Chiral symmetry breaking in the truncated Coulomb gauge: Nonconfining power law potentials

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In this paper we study the breaking of chiral symmetry with nonconfining powerlike potentials. The region of allowed exponents is identified and, after the previous study of confining (positive exponent) potentials, we now specialize in shorter range nonconfining potentials, with a negative exponent. These nonconfining potentials are close to the Coulomb potential, and they are also relevant as corrections to the linear confinement, and as models for the quark potential at the deconfinement transition. The mass-gap equation is constructed and solved, and the quark mass, the chiral angle, and the quark energy are calculated analytically with an exponent expansion in the neighborhood of the Coulomb potential. It is demonstrated that chiral symmetry breaking occurs, but only the chiral invariant false vacuum and a second nontrivial vacuum exist. Moreover chiral symmetry breaking is led by the ultraviolet part of the potential, with no infrared enhancement of the quark mass. Thus the breaking of chiral symmetry driven by nonconfining potentials differs from the one lead by confining potentials.

I. INTRODUCTION

The problem of the spontaneous chiral symmetry breaking (χSB) is one of the QCD cornerstones. While χSB driven by a confining potential has been studied in detail, we now explore the effect of shorter range nonconfining power-law potentials in χSB.

Continuing the well-defined mathematical problem of studying χSB driven by power-law potentials [1] $V(r) = \pm K_0^{1+\beta} r^\beta$, we now specialize in negative exponents $\beta < 0$. For different values of β, we find numerically chirally noninvariant possible vacua of the theory, solutions to the corresponding mass-gap equation. We exploit the potential model for QCD [1], whose origins can be traced back to QCD in the truncated Coulomb gauge and which is proven to be successful in studies of the low-energy phenomena in QCD [2]. This class of models can be indicated as Nambu and Jona-Lasinio type models [3] with the current-current quark interaction and the corresponding form factor coming from the bilocal gluonic correlator. A standard approximation in such models is to neglect the retardation and to approximate the gluonic correlator by an instantaneous potential of a certain form.

There are two different motivations to study χSB driven by a class of nonconfining potentials, the corrections to χSB due to nonconfining potentials, and χSB at the deconfinement transition. Notice that the quark-antiquark static potential computed in lattice QCD has clearly two distinct components, the confining potential (linearlike) and the shorter range Coulomb-like potential. In particular the shorter Coulomb-like range part of the quark-antiquark static potential may be more complicated than a pure $\frac{-\alpha}{r}$ Coulomb potential. There are at least two different Coulomb potentials, the perturbative Coulomb which includes logarithmic corrections and the Luscher Coulomb due to the confining string fluctuations [4]. Moreover the

matching of these two Coulomb potentials and of the long range linear potential may also be described by a nonconfining potential. Thus potentials in the neighborhood of a pure Coulomb potential, with $\beta \neq 1$, may also be phenomenologically relevant. Moreover, at the deconfinement, say at the deconfinement phase transition of QCD [5], or at a large number of flavors as in walking technicolor [6], the confining potential vanishes and it is then relevant to study the impact of potentials, shorter range than the confining potential, in the spontaneous χSB. To illustrate that different potentials may be relevant, in Fig. 1 we show different finite temperature T free energies computed in lattice QCD [7–11] by Kakzmarek *et al.* Notice that here we only address chiral symmetry breaking at zero T, nevertheless the present work may also be used as a
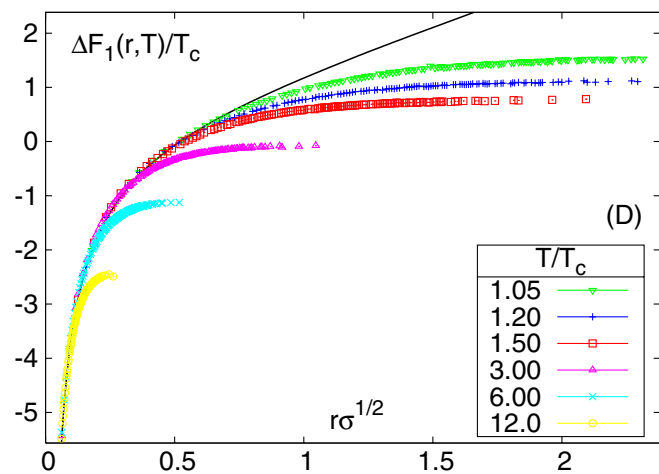
FIG. 1 (color online). We show examples of nonconfining potentials, in particular, the $T > T_c$ lattice QCD data for the free energy F_1, thanks to [7–11] Olaf Kaczmarek *et al.* The solid line represents the $T = 0$ static quark-antiquark potential.

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starting point for the study of χSB at finite T or at finite μ, where confinement is lost.

The problem of instability of the chirally invariant vacuum for powerlike confining potentials, i. e. for positive exponents, has already been studied thoroughly in the middle of 1980s by the Orsay group [12–15] and such an instability was proved for the range $0 \leq \beta < 3$. For numerical studies, the harmonic oscillator type potential $\beta = 2$ was chosen by these authors, as well as by the Lisbon group [16–18] and the Dubna [19] group, and a set of results for the hadronic properties was obtained in the framework of the given model. Adler and Davis, the Lisbon group, the Zagreb group, and the Rayleigh group also studied the linear potential many years ago [20–23].

Recently Bicudo and Nefediev [1] solved the mass-gap equations for $0 \leq \beta \leq 2$ explicitly, and demonstrated that the chiral angle, the vacuum energy density, and the chiral condensate are smooth slow functions of the form of the confining potential, so that the results obtained for the potential of a given form—the linear confinement being the most justified and phenomenologically successful choice [4,20–23]—have a universal nature for any quark-quark kernels of such a type. Following the set of recent publications devoted to possible multiple solutions for the chirally noninvariant vacuum in QCD [24] (see also [25] where a similar conclusion was made in a different approach), Bicudo and Nefediev also addressed the question of replica existence for various power laws r^β, and found that for the whole range of allowed powers $0 \leq \beta \leq 2$, replica solutions do exist similarly to the case of $\beta = 2$ studied in detail in [12].

This prompted us to extend the Coulomb potential with other negative exponents, and to study in detail its contribution to χSB. In Sec. II we extend the mass-gap equation for power-law potentials with negative exponents. In Sec. III we study analytically the mass-gap equation. In Sec. IV we solve algebraically the mass-gap equation in the chiral limit. Since the Coulomb potential leads to ultraviolet divergences, in Sec. V we briefly address the renormalization of the quark mass and of the quark energy. In Sec. VI we conclude.

II. THE MASS-GAP EQUATION FOR THE POWER-LAW POTENTIALS

We now derive the mass-gap equation for the power-law potentials. This extends the derivation of Bicudo and Nefediev [1] for positive exponents β.

The chiral model which we use for our studies is given by the Hamiltonian with the current-current interaction parametrized by the bilocal correlator $K_{\mu\nu}^{ab}$

$$H = \int d^3\mathbf{x}\, \bar{\psi}(\mathbf{x}, t)(-i\gamma \cdot \nabla)\psi(\mathbf{x}, t)$$
$$+ \frac{1}{2} \int d^3\mathbf{x} d^3\mathbf{y} J_\mu^a(\mathbf{x}, t) K_{\mu\nu}^{ab}(\mathbf{x} - \mathbf{y}) J_\nu^b(\mathbf{y}, t), \quad (1)$$

where the quark current is $J_\mu^a(\mathbf{x}, t) = \bar{\psi}(\mathbf{x}, t)\gamma_\mu \frac{\lambda^a}{2} \psi(\mathbf{x}, t)$ and the gluonic correlator is approximated by a density-density potential

$$K_{\mu\nu}^{ab}(\mathbf{x} - \mathbf{y}) = g_{\mu 0} g_{\nu 0} \frac{\delta^{ab}}{\frac{4}{3}} V_0(|\mathbf{x} - \mathbf{y}|), \quad (2)$$

where the denominator $\frac{4}{3}$ normalizes the Gell-Mann matrix contribution to the mass-gap equation. We now study the class of potentials with

$$V_0(|\mathbf{r}|) = -\alpha K_0^{\beta+1}|\mathbf{r}|^\beta, \quad (3)$$

where the only dimensional parameter of the model is the strength of the confining force K_0.

Previously Bicudo and Nefediev [1] studied the power-law confining potentials, using the notation $K_0^{\alpha+1}|\mathbf{x}|^\alpha$, adequate for the study of positive exponents, in particular, for the class of confining potentials $0 \leq \alpha \leq 2$, including the linear and the harmonic oscillator potentials. The results of Bicudo and Nefediev are show in Fig. 2.

Here we specialize in shorter range potentials, with a negative exponent. Since the Coulomb potential is frequently noted $-\alpha/r$, with dimensionless α, we now adopt the notation of Eq. (3), where the exponent is denoted β. Another difference to the previous work of Bicudo and Nefediev [1] is the sign of the potential. While the confining potentials are attractive, it is necessary to have a negative sign for the nonconfining potential for the potential to be attractive, and for the existence of bound states in the spectrum.
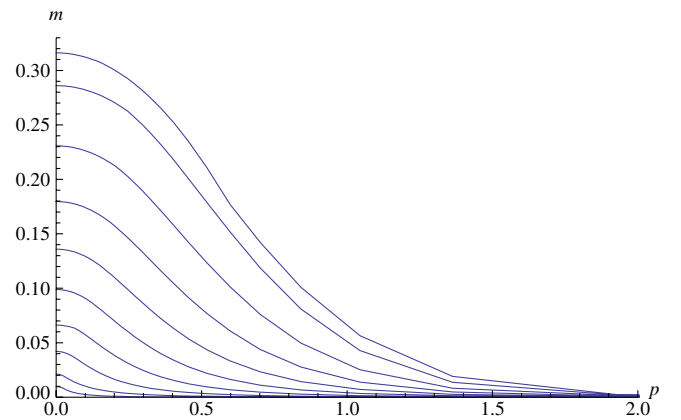

FIG. 2 (color online). We show the dynamical masses $m(p)$, solutions of the mass-gap equation with zero bare mass $m_0 = 0$, computed as in Bicudo and Nefediev [1]. The different solutions correspond to the positive exponents (from left and bottom to right and top) $\beta = 0.1, 0.3, 0.5, 0.7, 0.9, 1.1, 1.3, 1.5, 1.7$, and 1.9. Notice that the dynamical mass vanishes in the limit of $\beta \rightarrow 0$. The masses are a function of the momentum p and dimensionless units of $K_0 = 1$ are used. Also notice that the confining potentials enhance the masses in the IR.

The relativistic invariant Dirac-Feynman propagators [15] can be decomposed in the quark and antiquark Bethe-Goldstone propagators [18], used in the formalism of nonrelativistic quark models,

$$
\begin{aligned}
S_{\text{Dirac}}(k_0, \mathbf{k}) &= \frac{i}{\not k - m + i\epsilon} \\
&= \frac{i}{k_0 - E(k) + i\epsilon} \sum_s u_s u_s^\dagger \beta \\
&\quad - \frac{i}{-k_0 - E(k) + i\epsilon} \sum_s v_s v_s^\dagger \beta,
\end{aligned}
$$

$$
\begin{aligned}
u_s(\mathbf{k}) &= \left[\sqrt{\frac{1+S}{2}} + \sqrt{\frac{1-S}{2}} \hat{k} \cdot \boldsymbol{\alpha} \right] u_s(0), \\
v_s(\mathbf{k}) &= \left[\sqrt{\frac{1+S}{2}} - \sqrt{\frac{1-S}{2}} \hat{k} \cdot \boldsymbol{\alpha} \right] v_s(0), \\
&= -i\alpha_2 u_s^*(\mathbf{k}),
\end{aligned}
\tag{4}
$$

where the vector $\boldsymbol{\alpha}$ is a Dirac matrix, not to be confused with the parameter α of the potential. β is also a Dirac matrix, and not the exponent of the potential as it is in the remainder of this paper. We also define

$$
\begin{aligned}
S(k) &= \sin\varphi(k) = m(k)D(k) \\
C(k) &= \cos\varphi(k) = kD(k) \\
D(k) &= \frac{1}{\sqrt{k^2 + m(k)^2}},
\end{aligned}
\tag{5}
$$

where φ is the chiral angle and finally $m(k)$ is the aim of this paper, the constituent quark mass.

In the noncondensed vacuum, the quark mass is equal to the current quark mass m_0. In the physical vacuum, the constituent quark mass $m(k)$ is a variational function which is determined by the mass-gap equation. We illustrate in Fig. 2 examples of solutions, for the positive exponents $\beta \geq 0$.

There are three equivalent methods to derive the mass-gap equation for the true and stable vacuum, where constituent quarks acquire the constituent mass [26]. One method consists in assuming a quark-antiquark $^3P_0$ condensed vacuum, and in minimizing the vacuum energy density. A second method consists in rotating the quark and antiquark fields with a Bogoliubov-Valatin canonical transformation to diagonalize the terms in the Hamiltonian with two quark or antiquark second quantized fields. A third method consists in solving the Schwinger-Dyson equations for the propagators. Any of these methods leads to the same mass-gap equation and quark dispersion relation. Here we replace the propagator of Eq. (4) in the Schwinger-Dyson equation

$$
\begin{aligned}
0 = u_s^\dagger(k)\Big\{ &k\hat{k} \cdot \boldsymbol{\alpha} + m_0 \beta - \int \frac{dw'}{2\pi} \frac{d^3\mathbf{k}'}{(2\pi)^3} i\tilde{V}(k - k') \\
&\times \sum_{s'} \left[\frac{u(k')_{s'} u^\dagger(k')_{s'}}{w' - E(k') + i\epsilon} - \frac{v(k')_{s'} v^\dagger(k')_{s'}}{-w' - E(k') + i\epsilon} \right] \Big\} v_{s''}(k)
\end{aligned}
$$

$$
\begin{aligned}
E(k) = u_s^\dagger(k)\Big\{ &k\hat{k} \cdot \boldsymbol{\alpha} + m_0 \beta - \int \frac{dw'}{2\pi} \frac{d^3\mathbf{k}'}{(2\pi)^3} i\tilde{V}(k - k') \\
&\times \sum_{s'} \left[\frac{u(k')_{s'} u^\dagger(k')_{s'}}{w' - E(k') + i\epsilon} - \frac{v(k')_{s'} v^\dagger(k')_{s'}}{-w' - E(k') + i\epsilon} \right] \Big\} u_s(k),
\end{aligned}
\tag{6}
$$

where again β is a Dirac matrix, and not the exponent of the potential as it is in the remainder of this paper. The mass-gap equation and the quark energy are finally

$$
0 = +S(p)B(p) - C(p)A(p), \tag{7}
$$

$$
E(p) = +S(p)A(p) + C(p)B(p), \tag{8}
$$

where

$$
\begin{aligned}
A(p) &= m_c + \frac{1}{2} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \tilde{V}(\mathbf{p} - \mathbf{k}) S(k) \\
B(p) &= p + \frac{1}{2} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \tilde{V}(\mathbf{p} - \mathbf{k})(\hat{p} \cdot \hat{k}) C(k).
\end{aligned}
\tag{9}
$$

Using the Bogoliubov-Valatin transformation, the Hamiltonian (1) splits into the vacuum energy, the quadratic and the quartic parts in terms of the quark creation/annihilation operators. For the vacuum energy density one has

$$
\begin{aligned}
\mathcal{E}_{\text{vac}}[\varphi] &= \frac{1}{\text{Vol}} \langle 0| TH[\varphi] |0\rangle \\
&= -\frac{g}{2} \int \frac{d^3\mathbf{p}}{(2\pi)^3} ([A(p) + m_0]S(p) \\
&\quad + [B(p) + p]C(p)),
\end{aligned}
\tag{10}
$$

where Vol is the three-dimensional volume; the degeneracy factor g counts the number of independent quark degrees of freedom

$$
g = (2s + 1)N_C N_f, \tag{11}
$$

with $s = \frac{1}{2}$ being the quark spin; the number of colors N_C is put to three, and the number of light flavors N_f is two. Thus we find that $g = 12$.

To arrive at the mass-gap equation, we compute the Fourier transform of the potential. To regularize the infrared (IR) part of the potential, a modified version of the potential (3) [12] is convenient for $\beta \geq -1$,

$$
V_0(\mathbf{r}) = -\alpha K_0^{\beta+1} |\mathbf{r}|^\beta e^{-m|\mathbf{r}|}, \tag{12}
$$

where m plays the role of the regulator for the infrared behavior of the interaction, but the limit $m \to 0$ is understood. We get for the Fourier transform

$$\tilde{V}(\mathbf{k}) = -\alpha \int d^3\mathbf{r} K_0{}^{\beta+1} |\mathbf{r}|^\beta e^{-mr} e^{-i\mathbf{k}.\mathbf{r}}$$

$$= -\alpha \frac{4\pi}{k} \int_0^\infty dr K_0{}^{\beta+1} r^{\beta+1} e^{-mr} \sin kr$$

$$= -\alpha \frac{4\pi}{|k|} \frac{K_0{}^{\beta+1} \Gamma(\beta+2) \sin[(\beta+2) \arctan\frac{|k|}{m}]}{(k^2 + m^2)^{(\beta+2)/2}}$$

$$\rightarrow +\alpha \frac{4\pi K_0{}^{\beta+1} \Gamma(\beta+2) \sin\frac{\pi\beta}{2}}{|k|^{3+\beta}}, \qquad (13)$$

where the Fourier transform only exists for $\beta > -2$. For smaller exponents the Fourier transform is ultraviolet divergent.

For the generalized powerlike potential (12), the angular integrals necessary to compute the intermediate functions $A(p)$ and $B(p)$ are,

$$I_0 = \int_{-1}^1 d\omega \tilde{V}(k^2 + k'^2 - 2\omega kk')$$

$$= +\alpha 4\pi K_0{}^{\beta+1} \Gamma(\beta+2) \sin\frac{\pi\beta}{2}$$

$$\times \left\{ -\frac{2}{(1+\beta)} \frac{1}{2kk'} \left[\frac{1}{|k+k'|^{1+\beta}} - \frac{1}{|k-k'|^{1+\beta}} \right] \right\},$$

$$I_1 = \int_{-1}^1 d\omega \tilde{V}(k^2 + k'^2 - 2\omega kk')\omega$$

$$= +\alpha 4\pi K_0{}^{\beta+1} \Gamma(\beta+2) \sin\frac{\pi\beta}{2}$$

$$\times \left\{ \frac{2}{(1+\beta)} \frac{1}{2kk'} \left[\frac{1}{|k+k'|^{1+\beta}} + \frac{1}{|k-k'|^{1+\beta}} \right] \right.$$

$$+ \frac{4}{(-1+\beta)(1+\beta)} \frac{1}{(2kk')^2}$$

$$\times \left. \left[\frac{1}{|k+k'|^{-1+\beta}} - \frac{1}{|k-k'|^{-1+\beta}} \right] \right\}, \qquad (14)$$

and we get

$$\mathcal{C} = +\alpha \frac{K_0{}^{\beta+1}}{2\pi} \Gamma(1+\beta) \sin\frac{\pi\beta}{2}$$

$$A(p) = m_0 + \mathcal{C} \int_{-\infty}^\infty k^2 dk \left\{ \frac{1}{pk} \left[\frac{1}{|p-k|^{1+\beta}} \right] \right\} m(k) D(k)$$

$$B(p) = p + \mathcal{C} \int_{-\infty}^\infty k^2 dk \left\{ \frac{1}{pk} \left[+ \frac{1}{|p-k|^{1+\beta}} \right.\right.$$

$$+ \frac{1}{p^2 k^2} \left[- \frac{1}{|p-k|^{-1+\beta}} \right] \right\} k D(k), \qquad (15)$$

where, for the sake of convenience, we continued the integral to the negative values of k assuming that $m(p)$ is an even function, as it would happen in $1+1$ dimensions.

Consequently, the mass-gap Eq. (7) takes the form of a nonlinear integral equation for the constituent mass $m(p)$

$$m(p) = m_0 + \int_{-\infty}^\infty dk \mathcal{C} \frac{1}{p^3} D(k) \left\{ + \left[\frac{pk}{|p-k|^{1+\beta}} \right] \right.$$

$$\times [pm(k) - km(p)]$$

$$- \frac{1}{(-1+\beta)} \left[\frac{1}{|p-k|^{-1+\beta}} \right] km(p) \right\} \qquad (16)$$

and this is the main object of our studies.

III. ANALYTICAL PROPERTIES OF THE MASS-GAP EQUATION

We now analyze dimensionally the mass-gap equation, and study possible infrared (IR) and ultraviolet (UV) divergences.

In Sec. II the possible exponents β are already limited to $\beta > -2$, since in Eq. (13) the Fourier transform of the potential does not exist for smaller exponents. Now we address in a dimensional analysis the stability of a possible nontrivial vacuum. Let us assume that a solution $m(k)$ exists, minimizing the vacuum energy. Then we arbitrarily rescale the solution

$$m(k) \rightarrow m(\kappa k) \qquad (17)$$

and, if the vacuum energy does not have an absolute minimum for $\kappa = 1$, the vacuum is unstable and thus our assumption was wrong. We may now simply evaluate the vacuum energy as a function of the dimensionless factor κ. We get from Eq. (10)

$$\mathcal{E}_{\text{vac}}(\kappa) = c_1 \kappa^4 + c_2 m_0 \kappa^3 + c_3 K_0{}^{\beta+1} \kappa^{-\beta+3}, \qquad (18)$$

the vacuum energy density, with a dimension of the fourth power of momentum. In Eq. (18), the c_i are constants, equal to the different integrals in the vacuum energy density (10). Thus, in the chiral limit which is the one mattering here, the kinetic energy density scales like κ^4. We can show that c_1 is positive, and this prevents the vacuum from being UV unstable, providing the potential term has a smaller scaling power than the kinetic energy density. Thus for $\beta > -1$ there may be a solution. The nicer case is the one of the linear potential where the $\mathcal{E}_{\text{vac}}(\kappa)$ has a perfect Mexican hat shape. For $\beta < -1$, the potential always wins the kinetic term, moreover for an attractive potential we can show that the constant c_3 is negative, and thus the vacuum is unstable. For the Coulomb case $\beta = -1$ both terms scale equally and there is either a trivial solution $m(k) = 0$ if the $c_3 < c_1$ or the vacuum is unstable if $c_1 > c_3$. Thus we show that there may be a stable and nontrivial $m(k) > 0$ solution to the mass-gap equation only for $\beta > -1$. Since the present paper is specialized in negative exponents, we are interested in solving the mass-gap equation for $-1 < \beta < 0$.

In the present case of a negative exponent β, we now show that the gap equation (7) is IR finite. Notice that in the mass-gap equation any possible IR divergence may only occur in the two denominators with a power of $|p - k|$. The second denominator $\frac{1}{|p-k|^{-1+\beta}}$ is clearly IR finite since the exponent $1 - \beta > 1$. The first denominator $\frac{1}{|p-k|^{1+\beta}}$ tends to an IR divergence when $\beta \to 0$, however this divergence is canceled by the numerator $pm(k) - km(p)$. Thus the mass-gap equation (7) for negative exponents is quite different from the mass-gap with positive exponents, where an exact cancellation of the IR divergences of these two different terms would occur, but nevertheless would be technically harder to implement in the mass-gap equation.

In the present case of IR finiteness, we now focus in the UV sector of the equation. In what concerns the UV limit, each separate term in the integrand of the mass-gap equation (7) may be UV divergent in the limit of $\beta \to -1$, when the integrand momentum k tends to $\pm\infty$. To study whether the UV divergences cancel, it is convenient to perform momenta expansions in the integrand. There are two different expansions of interest, one where the momentum in the integrals is much larger than the external momentum, corresponding to the limit of $|k| \gg |p|$, where the mass is not limited. A second possible limit is the one of $k, p \gg m$ when we are interested in large external momenta, and we assume that the mass is limited. We start by expanding the integrand in the limit when $\frac{k}{p} \to 0$.

In order to utilize as much as possible the cancellations of UV divergences of the different terms in the mass-gap equation, we not only sum all the terms but also return to a momentum integral from 0 to ∞. Then, if $I(k)$ is the integrand of the mass-gap equation (16), for the expansion in $\frac{k}{p}$ of the mass independent terms we get

$$I(k) + I(-k) = \frac{\alpha}{2\pi} \Gamma(2 + \beta) \sin\frac{\pi\beta}{2} \frac{K_0^{1+\beta} D(k)}{|k|^{1+\beta}}$$
$$\times \left\{ \frac{2}{3} [(3 + \beta)m(p) - 3m(k)] \right.$$
$$+ \frac{1}{15} (3 + \beta)(2 + \beta)$$
$$\left. \times [(5 + \beta)m(p) - 5m(k)]\frac{p^2}{k^2} + o\left(\frac{p^4}{k^4}\right) \right\}.$$
(19)

Thus, at leading order in $\frac{k}{p}$, the mass-gap equation (16) can be rewritten as

$$\mathcal{C}' = \frac{\alpha K_0^{1+\beta}}{2\pi} \Gamma(2 + \beta) \sin\frac{\pi\beta}{2} \frac{2}{3}$$
$$m_0 = m(p) - \int_0^\infty dk\mathcal{C}' \frac{D(k)}{|k|^{1+\beta}} [(3 + \beta)m(p) - 3m(k)].$$
(20)

Equation (20) is UV divergent for $\beta \leq -1$ but it is indeed

UV finite for $-1 < \beta < 0$, and thus we can proceed with the search of its solution.

IV. ALGEBRAIC SOLUTION OF THE MASS-GAP EQUATION IN THE CHIRAL LIMIT

We now utilize algebraic methods to solve the mass-gap equation. We first set the mass-gap equation in a form close to an eigenvalue equation and show that the eigenvalues of this equation are real. The solutions of the mass-gap equation are the roots of the linearized eigenvalue equation, and this provides a fast convergence to the solution. We provide the solution in the limit where the UV contribution leads the mass-gap equation.

We consider the chiral limit of $m_0 \to 0$. We first address the mass-gap equation in three momentum dimensions. The three-dimensional mass-gap equation (7) can be rewritten as

$$0 = \left[\frac{1}{D(p)} \right] D(p)m(p)$$
$$+ \left[\frac{1}{2} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \tilde{V}(\mathbf{p} - \mathbf{k})(\hat{p} \cdot \hat{k}) \frac{kD(k)}{pD(p)} \right] D(p)m(p)$$
$$\times \left[-\frac{1}{2} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \tilde{V}(\mathbf{p} - \mathbf{k}) \right] D(k)m(k) \qquad (21)$$

and this is similar to a three-dimensional-like algebraic quasilinear equation with a symmetric matrix for a vector $D(k)m(k)$, since the integrand only depends on the distance $\mathbf{p} - \mathbf{k}$.

To solve the mass-gap equation, we start by fixing the denominator functions $D(k)$ inside the square brackets [ ] of Eq. (21), with our best initial guess. Then we apply the eigenvalue method to the resulting matrix equation. Clearly, the eigenvalues are real since the matrix is real and symmetric. A solution of the mass gap exists when the matrix has a root. In Eq. (21) we have three matrices. The first one is the identity and has positive eigenvalues growing with the mass function $m(p)$. The other two matrices have eigenvalues with little dependence on the mass function $m(k)$. If, in the limit of vanishing $m(k)$, there are one or more negative eigenvalues, then when we increase the mass $m(k)$, the first matrix increases and eventually it is able to cancel the negative eigenvalues of the second plus third matrices. In that case we find the desired roots, and we solve the mass-gap equation. The number of solutions is equal to the number of negative eigenvalues of the matrix computed in the massless limit.

To actually solve the mass-gap equation it is more convenient to solve the radial momentum version (one-dimensional) of the mass-gap equation. For the one-dimensional mass-gap we can start from Eq. (16), and rewrite it is a symmetric form,

$$0 = \left[\frac{1}{D(p)}\right]D(p)pm(p) + \int_{-\infty}^{\infty} dk \frac{\mathcal{C}}{p^3} k \frac{D(k)}{D(p)}$$
$$\times \left[+\frac{pk}{|p-k|^{1+\beta}} + \frac{1}{(-1+\beta)}\right.$$
$$\times \left.\frac{1}{|p-k|^{-1+\beta}}\right]D(p)pm(p)$$
$$+ \left[\int_{-\infty}^{\infty} dk\mathcal{C}\frac{-1}{|p-k|^{1+\beta}}\right]D(k)km(k), \qquad (22)$$

and again, in particular, we have two terms, one fully diagonal and another explicitly symmetric since it only depends on the diagonal distance $|p-k|$. Thus the matrix is Hermitian, and the eigenvalue equation now applies to a vector $D(k)km(k)$. Again, the number of solutions is equal to the number of negative eigenvalues of the matrix computed in the massless limit of $D(k) \to \frac{1}{k}$.

We now solve the mass-gap equation in the limit where it is led by the UV contribution. From the UV-led mass-gap Eq. (20) we get the matrix equation

$$0 = \left[1 - \int_0^{\infty} dk\mathcal{C}'\frac{D(k)}{|k|^{1+\beta}}(3+\beta)\right]m(p)$$
$$+ \left[\int_0^{\infty} dk\mathcal{C}'\frac{D(k)}{|k|^{1+\beta}}3\right]m(k). \qquad (23)$$

Let us consider that we discretize the momenta, for instance in a lattice of N points, where the correct solution is found in the limit $N \to \infty$. Then the mass-gap equation (23) is a $N \times N$ matrix equation. In this case the corresponding matrix is not symmetric, but actually we can find the eigenvalues exactly and show that they are real, since one of the matrices is a constant and the other is a projector.

In particular, the eigenvalue equation applies to the vectors $m(k)$. The first term in Eq. (23), proportional to $m(p)$, is constant and is thus proportional to the identity matrix. The term integrating in $m(k)$ does not depend on p and thus it is a projector on a constant vector $m_1(p) = $ const.

This implies that one eigenvector, say $\lambda_1(p)$, of the $N \times N$ matrix is constant,

$$\lambda_1 = \left\{1 - \int_0^{\infty} dk\mathcal{C}'\frac{D(k)}{|k|^{1+\beta}}(3+\beta)\right\}$$
$$+ \left\{\int_0^{\infty} dk\mathcal{C}'\frac{D(k)}{|k|^{1+\beta}}3\right\}$$
$$= \left\{1 - \int_0^{\infty} dk\mathcal{C}'\frac{D(k)}{|k|^{1+\beta}}\beta\right\}. \qquad (24)$$

And, since in the projector all the lines of the discretized matrix are identical, the matrix has $N - 1$ linear dependences and this implies that all the other $N - 1$ eigenvectors $\lambda_i(p)$, for $i > 1$, are canceled by the projector, and thus all their eigenvalues are identical, simply given by the

matrix proportional to the identity. The other eigenvalues $\lambda_i(p)$, $i > 1$ canceled by the projector are

$$\lambda_i = \left\{1 - \int_0^{\infty} dk\mathcal{C}'\frac{D(k)}{|k|^{1+\beta}}(3+\beta)\right\}. \qquad (25)$$

Now, notice that $\mathcal{C}' < 0$ if we consider an attractive, i.e. negative shorter range potential with $-\alpha < 0$, and with a negative $\beta \simeq -1$, and leading us to

$$\lambda_1 - 1 = -\int_0^{\infty} dk\mathcal{C}'\frac{D(k)}{|k|^{1+\beta}}\beta < 1,$$
$$\lambda_i - 1 = -\int_0^{\infty} dk\mathcal{C}'\frac{D(k)}{|k|^{1+\beta}}(3+\beta) > 1. \qquad (26)$$

Thus we can have one and only one root. The other eigenvalues are positive (and quite large if the integral is nearly UV divergent). Thus this differs from the confining potentials where a whole tower of replicas was found by Bicudo and Nefediev [1].

We now concentrate on the root to determine the constant eigenvector m that solves the mass-gap equation (20). It is convenient to rename the exponent to $\beta = -1 + \epsilon$, where we are interested in the exponent range of $0 < \epsilon < 1$. The mass-gap equation is then

$$0 = \lambda_1 = 1 - \int_0^{\infty} dk\mathcal{C}'\frac{1}{\sqrt{k^2 + m^2}|k|^{\epsilon}}$$
$$= 1 - \mathcal{C}'(-1+\epsilon)\frac{m^{-\epsilon}\Gamma(\frac{1-\epsilon}{2})\Gamma(\frac{\epsilon}{2})}{2\sqrt{\pi}}, \qquad (27)$$

and the solution is

$$m = \left[|\mathcal{C}'|(1-\epsilon)\frac{\Gamma(\frac{1-\epsilon}{2})\Gamma(\frac{\epsilon}{2})}{2\sqrt{\pi}}\right]^{1/\epsilon}$$
$$= K_0\left[\alpha\frac{\Gamma(1+\epsilon)}{2\pi}\sin\frac{\pi(1-\epsilon)}{2}\frac{1-\epsilon}{\frac{3}{2}}\frac{\Gamma(\frac{1-\epsilon}{2})\Gamma(\frac{\epsilon}{2})}{2\sqrt{\pi}}\right]^{1/\epsilon}$$
$$= K_0\left[\frac{\alpha}{3\pi\epsilon} - \alpha\frac{2+3\gamma+\psi(\frac{1}{2})}{6\pi} + o(\epsilon)\right]^{1/\epsilon}$$
$$= |K_0|\left(\frac{\alpha}{3\pi\epsilon}\right)^{1/\epsilon}e^{[-\alpha(2+3\gamma+\psi(1/2))/(6\pi)+o(\epsilon)]} \qquad (28)$$

and this diverges very fast in the limit of vanishing ϵ, but for finite ϵ it occurs that the negative β exponents indeed produce chiral symmetry breaking.

The solution of Eq. (28) for the dynamical quark mass is plotted in Fig. 3 as a function of ϵ. We find that a small but finite mass $m \simeq 0.05K_0$, almost independent of ϵ, in the range $0.15 < \epsilon < 0.9$. For $\epsilon \simeq 1$, corresponding to $\beta \simeq 0$ the mass vanishes. The mass explodes for $\epsilon \simeq 0$, close to the Coulomb potential.
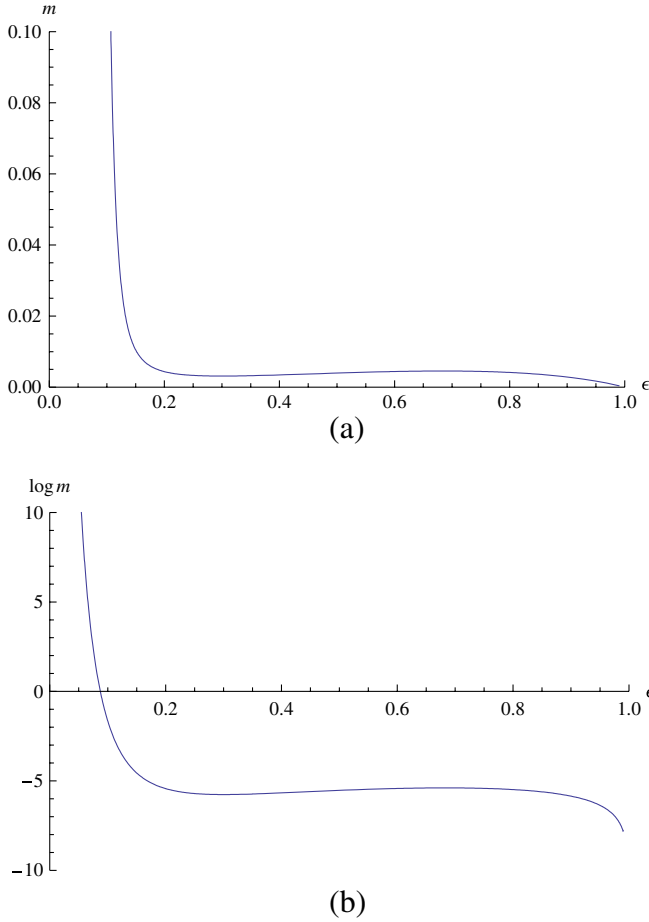
FIG. 3 (color online). We show (a) the quark mass m and (b) $\log(m)$ solution of the mass-gap equation in the chiral limit, plotted as a function of the exponent ϵ, in dimensionless units of $K_0 = 1$. We consider here $\alpha = \pi/12$ as in the Luscher term [4] computed in static lattice QCD potentials.

V. BRIEF RENORMALIZATION OF THE QUARK MASS AND ENERGY

Since the present framework of a power-law potential offers a possible regularization of the Coulomb potential divergences, which are expected to remain in any deconfined version of QCD (see Fig. 1), we also briefly sketch the renormalization of the quark energy and of the quark mass. Different methods to renormalize the Coulomb potential in the Coulomb gauge have been applied by Adler and Davis [20] and by Klabučar, Horvat, Kekez, and Palle [22,27,28] using the counterterm method and by Szczepaniak and Swanson [29] utilizing the similitude method of Glazek and Wilson [30]. For more detail, see these works. We now briefly outline how to renormalize the quark mass and energy using the counterterm method.

The quark mass is computed in Eq. (28); we now study the quark energy. It is first convenient to write the mass-gap equation (7) as

$$\frac{A(p)}{m(p)} = \frac{B(p)}{p}, \tag{29}$$

and then we get for the quark energy (8)

$$\begin{aligned} E(p) &= D(p)[m(p)A(p) + pB(p)] \\ &= \sqrt{m(p)^2 + p^2}\,\frac{A(p)}{m(p)} \\ &= \sqrt{m^2 + p^2}\,\frac{3}{1 - \epsilon}. \end{aligned} \tag{30}$$

This shows that, if the quark mass is divergent in the Coulomb limit of $\epsilon \to 0$, the quark energy is also divergent.

Let us now assume that the quark mass can be made small, since the light quarks u and v are quite close to the chiral limit. This corresponds to a different mass over momentum expansion from the one leading to Eq. (30). We now use $m \ll p, k$ and for simplicity we consider a constant $m(p) = m$. Including the current quark mass m_0, but with no counterterms yet, we get for the $A(p)$ and $B(p)$ integrals

$$\begin{aligned} \frac{A(p)}{m} &= \frac{m_0}{m} + \mathcal{C} \int_{-\infty}^{\infty} k^2 dk \left\{ \frac{2}{2pk} \left[\frac{1}{|p - k|^{1+\beta}} \right] \right\} \frac{1}{|k|} \\ &= \frac{m_0}{m} + \frac{\mathcal{C}}{p^\epsilon}\frac{2}{1 - \epsilon} = \frac{m_0}{m} + \alpha \left[\frac{1}{\pi\epsilon} + \frac{1 - \gamma - \log(\frac{p}{K_0})}{\pi} \right. \\ &\quad \left. + o(\epsilon) \right] \end{aligned}$$

$$\begin{aligned} \frac{B(p)}{p} &= 1 + \mathcal{C} \int_{-\infty}^{\infty} k^2 dk \left\{ \frac{2}{2pk} \left[+ \frac{1}{|p - k|^{1+\beta}} \right] \right. \\ &\quad \left. + \frac{4}{(-1 + \beta)}\frac{1}{(2pk)^2}\left[-\frac{1}{|p - k|^{-1+\beta}} \right] \right\} k\frac{1}{|k|}\frac{1}{p} \\ &= 1 + \frac{\mathcal{C}}{p^\epsilon}\frac{4}{(1 - \epsilon)(3 - \epsilon)} \\ &= 1 + \frac{2\alpha}{3}\left[\frac{1}{\pi\epsilon} + \frac{4 - 3\gamma - 3\log(\frac{p}{K_0})}{3\pi} + o(\epsilon) \right]. \tag{31} \end{aligned}$$

Thus there are two divergences, contributing in the mass-gap equation with $\frac{A(p)}{m(p)} - \frac{B(p)}{p}$ and in the quark energy with $\frac{A(p)}{m(p)} + \frac{B(p)}{p}$. To cancel these divergences, we renormalize the bare quark propagator, including counterterms in the inverse propagator. In our notation, as used in Eq. (6), we transform

$$k\hat{k} \cdot \boldsymbol{\alpha} + m_0\beta \to (1 - Z_2)k_0 + Z_2 k\hat{k} \cdot \boldsymbol{\alpha} + Z_m m_0\beta, \tag{32}$$

where β is a Dirac matrix, and not the exponent of the potential as it is in the remainder of this paper. In Eq. (32), Z_2 is the wave function renormalization factor and Z_m is the mass renormalization factor. Notice that the present framework, although relativistic, is not Lorentz invariant and thus we have to renormalize separately the temporal

and spatial components of the momentum in the propagator, splitting Z_2 into Z_0 and Z,

$$(1 - Z_2)k_0 + Z_2 k\hat{k} \cdot \boldsymbol{\alpha} \rightarrow (1 - Z_0)k_0 + Zk\hat{k} \cdot \boldsymbol{\alpha}. \quad (33)$$

Since we only need two counterterms to renormalize our two divergences, the simplest prescription consists in using $Z_0 = 1$. Then the equations for A and B become

$$\frac{A(p)}{m} = Z_m \frac{m_0}{m} + \alpha \left[\frac{1}{\pi\epsilon} + \frac{1 - \gamma - \log(\frac{p}{K_0})}{\pi} + o(\epsilon) \right]$$

$$\frac{B(p)}{p} = Z + \frac{2\alpha}{3} \left[\frac{1}{\pi\epsilon} + \frac{4 - 3\gamma - 3\log(\frac{p}{K_0})}{3\pi} + o(\epsilon) \right], \quad (34)$$

and thus the two divergences can be canceled separately. In any case we may choose, to leading order,

$$m = m_0, \quad (35)$$

since the mass divergence may be included in the renormalization factor Z_m. Assuming that the relativistic kinetic energy $E(p) = \sqrt{m^2 + p^2}$ is maintained, requiring $(A/m) = (B/p) = 1$, we get for the renormalization factors

$$Z_m = -\frac{\alpha}{\pi} \frac{1}{\epsilon} + 1 - \alpha \frac{1 - \gamma - \log(\frac{p}{K_0})}{\pi} + o(\epsilon)$$

$$Z = -\frac{2\alpha}{3\pi} \frac{1}{\epsilon} + 1 - \alpha \frac{8 - 6\gamma - 6\log(\frac{p}{K_0})}{9\pi} + o(\epsilon). \quad (36)$$

The renormalization factors in Eq. (36) are not constant, but it is standard to fix them at a given scale for the momentum p in the logarithms. Actually, the mass is not really a constant—this was just an approximation used to simplify the integrals in Eq. (15)—and the mass $m(p)$ runs logarithmically with the momentum p as is well known [20,22,27–30]. The proof that both m and E are finite is sufficient to briefly show that the renormalization of the Coulomb potential is possible with the regularization offered by the power-law potentials.

Notice that, once the renormalization of the UV divergences of the Coulomb potential has been achieved, we may add a UV finite potential to the Coulomb one. Then the current quark mass of the mass-gap equation due to the UV finite potential is the renormalized mass as discussed here. To simulate QCD with confinement, usually the potential added is the linear potential. To simulate QCD in a nonconfined phase, a power-law potential like the ones studied here may substitute the linear potential.

To illustrate an application of the renormalization, let us for instance consider a $T = 0$ quark-antiquark potential as in Fig. 1, fitted by a function

$$V(r) = V_0 - \frac{\alpha}{r} - \alpha\sqrt{\frac{K_0}{r}} + \sigma r, \quad (37)$$

where the $-\alpha\sqrt{\frac{K_0}{r}}$ potential improves a fit that would include the linear and Coulomb potentials only. The UV renormalization, as in Eq. (36), removes both the UV divergences and the finite contribution of the Coulomb potential (regularized with $-\alpha\frac{K_0^\epsilon}{r^{1-\epsilon}}$) to the mass-gap equation. The linear potential saturates when $T > T_c$ and here we do not include finite temperature effects which go beyond our aim. One is then left with a potential $-\alpha\sqrt{\frac{K_0}{r}}$, belonging to the class of potentials of Eq. (3). In the chiral limit, this leads to the same mass-gap equation of Eq. (16), resulting finally in the dynamical mass generation depicted in Fig. 3. Notice that, in this renormalized example, the relevant part of Fig. 3 is the central and finite plateau corresponding to $\epsilon = 0.5$, equivalent to $\beta = -0.5$.

VI. CONCLUSION

We study χSB driven by power-law potentials with a negative exponent $\beta < 0$. These potentials are nonconfining. We extend a previous study performed for confining potentials with a positive exponent [1]. In that study, chiral symmetry already vanishes when $\beta \rightarrow 0$; nevertheless, since the Coulomb potential is negative, it is natural to reverse the sign of the potential, and then χSB may occur for $\beta < 0$.

We work in momentum space and the existence of a Fourier transform is limited to $\beta > -2$. Then, with a dimensional analysis, we show that we may only have a stable χSB vacuum for $\beta \gtrsim -1$. Thus we study in detail the nonconfining potentials with $-1 < \beta < 0$, i.e. in the exponent range limited by the Coulomb potential and the logarithmic potential.

We find that χSB also occurs for the studied negative exponent power-law potentials generating dynamically a finite quark mass m.

Moreover we find qualitative differences to the mass generated with confining potentials. First, we find one and only one nontrivial solution of the mass-gap equation, whereas for confining potentials an infinite tower [1] of false, excited vacua, sometimes called replicas, is found. Also, the solution found has an approximately constant mass, as in Fig. 3, i.e. we find no IR enhancement of the quark mass, whereas the confining potentials studied previously [1] produce a significant IR enhancement of the quark mass, as in Fig. 2.

We also find an UV divergence of the dynamical mass m, in the Coulomb potential limit of the exponent $\beta \rightarrow -1$, and we outline how to renormalize the quark mass and energy, with the counterterm method.

This work may be a starting point for the study of χSB at the deconfinement transition, say with finite T, finite μ, or

large N_F. For instance, a potential of the form

$$V_0(r) = -\alpha\sqrt{\frac{K_0}{r}}, \qquad (38)$$

or with any other exponent $\beta \in]-1, 0[$, is not confining and yet it does produce chiral symmetry breaking. This suggests that it would be interesting to study the properties of the Goldstone bosons and of the remaining bosonic spectrum with such a potential.

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