

Black hole evaporation within a momentum-dependent metric

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We investigate the black hole thermodynamics in a “deformed” relativity framework where the energy-momentum dispersion law is Lorentz-violating and the Schwarzschild-like metric is momentum-dependent with a Planckian cutoff. We obtain net deviations of the basic thermodynamical quantities from the Hawking-Bekenstein predictions: actually, the black hole evaporation is expected to quit at a nonzero critical mass value (of the order of the Planck mass), leaving a zero temperature remnant, and avoiding a spacetime singularity. Quite surprisingly, the present semiclassical corrections to black hole temperature, entropy, and heat capacity turn out to be identical to the ones obtained within some quantum approaches.

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I. INTRODUCTION

Black holes are very peculiar physical systems since they are fully described only by classical theories, namely, general relativity and thermodynamics, as well as quantum gravity theories. The merging between the above theories is required, in particular, for primordial and for microscopic (“mini”) black holes rather than for stellar or galactic supermassive ones: this because of the extreme smallness of the Planck scale at which classical and quantum approaches appear to carry quite different predictions.

According to semiclassical Hawking-Bekenstein theory (HB) [1], which takes into account the quantum effects due to the very strong gravitational field on the black hole surface, we have emission of radiation out from the black hole as it becomes a black body at a given temperature, besides a negative-energy flux from the surface towards the interior of the black hole. As a consequence the total energy decreases in time, while the temperature increases more and more. This can be inferred from the following qualitative argument based on the Heisenberg indetermination relation. The energy of a photon [2] with a wavelength equal to the black hole radius can be assumed of the order of $1/R_S$, R_S being the Schwarzschild radius: we then expect a temperature of the order of $1/GM$ (hereafter $\hbar = c = k_B = 1$). Actually, the HB calculations lead to the following relation between black hole mass and temperature:

$$T = \frac{\kappa}{2\pi} = \frac{1}{8\pi GM} \sim 10^{-7} \left(\frac{M_\odot}{M} \right) [\text{K}], \quad (1)$$

where κ is the “surface gravitation” and $M_\odot \sim 10^{54}$ TeV is the solar mass. In the HB picture the evaporation goes on until the initial mass of the black hole is converted into radiation, and the process ends with an explosion since, as the temperature, also the mass loss rate dM/dt goes to

infinity. Besides the unphysical loss rate divergence, the total evaporation predicted by the HB theory entails other serious problems and inconsistencies such as the baryon and lepton number nonconservation, the “information paradox,” and the microscopic origin of the entropy [3,4].

A complete understanding of those problems is only possible within the framework of a quantum theory of gravity. However we think that the effective quantum behavior can be (at least qualitatively) predicted as well within a semiclassical theory such as the one we are going to propose in the next sections.

II. SPACETIME ENDOWED WITH A MOMENTUM-DEPENDENT METRIC

In recent times ultrahigh energy Lorentz symmetry violations have been investigated, both theoretically and experimentally, by means of quite different approaches, sometimes extending, sometimes abandoning the formal and conceptual framework of Einstein’s special relativity. Hereafter, for simplicity, we shall use the term “violation” of the Lorentz symmetry, but in some of the theories mentioned below—e.g., in the so-called “deformed” or “doubly” special relativity (DSR), where deformed 4-rotation generators are considered—although special relativity does not hold anymore; an underlying extended Lorentz invariance does exist. The most important consequence of a Lorentz violation is the modification of the ordinary momentum-energy dispersion law $E^2 = p^2 + m^2$, at energy scales usually assumed of the order of the Planck energy $E_P = M_P c^2 = \sqrt{\hbar c^5/G}$, by means of additional terms which vanish in the low momentum limit. Lorentz-breaking observable effects appear in grand unification theories [5], in string theories [6], in quantum gravity [7], in foamlike quantum spacetimes [8]; in spacetimes endowed with a nontrivial topology or with a discrete structure at the Planck length [9,10], or with a (canonical or noncanonical) noncommutative geometry [11–13]; in the so-called “extensions” of the standard model incorporat-

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ing breaking of Lorentz and CPT symmetries [14]; in theories with a variable speed of light or variable physical constants. In particular, the M-theory [6], the loop quantum gravity [9,10,15] and the causal dynamical triangulation [16] lead one to postulate an essentially discrete and quantized spacetime, where a fundamental mass-energy scale naturally arises, in addition to \hbar and c . An intrinsic length is directly correlated to the existence of a cutoff in the transferred momentum necessary to avoid the occurrence of “UV catastrophes” in quantum field theories. Moreover, some authors suspect that the Lorentz symmetry breaking may play a role in extreme astrophysical phenomena as, e.g., the observation of ultrahigh energy cosmic rays with energies [17] beyond the Greisen-Zatsepin-Kuzmin [18] cutoff, and of gamma ray bursts with energies beyond 20 TeV originated in distant galactic sources [19].

A natural extension of the standard dispersion law can be put in most cases under the general form

$$E^2 = p^2 + m^2 + p^2 f(p/M), \quad (2)$$

where M indicates a (large) mass scale characterizing the Lorentz violation. By using a series expansion for f , under the assumption that M is a very large quantity, we can consider only the lower order nonzero term in the expansion:

$$E^2 = p^2 + m^2 + \alpha p^2 \left(\frac{p}{M}\right)^n. \quad (3)$$

The most recurring exponent in the literature on Lorentz violation is the lowest one, i.e. $n = 1$ [20]:

$$E^2 = p^2 + m^2 + \alpha \frac{p^3}{M}. \quad (4)$$

An interesting theoretical approach to Lorentz symmetry violation is found in DSR [12,13,21] working in k -deformed Lie-algebra noncommutative (k -Minkowski) spacetimes, in which both the Planck scale and the speed of light act as characteristic scales of a 6-parameter group of spacetime 4-rotations with deformed but preserved Lorentz symmetries. In place of the ordinary constraint

$$E^2 - p^2 = m^2$$

in such theories the Lorentz-violating (LV) modified dispersion law can be put in the form

$$E^2 f^2(E) - p^2 g^2(p) = m^2. \quad (5)$$

For example in Ref. [12], where the Lie algebra is given by $[x_i, x_0] = i\lambda x_i$, $[x_i, x_k] = 0$ (λ being a very small length of the order of M_P^{-1}), the dispersion relation

$$E^2 = p^2 + m^2 + \lambda E p^2 \quad (6)$$

can be recovered taking

$$f^2(E) = 1 \quad g^2(p) = 1 + \lambda E. \quad (7)$$

In some DSR theories [22–25] a modified set of special

relativity principles is assumed: (a) the Galilean relativity principle; (b) the speed of light is energy-dependent, but in the small energy limit goes to the universal constant c for all inertial observers; (c) also the Planck energy-momentum is an absolute quantity, independent of the given inertial frame where it is measured. Let us quote some typical metric form factors appearing in the literature: in [26] it is assumed $f = 1$, $g = [1 - a(\lambda E)^n]^{-1}$; in [27] $f = 1$, $g = [1 - a(\lambda E)^n]$; in [22–24] $f = 1$, $g = (1 + \lambda E)^\gamma$, and in [25] $f = 1$, $g = 1 + (\lambda E)^\gamma$, with $\gamma \in \mathfrak{N}$. The above DSR theories have been generalized to curved spacetimes: this is the case of the so-called “doubly general relativity,” also called “rainbow’s gravity” [28]. The resulting metric depends on both probe energy and gravity field, with straightforward modifications to Einstein equations for field and matter.

Let us now propose a more symmetric form of constraint (5):

$$E^2 f^2(E) - p^2 g^2(p) = m^2. \quad (8)$$

We can easily see that if we make the most simple choice

$$f^2(E) = 1 \quad g^2(p) = 1 - \lambda p, \quad (9)$$

where $\lambda \simeq M_P^{-1}$ is the fundamental length scale of an underlying LV theory, we do recover the first order ($n = 1$) LV dispersion law (4). Notice that, because of the negative term $-\lambda p$, the energy vanishes when $p = \frac{1}{\lambda} \simeq M_P$. Hence quantity $1/\lambda$, in a sense, plays the role of the “maximal momentum” corresponding to the “granular” nature of space predicted in many of the models quoted in the beginning of this section. Equation (8) can be equivalently written as

$$\eta_{\mu\nu} p^\mu p^\nu = m^2 \quad (10)$$

with $\eta_{\mu\nu}$ diagonal in flat spacetime and [29]

$$\eta_{00} = f(E) = 1 \quad \eta_{ik} = -g(p)\delta_{ik} = -\sqrt{1 - \lambda p}\delta_{ik}. \quad (11)$$

Let us stress that, at variance with other metrics adopted in the literature (such as the ones previously quoted), metric (11) becomes singular at the Planck momentum. As we shall show, the presence of a very Planckian cutoff in the theory can overcome some of the above-mentioned problems when applying general relativity at the Planck scale.

Just as the properties of a crystal can depend on the energy of phonons propagating in it, analogously our spacetime geometry can depend on the moving particle energy. At low energies the phonons cannot see the discrete structure of crystals and behave like ordinary photons. At high energies, on the contrary, they become highly sensitive to the medium properties and exhibit a rather exotic behavior. We therefore expect that, like phonons, the Hawking radiation high energy photons should behave differently from the photons of the ordinary black body radiation. In the next section we shall study some conse-

quences of metric (11) on the thermodynamical evolution of a black hole.

III. MODIFIED BLACK HOLE THERMODYNAMICS

In general relativity the metric for a nonrotating, uncharged, spherical black hole, endowed with a mass M , is the Schwarzschild one:

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega, \quad (12)$$

while applying Eq. (11) we have the following modified metric [28]

$$ds^2 = -\frac{1}{f^2(E)}\left(1 - \frac{2GM}{r}\right)dt^2 + \frac{1}{g^2(p)}\left[\left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega\right]. \quad (13)$$

As said above, according to the HB theory the black hole temperature can be taken equal to $\frac{\kappa}{2\pi}$, where κ is the surface gravity. Because of the chosen metric, the black hole surface gravity and temperature do depend on the probe energy:

$$\kappa = -\lim_{r \rightarrow R_s} \sqrt{\frac{-g^{rr}}{g^{tt}}} \frac{(g^{tt})'}{g^{rr}} = \frac{g(p)}{f(E)} \frac{1}{4GM}, \quad (14)$$

$$T = \frac{\kappa}{2\pi} = \frac{g(p)}{f(E)} \frac{1}{8\pi GM}. \quad (15)$$

Hence, with our choice for $f(E)$ and $g(p)$, we get

$$T = \frac{\sqrt{1 - \lambda p}}{8\pi GM}. \quad (16)$$

Let us apply the ordinary uncertainty relation to photons near the event horizon:

$$p \simeq \delta p \sim \frac{1}{\delta x} \sim \frac{1}{4\pi R_s} = \frac{1}{8\pi GM}. \quad (17)$$

By inserting the previous equation in (16) we find

$$T = \frac{\sqrt{1 - \lambda p}}{8\pi GM} \sim \sqrt{1 - \frac{\lambda}{8\pi GM}}. \quad (18)$$

Being $p \leq 1/\lambda$ we shall consider only $M \geq M_{\text{cr}} \equiv \frac{\lambda}{8\pi G}$.

For $M \gg M_{\text{cr}}$ we can approximate the black hole temperature as follows:

$$T \sim \frac{1}{8\pi GM} \left[1 - \frac{1}{2} \frac{M_{\text{cr}}}{M} - \frac{1}{8} \left(\frac{M_{\text{cr}}}{M}\right)^2 + \mathcal{O}\left(\frac{M_{\text{cr}}^3}{M^3}\right) \right], \quad (19)$$

while, when M approaches M_{cr} , we can assume

$$T \sim \frac{1}{8\pi GM_{\text{cr}}^{3/2}} [\sqrt{M - M_{\text{cr}}} + M_{\text{cr}}^{-1/2} \mathcal{O}(M - M_{\text{cr}})]. \quad (20)$$

Therefore the black hole temperature, by contrast with the HB theory, does not diverge at $M = 0$, but has a finite maximum at $M = \frac{3}{2}M_{\text{cr}}$ (Fig. 1).

Assuming that the first principle of thermodynamics is still valid, namely $dQ = dM = TdS$, we can obtain the intrinsic entropy by inserting the temperature (15) into this relation, and then integrating

$$\begin{aligned} S &= \int \frac{dM}{T} \sim \int \frac{8\pi GM}{\sqrt{1 - \frac{M_{\text{cr}}}{M}}} dM \\ &= \frac{2\pi G}{\sqrt{1 - \frac{M_{\text{cr}}}{M}}} \left(2M^2 + MM_{\text{cr}} - 3M_{\text{cr}}^2 \right. \\ &\quad \left. + 3M_{\text{cr}}^2 \sqrt{1 - \frac{M_{\text{cr}}}{M}} \operatorname{arcsinh} \sqrt{\frac{M}{M_{\text{cr}}} - 1} \right). \end{aligned} \quad (21)$$

Of course, for $\lambda \rightarrow 0$ we recover the classical HB result

$$\lim_{\lambda \rightarrow 0} S = \frac{1}{4} A, \quad (22)$$

where A indicates the horizon area $16\pi GM^2$. The same classical behavior (quadratic in M), is obtained for very heavy black holes, i.e. for $M \gg M_{\text{cr}}$. Expanding Eq. (21) for $M \sim M_{\text{cr}}$ we obtain (see Fig. 2)

$$S \sim 4\pi GM_{\text{cr}}^{3/2} \sqrt{M - M_{\text{cr}}}. \quad (23)$$

Then the black hole entropy reaches its minimum (zero) together with the black hole mass, i.e., for the critical value M_{cr} .

For the sake of comparison let us recall that, still starting from modified dispersion laws and generalized Heisenberg uncertainty relations, some authors [30] obtain in the entropy formula a term logarithmically dependent on the

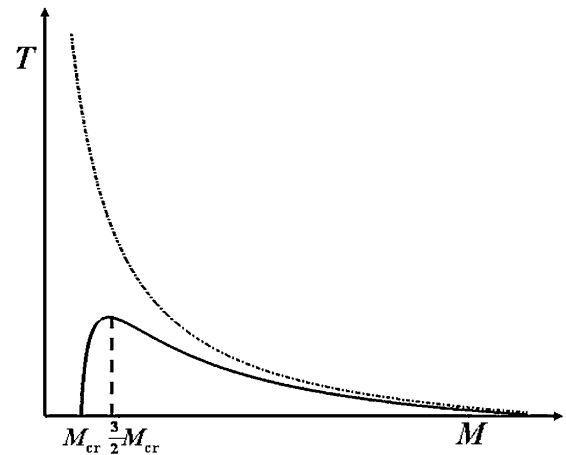


FIG. 1. Temperature as a function of the mass (compared with the standard HB plot).

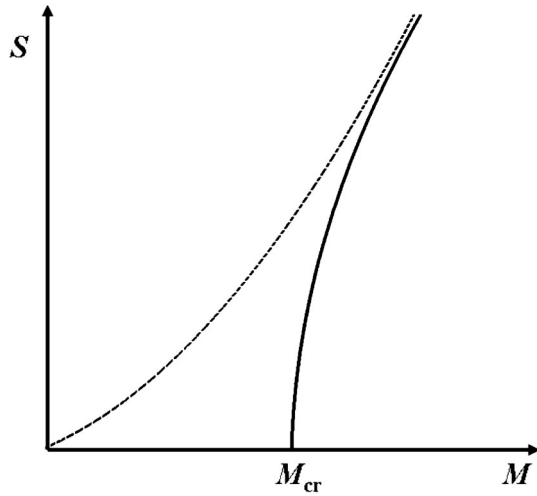


FIG. 2. Entropy as a function of the mass (compared with the standard HB plot).

black hole mass, a result found also in some string theory and loop quantum gravity computations.

Finally, let us calculate the black hole heat capacity. For $M \sim M_{\text{cr}}$ we get

$$C = T \frac{\partial S}{\partial T} = \left(\frac{\partial T}{\partial M} \right)^{-1} \sim 16\pi G \frac{M^{5/2} \sqrt{M - M_{\text{cr}}}}{3M_{\text{cr}} - 2M}. \quad (24)$$

Notice (Fig. 3) that the heat capacity diverges at $M = \frac{3}{2}M_{\text{cr}}$, corresponding to the maximum black hole temperature, and vanishes at the minimum mass $M = M_{\text{cr}}$.

IV. CONCLUSIONS

We have studied the black hole evaporation applying a momentum-dependent metric, corresponding to the lowest-order ($\sim p^3$) extension of the ordinary energy-momentum law dispersion. The HB inverse proportionality relation between mass and temperature is recovered only for early stages of the evaporation process; while in the final stage mass and temperature decrease together so that at the end we have a cold “extremal” black hole endowed with a critical mass $M = M_{\text{cr}}$ of the order of the Planck mass. Correspondingly the black hole entropy reaches its minimum. At variance with the standard predictions, we also find that at $M = \frac{3}{2}M_{\text{cr}}$ the temperature reaches a finite maximum (of the order of the Planck temperature) and the heat capacity diverges.

Some recent theoretical [31–33] pictures of the black hole thermodynamics and evaporation, which take into

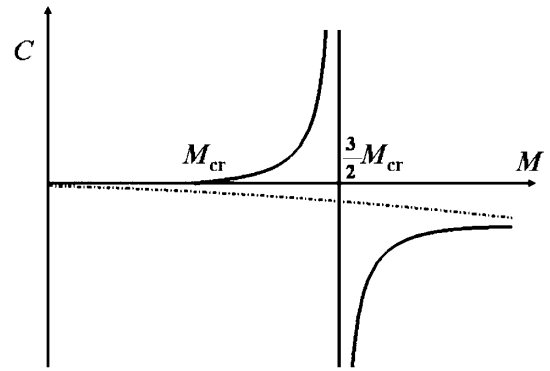


FIG. 3. Heat capacity as a function of the mass (compared with the standard HB plot).

account quantum effects expected when the energy scale approaches the Planck energy (that is, soon before the total collapse), result in avoiding undesired divergences and spacetime singularities. In [32] Bonanno and Reuter study the quantum gravity effects for a spherical black hole assuming a “running” Newton constant $G = G(k(r))$ obtained from the evolution for the effective scale-dependent gravitational action, by means of exact renormalization group equations. Using the “Einstein-Hilbert truncation method” they find an exact, nonperturbative solution to the evolution equation for $G(k)$. Actually, *the quantum computations performed by those authors and the present semiclassical analysis lead to the same physical predictions*. As a matter of fact, the behaviors of temperature, entropy, and heat capacity as functions of the mass obtained in Ref. [32] result to be (also analytically) identical to the ones found in this paper, then entailing a nonzero-mass zero-entropy $T = 0$ remnant as well. From direct comparison between Ref. [32] and the present theory we derive, as expected, that our mass scale λ is just of the order of the Planck mass.

We can conclude that, inside a classical noncommutative spacetime scenario which, as discussed in Sec. II, appears physically plausible on the grounds of various theoretical and experimental arguments, we have obtained a reliable picture of the black hole thermodynamics which overcomes some unphysical features of the HB theory: thus encouraging further theoretical studies in this direction.

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