

Breit-Wigner resonances and the quasinormal modes of anti-de Sitter black holes

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We show that the theory of Breit-Wigner resonances can be used as an efficient numerical tool to compute black hole quasinormal modes. For illustration, we focus on the Schwarzschild anti-de Sitter (SAdS) spacetime. The resonance method is better suited to small SAdS black holes than the traditional series expansion method, allowing us to confirm that the damping time scale of small SAdS black holes for scalar and gravitational fields is proportional to r_+^{-2l-2} , where r_+ is the horizon radius. The proportionality coefficients are in good agreement with analytic calculations. We also examine the eikonal limit of SAdS quasinormal modes, confirming quantitatively Festuccia and Liu's [arXiv:0811.1033] prediction of the existence of very long-lived modes. Our results are particularly relevant for the AdS/CFT correspondence, since long-lived modes presumably dominate the decay time scale of the perturbations.

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I. INTRODUCTION

It is well known that quasibound states manifest themselves as poles in the scattering matrix, and as Breit-Wigner resonances in the scattering amplitude. Chandrasekhar and Ferrari made use of the form of the scattering cross section near these resonances in their study of gravitational-wave scattering by ultracompact stars [1,2]. In geometrical units ($c = G = 1$), the Regge-Wheeler potential $V(r)$ describing odd-parity perturbations of a Schwarzschild black hole (BH) of mass M has a peak at $r \sim 3M$. Constant-density stellar models may have a radius $R/M < 3$ (but still larger than the Buchdal limit, $R/M > 2.25$). When $R/M \lesssim 2.6$, the radial potential describing odd-parity perturbations of the star (which reduces to the Regge-Wheeler potential for $r > R$) displays a local minimum as well as a maximum. If this minimum is sufficiently deep, quasistationary, “trapped” states can exist: gravitational waves can only leak out to infinity by “tunneling” through the potential barrier. Since the damping time of these modes is very long, Chandrasekhar and Ferrari dubbed them “slowly damped” modes [1].

For trapped modes of ultracompact stars, the asymptotic wave amplitude at spatial infinity, $\Psi \sim \alpha \cos \omega r + \beta \sin \omega r$, has a Breit-Wigner-type behavior close to the resonance:

$$\alpha^2 + \beta^2 \approx \text{const}[(\omega - \omega_R)^2 + \omega_I^2], \quad (1.1)$$

where ω_I^{-1} is the lifetime of the quasibound state and ω_R^2 its characteristic “energy.” The example of ultracompact stars shows that the search for weakly damped quasinormal modes (QNMs) corresponding to quasibound states ($\omega = \omega_R - i\omega_I$ with $\omega_I \ll \omega_R$) is extremely simplified. We locate the resonances by looking for minima of $\alpha^2 + \beta^2$ on the $\omega = \omega_R$ line, and the corresponding damping time ω_I can then be obtained by a fit to a parabola around the minimum [1,2].

Here we show that this “resonance method” can be used very successfully in BH spacetimes. The resonance method is particularly valuable in studies of asymptotically anti-de Sitter (AdS) BHs. The QNM spectrum of AdS BHs is related to thermalization time scales in a dual conformal field theory (CFT), according to the AdS/CFT conjecture [3]. Analytic studies of wave scattering in AdS BHs have previously hinted at the existence of resonances (see Fig. 9 in Ref. [4]); here we show that these are indeed Breit-Wigner resonances.

Various analytic calculations recently predicted the existence of long-lived modes in asymptotically AdS BH spacetimes [5–7]. These modes will presumably dominate the BH's response to perturbations, hence the thermaliza-

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tion time scale in the dual CFT. Since their existence may be very relevant for the AdS/CFT conjecture, we decided to investigate numerically these long-lived modes. In Sec. II we confirm the existence of quasibound states for small SAdS BHs, first predicted by Grain and Barrau [5], partially correcting some of their predictions. In Sec. III we reanalyze the eikonal limit of SAdS QNMs studied by Festuccia and Liu [6], finding excellent agreement with their calculations.

It may be useful to point out that a different but intimately related method (the complex angular momentum approach, a close kin of the theory of Regge poles in quantum mechanics) has been used in the past to study QNMs in asymptotically flat BH spacetimes [8–10]. Some aspects of the relation between the resonance method and the theory of Regge poles are illustrated, for example, in Ref. [2].

II. QUASIBOUND STATES IN SADS BLACK HOLES

In this paper we will focus on SAdS BHs in four space-time dimensions, but our results are trivially extended to higher dimensions. Scalar ($s = 0$), electromagnetic ($s = 1$), and vector-type (or Regge-Wheeler) gravitational perturbations ($s = 2$) of SAdS BHs are governed by a second-order differential equation for a master variable Ψ [3,11]:

$$f^2 \frac{d^2 \Psi}{dr^2} + f f' \frac{d\Psi}{dr} + (\omega^2 - V_{l,s}) \Psi = 0, \quad (2.1)$$

$$V_{l,s} = f \left[\frac{l(l+1)}{r^2} + (1-s^2) \left(\frac{2M}{r^3} + \frac{4-s^2}{2L^2} \right) \right], \quad (2.2)$$

where $f = r^2/L^2 + 1 - r_0/r$, L is the AdS radius, r_0 is related to the horizon radius r_+ through $r_0/L = (r_+/L)^3 + r_+/L$, and we assume that the perturbations depend on time as $e^{-i\omega t}$. As usual, we define a “tortoise” coordinate r_* by the relation $dr/dr_* = f$ (so that $r_* \rightarrow -\infty$ as $r \rightarrow r_+$).

The potential for scalar-field perturbations of SAdS BHs is shown in Fig. 1 for different values of r_+/L , ranging from “large” BHs with $r_+/L \sim 10^4$ to “small” BHs with $r_+/L \sim 10^{-4}$. Notice how a potential well of increasing depth and width develops in the small BH limit ($r_+/L \ll 1$).

Close to the horizon, where the potential $V_{l,s} \rightarrow 0$, we require ingoing-wave boundary conditions:

$$\Psi \sim e^{-i\omega r_*}, \quad r_* \rightarrow -\infty (r \rightarrow r_+). \quad (2.3)$$

Near spatial infinity ($r \rightarrow \infty$) the asymptotic behavior is

$$\Psi_{s=0} \sim Ar^{-2} + Br, \quad \Psi_{s=1,2} \sim A/r + B. \quad (2.4)$$

Regular scalar-field perturbations should have $B = 0$, corresponding to Dirichlet boundary conditions at infinity. The case for electromagnetic and gravitational perturbations is less clear, and there are indications that Robin

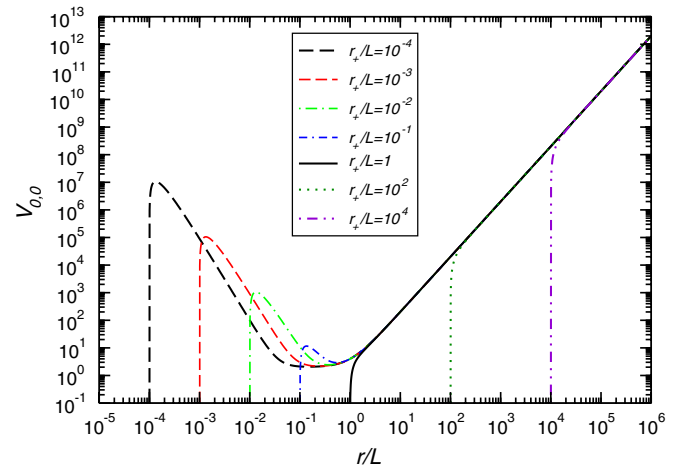


FIG. 1 (color online). Potential for scalar-field ($s = 0$) perturbations of a SAdS background with $l = 0$. Different lines refer to different values of r_+/L . A potential well develops for small BHs ($r_+/L < 1$).

boundary conditions may be more appropriate in the context of the AdS/CFT correspondence [12–15]. With this caveat, most calculations in the literature assume Dirichlet boundary conditions, so we choose to work with those.

In general, a solution with the correct boundary conditions at infinity behaves near the horizon ($r_* \rightarrow -\infty$) as

$$\Psi \sim A_{\text{in}} e^{-i\omega r_*} + A_{\text{out}} e^{i\omega r_*} \sim \alpha \cos \omega r_* + \beta \sin \omega r_*, \quad (2.5)$$

with $\alpha = A_{\text{out}} + A_{\text{in}}$, $\beta = i(A_{\text{out}} - A_{\text{in}})$. For increased numerical accuracy, in our calculations we use a higher-order expansion of the form

$$A_{\text{in}}(1 + a(r - r_+))e^{-i\omega r_*} + A_{\text{out}}(1 + a^*(r - r_+))e^{i\omega r_*}, \quad (2.6)$$

with

$$a = \frac{2l(l+1) + (s^2-1)((s^2-6)r_+^2/L^2 - 2)}{(2r_+/L)(1 + 3r_+^2/L^2 - 2i\omega r_+)}. \quad (2.7)$$

The problem is analogous to axial gravitational-wave scattering by compact stars, as long as we replace the “outgoing-wave boundary condition at infinity” in the stellar case by an “ingoing-wave boundary condition at the horizon” in the SAdS case (compare our Fig. 1 with Fig. 1 in Ref. [1]). Quasibound states for the potential (2.2) should show up as Breit-Wigner resonances of the form (1.1) for *real* ω .

Figure 2 shows a typical plot of $\alpha^2 + \beta^2$ as a function of $\omega R L$. The pronounced dips correspond to the location of a resonance, $\omega = \omega_R$, and the inset shows a zoomed-in view of one such particular resonance. Once a minimum in $\alpha^2 + \beta^2$ is located, the imaginary part ω_I can be found by a parabolic fit of $\alpha^2 + \beta^2$ to the Breit-Wigner expression (1.1). Alternative (but equivalent) expressions are

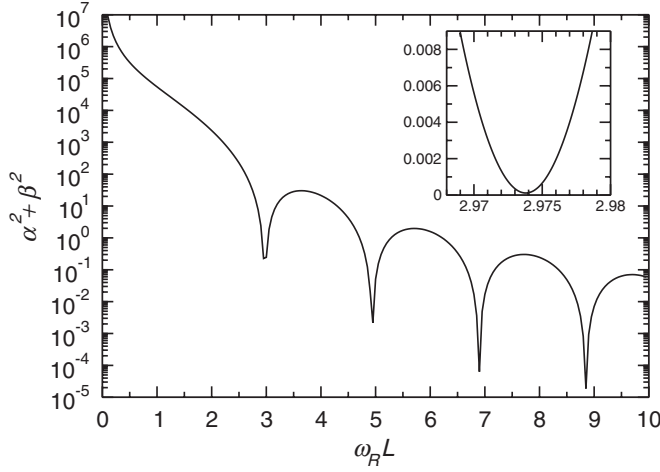


FIG. 2. A plot of $\alpha^2 + \beta^2$ for scalar-field SAdS perturbations with $l = 0$, $r_+/L = 10^{-2}$. Resonances are seen when $\omega_R \approx 3 + 2n$, i.e. close to the resonant frequencies of the pure AdS spacetime. In the inset we show the behavior near the minimum, which allows us to extract the decay time by a parabolic fit.

$\omega_l = -\beta/\alpha' = \alpha/\beta'$, where a prime stands for the derivative with respect to ω_R , evaluated at the minimum [1,2]. We used these three different expressions to estimate numerical errors in the computed quasinormal frequencies. In the next subsections we briefly report on our results for $s = 0$ and $s = 2$, respectively.

A. Scalar-field perturbations

The series solution method presented by Horowitz and Hubeny [3] was used by Konoplya in Ref. [16] to compute quasinormal frequencies of small SAdS BHs. The series has very poor convergence properties for $r_+/L < 1$, and QNM calculations in this regime take considerable computational time. As seen in Fig. 1, the potential for small SAdS BHs is able to sustain quasibound states, so we expect the resonance method to be well adapted to the study of small BHs.

In Table I we list QNMs for $l = 0$ scalar-field perturbations and for different BH sizes, comparing (where possible) results from the resonance method with Konoplya's series expansion calculation. A cubic fit of our data for $L/r_+ > 30$ yields $\omega_l L = 5.00r_+^2/L^2 + 47.70r_+^3/L^3$, a quartic fit yields $\omega_l = 5.09(r_+/L)^2 + 33.59(r_+/L)^3 + 485.09(r_+/L)^4$, and fits with higher-order terms basically leave a and b unchanged with respect to the quartic fit. The numerical results are consistent with Horowitz and Hubeny's prediction that $\omega_l \propto r_+^2$, and they are in very good agreement with analytic predictions by Cardoso and Dias [17], who derived a general expression for the resonant frequencies of small BHs in AdS for the $M\omega_R \ll 1$ regime. Setting $a = 0$ in Eq. (33) of Ref. [17], their result is

$$L\omega = l + 3 + 2n - i\omega_l L, \quad (2.8)$$

TABLE I. The fundamental $l = 0$ QNM frequencies for small SAdS BHs for selected values of r_+/L . The series solution data are taken from Ref. [16]. In the table, $\delta\omega_R L \equiv 3 - \omega_R L$.

L/r_+	Series		Resonance	
	$\delta\omega_R L^2/r_+$	$\omega_l L^3/r_+^2$	$\delta\omega_R L^2/r_+$	$\omega_l L^3/r_+^2$
12	3.064	9.533	2.992	9.662
20	2.922	7.720	2.912	7.840
30	2.805	6.660	2.802	6.714
50			2.700	5.952
100			2.610	5.471
200			2.580	5.266
500			2.560	5.158
1000			2.554	5.125
2000			2.550	5.109
5000			2.550	5.100

where n is a non-negative integer and

$$\omega_l L \approx -\gamma_0 [(l + 3 + 2n)(r_+/L)^{2l+2}]/\pi, \quad (2.9)$$

$$\gamma_0 \equiv \frac{2^{-1-6l}(l!)^2 \Gamma[-l - 1/2]^2 \Gamma[5 + 2l + 2n]}{(3 + 2n)(3 + 2l + 2n) \Gamma[l + 1/2]^2 \Gamma[2 + 2n]}.$$

For $l = n = 0$ one gets $\omega_l = 16(r_+/L)^2/\pi \sim 5.09(r_+/L)^2$, in excellent agreement with the fits. For general l Eq. (2.8) predicts an r_+^{2+2l} dependence, in agreement with our results for $l = 0, 1, 2$. Moreover, we find excellent agreement with the proportionality coefficient predicted by Eq. (2.9). Higher overtones are also well described by Eqs. (2.8) and (2.9).

Our results show that the resonant frequency ω_R always approaches the pure AdS value in the small BH limit, generally confirming the analysis by Grain and Barrau [5]. However, our numerics disagree with Grain and Barrau's semiclassical calculation of the monopole mode ($l = 0$). We find that all modes including the monopole reduce to pure AdS in the small BH limit. More precisely, as $r_+/L \rightarrow 0$ we find

$$\omega_R L = l + 3 + 2n - k_{ln} r_+/L, \quad n = 0, 1, 2, \dots \quad (2.10)$$

with $k_{l0} \sim 2.6, 1.7, 1.3$ for $l = 0, 1, 2$, respectively.

Our results for scalar-field perturbations are visually summarized in Fig. 3, where we combine results from the resonance method and from the series solution (compare e.g. Fig. 1 of Ref. [18]). Modes with different l coalesce in the large BH regime (top right in the plot), as long as $l \ll r_+/L$. As shown in Fig. 1, the potential for small SAdS BHs develops a well capable of sustaining quasistationary, long-lived modes. It should not be surprising that small and large BH QNMs have such a qualitatively different behavior.

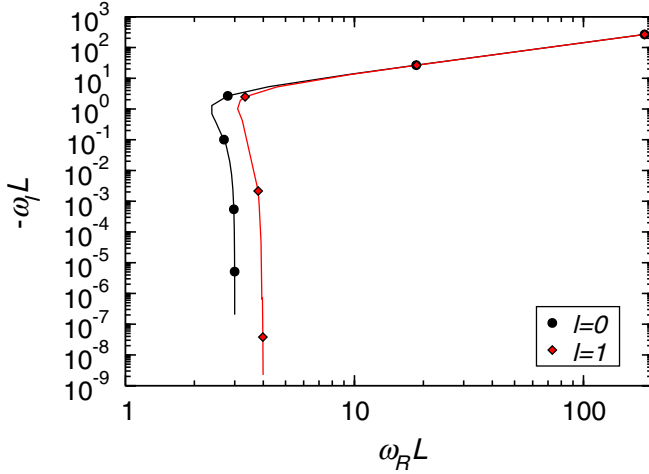


FIG. 3 (color online). Track traced in the complex plane $(\omega_R L, \omega_I L)$ by the fundamental $l = 0, 1$ scalar-field QNM frequencies as we vary the BH size r_+/L . Counterclockwise along these tracks, we mark by circles and diamonds the frequencies corresponding to decreasing decades in r_+/L ($r_+/L = 10^2, 10^1, 10^0, 10^{-1}, \dots$).

B. Gravitational perturbations

We have also searched for the modes of Regge-Wheeler or vector-type gravitational perturbations. The potential (2.2) for gravitational perturbations of small BHs does not develop a local minimum. Nevertheless, it develops a local maximum which, when imposing Dirichlet boundary conditions [see the discussion following Eq. (2.4)], can

sustain quasibound states. For this reason we expect the resonance method to be useful also in this case.

For small BHs our numerical results agree with the following analytic estimate, derived under the assumption that $M\omega_R \ll 1$ [19]:

$$\omega_I L \simeq -\gamma_2(l+2+2n)(r_+/L)^{2l+2}, \quad (2.11)$$

with

$$\gamma_2 \equiv \frac{(l+1)(2+l+2n)\Gamma[1/2-l]\Gamma[l]^3\Gamma[3+l]\Gamma[2(2+l+n)]}{(l-1)^2(2n)!\Gamma[l+3/2]\Gamma[2l+1]\Gamma[2l+2]\Gamma[-1/2-l-2n]\Gamma[7/2+l+2n]}. \quad (2.12)$$

For $n=0$ and $l=2$ this implies $\omega_I = 1024(r_+/L)^6/45\pi \simeq 7.24(r_+/L)^6$, while a fit of the numerics yields $\omega_I \sim 7.44(r_+/L)^6$. Again, $\omega_R L$ approaches the pure AdS value in the limit $r_+/L \rightarrow 0$:

$$\omega_R L = l+2+2n - k_{ln}r_+/L, \quad n = 0, 1, 2, \dots \quad (2.13)$$

For the fundamental $l=2$ mode we find $k_{20} \sim 1.4$.

III. LONG-LIVED MODES IN THE EIKONAL LIMIT

A recent study of the eikonal limit ($l \gg 1$) of SAdS BHs suggests that very long-lived modes should exist in this regime [6]. Define $r_b > r_c$ to be the two real zeros (turning points) of $\omega_R^2 - p^2 f/r^2 = 0$. Then the real part of a class of long-lived modes in four spacetime dimensions is given by the WKB condition

$$2 \int_{r_b}^{\infty} \frac{\sqrt{r^2 \omega_R^2 - p^2 f}}{r f} dr = \pi(2n+5/2), \quad (3.1)$$

where $p = l + 1/2$. Their imaginary part is given by

$$\omega_I = \frac{\gamma \Gamma}{8\omega_R}, \quad \log \Gamma = 2i \int_{r_b}^{r_c} \frac{\sqrt{r^2 \omega_R^2 - p^2 f}}{r f} dr. \quad (3.2)$$

The prefactor γ , not shown in Ref. [6], can be obtained by standard methods [20,21], with the result

$$\gamma = \left[\int_{r_b}^{\infty} \frac{\cos^2 \chi}{\sqrt{\omega_R^2 - p^2 f/r^2} f} dr \right]^{-1}, \quad (3.3)$$

$$\chi \equiv \int_r^{\infty} \sqrt{\omega_R^2 - p^2 f/r^2} \frac{dr}{f} - \frac{\pi}{4}.$$

The resonance method is well suited to analyze the eikonal limit, especially for small BHs (for large BHs the existence of a dip in the potential well requires very large values of l , which are numerically hard to deal with). In Table II we compare the WKB results against numerical results from the resonance method. Unfortunately, machine precision limitations do not allow us to extract extremely small imaginary parts. The agreement with the WKB condition of Ref. [6] is remarkable, even for relatively small values of l . Our numerics conclusively confirm the existence of very long-lived modes in the SAdS geometry, but the numerical results for the damping time scales disagree by orders of magnitude with the corresponding results by Grain and Barrau [5]. A reanalysis of

TABLE II. The QNM frequencies for an $r_+/L = 0.1$ BH for selected values of l , and comparison with the corresponding WKB prediction.

(l, n)	WKB		Resonance	
	$\omega_R L$	$\log \omega_I L$	$\omega_R L$	$\log \omega_I L$
3, 0	5.8668	-16.40	5.8734	-17.03
3, 1	7.6727	-12.61	7.6776	-12.97
3, 2	9.4189	-9.40	9.4219	-9.60
4, 0	6.8830	-22.13	6.8889	-22.84
4, 1	8.7139	-18.30	8.7184	-18.41
4, 2	10.4960	-14.65	10.4996	-14.72
5, 0	7.8945	-28.64	7.8997	-28.76
5, 1	9.7426	-23.97	9.7466	-24.02
5, 2	11.5482	-20.03	11.5516	-20.06

the assumptions implicit in their method would be useful to understand the cause of this disagreement.

IV. CONCLUSIONS AND OUTLOOK

The method described here provides a reliable and accurate alternative to the series solution method [3], to be used in regimes where the former has poor convergence

properties. Together, the two methods allow an almost complete characterization of the spectrum of BHs in AdS backgrounds, encompassing both small and large BHs. As an application of the method, we have explicitly confirmed the existence of the weakly damped modes predicted by Refs. [5,6].

Extensions of the resonance method to higher-dimensional [3] and charged geometries [18,22] should be trivial. Our techniques may be useful to verify the existence of the highly real modes predicted by Ref. [7]. Finally, it would be interesting to investigate whether the resonance method described here is of any use to investigate the eikonal limit of QNMs in asymptotically flat BH spacetimes.

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- [1] S. Chandrasekhar and V. Ferrari, Proc. R. Soc. A **434**, 449 (1991).
 - [2] S. Chandrasekhar and V. Ferrari, Proc. R. Soc. A **437**, 133 (1992).
 - [3] G. T. Horowitz and V. E. Hubeny, Phys. Rev. D **62**, 024027 (2000).
 - [4] T. Harmark, J. Natario, and R. Schiappa, arXiv:0708.0017.
 - [5] J. Grain and A. Barrau, Nucl. Phys. **B742**, 253 (2006).
 - [6] G. Festuccia and H. Liu, arXiv:0811.1033.
 - [7] R. G. Daghigh and M. D. Green, arXiv:0808.1596.
 - [8] N. Andersson and K. E. Thylwe, Classical Quantum Gravity **11**, 2991 (1994).
 - [9] N. Andersson, Classical Quantum Gravity **11**, 3003 (1994).
 - [10] Y. Decanini, A. Folacci, and B. Jensen, Phys. Rev. D **67**, 124017 (2003).
 - [11] V. Cardoso and J. P. S. Lemos, Phys. Rev. D **64**, 084017 (2001).
 - [12] I. G. Moss and J. P. Norman, Classical Quantum Gravity **19**, 2323 (2002).
 - [13] G. Michalogiorgakis and S. S. Pufu, J. High Energy Phys. **02** (2007) 023.
 - [14] I. Bakas, J. High Energy Phys. **01** (2009) 003.
 - [15] J. V. Rocha, J. High Energy Phys. **08** (2008) 075.
 - [16] R. A. Konoplya, Phys. Rev. D **66**, 044009 (2002).
 - [17] V. Cardoso and O. J. C. Dias, Phys. Rev. D **70**, 084011 (2004).
 - [18] E. Berti and K. D. Kokkotas, Phys. Rev. D **67**, 064020 (2003).
 - [19] V. Cardoso, O. J. C. Dias, and S. Yoshida, Phys. Rev. D **74**, 044008 (2006).
 - [20] G. Gamow, Z. Phys. **51**, 204 (1928).
 - [21] S. A. Gurvitz, Phys. Rev. A **38**, 1747 (1988).
 - [22] B. Wang, C. Y. Lin, and C. Molina, Phys. Rev. D **70**, 064025 (2004).