Particle phenomenology on noncommutative spacetime

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We introduce particle phenomenology on the noncommutative spacetime called the Groenewold-Moyal plane. The length scale of spacetime noncommutativity is constrained from the CPT violation measurements in the $K^0 - \bar{K}^0$ system and $g - 2$ difference of $\mu^+ - \mu^-$. The $K^0 - \bar{K}^0$ system provides an upper bound on the length scale of spacetime noncommutativity of the order of 10^{-32} m, corresponding to a lower energy bound E of the order of $E \gtrsim 10^{16}$ GeV. The $g - 2$ difference of $\mu^+ - \mu^-$ constrains the noncommutativity length scale to be of the order of 10^{-20} m, corresponding to a lower energy bound E of the order of $E \gtrsim 10^3$ GeV. We also present the phenomenology of the electromagnetic interaction of electrons and nucleons at the tree level on the noncommutative spacetime. We show that the distributions of charge and magnetization of nucleons are affected by spacetime noncommutativity. The analytic properties of electromagnetic form factors are also changed and it may give rise to interesting experimental signals.

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I. INTRODUCTION

Quantum field theories constructed on noncommutative spacetime provide a completely new perspective in building particle physics models beyond the standard model. (See [1] for general properties of noncommutative field theories). The noncommutative algebra of functions with the commutation relations between coordinate operators

$$[\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu}, \tag{1}$$

where $\theta_{\mu\nu}$ is a constant antisymmetric real matrix, can be used to model noncommutative spacetime. There are two major approaches to construct field theories on noncommutative spacetime. They are (i) the star-product formalism and (ii) the Seiberg-Witten map and enveloping algebra formalism. They both have a number of nice properties. We briefly discuss them below.

In the star-product formalism, to define a field theory on noncommutative spacetime, we replace the ordinary pointwise multiplication by a Moyal $\star$ product [2–4]

$$(f \star g)(x) = e^{(i/2)\theta_{\mu\nu}\partial_{x_\mu}\partial_{y_\nu}} f(x)g(y)|_{x=y}. \tag{2}$$

The replacement of pointwise products by star products has several nontrivial consequences in field theories. The so-called UV/IR mixing [5,6] is one such consequence. We can make the noncommutative field theory UV renormalizable by adding proper counterterms. But we still have an IR divergence problem. There have been a few proposals such as noncommutative hard resummation and (or) introducing a new way of regularization to resolve this problem [7–9]. The noncommutative gauge theories [10–12], the noncommutative version of real ϕ^4 theory [5,13–16] as well as the complex ϕ^4 theory [17], and the noncommutative version of QED [18,19] have been shown to be one-loop renormalizable. In [20] a noncommutative version of the standard model is constructed in the star-product approach.

The presence of UV/IR mixing in noncommutative gauge theories in the star-product formalism makes the low energy physics of these theories to depend crucially on the details of ultraviolet completion. This can make the photons in the theory to have contributions from a trace-$U(1)$, causing vacuum birefringence (polarization dependent propagation speed). In [21] limits on the energy scale of noncommutativity are obtained from bounds on vacuum birefringence.

In the star-product approach with twisted Poincaré invariance and deformed statistics [22–25] it was shown that UV/IR mixing is present in a non-Abelian gauge theory at one-loop level while it is absent in an Abelian gauge theory to all orders of perturbative expansion [26]. On using this formalism it was shown that the power spectrum for cosmic microwave backgrond (CMB) becomes direction dependent [27] and a lower energy bound for noncommutativity parameter θ is obtained from the CMB data [28].

An alternative method of noncommutative quantization was proposed in [29] based on enveloping algebra valued fields and Seiberg-Witten maps. The introduction of noncommutativity in gauge theories limits the choice of gauge group to that of a matrix representation of a $U(N)$ gauge group. But we need a more general gauge group like $SU(N)$. The use of enveloping algebra valued fields seems to be the easiest way, but it can make the model meaningless since this would imply an infinite number of degrees of freedom [29,30]. This problem can be solved by θ-expanding the model using the so-called Seiberg-Witten maps [31,32]. The Seiberg-Witten maps express noncom-

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mutative fields and parameters as local functions of the commutative fields and parameters. But problems still persist in the form of nonrenormalizability of these theories. The θ-expanded QED was shown to be power-counting nonrenormalizable [33]. Assuming that the noncommutative field theory under consideration is arising as the effective field theory of an unknown fundamental theory that is responsible for the noncommutativity of spacetime, the issue of renormalization can be tackled [34]. In [35] the gauge sector of the noncommutative standard model was shown to be one-loop renormalizable to first order in the expansion in θ. (See also [36–39].) In [40] it was shown that the photon self-energy is renormalizable to all orders in this approach.

Noncommutative non-Abelian gauge theories are constructed in [41]. See [34,42,43] for the constructions of the standard model on noncommutative spacetime using this approach.

Noncommutative gauge theories formulated using this approach may contain additional gauge anomalies which do not appear in ordinary commutative gauge theories. In [44] it was shown that noncommutative gauge theories with arbitrary compact gauge group have the same one-loop anomalies as their commutative counterparts. (See also [45]).

There has been much progress in applying the Seiberg-Witten map based noncommutative field theories in the context of high energy physics phenomenology of the noncommutative standard model [46–55], noncommutative neutrino physics [56–58], astrophysics [59], and cosmology [60,61].

In this paper we focus on the particle phenomenology of noncommutative field theories constructed using the star-product formalism along with twisted statistics [23–25]. In [22] it was shown that noncommutative spacetime with the commutation relations given in Eq. (1) can be interpreted in a Lorentz invariant way by invoking the concept of twisted Poincaré symmetry of the algebra of functions on a Minkowski spacetime. Twisting of the Poincaré symmetry leads to twisted statistics in noncommutative field theories [23–26,62–64].

Twisted noncommutative quantum field theories are shown to be Lorentz noninvariant [22–26,62–65], CPT violating [66], and nonlocal in nature [25,28,67,68]. The scattering matrix of these theories cannot be Lorentz invariant in general. They can depend upon the noncommutativity parameter $\theta^{\mu\nu}$ and the external four-momenta of the scattering particles. The incident and outgoing state vectors are also modified by the spacetime noncommutativity. The frame dependence of the S matrix gives rise to many interesting features in such quantum field theories. It gives rise to nontrivial effects such as corrections to the electric and magnetic properties of nucleons. In a general frame, the appearance of $\theta^{\mu\nu}$ in the S operator can also break the discrete symmetries P and CPT [66].

We organize the paper as follows: In Sec. II, we constrain the noncommutativity parameter from the CPT violation measurements in the neutral kaon ($K^0 - \bar{K}^0$) system [69]. In Sec. III, we put further constraint on the noncommutativity length scale from the CPT violation measurements on the $g-2$ of $\mu^+ - \mu^-$ [70–73]. We follow a phenomenological approach without invoking the microscopic structure of the underlying theory in detail. In Sec. IV, we show how the electron-nucleon scattering process at the tree level is affected by spacetime noncommutativity. The vertex function and thus the electromagnetic form factors of the nucleon are modified. They depend on the total incident and recoil four-momenta, and the noncommutativity parameter. Analytic properties of electromagnetic form factors are also changed, indicating the possibility of experimental signals. The paper concludes in Sec. V.

II. THE NEUTRAL KAON SYSTEM IN NONCOMMUTATIVE FRAMEWORK

In the standard (commutative) case, if we start off with a pure K^0 beam created in a strong interaction process at time $t = 0$, its intensity $I(K^0)$ oscillates with a frequency $\Delta m \equiv m_L - m_S$, where m_L (m_S) is the mass of the weak-interaction kaon eigenstate K_L (K_S). When kaons propagate through space they are distinguished by their mode of decay and thus by the weak-interaction eigenstates [74]. For K_S, the lowest mass intermediate states are two-pion states and they are expected to be smaller than one-pion intermediate state that occurs for K_L mass renormalization (see Fig. 1). We infer that K_L should be affected the most by spacetime noncommutativity.

In an arbitrary scattering diagram involving quark-quark-gluon (q-q-g) vertices, the space-space part of noncommutativity can be integrated out to zero from the S operator [25]. The S operator carries the time-space part of the noncommutativity through the twist factor [22,25,65,75] $\exp(\frac{1}{2}\overleftarrow{\partial}_0\vec{\theta}^0 \cdot \vec{P}_{\rm in})$, where $\overleftarrow{\partial}_0$ differentiates the appropriate time argument and $\vec{\theta}^0 = (\theta^{01}, \theta^{02}, \theta^{03})$.

The self-energy diagram for K_L with one-pion pole dominance should be affected by this twist factor. Such a self-energy diagram is depicted in Fig. 1. The coupling constant λ has dependence on $\vec{\theta}^0$ and the total incident momentum $\vec{P}_{\rm in}$ through the q-q-g vertices appearing in the microscopic version of the theory. We will not discuss the

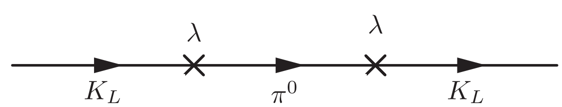

FIG. 1. A self-energy diagram for K_L with π^0 pole dominance. The coupling constant λ has dependence on $\vec{\theta}^0$ and the total incident momentum $\vec{P}_{\rm in}$.

details of the theory in the quark-gluon level as it is far too complicated.

The nonlocal nature of the theory in time and thus the appearance of $\vec{\theta}^0$ in scattering processes as mentioned above indicates that we should twist the mass and decay width of K_L:

$$m_L^\theta = m_L \exp\left(\frac{i}{2} m_{K^0} \vec{\theta}^0 \cdot \vec{P}_{\text{in}}\right), \tag{3}$$

$$\gamma_L^\theta = \gamma_L \exp\left(\frac{i}{2} m_{K^0} \vec{\theta}^0 \cdot \vec{P}_{\text{in}}\right), \tag{4}$$

where m_L and γ_L are the mass and width of the K_L eigenstate in the commutative case and m_{K^0} is the mass of the strong interaction eigenstate of K^0. Notice that these expressions recover their respective commutative forms when $\vec{\theta}^0 = 0$, $\vec{P}_{\text{in}} = 0$, or $\vec{\theta}^0 \cdot \vec{P}_{\text{in}} = 0$.

We can obtain an expression for the K^0 intensity at time t from the amplitudes of the states K_S and K_L:

$$A_S(t) = A_0 \exp[(-\tfrac{1}{2}\gamma_S + i m_S)t], \tag{5}$$

$$A_L(t) = A_0 \exp[(-\tfrac{1}{2}\gamma_L^\theta + i m_L^\theta)t], \tag{6}$$

where A_0 is the amplitude at time $t = 0$.

If the K^0 beam is pure with unit intensity when K^0's are created at $t = 0$, we have the expression for intensity at time t,

$$\begin{aligned} I_{K^0} &= \tfrac{1}{2}[A_S(t) + A_L(t)][A_S^*(t) + A_L^*(t)] \\ &= \tfrac{1}{4}[\exp(-\gamma_S t) + \exp(-\gamma_L \cos\alpha t + 2m_L \sin\alpha t) \\ &\quad + 2\exp(-\tfrac{1}{2}\gamma_S t - \tfrac{1}{2}\gamma_L \cos\alpha t + m_L \sin\alpha t)\cos\Delta m t], \end{aligned} \tag{7}$$

where

$$\Delta m = \frac{\gamma_L}{2}\sin\alpha + m_L\cos\alpha - m_S \tag{8}$$

and

$$\alpha = \tfrac{1}{2} m_{K^0} \vec{\theta}^0 \cdot \vec{P}_{\text{in}}. \tag{9}$$

From Eq. (7), we can define a width difference

$$\Delta\gamma = \gamma_S - \gamma_L\cos\alpha + 2m_L\sin\alpha. \tag{10}$$

These expressions also recover their standard forms in the $\theta^{\mu\nu} \to 0$ limit.

If we assume that the mass and width differences are arising purely due to spacetime noncommutativity, then we have $m_L - m_S = 0$ and $\gamma_S - \gamma_L = 0$ for the case $\theta^{\mu\nu} = 0$. In that case, $m_L = m_S = m_{K^0}$ and $\gamma_S = \gamma_L = \gamma_{K^0}$. Then we have the noncommutative expressions to the lowest order in $\vec{\theta}^0$:

$$\Delta m \simeq \frac{\gamma_{K^0}}{2}\left(\frac{m_{K^0}}{2}\vec{\theta}^0 \cdot \vec{P}_{\text{in}}\right), \tag{11}$$

$$\Delta\gamma \simeq 2m_{K^0}\left(\frac{m_{K^0}}{2}\vec{\theta}^0 \cdot \vec{P}_{\text{in}}\right). \tag{12}$$

In the standard phenomenological theory of kaons, the CPT violation complex parameter δ is defined as

$$\begin{aligned} \delta &= \frac{\Lambda_{\bar{K}^0\bar{K}^0} - \Lambda_{K^0K^0}}{2(\lambda_L - \lambda_S)} \\ &= \delta_\parallel \exp(i\phi_{\text{SW}}) + \delta_\perp \exp\left(i\left(\phi_{\text{SW}} + \frac{\pi}{2}\right)\right), \end{aligned} \tag{13}$$

where $\Lambda_{\bar{K}^0\bar{K}^0}$ and $\Lambda_{K^0K^0}$ are diagonal entries of the 2×2 matrix $\Lambda \equiv M - \frac{i}{2}\Gamma$, $\lambda_{L,S} = m_{L,S} - \frac{i}{2}\gamma_{L,S}$ are the eigenvalues of the matrix Λ, $\delta_\parallel$ and $\delta_\perp$ are, respectively, the projections of δ parallel and perpendicular to the superweak direction, $\phi_{\text{SW}} = \tan^{-1}(2\Delta m/\Delta\gamma)$.

The projections $\delta_\parallel$ and $\delta_\perp$ are related to the mass and width difference between the strong interaction eigenstates K^0 and $\bar{K}^0$:

$$\delta_\parallel = \frac{1}{4}\frac{\gamma_{K^0} - \gamma_{\bar{K}^0}}{\sqrt{\Delta m^2 + (\frac{\Delta\gamma}{2})^2}}, \qquad \delta_\perp = \frac{1}{2}\frac{m_{K^0} - m_{\bar{K}^0}}{\sqrt{\Delta m^2 + (\frac{\Delta\gamma}{2})^2}}. \tag{14}$$

From Eqs. (11) and (12) we see that the superweak angle ϕ_{SW} is not affected by noncommutativity to the lowest order. That is,

$$\phi_{\text{SW}} = \tan^{-1}(2\Delta m/\Delta\gamma) \simeq \tan^{-1}(\gamma_{K^0}/2m_{K^0}). \tag{15}$$

The real and imaginary parts of δ are known from kaon decay experiments. From [76,77] we have

$$\text{Re }\delta \simeq 2.9 \times 10^{-4}, \qquad \text{Im }\delta \simeq -0.2 \times 10^{-5}. \tag{16}$$

From Eq. (13), we have the expressions for $\delta_\parallel$ and $\delta_\perp$:

$$\delta_\parallel = \text{Re }\delta\cos(\phi_{\text{SW}}) + \text{Im }\delta\sin(\phi_{\text{SW}}), \tag{17}$$

$$\delta_\perp = -\text{Re }\delta\sin(\phi_{\text{SW}}) + \text{Im }\delta\cos(\phi_{\text{SW}}). \tag{18}$$

On using the superweak angle measured at the KTeV E731 experiment [69]

$$\phi_{\text{SW}} = 43.4° \pm 0.1°, \tag{19}$$

we obtain the values for $\delta_\parallel$ and $\delta_\perp$,

$$\delta_\parallel \simeq 20.93 \times 10^{-5}, \qquad \delta_\perp \simeq -20.07 \times 10^{-5}. \tag{20}$$

From Eq. (14) we have the mass difference

$$\begin{aligned} m_{K^0} - m_{\bar{K}^0} &= 2\delta_\perp\sqrt{\Delta m^2 + \left(\frac{\Delta\gamma}{2}\right)^2} \\ &\simeq \delta_\perp(m_{K^0}\vec{\theta}^0 \cdot \vec{P}_{\text{in}})\sqrt{\frac{1}{4}\gamma_{K^0}^2 + m_{K^0}^2}. \end{aligned}$$

On using $\tan(\phi_{\text{SW}}) \simeq \gamma_{K^0}/2m_{K^0}$ we have

$$m_{K^0} - m_{\bar{K}^0} \simeq \delta_\perp (m_{K^0}\vec{\theta}^0 \cdot \vec{P}_{\rm in}) m_{K^0}\sqrt{1 + \tan^2(\phi_{\rm SW})}. \tag{21}$$

Thus the *CPT* figure of merit takes the form

$$r_{K^0} \equiv \frac{|m_{K^0} - m_{\bar{K}^0}|}{m_{K^0}} \simeq \delta_\perp (m_{K^0}\vec{\theta}^0 \cdot \vec{P}_{\rm in})\sqrt{1 + \tan^2(\phi_{\rm SW})}. \tag{22}$$

The noncommutativity parameter measured in the laboratory frame can in fact vary with time due to the rotation of the Earth. Let us denote the laboratory frame by $(\hat{x}, \hat{y}, \hat{z})$ and the nonrotating frame (compatible with the Earth's celestial equatorial coordinates) by $(\hat{X}, \hat{Y}, \hat{Z})$. The components of the noncommutativity parameter measured in the laboratory frame at time t, $(\theta^x(t), \theta^y(t), \theta^z(t))$ can be connected to the components in the nonrotating frame $(\theta^X, \theta^Y, \theta^Z) \equiv (\theta^{01}, \theta^{02}, \theta^{03})$.

The relations connecting the components of a vector between these two frames are known in the literature. From [78] we have

$$\theta^x(t) = \theta^X \cos\chi \cos\Omega t + \theta^Y \cos\chi \sin\Omega t - \theta^Z \sin\chi, \tag{23}$$

$$\theta^y(t) = -\theta^X \sin\Omega t + \theta^Y \cos\Omega t, \tag{24}$$

$$\theta^z(t) = \theta^X \sin\chi \cos\Omega t + \theta^Y \sin\chi \sin\Omega t + \theta^Z \cos\chi, \tag{25}$$

where $\chi = \cos^{-1}(\hat{z} \cdot \hat{Z})$ and Ω is the sidereal frequency of the Earth.

In the laboratory frame we have the relation

$$\vec{\theta}^{\,0}(t) \cdot \vec{P}_{\rm in} \simeq \left(\frac{|m_{K^0} - m^0_{\bar{K}}|}{m_{K^0}}\right) \frac{1}{\delta_\perp m_{K^0}\sqrt{1 + \tan^2(\phi_{\rm SW})}}. \tag{26}$$

The KTeV experiment at Fermilab involves a highly collimated kaon beam with an average boost factor $\bar{\gamma}$ of the order of 100 and $\beta = v/c \simeq 1$. The $\hat{z}$-axis of the laboratory frame is chosen along the beam direction such that the kaon three-velocity reduces to $\vec{\beta} = (0, 0, \beta)$. In that case the above equation reduces to

$$\theta^z(t) P^z_{\rm in} \simeq \left(\frac{|m_{K^0} - m^0_{\bar{K}}|}{m_{K^0}}\right) \frac{1}{\delta_\perp m_{K^0}\sqrt{1 + \tan^2(\phi_{\rm SW})}}. \tag{27}$$

In the nonrotating frame,

$$(\theta^X \cos\Omega t + \theta^Y \sin\Omega t)\sin\chi + \theta^Z \cos\chi \simeq \left(\frac{|m_{K^0} - m^0_{\bar{K}}|}{m_{K^0}}\right) \frac{1}{\delta_\perp m_{K^0} P^z_{\rm in}\sqrt{1 + \tan^2(\phi_{\rm SW})}}. \tag{28}$$

The first two terms in the left-hand side oscillate in time with a frequency Ω. Since experiments are performed over extended time periods, we may disregard the time dependence and thus take the time averaged form of the above expression. Thus we have

$$\theta^Z \cos\chi \simeq \left(\frac{|m_{K^0} - m^0_{\bar{K}}|}{m_{K^0}}\right) \times \frac{1}{\delta_\perp (m_{0K^0})^2 \bar{\gamma}^2 \bar{\beta} c \sqrt{1 + \tan^2(\phi_{\rm SW})}}, \tag{29}$$

where we have replaced the incident momentum $P^z_{\rm in}$ by $m_{0K^0}\bar{\gamma}\,\bar{\beta}\,c$, m_{K^0} by $m_{0K^0}\bar{\gamma}$, with m_{0K^0} the kaon rest mass. $\bar{\gamma}$ and $\bar{\beta}$ are averages of β and γ. This expression gives a bound for the z component of the noncommutativity parameter in the nonrotating frame.

The detector geometry in the KTeV experiment has $\cos\chi = 0.6$. Thus on using the results from the experiments on kaons we obtain

$$\theta^Z \simeq \left(\frac{|m_{K^0} - m^0_{\bar{K}}|}{m_{K^0}}\right) \times \frac{\hbar^2/c}{\delta_\perp \cos\chi (m_{0K^0})^2 \bar{\gamma}^2 \bar{\beta} c \sqrt{1 + \tan^2(\phi_{\rm SW})}} \lesssim 0.0829 \times 10^{-63}\ {\rm m}^2. \tag{30}$$

This gives an upper bound for the noncommutativity length scale

$$\sqrt{\theta^Z} \lesssim 10^{-32}\ {\rm m}, \tag{31}$$

corresponding to a lower bound for energy E associated with spacetime noncommutativity

$$E \gtrsim 10^{16}\ {\rm GeV}. \tag{32}$$

III. NONCOMMUTATIVITY BOUND FROM $g - 2$ DIFFERENCE OF μ^+ AND μ^-

We can put further constraint on the noncommutativity length scale from the *CPT* violation measurements on the $g - 2$ of positive and negative muons [70–73]. (It is also possible to put a bound on spacetime noncommutativity from the $g - 2$ difference between electron and positron [79,80]. It turns out that this bound does not constrain the noncommutativity parameter very well due to the small mass of the electron).

In twisted noncommutative field theories on the Groenewold-Moyal (GM) plane (noncommutative spacetime modeled by the star-product approach with deformed statistics) gauge-matter field vortices are in general affected by spacetime noncommutativity. It is shown elsewhere [62] that the S operator in an Abelian gauge-matter theory (say, QED) is unaffected by spacetime noncommutativity. However, higher order hadronic (and thus non-Abelian) loop corrections to the QED scattering diagrams can in fact carry a nontrivial dependence on spacetime noncommutativity. Such a diagram is pictured in Fig. 2, where the $g - 2$ of a lepton $l^\pm$ ($\mu^\pm$ or $e^\pm$) receives a lowest

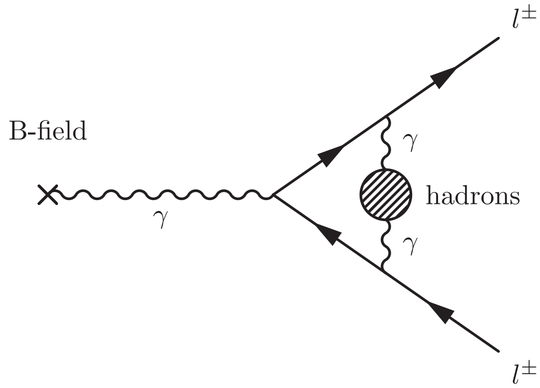

FIG. 2. Lowest order hadronic contribution to $g-2$ of a lepton $l^{\pm}$ ($\mu^{\pm}$ or $e^{\pm}$) in a uniform magnetic field B.

order hadronic loop contribution when it is moving in an external electromagnetic field. The experiments performed at CERN and BNL [70–73] to measure the muon $g-2$ use a storage ring magnet in which muons are circulating in a uniform magnetic field. (The Penning trap experiments to determine electron-positron g factors [79,80] use a strong homogeneous magnetic field and a quadrupole electric field.)

The interaction Hamiltonian density of the noncommutative field theory in general splits into two parts, a part with matter-gauge couplings and pure matter coupling $\mathcal{H}^{M,G}_{I(\theta)}$ and a pure gauge part $\mathcal{H}^{G}_{I(\theta)}$ [25,66],

$$\mathcal{H}_{I(\theta)} = \mathcal{H}^{M,G}_{I(\theta)} + \mathcal{H}^{G}_{I(\theta)}, \tag{33}$$

where

$$\mathcal{H}^{M,G}_{I(\theta)} = \mathcal{H}^{M,G}_{I(0)} e^{(1/2)\overleftarrow{\partial}\wedge P}, \qquad \mathcal{H}^{G}_{I(\theta)} = \mathcal{H}^{G}_{I(0)}. \tag{34}$$

In a non-Abelian gauge theory, $\mathcal{H}^{G}_{I(\theta)} = \mathcal{H}^{G}_{I(0)} \neq 0$, so that the S operator $S^{M,G}_{(\theta)}$ of the theory

$$S^{M,G}_{(\theta)} \neq S^{M,G}_{(\theta=0)} = S^{M,G}. \tag{35}$$

This can in fact affect the lepton-photon vertex function through the contribution from hadronic (and thus non-Abelian) loops. (See Fig. 2.)

The S operator $S^{M,G}_{(\theta)}$ depends only on θ^{0i} [66],

$$S^{M,G}_{(\theta)} = S^{M,G}_{\theta^{0i}}. \tag{36}$$

Charge conjugation C and time reversal T on $S^{MG}_{(\theta)}$ do not affect θ^{0i}, while parity P changes its sign [66]. Thus a nonzero θ^{0i} contributes to P and thus CPT violation.

The appearance of the term $\theta^{0i}P_i^{\mathrm{inc}}$, where P_i^{inc} is the total incident momentum, in the S operator suggests that the amount of CPT violation to the leading order in $\theta^{\mu\nu}$ should be $\mathcal{O}(\vec{\theta}^0 \cdot \vec{P}_{\mathrm{inc}})$.

From the CERN experiments on positive and negative muon g factors, the standard CPT figure of merit is given by [71]

$$r_g^{\mu} \equiv \frac{|g_{\mu^+} - g_{\mu^-}|}{g_{\mu}^{\mathrm{avg}}} \lesssim 10^{-8}. \tag{37}$$

To the leading order in $\theta^{\mu\nu}$

$$r_g^{\mu} \approx m_{\mu}^{\mathrm{avg}} \vec{\theta}^0 \cdot \vec{P}_{\mathrm{inc}}, \tag{38}$$

where m_{μ}^{avg} is the average mass of the positive and negative muons, the only relevant mass scale in the theory.

We get the maximum bound when $\vec{\theta}^0$ and $\vec{P}_{\mathrm{inc}}$ are parallel. In that case

$$|\vec{\theta}| \lesssim \frac{10^{-8}}{(m_{\mu}\gamma)^2}, \tag{39}$$

where γ is the relativistic factor. The muon $g-2$ experiments at CERN and BNL were performed at a specific value of the relativistic factor, $\gamma = 29.3$.

Equation (39) gives an upper bound for the length scale of noncommutativity

$$|\vec{\theta}| \lesssim \frac{10^{-8}}{(m_{\mu}\gamma)^2} \simeq 1.61 \times 10^{-39}\ \mathrm{m}^2 \rightarrow \sqrt{\theta} \lesssim 10^{-20}\ \mathrm{m}. \tag{40}$$

This corresponds to a lower bound for the energy scale $E \gtrsim 10^3$ GeV.

The Penning trap experiments measure the difference in electron and positron g factors [79,80], $r_g^e \equiv |g_{e^+} - g_{e^-}|/g_e^{\mathrm{avg}} \lesssim 10^{-12}$. This CPT figure of merit is more precise compared to that of the muon. However, it is much less sensitive to the new physics arising from spacetime noncommutativity as the electron has lower mass compared to muons and kaons.

IV. NONCOMMUTATIVE CORRECTIONS TO ELASTIC ELECTRON-NUCLEON SCATTERING

In this section we study how the spacetime noncommutativity affects the elastic electron-nucleon scattering process at the tree level. We show that the nucleon vertex function and the electromagnetic form factors are affected by noncommutativity. The spatial distributions of the charge and magnetization carried by the nucleon are also modified.

A. Phenomenology of electron-nucleon interaction on the commutative spacetime

The scattering process of an electron from a nucleon through the exchange of a virtual photon is represented by the Feynman diagram in Fig. 3. Such a process is contained in the matrix element

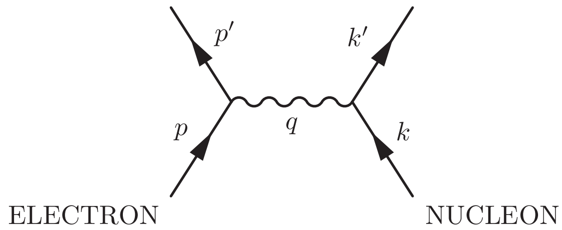

FIG. 3. Feynman diagram of electron-nucleon scattering caused by the exchange of a virtual photon.

$$\langle p', k'|S^{(2)}|p, k\rangle = \langle p', k'|\mathbf{T}\bigg(\frac{(-i)^2}{2!} \times \int d^4x d^4y j^{p,n}_\mu(x)A^\mu(x)j^e_\mu(y)A^\mu(y)\bigg)|p, k\rangle, \tag{41}$$

where $j^{p,n}_\mu$, j^e_μ, respectively, are the electron and nucleon currents, A^μ is the photon field, and p, p' and k, k' are the four-momenta of the incident and scattered electron and nucleon, respectively.

In momentum space, up to a possible minus sign, it takes the form

$$\int \frac{d^4q}{(2\pi)^4}(2\pi)^4\delta^{(4)}(p' - p + q)(2\pi)^4 \times \delta^{(4)}(k' - k - q)j^{p,n}_\mu(p', p)\frac{ig^{\mu\nu}}{q^2}j^e_\nu(k', k), \tag{42}$$

where $g^{\mu\nu}$ is the metric tensor. The factor $1/q^2$ represents the propagation of a virtual photon of four-momentum q_μ between the electron and nucleon. It is

$$q_\mu = (p'_\mu - p_\mu) = -(k'_\mu - k_\mu). \tag{43}$$

This expression takes the form $i\mathcal{M}(2\pi)^4\delta^{(4)}(p' + k' - p - k)$, after a delta function integration, with

$$i\mathcal{M} = j^{p,n}_\mu(p', p)\bigg(\frac{ig^{\mu\nu}}{q^2}\bigg)j^e_\nu(k', k). \tag{44}$$

The electron charge-current density j^e_μ, which, assuming that the electron has no internal structure, is given by

$$j^e_\mu(p', p) = -ie\bar{u}(p')\gamma_\mu u(p), \tag{45}$$

where $\bar{u}$ and u are the Dirac spinors of the electron.

The nucleon charge-current density $j^{p,n}_\mu$ (proton or neutron) is given by

$$j^{p,n}_\mu(k', k) = -ie\bar{N}(k')\Gamma^\mu(k', k)N(k), \tag{46}$$

where $\bar{N}$ and N are the Dirac spinors of the nucleon and Γ^μ, called the vertex function, includes all the effects due to the internal structure of the nucleon.

In terms of the electron and nucleon spinors, the matrix element for elastic scattering takes the form

$$-i\mathcal{M} = \frac{-ig_{\mu\nu}}{q^2}(ie\bar{u}(p')\gamma^\nu u(p)) \times (-ie\bar{N}(k')\Gamma^\mu(k', k)N(k)). \tag{47}$$

If we assume that (i) $j^{p,n}_\mu$ transforms as a four-vector (relativistic covariance), (ii) $j^{p,n}_\mu$ is conserved, and (iii) the nucleon is a Dirac particle, then the nucleon charge-current density is constrained to be of the form [81,82]

$$j^{p,n}_\mu(p', p) = ie\bar{N}(p')\bigg[\gamma_\mu F^{p,n}_1(q^2) + \frac{\kappa^{p,n}}{2m^{p,n}}\sigma_{\mu\nu}q_\nu F^{p,n}_2(q^2)\bigg]N(p). \tag{48}$$

In Eq. (48), $\kappa^{p,n}$ is the anomalous magnetic moment of the nucleon in nuclear magnetons, $m^{p,n}$ is the mass of the nucleon, $\sigma_{\mu\nu} = (1/2i)[\gamma_\mu, \gamma_\nu]$, and the functions $F^{p,n}_{1,2}$ (known as the electromagnetic form factors) describe the internal structure of the nucleon.

The form factors essentially measure how strongly the nucleon "holds together" and recoils when a momentum q_μ is exchanged in the scattering process. They give the spatial distributions of charge and magnetic moment of the nucleon, $G_{E_{p,n}}$ and $G_{M_{p,n}}$, respectively,

$$G_{E_{p,n}}(q^2) = F^{p,n}_1(q^2) - \frac{q^2}{4m^2_{p,n}}F^{p,n}_2(q^2), \tag{49}$$

$$G_{M_{p,n}}(q^2) = F^{p,n}_1(q^2) + F^{p,n}_2(q^2), \tag{50}$$

G_{E_p}, G_{M_p}, G_{E_n}, G_{M_n} are called the electric and magnetic Sachs form factors for the proton and neutron, respectively [83–85].

B. Phenomenology of electron-nucleon interaction on the noncommutative spacetime

Matter fields on the GM plane obey twisted (statistics deformed) commutation relations [24,62,63]. The noncommutative spinor field $\psi^{(\theta)}$ is related to its commutative counterpart ψ through the relation [24,66]

$$\psi^{(\theta)}(\mathbf{x}, t) = \psi(\mathbf{x}, t)e^{(1/2)\overleftarrow{\partial}\wedge P}, \tag{51}$$

where $\overleftarrow{\partial}\wedge P = \overleftarrow{\partial}_\mu\theta^{\mu\nu}P_\nu$ and P_ν is the total momentum operator.

Matter fields on the GM plane must be transported by the connection compatibly with Eq. (51). It imposes a natural choice on the covariant derivatives [24,66], making the gauge sector of the theory commutative. Thus the gauge field $A^{(\theta)}_\mu(\mathbf{x}, t)$ on the GM plane is the same as its commutative counterpart

$$A^{(\theta)}_\mu(\mathbf{x}, t) = A_\mu(\mathbf{x}, t). \tag{52}$$

The Feynman rules for arbitrary noncommutative scattering processes are investigated in [86]. We focus, here, on a simple way to find the noncommutative corrections to the tree-level electron-nucleon elastic scattering process.

Since we are considering a $U(1)$ gauge theory, we have $\mathcal{H}^G_{I(\theta)} = \mathcal{H}^G_{I(0)} = 0$. It is shown elsewhere [23,25,62] that the S operators of such theories are the same as those of their commutative counterparts. In an Abelian gauge theory such as QED,

$$S_{(\theta)} = S_{(\theta=0)} = S. \tag{53}$$

To find the S-matrix element for the electron-nucleon scattering at the tree level, we look at the noncommutative version of the matrix element given in Eq. (41),

$${}_{(\theta)}\langle p', k'|S^{(2)}_{(\theta)}|p, k\rangle_{(\theta)} = {}_{(\theta)}\langle p', k'|S^{(2)}|p, k\rangle_{(\theta)}. \tag{54}$$

The noncommutative matrix element differs from its commutative counterpart through the appearance of twisted incident and outgoing state vectors.

We write down the twisted incident and outgoing state vectors in terms of the twisted fields acting on the vacuum

$$\begin{aligned} |p, k\rangle_{(\theta)} &= a^\dagger_N(p)a^\dagger_e(k)|0\rangle, \\ {}_{(\theta)}\langle p', k'| &= \langle 0|a_N(p')a_e(k'). \end{aligned} \tag{55}$$

The map from noncommutative creation and annihilation operators $a^\dagger_{e,N}(k)$, $a_{e,N}(k)$ to the corresponding commutative creation and annihilation operators $c^\dagger_{e,N}(k)$, $c_{e,N}(k)$ (called the "dressing transformation") can be used to untwist the incident and outgoing state vectors. The map is [63]

$$\begin{aligned} a^\dagger_{e,N}(k) &= c^\dagger_{e,N}(k)e^{(i/2)k\wedge P}, \\ a_{e,N}(k) &= c_{e,N}(k)e^{-(i/2)k\wedge P}. \end{aligned} \tag{56}$$

The twisted incident and outgoing state vectors can be expressed in terms of the untwisted incident and outgoing state vectors

$$|p, k\rangle_{(\theta)} = e^{(i/2)p\wedge k}c^\dagger_N(p)c^\dagger_e(k)|0\rangle = e^{(i/2)p\wedge k}|p, k\rangle \tag{57}$$

and

$$\begin{aligned} {}_{(\theta)}\langle p', k'| &= e^{-(i/2)p'\wedge k'}\langle 0|c_N(p')c_e(k') \\ &= e^{-(i/2)p'\wedge k'}\langle p', k'|. \end{aligned} \tag{58}$$

The second order term in the S-operator expansion $S^{(2)}_{(\theta)}$ is independent of θ [23,25,62]. Hence

$$S^{(2)}_{(\theta)} = S^{(2)}_{(\theta=0)} = S^{(2)}, \tag{59}$$

the commutative S operator.

We write down the S-matrix element in noncommutative spacetime (given in Eq. (54)) in terms of the commutative matrix element,

$${}_{(\theta)}\langle p', k'|S^{(2)}_{(\theta)}|p, k\rangle_{(\theta)} = e^{-(i/2)p'\wedge k'}e^{(i/2)p\wedge k}\langle p', k'|S^{(2)}|p, k\rangle. \tag{60}$$

Thus the noncommutative matrix element is

$$\begin{aligned} i\mathcal{M}_{(\theta)} &= e^{-(i/2)p'\wedge k'}e^{(i/2)p\wedge k}j^{p,n}_\mu(p', p)\left(\frac{ig^{\mu\nu}}{q^2}\right)j^e_\nu(k', k) \\ &= \frac{-ig_{\mu\nu}e^{(i/2)(p+k)\wedge q}}{q^2}[ie\bar{u}(k')\gamma^\nu u(k)] \\ &\quad \times [-ie\bar{N}(p')\Gamma^\mu(p', p)N(p)]. \end{aligned} \tag{61}$$

In the tree-level scattering process we assumed that the electron is a point particle and the nucleon has an internal structure. The vertex function Γ^μ contains all the details of the internal structure of the nucleon. We infer that the additional $\theta^{\mu\nu}$ dependent factor represents the noncommutative modification of the internal structure of the nucleon.

The nucleon charge-current density given in Eq. (48) is effectively modified in the noncommutative case, and it takes the form

$$\begin{aligned} j^{p,n(\theta)}_\mu(k, p, q) = iee^{(i/2)(p+k)\wedge q}\bar{N}(p')\Big[&\gamma_\mu F^{p,n}_1(q^2) \\ &+ \frac{\kappa^{p,n}}{2m}\sigma_{\mu\nu}q_\nu F^{p,n}_2(q^2)\Big]N(p). \end{aligned} \tag{62}$$

This shows that the electromagnetic form factors are modified

$$F^{p,n(\theta)}_{1,2}(k, p, q) = e^{(i/2)(p+k)\wedge q}F^{p,n}_{1,2}(q^2). \tag{63}$$

They are dependent on the total incident four-momentum $p_\mu + k_\mu$ and the recoil four-momentum q_ν of the scattering particles and the noncommutativity parameter $\theta^{\mu\nu}$.

In the noncommutative case, the spatial distributions of charge and magnetic moment of the nucleon (Sachs form factors), $G^{(\theta)}_{E_{p,n}}$ and $G^{(\theta)}_{M_{p,n}}$, respectively, are

$$G^{(\theta)}_{E_{p,n}}(k, p, q) = e^{(i/2)(p+k)\wedge q}\left(F^{p,n}_1(q^2) - \frac{q^2}{4m^2_{p,n}}F^{p,n}_2(q^2)\right), \tag{64}$$

$$G^{(\theta)}_{M_{p,n}}(k, p, q) = e^{(i/2)(p+k)\wedge q}(F^{p,n}_1(q^2) + F^{p,n}_2(q^2)). \tag{65}$$

They are now functions of k, p, q and direction dependent, unlike the commutative case. Possible experimental signals due to these effects should be explored further.

V. CONCLUSIONS

In this paper, we have constrained the spacetime noncommutativity parameter using the CPT figure of merit measured in the $K^0 - \bar{K}^0$ system and $g - 2$ difference of positive and negative muons following a phenomenological approach. We get the noncommutativity length scale bounds $\lesssim 10^{-32}$ m and $\lesssim 10^{-20}$ m and energy bounds $\gtrsim$

10^{16} GeV and $\gtrsim 10^3$ GeV, respectively, from the $K^0 - \bar{K}^0$ system and $g - 2$ of $\mu^+ - \mu^-$.

We have also shown that the electromagnetic form factors and thus the distributions of charge and magnetization of the nucleon are modified. The form factors are no longer analytic functions of q^2 as in the commutative case. They are direction dependent in the noncommutative case, indicating the possibility of Lorentz violation. It may lead to interesting experimental signals.

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