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Generalized constraints on the curvature perturbation from primordial black holes

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Primordial black holes (PBHs) can form in the early Universe via the collapse of large density perturbations. There are tight constraints on the abundance of PBHs formed due to their gravitational effects and the consequences of their evaporation. These abundance constraints can be used to constrain the primordial power spectrum, and hence models of inflation, on scales far smaller than those probed by cosmological observations. We compile, and where relevant update, the constraints on the abundance of PBHs before calculating the constraints on the curvature perturbation, taking into account the growth of density perturbations prior to horizon entry. We consider two simple parametrizations of the curvature perturbation spectrum on the scale of interest: constant and power-law. The constraints from PBHs on the amplitude of the power spectrum are typically in the range 10^{-2}–10^{-1} with some scale dependence.

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I. INTRODUCTION

Primordial black holes (PBHs) can form in the early Universe via the collapse of large density perturbations [1,2]. If the density perturbation at horizon entry in a given region exceeds a threshold value, of order unity, then gravity overcomes pressure forces and the region collapses to form a PBH with mass of order the horizon mass. There are a number of limits, spanning a wide range of masses, on the PBH abundance. PBHs with mass $M_{\rm PBH} \lesssim 5 \times 10^{14}$ g will have evaporated by the present day [3,4], and their abundance is constrained by the consequences of the Hawking radiation emitted. More massive PBHs are constrained by their present-day gravitational effects. The resulting limits on the initial mass fraction of PBHs are very tight, $\beta \equiv \rho_{\rm PBH}/\rho_{\rm tot} < \mathcal{O}(10^{-20})$, and can be used to constrain the power spectrum of the primordial density, or curvature, perturbations (see e.g. Ref. [5]).

The power spectrum of the primordial curvature perturbation, $\mathcal{P}_{\mathcal{R}}(k)$, on cosmological scales is now accurately measured by observations of the cosmic microwave background (CMB) [6] and large-scale structure [7,8]. These measurements can be used to constrain, and in some cases exclude, inflation models (c.f. Ref. [9]). Cosmological observations span a relatively small range of scales (comoving wave numbers between $k \sim 1$ Mpc^{-1} and $k \sim 10^{-3}$ Mpc^{-1}), and hence probe a limited region of the inflaton potential. The PBH constraints on the curvature power spectrum are fairly weak; the upper limit is many orders of magnitude larger than the measurements on cosmological scales. They do, however, apply over a very wide range of scales (from $k \sim 10^{-2}$ Mpc^{-1} to $k \sim 10^{23}$ Mpc^{-1}) and therefore provide a useful constraint on models of inflation [10].

The simplest assumption for the power spectrum is a scale-free power law with constant spectral index, n:

$$\mathcal{P}_{\mathcal{R}}(k) \equiv \frac{k^3}{2\pi^2}\langle|\mathcal{R}_k|^2\rangle = \mathcal{P}_{\mathcal{R}}(k_0)\left(\frac{k}{k_0}\right)^{n-1}, \qquad (1)$$

where k_0 is a suitably chosen normalization scale. In this case the PBH abundance constraints require $n < 1.25$–1.30 [11–14]. The spectral index on cosmological scales is, however, now accurately measured: $n = 0.963^{+0.014}_{-0.015}$ [6]. In other words, if the power spectrum is a pure power law then the number of PBHs formed will be completely negligible. However, as we will now outline, if the primordial perturbations are produced by inflation then the power spectrum is not expected to be an exact power law over all scales.

The power spectrum produced by slow-roll inflation can be written as an expansion about a wave number k_0 (e.g. Ref. [15])

$$\ln\mathcal{P}_{\mathcal{R}}(k) \approx \ln\mathcal{P}_{\mathcal{R}}(k_0) + [n(k_0) - 1]\ln\left(\frac{k}{k_0}\right) + \frac{1}{2}\alpha(k_0)\ln^2\left(\frac{k}{k_0}\right) + \ldots, \qquad (2)$$

where the spectral index and its running, $\alpha(k) \equiv d\ln n/d\ln k$, are evaluated at k_0 and can be expressed in terms of the slow-roll parameters. This expansion is valid provided $\ln(k/k_0)$ is small for the relevant k values. This is the case for cosmological observations, but not for the wide range of scales probed by PBH constraints.

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In fact only very specific, and contrived, inflaton potentials produce a constant spectral index [16]. It is possible for the power spectrum to vary sufficiently with scale so that PBHs can potentially be overproduced. For instance in the running-mass inflation model [17,18], the power spectrum is strongly scale-dependent and PBH constraints exclude otherwise viable regions of parameter space [19,20]. More generally, Peiris and Easther [21] recently found, using slow-roll reconstruction, that inflation models which are consistent with cosmological data can overproduce PBHs.

Motivated by this, we compile and update the constraints on the abundance of PBHs (Sec. II). We then translate the abundance constraints into detailed generalized constraints on the power spectrum of the curvature perturbations (Sec. III), taking into account the evolution of the density perturbations prior to horizon entry.

II. PBH ABUNDANCE CONSTRAINTS

The PBH constraints can, broadly, be split into two classes: those that arise from their present-day gravitational consequences and those that arise from the products of their evaporation. In both cases, in order to constrain the primordial density or curvature perturbation, we need to translate the constraints into limits on the initial PBH mass fraction.

Throughout we will assume that the PBHs form at a single epoch and their mass is a fixed fraction, f_M, of the horizon mass $M_{\rm PBH} = f_M M_{\rm H}$, where $f_M \approx (1/3)^{3/2}$ [22]. A scale-invariant power spectrum produces an extended PBH mass function $dn_{\rm PBH}/dM_{\rm PBH} \propto M_{\rm PBH}^{-5/2}$ [2,23], however (as discussed in the Introduction above) in this case the number density of PBHs would be completely negligible [10,24]. For scale-dependent power spectra which produce an interesting PBH abundance it can be assumed that all PBHs form at a single epoch [25]. As a consequence of near critical phenomena in gravitational collapse [26–28] the PBH mass may, however, depend on the size of the fluctuation from which it forms [29–31] in which case the mass function has finite width. Most of the constraints that we discuss below effectively apply to the mass function integrated over a range of masses. The range of applicability is usually significantly larger than the width of the mass function produced by critical collapse, so in the absence of a concrete prediction or model for the primordial power spectrum in most cases it is reasonable to approximate the mass function as a delta-function. The constraints from cosmic-rays and gamma-rays produced by recently evaporating PBHs are an exception to this. These constraints depend significantly on the PBH mass function and therefore need to be calculated on a case by case basis [32–36]. We therefore do not include these constraints in our calculation of generalized constraints on the curvature perturbation power spectrum.

Taking into account the cosmological expansion, the initial PBH mass fraction, $\beta(M_{\rm PBH})$, is related to the present-day PBH density, $\Omega^0_{\rm PBH}$, by

$$\beta(M_{\rm PBH}) \equiv \frac{\rho^{\rm i}_{\rm PBH}}{\rho^{\rm i}_{\rm crit}} = \frac{\rho^{\rm eq}_{\rm PBH}}{\rho^{\rm eq}_{\rm crit}}\left(\frac{a_{\rm i}}{a_{\rm eq}}\right) \approx \Omega^0_{\rm PBH}\left(\frac{a_{\rm i}}{a_{\rm eq}}\right), \tag{3}$$

where a is the scale factor, "eq" refers to matter-radiation equality and $\rho_{\rm crit}$ is the critical energy density. Using the constancy of the entropy, ($s = g_{*s} a^3 T^3$), where $g_{\star s}$ refers to the number of entropy degrees of freedom, we relate the scale factor to the temperature of the Universe and using the radiation density, $\rho = \frac{\pi^2}{30} g_\star T^4$, and horizon mass, $M_{\rm H} = \frac{4\pi}{3}\rho H^{-3}$, we obtain

$$\beta(M_{\rm PBH}) = \Omega^0_{\rm PBH}\left(\frac{g^{\rm eq}_\star}{g^{\rm i}_\star}\right)^{1/12}\left(\frac{M_{\rm H}}{M^{\rm eq}_{\rm H}}\right)^{1/2}, \tag{4}$$

where $g_\star$ is the total number of effectively massless degrees of freedom and we have taken $g_{\star s} \approx g_\star$. The horizon mass at matter-radiation equality is given by (c.f. Ref. [37])

$$M^{\rm eq}_{\rm H} = \frac{4\pi}{3}\rho_{\rm eq} H^{-3}_{\rm eq} = \frac{8\pi}{3}\frac{\rho^0_{\rm rad}}{a_{\rm eq} k^3_{\rm eq}}. \tag{5}$$

Inserting numerical values, $g^{\rm i}_\star \approx 100$, $g^{\rm eq}_\star \approx 3$, $\Omega^0_{\rm rad}h^2 = 4.17 \times 10^{-5}$, $\rho_{\rm crit} = 1.88 \times 10^{-29} h^2\ {\rm g\,cm^{-3}}$, $k_{\rm eq} = 0.07\Omega^0_{\rm m} h^2\ {\rm Mpc^{-1}}$, $a^{-1}_{\rm eq} = 24000\Omega^0_{\rm m} h^2$ and $\Omega^0_{\rm m} h^2 = 0.1326 \pm 0.0063$ [6] gives $M^{\rm eq}_{\rm H} = 1.3 \times 10^{49} (\Omega_{\rm m} h^2)^{-2}$ g so that

$$\beta(M_{\rm PBH}) = 6.4 \times 10^{-19}\Omega^0_{\rm PBH}\left(\frac{M_{\rm PBH}}{f_M 5 \times 10^{14}\ {\rm g}}\right)^{1/2}. \tag{6}$$

As first shown by Hawking [38] a black hole with mass $M_{\rm PBH}$ emits thermal radiation with temperature

$$T_{\rm BH} = \frac{1}{8\pi G M_{\rm PBH}} \approx 1.06\left(\frac{10^{13}\ {\rm g}}{M_{\rm PBH}}\right)\ {\rm GeV}. \tag{7}$$

The current understanding of PBH evaporation [39] is that PBHs directly emit all particles which appear elementary at the energy scale of the PBH and have rest mass less than the black hole temperature. Thus if the black hole temperature exceeds the QCD confinement scale, quark and gluon jets are emitted directly. The quark and gluon jets then fragment and decay producing astrophysically stable particles: photons, neutrinos, electrons, protons and their antiparticles. Taking into account the number of emitted species the mass loss rate can be written as [40]

$$\frac{dM_{\rm PBH}}{dt} = -5.34 \times 10^{25}\phi(M_{\rm PBH}) M^{-2}_{\rm PBH}\ {\rm g\,s^{-1}}, \tag{8}$$

where $\phi(M_{\rm PBH})$ takes into account the number of directly emitted species ($\phi(M_{\rm PBH}) = 0.267 g_0 + 0.147 g_{1/2} + 0.06 g_1 + 0.02 g_{3/2} + 0.007 g_2$ where g_s is the number of degrees of freedom with spin s) and is normalized to one for PBHs with mass $M_{\rm PBH} \gg 10^{17}$ g which can only emit photons and neutrinos. For lighter PBHs $\phi(5\times10^{14}\ \mathrm{g} < M_{\mathrm{PBH}} < 10^{17}\ \mathrm{g}) = 1.569$. The PBH lifetime is then given by [40]

$$\tau \approx 6.24\times10^{-27} M_{\mathrm{PBH}}^3 \phi(M_{\mathrm{PBH}})^{-1}\ \mathrm{s}. \tag{9}$$

From the Wilkinson Microwave Anisotropy Probe (WMAP) 5 yr data [6] the present age of the Universe is $t_0 = 13.69 \pm 0.13$ Gyr. The initial mass of a PBH which is evaporating today is therefore $M_{\mathrm{PBH}} \approx 5\times10^{14}$ g [4], while less massive PBHs will have evaporated by the present day.

We will now compile, and where relevant update, the PBH abundance constraints. We divide the constraints into two classes: those for PBHs with $M_{\mathrm{PBH}} > 5\times10^{14}$ g, arising from their gravitational consequences (Sec. II A) and those for $M_{\mathrm{PBH}} < 5\times10^{14}$ g arising from their evaporation (Sec. II B).

A. Gravitational constraints

1. Present-day density

The present-day density of PBHs with $M_{\mathrm{PBH}} > 5\times10^{14}$ g which have not evaporated by today must be less than the upper limit on the present-day cold dark matter (CDM) density. Using the 5 yr WMAP measurements [6], $\Omega^0_{\mathrm{CDM}}h^2 = 0.1099 \pm 0.0062$, $h = 0.719^{+0.026}_{-0.027}$, gives

$$\Omega^0_{\mathrm{PBH}} < 0.25, \tag{10}$$

which, using Eq. (6), leads to

$$\beta(M_{\mathrm{PBH}}) < 1.6\times10^{-19}\left(\frac{M_{\mathrm{PBH}}}{f_M 5\times10^{14}\ \mathrm{g}}\right)^{1/2} \quad \text{for } M_{\mathrm{PBH}} > 5\times10^{14}\ \mathrm{g}. \tag{11}$$

2. Lensing of cosmological sources

If there is a cosmologically significant density of compact objects then the probability that a distant point source is lensed is high [41]. The limits as given below have been calculated assuming an Einstein de Sitter Universe, $\Omega_{\mathrm{m}} = 1$, and a uniform density of compact objects. The recalculation of the constraints for a Λ dominated Universe would be nontrivial. The constraints would, however, be tighter (due to the increased path length and the larger optical depth to a given redshift) [42], and the constraints given below are therefore conservative and valid to within a factor of order unity.

Gamma-ray bursts.—Light compact objects can femtolens gamma-ray bursts (GRBs), producing a characteristic interference pattern [43]. A null search using BATSE data leads to a constraint [44]

$$\Omega_c < 0.2 \quad \text{for } 10^{-16}M_\odot < M_{\mathrm{PBH}} < 10^{-13}M_\odot, \tag{12}$$

where Ω_c is the density of compact objects, assuming a mean GRB redshift of one.

Quasars.—Compact objects with mass $10^{-3}M_\odot < M_{\mathrm{PBH}} < 300M_\odot$ can microlens quasars, amplifying the continuum emission without significantly changing the line emission [45]. Limits on an increase in the number of small equivalent width quasars with redshift lead to the constraint [42]:

$$\Omega_c < 0.2 \quad \text{for } 0.001M_\odot < M_{\mathrm{PBH}} < 60M_\odot, \tag{13}$$

assuming $\Omega_{\mathrm{tot}} = \Omega_c$.

Radio sources.—Massive compact objects, $10^6M_\odot < M_{\mathrm{PBH}} < 10^8M_\odot$, can millilens radios sources producing multiple sources with milliarcsec separation [46]. A null search using 300 compact radio sources places a constraint [47]

$$\Omega_c < 0.013 \quad \text{for } 10^6M_\odot < M_{\mathrm{PBH}} < 10^8M_\odot. \tag{14}$$

3. Halo fraction constraints

There are also constraints from the gravitational consequences of PBHs within the Milky Way halo. They are typically expressed in terms of the fraction of the mass of the Milky Way halo in compact objects, $f_{\mathrm{h}} = M^{\mathrm{MW}}_{\mathrm{PBH}}/M^{\mathrm{MW}}_{\mathrm{tot}}$. They require some modeling of the Milky Way halo (typically the density and/or velocity distribution of the halo objects). Consequently there is a factor of a few uncertainty in the precise values of the constraints.

Assuming that PBHs make up the same fraction of the halo dark matter as they do of the cosmological cold dark matter, and ignoring the uncertainties in the CDM density (since this is negligible compared with the uncertainties in halo fraction limit calculations), we can relate the halo fraction to the PBH cosmological density:

$$f_h \equiv \frac{M^{\mathrm{MW}}_{\mathrm{PBH}}}{M^{\mathrm{MW}}_{\mathrm{CDM}}} \approx \frac{\rho^0_{\mathrm{PBH}}}{\rho^0_{\mathrm{CDM}}} = \frac{\Omega^0_{\mathrm{PBH}}h^2}{\Omega^0_{\mathrm{CDM}}h^2} \approx 5\Omega^0_{\mathrm{PBH}}. \tag{15}$$

Microlensing.—Solar and planetary mass compact objects in the Milky Way halo can microlens stars in the Magellanic Clouds, causing temporary one-off brightening of the microlensed star [48]. The relationship between the observed optical depth, τ, (the probability that a given star is amplified by more than a factor of 1.34) and the fraction of the halo in MACHOs depends on the distribution of MACHOs in the Milky Way halo. For the "standard" halo model used by the microlensing community (a spherical cored isothermal sphere) $\tau \approx 5\times10^{-7}f_h$ [49,50], with the derived value of limits on f_h varying by factors of order unity for other halo models.

The EROS Collaboration find a 95% upper confidence limit $\tau < 0.36\times10^{-7}$ which they translate into limits on the halo fraction [51]:

$$f_h < 0.04 \quad \text{for } 10^{-3}M_\odot < M_{\mathrm{PBH}} < 10^{-1}M_\odot, \tag{16}$$

or

$$f_h < 0.1 \quad \text{for } 10^{-6}M_\odot < M_{\rm PBH} < M_\odot. \tag{17}$$

Combined EROS and MACHO Collaboration limits on short duration events constrain the abundance of light MACHOs [52]

$$f_h < 0.25 \quad \text{for } 10^{-7}M_\odot < M_{\rm PBH} < 10^{-3}M_\odot, \tag{18}$$

while a dedicated search by the MACHO Collaboration for long (> 150 days) duration events leads to limits on more massive MACHOs [53]:

$$f_h < 1.0 \quad \text{for } 0.3M_\odot < M_{\rm PBH} < 30M_\odot, \tag{19}$$

or

$$f_h < 0.4 \quad \text{for } M_{\rm PBH} < 10M_\odot. \tag{20}$$

Combined, these limits give

$$f_h < 0.25 \quad \text{for } 10^{-7}M_\odot < M_{\rm PBH} < 10^{-6}M_\odot, \tag{21}$$

$$f_h < 0.1 \quad \text{for } 10^{-6}M_\odot < M_{\rm PBH} < M_\odot, \tag{22}$$

$$f_h < 0.4 \quad \text{for } M_\odot < M_{\rm PBH} < 10M_\odot. \tag{23}$$

Wide binary disruption.—More massive compact objects would affect the orbital parameters of wide binaries [54,55]. Comparison of the separations of observed halo binaries with simulations of encounters between compact objects and wide binaries lead to a constraint [56]

$$f_h < 0.2 \quad \text{for } 10^3M_\odot < M_{\rm PBH} < 10^8M_\odot. \tag{24}$$

See however Ref. [57] for a recent reexamination of this constraint which leads to a somewhat weaker limit.

Disk heating.—Massive halo objects traversing the Galactic disk will heat the disk, increasing the velocity dispersion of the disk stars [58]. This leads to a limit, from the observed stellar velocity dispersions, on the halo fraction in massive objects [59]

$$f_h < \frac{M_{\rm disk,lim}}{M_{\rm PBH}},$$
$$M_{\rm disk,lim} = 3\times 10^6\left(\frac{\rho_h}{0.01M_\odot\ {\rm pc}^{-3}}\right)^{-1}\left(\frac{\sigma_{\rm obs}}{60\ {\rm km\,s}^{-1}}\right)^2 \times\left(\frac{t_{\rm s}}{10^{10}\ {\rm yr}}\right)^{-1}M_\odot, \tag{25}$$

where ρ_h is the local halo density and $\sigma_{\rm obs}$ and $t_{\rm s}$ are the velocity dispersion and age of the halo stars, respectively.

B. Evaporation constraints

1. Diffuse gamma-ray background

PBHs with masses in the range $2\times 10^{13} < M_{\rm PBH} < 5\times 10^{14}$ g evaporate between $z \approx 700$ and the present day and can contribute to the diffuse gamma-ray background [23,24,32,60–62]. As discussed above, these constraints depend significantly on the PBH mass function and hence we will not consider them further.

2. Cosmic-rays

The abundance of PBHs evaporating around the present day can also be constrained by limits on the abundance of cosmic-rays (in particular positrons and antiprotons) [23,63]. The constraints from antiprotons have been calculated for several mass functions and are essentially equivalent to those from the diffuse gamma-ray background [35,64].

3. Neutrinos

Neutrinos produced by PBH evaporation contribute to the diffuse neutrino background. The neutrino spectrum, and hence the resulting PBH abundance constraints, depend strongly on the PBH mass function, but the constraints are typically weaker than those from the diffuse gamma-ray background [33,34].

4. Hadron injection

PBHs with mass $M_{\rm PBH} < 10^{10}$ g have a lifetime $\tau < 10^3$ s and evaporate before the end of nucleosynthesis, and can therefore affect the light element abundances [65–68]. The constraints from hadron injection have been reevaluated (see Ref. [69]), taking into account the emission of fundamental particles and using more up-to-date measurements of the Deuterium and ^{4}He abundances ($D/H \le 4.0\times 10^{-5}$, $Y_p \le 0.252$ respectively):

$$\beta(M_{\rm PBH}) < 10^{-20} \quad \text{for } 10^8\ {\rm g} < M_{\rm PBH} < 10^{10}\ {\rm g}, \tag{26}$$

$$\beta(M_{\rm PBH}) < 10^{-22} \quad \text{for } 10^{10}\ {\rm g} < M_{\rm PBH} < 3\times 10^{10}\ {\rm g}. \tag{27}$$

5. Photodissociation of deuterium

The photons produced by PBHs which evaporate between the end of nucleosynthesis and recombination can photodissociate deuterium [70]. The resulting constraints on the PBH abundance have been updated, in the context of braneworld cosmology in Ref. [71]. Adapting that calculation to the standard cosmology we find:

$$\beta(M_{\rm PBH}) < 3\times 10^{-22}\left(\frac{M_{\rm PBH}}{f_M 10^{10}\ {\rm g}}\right)^{1/2} \quad \text{for } 10^{10}\ {\rm g} < M_{\rm PBH} < 10^{13}\ {\rm g}. \tag{28}$$

6. CMB distortion

Photons emitted by PBHs which evaporate between $z \sim 10^6$ and recombination at $z \sim 10^3$ can produce distortions in the cosmic microwave background radiation [72]. Using the COBE/FIRAS limits on spectral distortions of the CMB from a black body spectrum [73], Ref. [74] finds

$$\beta(M_{\rm PBH}) < 10^{-21} \quad \text{for } 10^{11}\ {\rm g} < M_{\rm PBH} < 10^{13}\ {\rm g}. \tag{29}$$

TABLE I. Summary of constraints on the initial PBH abundance, $\beta(M_{PBH})$.

Description	Mass range	Constraint on $\beta(M_{PBH})$
	Gravitational constraints	
Present-day PBH density	$M_{PBH} > 5\times10^{14}$ g	$<2\times10^{-19}(\frac{M_{PBH}}{f_M 5\times10^{14}\text{ g}})^{1/2}$
GRB femtolensing	$10^{-16}M_\odot < M_{PBH} < 10^{-13}M_\odot$	$<1\times10^{-19}(\frac{M_{PBH}}{f_M 5\times10^{14}\text{ g}})^{1/2}$
Quasar microlensing	$0.001M_\odot < M_{PBH} < 60M_\odot$	$<1\times10^{-19}(\frac{M_{PBH}}{f_M 5\times10^{14}\text{ g}})^{1/2}$
Radio source microlensing	$10^6M_\odot < M_{PBH} < 10^8M_\odot$	$<6\times10^{-20}(\frac{M_{PBH}}{f_M 5\times10^{14}\text{ g}})^{1/2}$
	Halo density[a]	
LMC Microlensing	$10^{-7}M_\odot < M_{PBH} < 10^{-6}M_\odot$	$<3\times10^{-20}(\frac{M_{PBH}}{f_M 5\times10^{14}\text{ g}})^{1/2}$
	$10^{-6}M_\odot < M_{PBH} < M_\odot$	$<1\times10^{-20}(\frac{M_{PBH}}{f_M 5\times10^{14}\text{ g}})^{1/2}$
	$M_\odot < M_{PBH} < 10M_\odot$	$<5\times10^{-20}(\frac{M_{PBH}}{f_M 5\times10^{14}\text{ g}})^{1/2}$
Wide binary disruption	$10^3M_\odot < M_{PBH} < 10^8M_\odot$	$<3\times10^{-20}(\frac{M_{PBH}}{f_M 5\times10^{14}\text{ g}})^{1/2}$
Disk heating	$M_{PBH} > 3\times10^6M_\odot$	$<2\times10^6\frac{1}{f_M^{1/2}}(\frac{M_{PBH}}{5\times10^{14}\text{ g}})^{-1/2}$
	Evaporation	
Diffuse gamma-ray background	$2\times10^{13}\text{ g} < M_{PBH} < 5\times10^{14}$ g	*depends on* PBH *mass function*
Cosmic-rays	similar to DGRB	*depends on* PBH *mass function*
Neutrinos	similar to DGRB	*depends on* PBH *mass function*
Hadron injection	$10^8\text{ g} < M_{PBH} < 10^{10}$ g	$<10^{-20}$
	$10^{10}\text{ g} < M_{PBH} < 3\times10^{10}$ g	$<10^{-22}$
Photodissociation of deuterium	$10^{10}\text{ g} < M_{PBH} < 10^{13}$ g	$<3\times10^{-22}(\frac{M_{PBH}}{f_M 10^{10}\text{ g}})^{1/2}$
CMB distortion	$10^{11}\text{ g} < M_{PBH} < 10^{13}$ g	$<10^{-21}$
(Quasi-)stable massive particles[b]	$M_{PBH} < 10^{11}$ g	$<\sim10^{-18}(\frac{f_M M_{PBH}}{10^{11}\text{ g}})^{-1/2}$
Present-day relic density[c]	$M_{PBH} < 5\times10^{14}$ g	$<4\frac{1}{f_M^{1/2}f_{rel}}(\frac{M_{PBH}}{5\times10^{14}\text{ g}})^{3/2}$

[a]These constraints depend on the PBH distribution within the Milky Way halo and hence have a factor of order a few uncertainty.
[b]Conservative summary, depends on physics beyond the standard model of particle physics.
[c]Only applies if evaporation leaves stable relic.

7. (Quasi-)stable massive particles

In extensions of the standard model there are generically stable or long-lived massive ($\mathcal{O}(100\text{ GeV})$) particles. Light PBHs with mass $M_{PBH} \lesssim 10^{11}$ g can emit these particles and their abundance is hence limited by the present-day abundance of stable massive particles [75] and the decay of long-lived particles [76,77].[1]

Gravitinos in supergravity theories and moduli in string theories are generically quasistable and decay after big bang nucleosynthesis, potentially altering the light element abundances. The effect of their decay on the products of big bang nucleosynthesis leads to a constraint on the initial PBH fraction [76]:

$$\beta(M_{PBH}) < 5\times10^{-19}\left(\frac{g_\star^{i}}{200}\right)^{1/4}\left(\frac{\alpha}{3}\right)\left(\frac{x_\phi}{6\times10^{-3}}\right)^{-1} \times\left(\frac{f_M M_{PBH}}{10^9\text{ g}}\right)^{-1/2}\left(\frac{\bar{Y}_\phi}{10^{-14}}\right) \quad \text{for } M_{PBH} < 10^9\text{ g}, \tag{30}$$

where x_ϕ is the fraction of the luminosity going into quasistable massive particles, $g_\star^{i}$ is the initial number of degrees of freedom (taking into account supersymmetric particles), α is the mean energy of the particles emitted in units of the PBH temperature and $\bar{Y}_\phi$ is the limit on the quasistable massive particle number density to entropy density ratio.

In supersymmetric models, in order to avoid the decay of the proton, there is often a conserved quantum number R-parity, which renders the Lightest Supersymmetric Particle (LSP) stable, and the present-day density of such stable particles produced via PBH evaporation must not exceed the upper limit on the present-day CDM density [75]. This leads to a constraint on the initial PBH fraction (c.f. Ref. [76]):

$$\beta(M_{PBH}) < 6\times10^{-19}h^2\left(\frac{g_\star^{i}}{200}\right)^{1/4}\left(\frac{\alpha}{3}\right)\left(\frac{x_{LSP}}{0.6}\right)^{-1} \times\left(\frac{f_M M_{PBH}}{10^{11}\text{ g}}\right)^{-1/2}\left(\frac{m_{LSP}}{100\text{ GeV}}\right)^{-1} \text{for } M_{PBH} < 10^{11}\text{ g}\left(\frac{100\text{ GeV}}{m_{LSP}}\right), \tag{31}$$

[1]More massive PBHs can also emit these particles in the late stages of their evaporation, when their mass drops below $\sim10^9$ g. However the resulting constraints are substantially weaker than those from hadron injection during nucleosynthesis.

where $m_{\rm LSP}$ is the mass of the LSP and $x_{\rm LSP}$ is the fraction of the luminosity carried away by the LSP.

These constraints depend on the (uncertain) details of physics beyond the standard model, and we therefore summarize them conservatively as

$$\beta(M_{\rm PBH}) \lesssim 10^{-18}\left(\frac{f_M M_{\rm PBH}}{10^{11}\ {\rm g}}\right)^{-1/2} \quad \text{for } M_{\rm PBH} < 10^{11}\ {\rm g}. \tag{32}$$

8. *Present-day relic density*

It has been argued that black hole evaporation could leave a stable Planck mass relic [78–80], in which case the present-day density of relics must not exceed the upper limit on the CDM density

$$\Omega^0_{\rm rel} < 0.25. \tag{33}$$

Writing the relic mass as $M_{\rm rel} = f_{\rm rel} M_{\rm Pl}$ this gives a tentative constraint

$$\beta(M_{\rm PBH}) < 4\frac{1}{f_M^{1/2} f_{\rm rel}}\left(\frac{M_{\rm PBH}}{5\times 10^{14}\ {\rm g}}\right)^{3/2} \quad \text{for } M_{\rm PBH} < 5\times 10^{14}\ {\rm g}. \tag{34}$$

The constraints are summarized in Table I and are displayed in Fig. 1. As can be seen from Fig. 1, the constraints probe a very large range of scales, and in some cases several constraints overlap across particular mass ranges. The solid line indicates the strongest constraints for each mass scale, and we consider only these when constraining the primordial power spectrum in Sec. III.

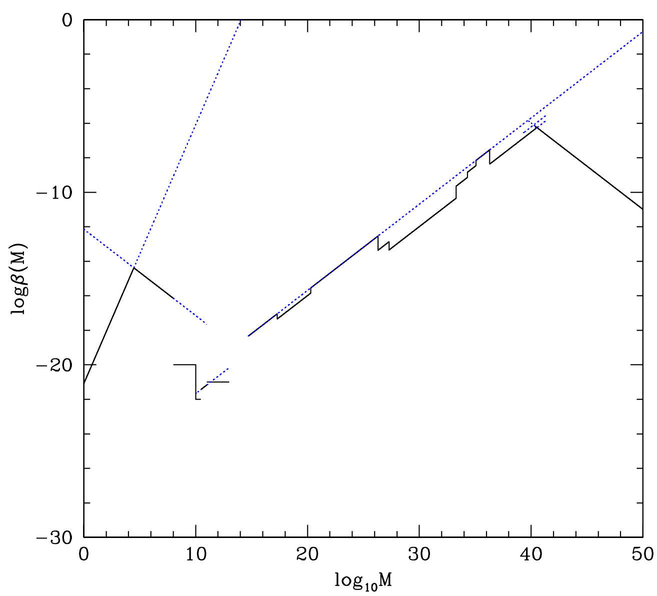

FIG. 1 (color online). The limits on the initial mass fraction of PBHs as a function of PBH mass (in grams). The solid lines represent the tightest limits for each mass range and the dotted lines are the weaker limits where there is an overlap between constraints. As discussed in Sec. II we have not considered the diffuse gamma-ray background constraint which applies for $2\times 10^{13}\ {\rm g} < M_{\rm PBH} < 5\times 10^{14}$ g as it depends significantly on the PBH mass function.

III. CONSTRAINTS ON THE CURVATURE PERTURBATION POWER SPECTRUM

We focus in the following on the standard case of PBH formation, which applies to scales which have left the horizon at the end of inflation. It has recently been shown [81,82] that PBHs can also form on scales which never leave the horizon during inflation, and therefore never become classical. We also only consider Gaussian perturbations and a trivial initial radial density profile, and refer to Ref. [83] for the effects of non-Gaussian perturbations and to Refs. [84,85] for estimates on the effect of deviations from a trivial initial density profile.

A region will collapse to form a PBH if the smoothed density contrast, in the comoving gauge, at horizon crossing ($R = (aH)^{-1}$), $\delta_{\rm hor}(R)$, satisfies the condition $\delta_{\rm c} \leq \delta_{\rm hor}(R) \leq 1$ [22], where $\delta_{\rm c} \sim 1/3$. The mass of the PBH formed is approximately equal to the horizon mass at horizon entry, $M_{\rm PBH} = f_M M_{\rm H}$, which is related to the smoothing scale, R, by [37]

$$M_{\rm H} = M_{\rm H}^{\rm eq}(k_{\rm eq}R)^2\left(\frac{g_{\star,}^{\rm eq}}{g_\star}\right)^{1/3}, \tag{35}$$

where $M_{\rm H}^{\rm eq} = 1.3\times 10^{49}(\Omega_{\rm m}h^2)^{-2}$ g is the horizon mass at matter-radiation equality.

Taking the initial perturbations to be Gaussian, the probability distribution of the smoothed density contrast, $P(\delta_{\rm hor}(R))$, is given by (e.g. Ref. [86])

$$P(\delta_{\rm hor}(R)) = \frac{1}{\sqrt{2\pi}\sigma_{\rm hor}(R)}\exp\left(-\frac{\delta^2_{\rm hor}(R)}{2\sigma^2_{\rm hor}(R)}\right), \tag{36}$$

where $\sigma(R)$ is the mass variance

$$\sigma^2(R) = \int_0^\infty W^2(kR)\mathcal{P}_\delta(k,t)\frac{dk}{k}, \tag{37}$$

and $W(kR)$ is the Fourier transform of the window function used to smooth the density contrast. We assume a Gaussian window function for which $W(kR) = \exp(-k^2R^2/2)$.

The fraction of the energy density of the Universe contained in regions dense enough to form PBHs is then given, as in Press-Schechter theory [87] by,

$$\beta(M_{\rm PBH}) = 2\frac{M_{\rm PBH}}{M_H}\int_{\delta_{\rm c}}^1 P(\delta_{\rm hor}(R))d\delta_{\rm hor}(R). \tag{38}$$

This leads to a relationship between the PBH initial mass fraction and the mass variance,

$$\beta(M_{\rm PBH}) = \frac{2f_{\rm M}}{\sqrt{2\pi}\sigma_{\rm hor}(R)}\int_{\delta_{\rm c}}^{1}\exp\left(-\frac{\delta_{\rm hor}^2(R)}{2\sigma_{\rm hor}^2(R)}\right)d\delta_{\rm hor}(R),$$
$$\approx f_M {\rm erfc}\left(\frac{\delta_{\rm c}}{\sqrt{2}\sigma_{\rm hor}(R)}\right). \quad (39)$$

The constraints on the PBH initial mass fraction can therefore be translated into constraints on the mass variance by simply inverting this expression.

In order to calculate the mass variance we need the density contrast in the comoving, or total matter, gauge as a function of time and scale (c.f. Refs. [14,36]). To calculate this we take the expressions for the evolution of perturbations in the conformal Newtonian gauge, valid on both sub- and super-horizon scales, and carry out a gauge transformation to the total matter gauge (for further details see the Appendix). We find

$$\delta(k,t) = -\frac{4}{\sqrt{3}}\left(\frac{k}{aH}\right)j_1(k/\sqrt{3}aH)\mathcal{R}, \quad (40)$$

where j_1 is a spherical Bessel function and $\mathcal{R}$ is the primordial curvature perturbation. Hence the power spectrum of the density contrast is given by

$$\mathcal{P}_{\delta}(k,t) = \frac{16}{3}\left(\frac{k}{aH}\right)^2 j_1^2(k/\sqrt{3}aH)\mathcal{P}_{\mathcal{R}}(k). \quad (41)$$

Substituting this into Eq. (37), and setting $R=(aH)^{-1}$, gives

$$\sigma_{\rm hor}^2(R) = \frac{16}{3}\int_0^\infty (kR)^2 j_1^2(kR/\sqrt{3})\exp(-k^2R^2)\mathcal{P}_{\mathcal{R}}(k)\frac{dk}{k}. \quad (42)$$

Since the integral is dominated by scales $k \sim 1/R$ we assume that, *over the scales probed by a specific PBH abundance constraint*, the curvature power spectrum can be written as a power law

$$\mathcal{P}_{\mathcal{R}}(k) = \mathcal{P}_{\mathcal{R}}(k_0)\left(\frac{k}{k_0}\right)^{n(k_0)-1}. \quad (43)$$

This assumption is valid for general slow-roll inflation models such as those considered in Refs. [19–21]. Using Eqs. (39) and (42) we can translate the PBH abundance constraints in Sec. II into constraints on the amplitude of the curvature perturbation spectrum. For each constraint we take the pivot point, k_0, to correspond to the scale of interest, $k_0 = 1/R$, and consider a range of values for $n(k_0)$ consistent with slow-roll inflation, $0.9 < n(k_0) < 1.1$. The resulting constraints for $n(k_0) = 1$ are displayed in Fig. 2. For $n(k_0) = 0.9$ and 1.1 the constraints are weakened or strengthened, respectively, at the order of 2%. This indicates that, for slow-roll inflation models, the constraints are not particularly sensitive to the exact shape of the power spectrum in the vicinity of the scale of interest. Thelarge-scale constraints (small k) come from various astrophysical sources such as Milky Way disk heating, wide binary disruption and a variety of lensing effects. The small-scale constraints generally arise from the consequences of PBH evaporation, in particular, on nucleosynthesis and the CMB. These evaporation constraints lead to tighter constraints on the abundance of PBHs and therefore the primordial power spectrum is more tightly constrained on these scales. In general the constraints on the amplitude of the primordial power spectrum span the range $\mathcal{P}_{\mathcal{R}} < 10^{-2}$–$10^{-1}$ with some scale dependence.

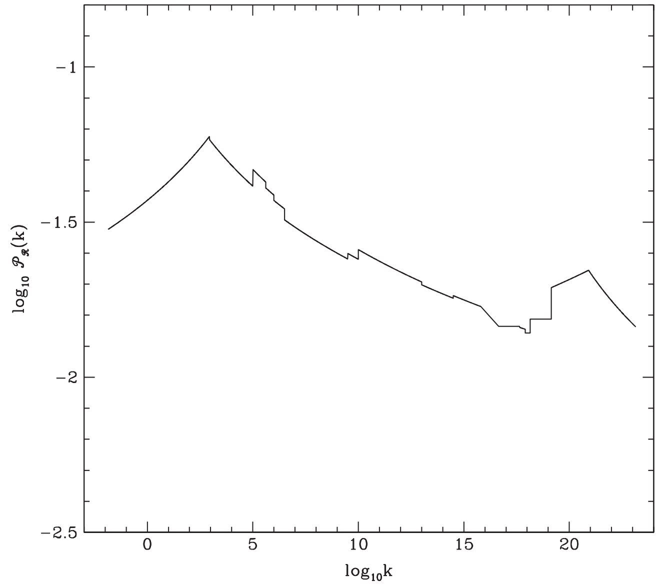

FIG. 2. Generalized constraints on the amplitude of the power spectrum of primordial curvature perturbations as a function of comoving wave number (in units of Mpc^{-1}). We have assumed that the power spectrum is scale-invariant over the (relatively small) range of scales which contributes to a given constraint. Deviations from scale-invariance consistent with slow-roll inflation lead to small changes in the constraints (see text for further details).

IV. SUMMARY

We have compiled, and where relevant updated, the observational limits on the initial abundance of primordial black holes. We then translated these limits into generalized constraints on the power spectrum of the primordial curvature perturbation, taking into account the full time evolution of the density contrast. The constraints on the amplitude of the power spectrum are typically in the range $\mathcal{P}_{\mathcal{R}} < 10^{-2}$–$10^{-1}$ with some scale dependence. This is slightly weaker than the $\mathcal{P}_{\mathcal{R}} < 10^{-3}$–$10^{-2}$ assumed in Ref. [21]. These more accurate generalized constraints could be used to more accurately constrain the parameter space of slow-roll inflation models (c.f. [11–14,19–21]).

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APPENDIX: DENSITY CONTRAST CALCULATION

The general, scalar, perturbed metric can be written as (c.f. [86]):

$$ds^2 = a^2(\tau)\{-(1+2\phi)d\tau^2 + 2B_i^{(s)}d\tau dx^i + [(1-2\psi)\delta_{ij} + 2E_{ij}^{(s)}]dx^i dx^j\}, \tag{A1}$$

where $B_i^{(s)}$ and $E_{ij}^{(s)}$ can be written as

$$B_i^{(s)} = -\frac{ik_i}{k}B, \tag{A2}$$

$$E_{ij}^{(s)} = \left(-\frac{k_i k_j}{k^2} + \frac{1}{3}\delta_{ij}\right)E, \tag{A3}$$

where ϕ, B, ψ, E are arbitrary scalar functions which describe the perturbations on a Friedmann-Robertson-Walker (FRW) background. Performing a first-order coordinate change, $\tilde{\tau} = \tau + \xi^0$, $\tilde{x}^i = x^i + \xi^i$, the scalar metric variables in the new gauge (denoted by a tilde) are then given by

$$\tilde{\phi} = \phi - \xi^{0\prime} - h\xi^0, \tag{A4}$$

$$\tilde{B} = B - \xi' + k\xi^0, \tag{A5}$$

$$\tilde{\psi} = \psi + h\xi^0, \tag{A6}$$

$$\tilde{E} = E - k\xi, \tag{A7}$$

where the scalar part of $\xi^i(\tau, x^i)$ is defined as $\xi^{i(s)} = -\frac{ik^i}{k}\xi$, primes denote derivatives with respect to conformal time τ, and $h = a'/a = aH$. The density contrast and the velocity perturbation transform as

$$\tilde{\delta} = \delta + 3h(1+w)\xi^0, \tag{A8}$$

$$\tilde{V} = V + \xi', \tag{A9}$$

where V is related to the scalar part of the 3-velocity vector, $v^{i(s)} = -\frac{ik^i}{k}V$.

Reference [88] calculated the evolution of the density and velocity perturbations in the conformal Newtonian gauge (which has $B_N = E_N = 0$).

They found that during radiation domination ($w = 1/3$) for a fluid with vanishing anisotropic stress (and hence $\phi_N = \psi_N$), the remaining perturbations evolve according to

$$\phi_N = \frac{j_1(\kappa)}{\sqrt{3}\kappa}C, \tag{A10}$$

$$\delta_N = \frac{2}{\sqrt{3}}\left(2\frac{j_1(\kappa)}{\kappa} - j_0(\kappa) - \kappa j_1(\kappa)\right)C, \tag{A11}$$

$$V_N = \left(j_1(\kappa) - \frac{\kappa}{2}j_0(\kappa)\right)C, \tag{A12}$$

where $\kappa = k/\sqrt{3}aH$ and C is a normalization constant.

For the PBH abundance calculation we need the density perturbation in the total matter gauge (T), which is defined by $B_T + V_T = 0$, and $E_T = 0$. Since the comoving curvature perturbation, $\mathcal{R}$, is identical to the curvature perturbation in the total matter gauge, ψ_T (see e.g. [89]), we get using Eqs. (A4)–(A7),

$$\mathcal{R} = \phi_N - \frac{h}{k}V_N. \tag{A13}$$

Similarly, the density contrast in the total matter gauge (during radiation domination) is

$$\delta_T = \delta_N - 4\frac{h}{k}V_N. \tag{A14}$$

Using the solutions Eqs. (A10) and (A12) above we get

$$\mathcal{R} = \frac{C}{2\sqrt{3}}j_0(\kappa), \tag{A15}$$

which reduces in the small κ (large-scale) limit to $C = 2\sqrt{3}\mathcal{R}$, allowing us to replace the normalization constant C with the large-scale limit of $\mathcal{R}$. Equation (A14) then becomes, using Eqs. (A11) and (A12),

$$\delta_T = -\frac{4}{\sqrt{3}}\left(\frac{k}{aH}\right)j_1(k/\sqrt{3}aH)\mathcal{R}. \tag{A16}$$

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