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Newman-Penrose constants of stationary electrovacuum space-times

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A theorem related to the Newman-Penrose constants is proven. The theorem states that all the Newman-Penrose constants of asymptotically flat, stationary, asymptotically algebraically special electrovacuum space-times are zero. Straightforward application of this theorem shows that all the Newman-Penrose constants of the Kerr-Newman space-time must vanish.

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I. INTRODUCTION

Newmen-Penrose (NP) constants are very interesting and useful quantities in the study of asymptotic flat space-times. They were first found by E. T. Newman and R. Penrose in 1968 [1] and then discussed by many other authors [2–7]. Although the NP constants have been found for 40 years, their physical interpretation remains an open question. One reason is that the computation of these constants for a general asymptotically flat space-time is not easy. In stationary vacuum cases, these constants can be viewed as a combination of multipole moments of space-times [8–11]. Calculations of the NP constants for vacuum solutions have been made by many authors [12–17]. People used to conjecture that the algebraically special condition (ASC) leads to the vanishing of NP constants. However, Kinnersly and Walker [12] provided a counterexample. Recently, some authors [15] proposed the asymptotically algebraically special condition (AASC), and proved that the NP constants vanish for vacuum, stationary, asymptotic algebraically special space-times. In fact, the two conditions are closely related. It is well known that the ASC implies that the Weyl curvature possesses a multiple principle null direction. This condition can be expressed in terms of two geometric invariants I and J, defined by $I = \Psi_0\Psi_4 - 4\Psi_1\Psi_3 + 3(\Psi_2)^2$ and $J = \Psi_4\Psi_2\Psi_0 + 2\Psi_3\Psi_2\Psi_1 - (\Psi_2)^3 - (\Psi_3)^2\Psi_0 - (\Psi_1)^2\Psi_4$ [18,19]. A space-time is said to be algebraically special if $I^3 - 27J^2 = 0$. It has been shown that a general asymptotically flat, stationary space-time satisfies $I^3 - 27J^2 \sim O(r^{-21})$ near future null infinity [15]. Thus, $I^3 - 27J^2$ will peel off very quickly for a general stationary vacuum asymptotically flat space-time although the space-time may not be algebraically special. A space-time is said to be "asymptotically algebraically special" if $I^3 - 27J^2 \sim O(r^{-22})$ [15], i.e., one order faster than general cases. From the geometric point of view, this indicates that one pair of principle null directions coincides near null infinity. By imposing this condition, the authors of [15] showed that the NP constants vanish for vacuum, stationary space-times. Based on the result of [15], NP constants can be seen as a combination of Janis-Newman multipoles of gravitational field [20]. An intriguing question is whether the Janis-Newman multipoles of matter field will contribute to the NP constants. In this paper, we extend the discussion to the electrovacuum case. By imposing the AASC, we show that the NP constants still vanish in the presence of a stationary Maxwell field. If the Maxwell field is not stationary, the multipole moments of the Maxwell field will contribute to the NP constants.

This paper is organized as follows: In Sec. II, we apply the method of Taylor expansion to a stationary electrovacuum space-time. With the help of the Killing equation, we reduce the dynamical freedom of gravitational field into a set of arbitrary constants. Detailed expressions are given up to order $O(r^{-6})$. We then prove that all the NP constants of a stationary asymptotically algebraically special electrovacuum space-time are zero. Finally, we make some concluding remarks in Sec. III.

II. THE NEWMAN-PENROSE CONSTANTS OF STATIONARY ASYMPTOTICALLY ALGEBRAICALLY SPECIAL ELECTROVACUUM SPACE-TIMES

In an asymptotically flat space-time, the Newman-Penrose constants are defined by [18]

$$G_m = \int_{S_\infty} {}_2Y_{2,m}\Psi_0^1 dS,$$

where ${}_2Y_{2,m}$ is a spin-weight harmonic function and Ψ_0^1 is a component of the Weyl tensor. Since the integral is performed on a two-sphere at infinity, we only need the asymptotic form of the Weyl tensor in the calculation. According to the peeling off theorem given by Sachs [18], we may express the Weyl tensor and Maxwell field as

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$$\Psi_n \sim O(r^{n-5}) \qquad n = 0, 1, 2, 3, 4,$$
$$\phi_m \sim O(r^{m-3}) \qquad m = 0, 1, 2. \tag{1}$$

The vacuum case has been studied previously [7,13–15]. An interesting issue is to consider the effect of matter fields on NP constants. In this paper, we shall concentrate on the electromagnetic field. Like in the vacuum case, we require the space-time to be stationary. Obviously, there is no Bondi energy flux in such a space-time, i.e., $\dot{\sigma}^0 = 0$. In this case, we can choose some suitable coordinates, such that the asymptotic shear σ^0 is zero. Similarly, the stationary condition has eliminated the freedom of the news function. We also demand the Weyl tensor satisfy the asymptotically algebraically special condition, which has been discussed above. The main purpose of this paper is to prove the following theorem:

Theorem 1. All the NP constants of an asymptotically flat, stationary, asymptotically algebraically special electrovacuum space-time are zero.

Note that Kerr-Newman solution satisfies all the conditions in the theorem. It follows immediately that all the NP constants in a Kerr-Newman space-time must vanish.

Proof of the theorem. We choose the standard Bondi-Sachs's coordinates and construct the standard Bondi null tetrad [15,21]. With the gauge choice in [18,22], we can write down the NP coefficients and null tetrad of the stationary electrovacuum space-time. Some low order terms have been calculated and can be found in [18]. Calculation of the NP constants requires higher order terms in the expansions. Consider the following NP equations:

$$\delta\lambda - \bar{\delta}\mu = \bar{\tau}\mu + (\bar{\alpha} - 3\beta)\lambda - \Psi_3 + \Phi_{21}, \tag{2}$$

$$\Delta\lambda - \bar{\delta}\nu = 2\alpha\nu + (\bar{\gamma} - 3\gamma - \mu - \bar{\mu})\lambda - \Psi_4, \tag{3}$$

where $\Phi_{ij} = 8\pi\phi_i\bar{\phi}_j$ is the Maxwell stress tensor. The coefficient of r^{-2} in Eq. (2) yields $\Psi_3^0 = 0$. Expanding Eq. (3) up to $O(r^{-3})$, we obtain $\Psi_4^0 = \Psi_4^1 = \Psi_4^2 = 0$.

Now we shall use the Killing equation to reduce other dynamical freedoms and get a general asymptotic expansion of the stationary electrovacuum space-time. We write down the timelike Killing vector as

$$t^a = Tl^a + n^a + \bar{A}m^a + A\bar{m}^a.$$

The Killing equations are given by

$$-DT + (\gamma + \bar{\gamma}) + \bar{\tau}A + \tau\bar{A} = 0, \tag{4}$$

$$DA + \tau + \bar{\rho}A + \sigma\bar{A} = 0, \tag{5}$$

$$-D'T - (\gamma + \bar{\gamma})T - \nu A - \bar{\nu}\bar{A} = 0, \tag{6}$$

$$-\tau T + \bar{\nu} + D'A + (\bar{\gamma} - \gamma)A - \delta T - \tau T - \mu A - \bar{\lambda}\bar{A} = 0, \tag{7}$$

$$-\sigma T + \bar{\lambda} + \delta A + (\bar{\alpha} - \beta)A = 0, \tag{8}$$

$$-\rho T + \mu + \delta\bar{A} - (\bar{\alpha} - \beta)\bar{A} - \bar{\rho}T + \bar{\mu} + \bar{\delta}A - (\alpha - \bar{\beta})A = 0. \tag{9}$$

Similar to the analysis in [15], assuming the asymptotic behaviors of T and A as

$$T = T^0 + \frac{T^1}{r} + \cdots, \qquad A = A^0 + \frac{A^1}{r} + \cdots, \tag{10}$$

we can solve the Killing equations order by order. The stationary condition implies $\dot{\sigma}^0 = 0$. It has been found that the Maxwell field does not change the lowest two powers of $1/r$ in the Killing equations. So the constant terms in the Killing equations yield the same result as in the vacuum case, i.e., $T^0 = \frac{1}{2}$, $\dot{T}^1 = 0$, $\dot{A}^1 = 0$. The coefficients of r^{-1} in the Killing equations give rise to $\sigma^0 = 0$, $A^1 = 0$, $\dot{T}^2 = 0$, $\Psi_2^0 = \bar{\Psi}_2^0$, $T^1 = \frac{1}{2}(\Psi_2^0 + \bar{\Psi}_2^0)$, $\dot{A}^2 = -\frac{1}{2}\eth\Psi_2^0 + \frac{1}{2}\delta_0(\Psi_2^0 + \bar{\Psi}_2^0)$, and

$$\dot{\Psi}_2^0 = 0. \tag{11}$$

From the r^{-2} terms in the NP equation

$$\delta\nu - \Delta\mu = \gamma\mu - 2\nu\beta + \bar{\gamma}\mu + \mu^2 + |\lambda|^2 + \Phi_{22}, \tag{12}$$

we find $8\pi|\phi_2^0|^2 = \dot{\Psi}_2^0 = 0$. Hence $\phi_2^0 = 0$. From the r^{-5} terms in the NP equation

$$\delta\rho - \bar{\delta}\sigma = \tau\rho + (\bar{\beta} - 3\alpha)\sigma + (\rho - \bar{\rho})\tau - \Psi_1 + \Phi_{01}, \tag{13}$$

we have

$$\tfrac{1}{6}(\bar{\eth}\Psi_0^0 - 40\pi\phi_0^0\bar{\phi}_1^0) + \tfrac{1}{2}\bar{\eth}\Psi_0^0 = -\tfrac{1}{3}(\bar{\eth}\Psi_0^0 - 40\pi\phi_0^0\bar{\phi}_1^0) + \bar{\eth}\Psi_0^0 - 16\pi\phi_0^0\bar{\phi}_1^0 \tag{14}$$

which implies

$$\phi_0^0\bar{\phi}_1^0 = 0. \tag{15}$$

This equation will play an important role in our proof, which gives $\phi_0^0 = 0$ or $\phi_1^0 = 0$. Now we discuss the two cases, respectively.

(1) $\phi_0^0 = 0$. Consider the Maxwell equations

$$D\phi_1 - \bar{\delta}\phi_0 = -2\alpha\phi_0 + 2\rho\phi_1, \tag{16}$$

$$D\phi_2 - \bar{\delta}\phi_1 = -\lambda\phi_0 + \rho\phi_2. \tag{17}$$

The coefficients of r^{-4} in these equations yield $\phi_1^1 = \phi_2^2 = 0$.

Consider the other two Maxwell equations

$$\delta\phi_1 - \Delta\phi_0 = (\mu - 2\lambda)\phi_0 + 2\tau\phi_1 - \sigma\phi_2, \tag{18}$$

$$\delta\phi_2 - \Delta\phi_1 = -\nu\phi_0 + 2\mu\phi_1 + (\tau - 2\beta)\phi_2. \tag{19}$$

The r^{-2} terms in Eq. (18) and the r^{-3} terms in Eq. (19) yield

$$\dot{\phi}_1^0 = 0 \tag{20}$$

$$\dot{\phi}_0^0 = \eth\phi_1^0 = 0 \tag{21}$$

where "·" denotes $\frac{\partial}{\partial u}$. Combining these two equations, we have $\phi_1^0 = \text{const}$. So from the r^{-3} terms of Eq. (17), we obtain $\phi_2^1 = -\bar{\eth}\phi_1^0 = 0$.

(2) $\phi_1^0 = 0$. Again, from the r^{-3} terms of Eq. (17), we have $\phi_2^1 = -\bar{\eth}\phi_1^0 = 0$.

Thus, in both cases we have $\phi_2^1 = 0$. Note that it is Φ_{ij} instead of ϕ_i that appear in the NP equations. The fact that $\phi_i = O(r^{-3})$ [except $\phi_1 \sim O(r^{-2})$ in case (1)] shows that the presence of the electromagnetic field does not contribute to r^{-1} and r^{-2} terms. The electromagnetic field makes contribution only to order r^{-3} and higher orders in the expansions. Combining these results, we obtain the reduced N-P coefficients

$$\begin{aligned}
\rho &= -\frac{1}{r} + \frac{8\pi\phi_0^0\bar{\phi}_0^0}{3r^5} + O(r^{-6}),\\
\sigma &= -\frac{\Psi_0^0}{2r^4} - \frac{\Psi_0^1}{3r^5} + O(r^{-6}),\\
\alpha &= \frac{\alpha^0}{r} - \frac{\bar{\alpha}^0\bar{\Psi}_0^0}{6r^4} + \frac{\alpha^0 8\pi\phi_0^0\bar{\phi}_0^0 - \bar{\alpha}^0\bar{\Psi}_0^1 - 24\pi(\phi_1^0\bar{\phi}_0^1 + \phi_1^1\bar{\phi}_0^0)}{12r^5} + O(r^{-6}),\\
\beta &= -\frac{\bar{\alpha}^0}{r} - \frac{\Psi_1^0}{2r^3} + \frac{\alpha^0\Psi_0^0 + 2\bar{\eth}\Psi_0^0}{6r^4} - \frac{3\Psi_1^2 + 8\pi\bar{\alpha}^0\phi_0^0\bar{\phi}_0^0 - \alpha^0\Psi_0^1}{12r^5} + O(r^{-6}),\\
\tau &= -\frac{\Psi_1^0}{2r^3} + \frac{\bar{\eth}\Psi_0^0}{3r^4} + \frac{\bar{\eth}\Psi_0^1 - 8\pi\eth(\phi_0^0\bar{\phi}_0^0) - 48\pi(\phi_0^1\bar{\phi}_1^0 + \phi_0^0\bar{\phi}_1^1)}{8r^5} + O(r^{-6}),\\
\lambda &= -\frac{\bar{\Psi}_0^0}{12r^4} - \frac{3\bar{\Psi}_0^0\Psi_2^0 + \bar{\Psi}_0^1 + 48\pi\phi_2^2\bar{\phi}_0^0}{24r^5} + O(r^{-6}),\\
\mu &= -\frac{1}{2r} - \frac{\Psi_2^0}{r^2} + \frac{\bar{\eth}\Psi_1^0 - 16\pi\phi_1^0\bar{\phi}_1^0}{2r^3} - \frac{\bar{\eth}^2\Psi_0^0}{6r^4} - \frac{6\Psi_2^3 + 8\pi\phi_0^0\bar{\phi}_0^0}{24r^5} + O(r^{-6}),\\
\gamma &= -\frac{\Psi_2^0}{2r^2} + \frac{2\bar{\eth}\Psi_1^0 - 48\pi\phi_1^0\bar{\phi}_1^0 + \alpha^0\Psi_1^0 - \bar{\alpha}^0\bar{\Psi}_1^0}{6r^3} - \frac{1}{24}[2(\alpha^0\bar{\eth}\Psi_0^0 - \bar{\alpha}^0\eth\bar{\Psi}_0^0) + 3\bar{\eth}^2\Psi_0^0]r^{-4}\\
&\quad + \frac{1}{20}[\alpha^0 8\pi(\phi_0^0\bar{\phi}_1^1 + \phi_0^1\bar{\phi}_1^0) + \alpha^0\Psi_1^2 - \bar{\alpha}^0 8\pi(\phi_1^0\bar{\phi}_0^1 + \phi_1^1\bar{\phi}_0^0) - \bar{\alpha}^0\bar{\Psi}_1^2 - |\Psi_1^0|^2 - 4\Psi_2^3\\
&\quad - 32\pi(\phi_1^0\bar{\phi}_1^2 + \phi_1^1\bar{\phi}_1^1 + \phi_1^2\bar{\phi}_1^0)]r^{-5} + O(r^{-6}),\\
\nu &= -\frac{1}{12}[\bar{\Psi}_1^0 + 2\bar{\eth}^2\Psi_1^0]r^{-3} + \frac{1}{24}[\eth\bar{\Psi}_0^0 + \bar{\eth}^3\Psi_0^0]r^{-4} - \frac{1}{120}[6\Psi_2^1\bar{\Psi}_1^0 - 8\Psi_2^0\eth\bar{\Psi}_0^0 + 24\pi(\phi_1^0\bar{\phi}_0^1 + \phi_1^1\bar{\phi}_0^0) + 3\bar{\Psi}_1^2\\
&\quad + 24\Psi_3^4 + 192\pi\phi_2^2\bar{\phi}_1^1]r^{-5} + O(r^{-6}),
\end{aligned} \tag{22}$$

and the null tetrad

$$\begin{aligned}
l^a &= \frac{\partial}{\partial r},\\
n^a &= \frac{\partial}{\partial u} + \bigg[-\frac{1}{2} - \frac{\Psi_2^0}{r} + \frac{\bar{\eth}\Psi_1^0 + \eth\bar{\Psi}_1^0 + 64\pi\phi_1^0\bar{\phi}_1^0}{6r^2} - \frac{\bar{\eth}^2\Psi_0^0 + \eth^2\bar{\Psi}_0^0}{24r^3} - \frac{1}{20}(3|\Psi_1^0|^2 + \Psi_2^3 + \bar{\Psi}_2^3\\
&\quad + 16\pi(\phi_1^0\bar{\phi}_1^2 + \phi_1^1\bar{\phi}_1^1 + \phi_1^2\bar{\phi}_1^0)r^{-4} + O(r^{-5})\bigg]\frac{\partial}{\partial r} + \bigg[\frac{1+\zeta\bar{\zeta}}{6\sqrt{2}r^3}\Psi_1^0 - \frac{1+\zeta\bar{\zeta}}{12\sqrt{2}r^4}\bar{\eth}\Psi_0^0 + O(r^{-5})\bigg]\frac{\partial}{\partial\zeta}\\
&\quad + \bigg[\frac{1+\zeta\bar{\zeta}}{6\sqrt{2}r^3}\bar{\Psi}_1^0 - \frac{1+\zeta\bar{\zeta}}{12\sqrt{2}r^4}\eth\bar{\Psi}_0^0 + O(r^{-5})\bigg]\frac{\partial}{\partial\bar{\zeta}},\\
m^a &= \bigg[-\frac{\Psi_1^0}{2r^2} + \frac{\bar{\eth}\Psi_0^0}{6r^3} - \frac{\Psi_1^2 + 8\pi(\phi_0^1\bar{\phi}_1^0 + \phi_0^0\bar{\phi}_1^1)}{12r^4} + O(r^{-5})\bigg]\frac{\partial}{\partial r} + \bigg[\frac{1+\zeta\bar{\zeta}}{6\sqrt{2}r^4}\Psi_0^0 + O(r^{-5})\bigg]\frac{\partial}{\partial\zeta} + \bigg[\frac{1+\zeta\bar{\zeta}}{\sqrt{2}r} + O(r^{-5})\bigg]\frac{\partial}{\partial\bar{\zeta}},
\end{aligned} \tag{23}$$

where $\delta_0 = \frac{(1+\zeta\bar{\zeta})}{\sqrt{2}} \frac{\partial}{\partial\bar{\zeta}}$, $\zeta = e^{i\phi}\cot\frac{\theta}{2}$, $\eth f = (\delta_0 + 2s\bar{\alpha}^0)f$ (s is the spin weight of f). The differential operators $\eth$ and $\bar{\eth}$ are defined in [18,23].

Then the components of the Weyl curvature and the electromagnetic tensor reduce to

$$
\begin{aligned}
\Psi_0 &= \frac{\Psi_0^0}{r^5} + \frac{\Psi_0^1}{r^6} + O(r^{-7}),\\
\Psi_1 &= \frac{\Psi_1^0}{r^4} + \frac{\Psi_1^1}{r^5} + \frac{\Psi_1^2}{r^6} + O(r^{-7}),\\
\Psi_2 &= \frac{\Psi_2^0}{r^3} + \frac{\Psi_2^1}{r^4} + \frac{\Psi_2^2}{r^5} + \frac{\Psi_2^3}{r^6} + O(r^{-7}),\\
\Psi_3 &= \frac{\Psi_3^2}{r^4} + \frac{\Psi_3^3}{r^5} + \frac{\Psi_3^4}{r^6} + O(r^{-7}),\\
\Psi_4 &= \frac{\Psi_4^3}{r^4} + \frac{\Psi_4^4}{r^5} + \frac{\Psi_4^5}{r^6} + O(r^{-7}),\\
\phi_0 &= \frac{\phi_0^0}{r^3} + \frac{\phi_0^1}{r^4} + \frac{\phi_0^2}{r^5} + O(r^{-6}),\\
\phi_1 &= \frac{\phi_1^0}{r^2} + \frac{\phi_1^1}{r^3} + \frac{\phi_1^2}{r^4} + \frac{\phi_1^3}{r^5} + O(r^{-6}),\\
\phi_2 &= \frac{\phi_2^2}{r^3} + \frac{\phi_2^3}{r^4} + \frac{\phi_2^4}{r^5} + O(r^{-6}).
\end{aligned}
\tag{24}
$$

The Bianchi identity takes the form

$$
\begin{aligned}
&\bar{\delta}\Psi_0 - D\Psi_1 + D\Phi_{01} - \delta\Phi_{00}\\
&\quad = 4\alpha\Psi_0 - 4\rho\Psi_1 - 2\tau\Phi_{00} + 2\rho\Phi_{01} + 2\sigma\Phi_{10}.
\end{aligned}
\tag{25}
$$

The coefficient of r^{-6} in Eq. (25) yields $\Psi_1^1 = -\bar{\eth}\Psi_0^0$.

Similarly, the other components of the Bianchi identity and the Maxwell equations lead to

$$
\begin{aligned}
&\phi_1^1 = -\bar{\eth}\phi_0^0, \qquad \phi_1^2 = -\frac{1}{2}\bar{\eth}\phi_0^1,\\
&\phi_1^3 = -\frac{1}{3}\bar{\eth}\phi_0^2 - \frac{1}{2}\bar{\Psi}_1^0\phi_0^0. \qquad \phi_2^2 = \frac{1}{2}\bar{\eth}^2\phi_0^0,\\
&\phi_2^3 = \frac{1}{6}\bar{\eth}^2\phi_0^1, \qquad \phi_2^4 = \frac{1}{12}\bar{\eth}^2\phi_0^2 + \frac{1}{12}\eth\bar{\Psi}_0^0 + \frac{1}{2}\bar{\Psi}_1^0\bar{\eth}\phi_0^0\\
&\Psi_1^1 = -\bar{\eth}\Psi_0^0, \qquad \Psi_1^2 = -\frac{1}{2}\bar{\eth}\Psi_0^1 + 16\pi(\phi_0^0\bar{\phi}_1^1 + \phi_0^1\bar{\phi}_1^0) + 4\pi\eth(\phi_0^0\bar{\phi}_0^0),\\
&\Psi_2^1 = -\bar{\eth}\Psi_1^0 + 16\pi\phi_1^0\bar{\phi}_1^0, \qquad \Psi_2^2 = \frac{1}{2}\bar{\eth}^2\Psi_0^0,\\
&\Psi_2^3 = -\frac{2}{3}|\Psi_1^0|^2 - \frac{1}{3}\bar{\eth}\Psi_1^2 + \frac{16}{9}\pi\eth(\phi_1^0\bar{\phi}_0^1 + \phi_1^1\bar{\phi}_0^0) - \frac{8}{9}\pi\bar{\eth}(\phi_0^0\bar{\phi}_1^1 + \phi_0^1\bar{\phi}_1^0) - \frac{20}{9}\pi\phi_0^0\bar{\phi}_0^0\\
&\qquad + \frac{80}{9}\pi(\phi_1^0\bar{\phi}_1^2 + \phi_1^1\bar{\phi}_1^1 + \phi_1^2\bar{\phi}_1^0) + \frac{8}{9}\pi\frac{\partial}{\partial u}(\phi_0^0\bar{\phi}_0^1 + \phi_0^1\bar{\phi}_0^0),\\
&\Psi_3^2 = \frac{1}{2}\bar{\eth}^2\Psi_1^0, \qquad \Psi_3^3 = -\frac{1}{2}\bar{\Psi}_1^0\Psi_2^0 - \frac{1}{6}\bar{\eth}^3\Psi_0^0,\\
&\Psi_3^4 = -\frac{1}{4}\bar{\eth}\Psi_2^3 + \frac{1}{8}\Psi_2^0\eth\Psi_0^0 + \frac{1}{2}\bar{\Psi}_1^0\bar{\eth}\Psi_1^0 + \frac{1}{12}k\eth(\phi_2^2\bar{\phi}_0^0) - \frac{4}{3}\pi\bar{\eth}(\phi_1^0\bar{\phi}_1^2 + \phi_1^1\bar{\phi}_1^1\phi_1^2\bar{\phi}_1^0) + 4\pi(\phi_2^2\bar{\phi}_1^1 + \phi_2^3\bar{\phi}_1^0)\\
&\qquad + 4\pi(\phi_1^0\bar{\phi}_0^1 + \phi_1^1\bar{\phi}_0^0) + 4\pi\bar{\Psi}_1^0\phi_1^0\bar{\phi}_1^0 + \frac{4}{3}\pi\frac{\partial}{\partial u}(\phi_1^1\bar{\phi}_0^1 + \phi_1^2\bar{\phi}_0^0),\\
&\Psi_4^3 = -\frac{1}{6}\bar{\eth}^3\Psi_1^0, \qquad \Psi_4^4 = -\frac{1}{24}\bar{\eth}^4\Psi_0^0,\\
&\Psi_4^5 = -\frac{1}{5}\bar{\eth}\Psi_3^4 - \frac{8}{5}\pi\bar{\eth}(\phi_2^2\bar{\phi}_1^1 + \phi_2^3\bar{\phi}_1^0) - \frac{1}{5}\bar{\Psi}_1^0\bar{\eth}^2\Psi_1^0 - \frac{1}{20}\Psi_2^0\bar{\Psi}_0^0 + 4\pi(\phi_2^2\bar{\phi}_0^0) + \frac{8}{5}\pi\frac{\partial}{\partial u}(\phi_2^2\bar{\phi}_0^1 + \phi_2^3\bar{\phi}_0^0).
\end{aligned}
\tag{26}
$$

Similarly to the treatment in [15], the r^{-3} terms in the Killing equations lead to $\eth\Psi_1^0 = 0$. Thus, we have

$$\Psi_1^0 = \sum_{m=-1}^{1} B_{m1} Y_{1,m}, \qquad \Psi_2^0 = C. \tag{27}$$

The coefficient of r^{-3} in Eq. (2) gives $\Psi_3^1 = \bar{\delta}\Psi_2^0 = 0$.

In order to find more restrictions on Ψ_0, we need to compute higher order terms of the Killing equations. The terms of order r^{-4} of the Killing equations yield

$$3T^3 + (\gamma^4 + \bar{\gamma}^4) = 0, \tag{28}$$

$$4A^3 = \tfrac{1}{3}\bar{\eth}\Psi_0^0, \tag{29}$$

$$\dot{T}^4 + \tfrac{8}{3}\pi\phi_1^0\bar{\phi}_1^0 = 0, \tag{30}$$

$$\tfrac{1}{2}\Psi_1^0 T^1 - \tau^4 + \bar{\nu}^4 + \dot{A}^4 + (\Psi_2^0 + \bar{\Psi}_2^0)A^2 + 2A^3 - \delta_0 T^3 + \Psi_2^0 A^2 = 0, \tag{31}$$

$$\tfrac{1}{6}\Psi_0^0 + \eth A^3 = 0, \tag{32}$$

$$2T^3 + \mu^4 + \bar{\mu}^4 + \eth\bar{A}^3 + \bar{\eth}A^3 = 0. \tag{33}$$

Equations (29) and (32) imply

$$\Psi_0^0 = \sum_{m=-2}^{2} A_m(u)\,{}_2Y_{2,m}. \tag{34}$$

Equation (27) and $\dot{T}^3 = 0$ (which comes from the r^{-3} terms in the Killing equations) imply that Ψ_0^0 is independent of u.

Combining Eqs. (24), (26), (27), and (34), one finds

$$I^3 - 27J^2 \sim O(r^{-21}). \tag{35}$$

This result holds for a general asymptotically flat stationary space-time. As mentioned in the Introduction, the AASC requires

$$I^3 - 27J^2 \sim O(r^{-22}), \tag{36}$$

which is just one order faster than the falloff rate of a general asymptotically flat space-time. This means that the AASC is a weak requirement and as demonstrated at the end of this section, there exist many asymptotic flat space-times which satisfy this condition.

Our purpose is to calculate the Newman-Penrose constants, which are contained in the coefficients of Ψ_0^1. From the r^{-5} terms in the Killing equations, we have

$$4T^4 + (\gamma^5 + \bar{\gamma}^5) - \tfrac{1}{2}\bar{\Psi}_1^0 A^2 - \tfrac{1}{2}\Psi_1^0\bar{A}^2 = 0, \tag{37}$$

$$\begin{aligned} A^4 &= \tfrac{1}{5}\tau^5 \\ &= \tfrac{1}{40}[\bar{\eth}\Psi_0^1 - 48\pi(\phi_0^0\bar{\phi}_1^1 + \phi_0^1\bar{\phi}_1^0) - 8\pi\eth(\phi_0^0\bar{\phi}_0^0)], \end{aligned} \tag{38}$$

$$\tfrac{1}{8}\Psi_0^1 + \tfrac{3}{8}\Psi_0^0\Psi_2^0 - 2\pi\phi_0^0\bar{\phi}_2^2 - \tfrac{1}{4}(\Psi_1^0)^2 + \eth A^4 = 0, \tag{39}$$

$$-\rho^5 + 2T^4 + (\mu^5 + \bar{\mu}^5) + \tfrac{3}{2}\Psi_1^0\bar{A}^2 + \tfrac{3}{2}\bar{\Psi}_1^0 A^2 + \eth\bar{A}^4 + \bar{\eth}A^4 = 0. \tag{40}$$

Equations (38) and (39) yield:

$$\eth\bar{\eth}\Psi_0^1 + 5\Psi_0^1 = 10(\Psi_1^0)^2 - 15\Psi_0^0\Psi_2^0 + 80\pi\phi_0^0\bar{\phi}_2^2 + 48\pi\eth(\phi_0^0\bar{\phi}_1^1 + \phi_0^1\bar{\phi}_1^0) + 8\pi\eth^2(\phi_0^0\bar{\phi}_0^0). \tag{41}$$

The terms of ϕ_j^i on the right-hand side of Eq. (41) are the contribution from the Maxwell field [15]. To simplify this equation, we need to investigate the electromagnetic field in more detail.

Since the electromagnetic field is stationary, we have $\mathcal{L}_t F_{ab} = 0$, where t^c is the Killing vector. Noting that $\phi_0 = F_{lm}$ and using the expansion of t^c, we have

$$\begin{aligned} \mathcal{L}_t\phi_0 &= \mathcal{L}_t F_{ab} l^a m^b = (Tl^c + n^c + \bar{A}m^c + A\bar{m}^c)\phi_0 \\ &= F_{ab} l^a [t, m]^b + F_{ab} m^b [t, l]^a \\ &= (\gamma + \bar{\gamma} + \bar{A}\,\bar{\tau} + A\tau)\phi_0 \\ &\quad - (\tau + \bar{A}\sigma + A\rho)(\phi_1 - \bar{\phi}_1) \\ &\quad + [T\bar{\varrho} - \mu + \gamma + \bar{\gamma} - A(\bar{\beta} - \alpha)]\phi_0 \\ &\quad + [T\sigma - \bar{\lambda} - A(\bar{\alpha} - \beta)]\bar{\phi}_0 \end{aligned} \tag{42}$$

where $[t, m]^b$ denotes the commutator of t^c and m^b. So we obtain

$$\begin{aligned} &(Tl^c + n^c + \bar{A}m^c + A\bar{m}^c)\phi_0 \\ &= (\gamma + \bar{\gamma} + \bar{A}\,\bar{\tau} + A\tau)\phi_0 - (\tau + \bar{A}\sigma + A\rho)(\phi_1 - \bar{\phi}_1) \\ &\quad + [T\bar{\varrho} - \mu + \gamma + \bar{\gamma} - A(\bar{\beta} - \alpha)]\phi_0 \\ &\quad + [T\sigma - \bar{\lambda} - A(\bar{\alpha} - \beta)]\bar{\phi}_0. \end{aligned} \tag{43}$$

Substituting (23) into (43) yields:

$$\begin{aligned}
&\frac{\partial}{\partial u}\phi_0+\left[-\frac{1}{2}-\frac{\Psi_2^0}{r}+O(r^{-2})\right]\frac{\partial}{\partial r}\phi_0+\left[\frac{1+\zeta\bar{\zeta}}{6\sqrt{2}r^3}\Psi_1^0+O(r^{-4})\right]\frac{\partial}{\partial\zeta}\phi_0+\left[\frac{1+\zeta\bar{\zeta}}{6\sqrt{2}r^3}\bar{\Psi}_1^0+O(r^{-4})\right]\frac{\partial}{\partial\bar{\zeta}}\phi_0+T\frac{\partial}{\partial r}\phi_0\\
&\quad-\bar{A}\left[\frac{\Psi_1^0}{2r^2}+O(r^{-3})\right]\frac{\partial}{\partial r}\phi_0+\bar{A}\left[\frac{1+\zeta\bar{\zeta}}{6\sqrt{2}r^4}\Psi_0^0+O(r^{-5})\right]\frac{\partial}{\partial\zeta}\phi_0+\bar{A}\left[\frac{1+\zeta\bar{\zeta}}{\sqrt{2}r}+O(r^{-5})\right]\frac{\partial}{\partial\bar{\zeta}}\phi_0\\
&\quad-A\left[\frac{\bar{\Psi}_1^0}{2r^2}+O(r^{-3})\right]\frac{\partial}{\partial r}\phi_0+A\left[\frac{1+\zeta\bar{\zeta}}{6\sqrt{2}r^4}\bar{\Psi}_0^0+O(r^{-5})\right]\frac{\partial}{\partial\bar{\zeta}}\phi_0+A\left[\frac{1+\zeta\bar{\zeta}}{\sqrt{2}r}+O(r^{-5})\right]\frac{\partial}{\partial\zeta}\phi_0\\
&=(\gamma+\bar{\gamma}+\bar{A}\,\bar{\tau}+A\tau)\phi_0-(\tau+\bar{A}\sigma+A\rho)(\phi_1-\bar{\phi}_1)+[T\bar{\varrho}-\mu+\gamma+\bar{\gamma}-A(\bar{\beta}-\alpha)]\phi_0\\
&\quad+[T\sigma-\bar{\lambda}-A(\bar{\alpha}-\beta)]\bar{\phi}_0. \qquad (44)
\end{aligned}$$

Again, we compute the ϕ_j^i terms in Eq. (41) in the two cases.

For case (1) $\phi_0^0=0$, computing the coefficient of r^{-5} in Eq. (44), we obtain

$$\dot{\phi}\,_0^2=-3\Psi_2^0\phi_0^0=0. \qquad (45)$$

The coefficient of r^{-5} of Eq. (18) gives

$$\eth\phi_1^1-\dot{\phi}_0^2-2\phi_0^1-3\Psi_2^0\phi_0^0=-\tfrac{1}{2}\phi_0^1-\Psi_1^0\phi_1^0. \qquad (46)$$

Using $\phi_0^0=0$ and $\dot{\phi}_0^2=0$, we get

$$\phi_0^1=\tfrac{2}{3}\phi_1^0\Psi_1^0. \qquad (47)$$

By taking $\eth$ on both sides and using $\eth\Psi_1^0=0$, we have immediately

$$\eth\phi_0^1=\tfrac{2}{3}\phi_1^0\eth\Psi_1^0=0. \qquad (48)$$

Then the ϕ_j^i terms in Eq. (41) become

$$\begin{aligned}
&80\pi\phi_0^0\bar{\phi}_2^2+48\pi\eth(\phi_0^0\bar{\phi}_1^1+\phi_0^1\bar{\phi}_1^0)+8\pi\eth^2(\phi_0^0\bar{\phi}_0^0)\\
&\quad=48\pi\eth(\phi_0^1\bar{\phi}_1^0)=(48\pi\eth\phi_0^1)\bar{\phi}_1^0+\phi_0^1(48\pi\eth\bar{\phi}_1^0)=0,
\end{aligned} \qquad (49)$$

where Eqs. (21) and (48) have been used in the last step.

For case (2) $\phi_1^0=0$, the coefficient of r^{-4} in Eq. (44) leads to

$$0=\dot{\phi}_0^1-3T^0\phi_0^0+\tfrac{3}{2}\phi_0^0=\dot{\phi}_0^1. \qquad (50)$$

Because the spin weight of ϕ_0 is 1, we can expand ϕ_0^0 as $\phi_0^0=\sum_{l=1}^{\infty}\sum_{m=-l}^{l}d_{l,m1}Y_{l,m}$, where $d_{l,m}$ are some constants. The r^{-4} terms in (18) yield

$$\dot{\phi}\,_0^1=-\bar{\eth}\eth\phi_0^0=\frac{1}{2}\sum_{l=1}^{\infty}(l+2)(l-1)\sum_{m=-l}^{l}d_{l,m1}Y_{l,m}. \qquad (51)$$

Combining (50) and (51) and using the fact that spin-weight harmonic function components are linearly independent, we obtain $l=1$. Consequently,

$$\phi_0^0=\sum_{m=-1}^{1}d_{m1}Y_{1,m}, \qquad (52)$$

where d_m are constants. By expanding ϕ_0^0, we find $\eth\phi_0^0=0$. The contribution from the Maxwell field in Eq. (41) then leads to:

$$\begin{aligned}
&80\pi\phi_0^0\bar{\phi}_2^2+48\pi\eth(\phi_0^0\bar{\phi}_1^1)+8\pi\eth^2(\phi_0^0\bar{\phi}_0^0)\\
&=40\pi\phi_0^0\eth^2\bar{\phi}_0^0-48\pi\eth(\phi_0^0\eth\bar{\phi}_0^0)+8\pi\eth(\phi_0^0\eth\bar{\phi}_0^0+\bar{\phi}_0^0\eth\phi_0^0)\\
&=40\pi\phi_0^0\eth^2\bar{\phi}_0^0-48\pi\eth\phi_0^0\eth\bar{\phi}_0^0-48\pi\phi_0^0\eth^2\bar{\phi}_0^0\\
&\quad+8\pi\eth\phi_0^0\eth\bar{\phi}_0^0+8\pi\eth\bar{\phi}_0^0\eth\phi_0^0+8\pi\phi_0^0\eth^2\bar{\phi}_0^0\\
&\quad+8\pi\bar{\phi}_0^0\eth^2\phi_0^0=-32\pi\eth\phi_0^0\eth\bar{\phi}_0^0+8\pi\bar{\phi}_0^0\eth^2\phi_0^0=0
\end{aligned} \qquad (53)$$

where we have used $\phi_1^1=-\bar{\eth}\phi_0^0$ and $\phi_2^2=\frac{1}{2}\bar{\eth}^2\phi_0^0$. Therefore, the electromagnetic field makes no contribution to the equation of Ψ_0^1. So in either case, the equation of Ψ_0^1 reduces to

$$\eth\bar{\eth}\Psi_0^1+5\Psi_0^1=10(\Psi_1^0)^2-15\Psi_0^0\Psi_2^0, \qquad (54)$$

which is exactly the same equation as that in the vacuum case. Then by imposing the AASC, it is shown in [15] that Eq. (54) implies that all the Newman-Penrose constants must be zero. This completes the proof of our theorem.

Remark. The asymptotically algebraically special condition has played an important role in the proof of this paper and in [15]. Obviously, this condition is satisfied by the Kerr-Newman solution. The following arguments show that the AASC is a rather weak condition imposed on a general asymptotically flat space-time. Note that the Kerr-Newman space-time is axisymmetric. Such symmetry is not required in our theorem. From Eq. (27), we can see that Ψ_1^0 contains ${}_1Y_{1,1}$ and ${}_1Y_{1,-1}$ components that do not appear in the Kerr-Newman solution. Simple calculation shows that span$\{{}_1Y_{1,1},{}_1Y_{1,0},{}_1Y_{1,-1}\}$ is not a representative space of $SO(3)$. Thus we cannot cancel such components by a rotation. Based on the characteristic initial value method [24], it is not difficult to construct exact solutions with nonzero B_1 and B_{-1}. Furthermore, the spin-weight components of Ψ_0^k are just the Janis-Newman multipoles of gravitational field [20]. The AASC only gives a restriction between Janis-Newman's dipoles and quadrupoles [15]. Since there is no restriction on higher order multipoles, it is easy to see that there are many solutions which satisfy the conditions of our theorem and are not equivalent to the Kerr-Newman solution.

III. CONCLUDING REMARKS

We have proven that all the NP constants of an asymptotic flat, stationary, asymptotically algebraically special electrovacuum space-time are zero. The Kerr-Newman solution manifestly satisfies all the conditions. So our theorem implies that all the NP constants of the Kerr-Newman solution are zero. This result has been obtained recently [25] by other authors. In the proof of the theorem, we have assumed that the Maxwell field is stationary. If this condition is not imposed, $\dot{\phi}_0^1$ will not be zero. Then Eq. (51) tells us ϕ_0^0 will contain other components of the spin-weight spherical functions. These terms correspond to the Janis-Newman multipole of Maxwell field [20]. In the presence of these terms, the NP constants may not vanish. Last but not least, an interesting issue is to single out the Kerr-Newman solution from solutions which satisfy the conditions of our theorem. From the discussion of the last section, we find that the AASC is not enough to uniquely determine the Kerr-Newman solution. It seems that more restrictions on the Maxwell field are needed. This will be discussed in our future work.

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