alpha'' = \lambda^2 e^{2\alpha}. \tag{31}$$

The latter is a Liouville equation whose solution may be written as

$$e^{-\alpha} = \begin{cases} (\lambda/h)\sinh[h(u-u_1)], & h > 0, \\ \lambda(u-u_1), & h = 0, \\ (\lambda/h)\sin[h(u-u_1)], & h < 0, \end{cases} \tag{32}$$

where h and u_1 are integration constants. Equation (28) leads to a relation among the constants:

$$h^2\,\text{sign}h = N^2 + \frac{3}{2}\varepsilon\varkappa C^2. \tag{33}$$

This, together with (25), completes the solution.

Within this general solution, let us now single out wormhole solutions, i.e., those in which $\beta(u)$ tends to infinity at both ends of the range of u. For convenience and without loss of generality we put $u_1 = 0$ and assume $u > 0$. It is then easy to see that in the limit $u \to 0$ we have $\beta \to \infty$ for any values of h since $\alpha \approx -\ln u \to \infty$. Let us look when $\beta \to \infty$ at large u.

If $h > 0$ [by (33), it is always the case if there is a normal scalar field, $\varepsilon = +1$, $C \neq 0$, but is also possible with a phantom scalar, $\varepsilon = -1$, $C \neq 0$, and in the absence of a scalar, $C = 0$], then $\alpha \approx -hu$ as $u \to \infty$, hence $3\beta \approx (A + B - h)u$, and we obtain a wormhole if $A + B > h$.

The value $h = 0$ may appear without a scalar field ($C = 0$, a pure vacuum solution with negative Λ), it corresponds to $A = B = 0$, and one can verify that this is a cylindrically symmetric version of the anti-de Sitter metric which is not wormhole. However, by (33), we may have $h = 0$ with nonzero A or B (or both) if there is a phantom scalar ($\varepsilon = -1$, $C \neq 0$). Then, as $u \to \infty$, $\alpha \sim -\ln u \to -\infty$, but its contribution is inessential in the expressions (25), and we have $\beta \to \infty$, hence a wormhole, if $A + B > 0$.

Lastly, if $h < 0$, which can only happen in the presence of a phantom field, the other end of u range is $u \to \pi/|h|$, and it is quite similar to $u = 0$. Thus *all* solutions with $h < 0$ are wormhole.

So, we have a large family of wormhole solutions with a negative cosmological constant, with or without massless scalar fields, both normal and phantom. However, none of these solutions are asymptotically regular.

Among these solutions, there is a symmetric subfamily: it corresponds to $\varepsilon = -1$, $C \neq 0$, $A = B = 0$, $h < 0$. In accord with Proposition 2, it has a negative energy density, $T^0_0 = -\frac{1}{2}\phi'^2 e^{-2\alpha} + \Lambda/\varkappa$. In terms of the Gaussian coordinate $l = (1/\lambda)\log\tan(hu/2)$ ($l \in \mathbb{R}$ is a length in the radial direction), the metric in this case can be written in the simple-looking form

$$ds^2 = -dl^2 + \left(\frac{|h|}{\lambda}\cosh\frac{l}{\lambda}\right)^{2/3}(dt^2 - dz^2 - d\varphi^2). \tag{34}$$

D. Spinor fields: General considerations

Spinor fields of sufficiently general nature in static, cylindrically symmetric configurations have been considered in Ref. [12], and we here follow this paper. The Lagrangian

$$L_{\rm sp} = \frac{i}{2}(\bar{\psi}\gamma^\mu\nabla_\mu\psi - (\nabla_\mu\bar{\psi})\gamma^\mu\psi) - m\bar{\psi}\psi - F(S), \tag{35}$$

where $F(S)$ is an arbitrary function of the invariant $S = \bar{\psi}\psi$, describes as a special case the Dirac spinor field ψ of arbitrary mass m as well as a general class of nonlinearities. The equations for ψ and $\bar{\psi}$ and the spinor field SET are [13]

$$i\gamma^\mu\nabla_\mu\psi - m\psi - \partial F/\partial\bar{\psi} = 0, \tag{36}$$

$$i\nabla_\mu\bar{\psi}\gamma^\mu + m\bar{\psi} + \partial F/\partial\psi = 0, \tag{37}$$

$$T^\rho_{\mu\,{\rm sp}} = \frac{i}{4}g^{\rho\nu}(\bar{\psi}\gamma_\mu\nabla_\nu\psi + \bar{\psi}\gamma_\nu\nabla_\mu\psi - \nabla_\mu\bar{\psi}\gamma_\nu\psi - \nabla_\nu\bar{\psi}\gamma_\mu\psi) - \delta^\rho_\mu L_{\rm sp}, \tag{38}$$

where $\nabla_\mu\psi$ is the covariant derivative of the spinor field

$$\nabla_\mu\psi = \partial\psi/\partial x^\mu - \Gamma_\mu\psi, \tag{39}$$

with $\Gamma_\mu(x)$ being the Fock-Ivanenko spinor affine connection matrices. The γ^μ matrices are related to the flat-space Dirac matrices $\tilde{\gamma}^\mu$ by $\gamma_\mu(u) = e^a_\mu\tilde{\gamma}_a$, where e^a_μ are the tetrad vectors, so that $g_{\mu\nu} = e^a_\mu e^b_\nu\eta_{ab}$, where $\eta_{ab} = {\rm diag}(1, -1, -1, -1)$.

In our static, cylindrically symmetric case, with the metric (2) and $\psi = \psi(u)$, we have [12] $L_{\rm sp} = SF_S - F(S)$, $F_S := dF/dS$. The SET components of the spinor field are

$$T^0_0 = T^2_2 = T^3_3 = F(S) - SF_S, \tag{40}$$

$$T^1_1 = \frac{i}{2}(\bar{\psi}\gamma^1\nabla_1\psi - \nabla_1\bar{\psi}\gamma^1\psi) + SF_S - F(S), \tag{41}$$

so that Eq. (23) holds, along with its consequences such as the expressions (25) for the metric functions and Propositions 2 and 3 which restrict the possible wormhole existence.

Equation (36) may be written in the form [12]

$$i{\rm e}^{-\alpha}\tilde{\gamma}^1\left(\partial_u + \frac{1}{2}\alpha'\right)\psi - m\psi - F_S\psi = 0. \tag{42}$$

Combining it with its conjugate, we arrive at the equation $S' + \alpha' S = 0$, giving

$$S(u) = c_0{\rm e}^{-\alpha(u)}, \qquad c_0 = {\rm const}. \tag{43}$$

Then F and F_S are expressed in terms of ${\rm e}^{-\alpha(u)}$. Moreover, Eq. (42) and its conjugate allow one to reexpress T^1_1 as

$$T^1_1 = mS + F(S) =: M(S). \tag{44}$$

The only remaining Einstein equation to be solved is

$$\alpha'^2 - N^2 = -3\varkappa{\rm e}^{2\alpha}M(S), \qquad N^2 := \frac{1}{3}(A^2 + AB + B^2), \tag{45}$$

Since by (43) $\alpha' = -S'/S$, Eq. (45) is rewritten as

$$\left(\frac{dS}{du}\right)^2 = N^2S^2 - \frac{3\varkappa c_0^2}{M(S)}, \tag{46}$$

which is easily solved by quadratures. Thus, given $F(S)$, the Einstein equations are solved in a general form even without entirely integrating the nonlinear spinor equations [12].

With (40) and (44), the condition (9) implies

$$2M - SM_S < 0 \tag{47}$$

at a wormhole throat. This condition is similar to the $V < 0$ condition for scalar fields.

E. Spinor field: Example of a wormhole solution

A simple example of a wormhole solution can be obtained using the inverse problem method: choosing the form of $\alpha(u)$, we easily find both $S(u)$ and $M(u)$, hence $M(S)$. So, let us put

$${\rm e}^{\alpha} = A_0\cosh ku, \qquad A_0, k = {\rm const} > 0, \tag{48}$$

to obtain, according to (43) and (45),

$$S(u) = c_0/(A_0\cosh ku), \tag{49}$$

$$3\varkappa A_0^2M(S) = \frac{N^2 - k^2}{\cosh^2 ku} + \frac{k^2}{\cosh^4 ku} = (N^2 - k^2)\left(\frac{A_0}{c_0}\right)^2S^2 + k^2\left(\frac{A_0}{c_0}\right)^4S^4. \tag{50}$$

The solution as a whole is determined by Eqs. (25), (48), and (49). It is regular at all $u \in \mathbb{R}$ and, since the asymptotics of $\beta(u)$ are

$$\beta \approx \begin{cases} (k + A + B)u, & u \to +\infty, \\ (k - A - B)|u|, & u \to -\infty, \end{cases} \tag{51}$$

it describes a wormhole if $|A + B| < k$. The wormhole is symmetric if $A = B = 0$, which corresponds to a nonlinear spinor field with

$$M(S) = -\frac{k^2S^2}{3\varkappa c_0^2}\left[1 - \left(\frac{A_0S}{c_0}\right)^2\right].$$

Its metric can be written in terms of the Gaussian coordinate $l = (A_0/k)\sinh ku \in \mathbb{R}$) as

$$ds^2 = -dl^2 + (A_0^2 + k^2l^2)^{1/3}(dt^2 - dz^2 - d\phi^2). \tag{52}$$

It is easy to verify that, as in the scalar case, the density T^0_0 is negative near the throat $u = 0$, $l = 0$.

F. Einstein-Maxwell fields and nonlinear electrodynamics (NED)

Electromagnetic fields $F_{\mu\nu}$, compatible with the geometry (2), can have three different directions:
Radial (R): electric, $F_{01}(u)$ ($E^2 = F_{01}F^{10}$), and magnetic, $F_{23}(u)$ ($B^2 = F_{23}F^{23}$).
Azimuthal (A): electric, $F_{03}(u)$ ($E^2 = F_{03}F^{30}$), and magnetic, $F_{12}(u)$ ($B^2 = F_{12}F^{12}$).
Longitudinal (L): electric, $F_{02}(u)$ ($E^2 = F_{02}F^{20}$), and magnetic, $F_{13}(u)$ ($B^2 = F_{13}F^{13}$).

Here E and B are the absolute values of the electric field strength and magnetic induction, respectively. Self-gravitating static, cylindrically symmetric configurations of such electromagnetic fields have been considered in the framework of the Einstein-Maxwell theory in [14] and in the Einstein-NED theory with gauge-invariant NED Lagrangians of the form

$$L_e = -\Phi(F)/(16\pi), \qquad F := F^{\mu\nu}F_{\mu\nu} \tag{53}$$

in [15]. The Maxwell Lagrangian is recovered by putting $\Phi(F) \equiv F$. The Lagrangian (53) leads to the SET

$$T^\nu_\mu = \frac{1}{16\pi}[-4F^{\nu\alpha}F_{\mu\alpha}\Phi_F + \delta^\nu_\mu \Phi], \tag{54}$$

where $\Phi_F = d\Phi/dF$. For a radial electromagnetic field in the geometry (2) this gives

$$T^0_0 = T^1_1 = \frac{1}{16\pi}(4E^2\Phi_F + \Phi), \tag{55}$$

$$T^2_2 = T^3_3 = \frac{1}{16\pi}(-4B^2\Phi_F + \Phi). \tag{56}$$

Similar relations for a longitudinal field are obtained by replacing $1 \leftrightarrow 2$ in the indices and for an azimuthal field by replacing $1 \leftrightarrow 3$.

Now, let us find out which kind of electromagnetic field is suitable for obtaining wormholes. It is easy to verify that the expression (9), which should be negative at and near a wormhole throat, has the form

$$T^* = (4E^2\Phi_F + \Phi)/(8\pi) \qquad (\text{R and L fields}), \tag{57}$$

$$T^* = (-4B^2\Phi_F + \Phi)/(8\pi) \qquad (\text{A fields}). \tag{58}$$

In Maxwell electrodynamics this gives $T^* = (E^2 + B^2)/(4\pi)$ for L and R fields and $T^* = -(E^2 + B^2)/(4\pi)$ for A fields. Thus wormholes are only possible with azimuthal fields. Indeed, the corresponding exact solution to the Einstein-Maxwell equations has the form [14]

$$ds^2 = \frac{\cosh^2(hu)}{Kh^2}[e^{2au}dt^2 - e^{2(a+b)u}du^2 - e^{2bu}d\varphi^2] - \frac{Kh^2}{\cosh^2(hu)}dz^2, \tag{59}$$

where $K = [G(i_e^2 + i_m^2)]^{-1}$, $h^2 = ab$, $a, b = \text{const}$, $a > 0$, $b > 0$, and the electromagnetic field is given by

$$F_{03} = i_m = \text{const}; \qquad F^{12} = i_e e^{-2\alpha}, \qquad i_e = \text{const}, \tag{60}$$

where i_e and i_m are the effective currents of electric and magnetic charges along the z axis, respectively. This solution is written under the coordinate condition (7), and evidently, $r = e^\beta \sim e^{(h\pm b)|u|}$ as $u \to \pm\infty$. Thus it is a wormhole solution if $b < h$. It is easy to see that such a wormhole is neither symmetric nor asymptotically regular.

Let us now turn to NED; see Eq. (53). It is clear that $\Phi(F)$ may be chosen so that the expression (57) for R and L fields will be negative at some F, but this will simultaneously mean that the energy density T^0_0 will be negative. Such cases are yet to be studied. On the other hand, with A fields it is easy to obtain wormhole solutions like (59) and (60), and moreover one can show that no such solutions have two regular asymptotics. Indeed, since in this case $T^0_0 = T^3_3$, the corresponding component of the Einstein equations in the coordinates (7) gives $\beta'' = \gamma''$, whence $\beta = \gamma + bu + b_0$ with b, $b_0 = \text{const}$. A regular asymptotic at $u = \infty$ requires $\gamma \to \text{const}$ and $b > 0$ whereas a regular asymptotic at $u = \infty$ requires $\gamma \to \text{const}$ and $b < 0$. Thus a regular asymptotic can appear only at one "end," and, moreover, it can be only achieved at the expense of a non-Maxwell behavior of $\Phi(F)$ at small F [15].

IV. CONCLUDING REMARKS

We have seen that the existence conditions for cylindrically symmetric wormholes can be satisfied without violating the weak or null energy conditions near the throat. We have presented a number of explicit examples of wormhole solutions with nonphantom sources.

However, as is always the case when dealing with cylindrically symmetric systems, it is rather hard to obtain solutions with regular (i.e., flat or string) asymptotics: indeed, even the Levi-Civita vacuum solution has such an asymptotic only in a special case. Of course, such asymptotic behaviors are necessary if we wish to describe a wormhole in a flat or weakly curved background universe. We have proved that if one wishes to have this behavior at both sides of the throat, it is necessary to invoke matter with negative energy density.

This problem is still more important if we try to apply cylindrically symmetric solutions as an approximate description of toroidal systems, e.g., like those discussed by Gonzalez-Diaz [16]. This approximation must work well if a torus containing matter and significant curvature is thin, like a circular string, i.e., its larger radius is much greater than the smaller radius. In this case, small segments along such a "string" are approximately cylindrically symmetric. But this means that sufficiently far from the thin ring, in any direction (to or from the center of the ring or in any other), the space-time should be almost flat.

Thus, in future studies, it is desirable either to find asymptotically regular cylindrically symmetric geometries with more or less realistic material sources or to consider configurations consisting of at least three layers: one responsible for the wormhole throat and its neighborhood and two others (on each side of the throat) providing regular asymptotics. One can note that the required negative energy densities may appear due to quantum effects, such as vacuum polarization and the Casimir effect.

A challenging problem is to clear up a relationship between the throat topology and the energy conditions. A point of interest is that in cylindrically symmetric spacetimes there can be at least two different definitions of a wormhole throat: the one we have been using, in terms of the radius $r(u)$, and the alternative one, in terms of the area function $a(r)$; moreover, they lead to different wormhole existence conditions in terms of the SET, but both of them differ from the conditions for spherically symmetric wormholes or those with a spherical topology of throats. Thus the topological issue is probably crucial for formulating the general properties of matter sources for wormhole geometries. And it is yet to be ascertained what are the similar conditions at throats of toroidal and more complex topologies. It seems plausible that a toroidal throat should behave like a cylindrical one since the condition sought for should be of local nature, but a rigorous proof is so far lacking.

There are quite a number of results indicating wormhole properties of many stringlike configurations. One can mention, in particular, Clement's "flat wormholes" obtained from properly moving straight cosmic strings, the wormhole nature of Kerr and Kerr-Newman space-times with large charges and/or angular momenta, where naked ring singularities have certain string properties (see [17,18] and references therein), and their static counterparts [19,20].[4] Rotating cylindrically symmetric configurations also tend to show wormhole properties [21]. There are also toroidal wormhole solutions in anti-de Sitter space-times built using the cut and paste technique [22]. So one can be rather optimistic about the existence of realistic cylindrical or toroidal wormhole models.

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[4]There is a fundamental difference between "ring wormholes" described by Kerr, Zipoy and similar metrics and the cylindrical and toroidal wormholes discussed here: in the first case, one gets from one "universe" to another by threading the ring, i.e., crossing the disc that subtends it, while in the second case, to traverse the wormhole, it is necessary to approach the string itself.

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