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Noncommutative Landau problem in Podolsky's generalized electrodynamics

Marco Maceda* and Alfredo Macías†

Depto. de Física, Universidad Autónoma Metropolitana-Iztapalapa, A.P. 55-534, C.P. 09340, México D.F., México

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We discuss the problem of an electron in the presence of a magnetic field in Podolsky's generalized electrodynamics. The problem of the energy levels is then formulated using noncommutative coordinates, and its physical consequences are analyzed.

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I. INTRODUCTION

The idea of quantizing Minkowski space-time while preserving Lorentz invariance was put forward by H. S. Snyder [1,2] as a result of his investigations on a viable solution to the infinities arising in the calculations of quantum electrodynamics. The quantization of space-time in general and not only restricted to Minkowski space-time has, since then, been an active field of research leading to the concept of noncommutative geometry [3,4].

There has been an increased interest in past years on the formulation of quantum mechanics on noncommutative space [5–10]. The main motivation in studying such modification of quantum mechanics comes from the close relation observed between the form of both the Hamiltonian in noncommutative quantum mechanics and the classical Landau problem [6,9,10]. The Landau problem in noncommutative space has been studied as part of this line of research. As stated in [9], it concerns the description of a particle moving in the noncommutative plane in the presence of a constant magnetic field derived from a vector potential $\vec{A}$.

In this work we shall be interested in using for the Landau problem, a vector potential $\vec{A}$ obtained from Podolsky's generalized electrodynamics [11–13], which is a higher derivative theory.

Podolsky's generalized electrodynamics is interesting to consider because of two complementary aspects: first, it contains a natural length parameter and several phenomenological consequences can be worked out, in particular, upper bounds on the value of the length scale can be deduced from experimental data of scattering processes [14,15]. The second aspect concerns the fact that theories involving higher derivatives terms are usually related to the Cauchy problem (higher time derivatives) or to nonlocality issues (higher spatial derivatives). In the case of Podolsky's electrodynamics, the issue of nonlocality arises due to the presence of a tildon field, whose wave number depends on the natural length scale, but which actually never appears in real processes [14].

*mmac@xanum.uam.mx
†amac@xanum.uam.mx

The outline of the paper is as follows. In Sec. II we review some aspects of the Landau problem in both in commutative and noncommutative quantum mechanics. The inclusion of higher derivatives effects will be studied in Sec. III. Finally, the interplay between higher derivatives and noncommutativity will be examined in Sec. IV.

II. LANDAU PROBLEM

Consider a charged particle of electric charge q and mass m moving in a two-dimensional plane with coordinates x and y. In the presence of a constant magnetic field B perpendicular to the xy-plane, the Hamiltonian associated with this system is given by (see, for example, [16])

$$H = \frac{1}{2m}\left[\left(p_x + \frac{qB}{2}y\right)^2 + \left(p_y - \frac{qB}{2}x\right)^2 + p_z^2\right] = \frac{1}{2m}(p_x^2 + p_y^2) + \frac{1}{2}m\omega_L^2(x^2 + y^2) - \omega_L L_z + \frac{1}{2m}p_z^2, \tag{1}$$

where B is the magnetic field, $L_z = xp_y - yp_z$ is the z component of the angular momentum, and $\omega_L = qB/2m$ is Larmor's frequency.

The solution to the eigenvalue problem with the Hamiltonian H is based on the decomposition $H = H_{xy} + H_{L_z} + H_z$. The fact that H_{xy}, H_{L_z}, and H_z commute with each other allows one to look for a common set of eigenvectors so that the eigenvalues of H are given simply by the relation $E = E_{xy} + E_{L_z} + E_z$. In fact one has

$$E = (N + 1)\hbar\omega_L - n\hbar\omega_L + \frac{\hbar^2 k^2}{2m}, \tag{2}$$

where $N = (2n' + |n|)$, $n' = 0, 1, 2, \ldots$, n is the magnetic quantum number and k the wave number. The eigenfunctions in cylindrical coordinates are of the form

$$\psi_{n'nk}(\rho, \phi, z) = \rho^{|m|}{}_2F_1(-n', |n| + 1; \rho^2/d^2) \times e^{-\rho^2/2d^2} e^{in\phi} e^{ikz}, \tag{3}$$

where ${}_2F_1(a, b; c)$ is the confluent hypergeometric function [17], $\rho^2 := x^2 + y^2$, and $d := \sqrt{\hbar/m\omega_L}$ is the characteristic length of a harmonic oscillator of frequency ω_L.

The noncommutative Landau problem has been discussed previously in the literature [6,9,10], and it is based on the introduction of the commutation relation $[x, y]_\star = i\theta$ between spatial coordinates x and y. Here θ is a constant deformation parameter, and $\star$ means the Groenewold-Moyal (GM) product [18,19].

The Landau problem either in the noncommutative plane or in the noncommutative phase space can be solved explicitly using the so-called Bopp shifts. In our case the Bopp shifts are given by

$$x \rightarrow x - \frac{\theta}{2} p_y = x + \frac{i\hbar\theta}{2}\partial_y, \qquad y \rightarrow y + \frac{\theta}{2} p_x = y - \frac{i\hbar\theta}{2}\partial_x, \tag{4}$$

where now x and y are considered as commuting variables. Using these redefinitions the energy eigenvalues for the noncommutative Landau problem are found to be

$$E = (N+1)\hbar\omega'_L - n\hbar\omega'_L + \frac{\hbar^2 k^2}{2m}, \tag{5}$$

where

$$B' := \frac{B}{1 + \frac{qB}{4\hbar}\theta}, \qquad m' := \frac{m}{(1 + \frac{qB}{4\hbar}\theta)^2}, \qquad \omega'_L :== \omega_L\left(1 + \frac{qB}{4\hbar}\theta\right). \tag{6}$$

The role of the noncommutative parameter θ is reflected on the new effective values of the magnetic field, mass, and Larmor's frequency.

III. PODOLSKY'S GENERALIZED ELECTRODYNAMICS

The Lagrangian for the electromagnetic field in Podolsky's theory is given by the expression

$$\mathcal{L}_{\text{E.M.}} = -\frac{1}{8\pi}\left[\frac{1}{2}F_{\alpha\beta}F^{\alpha\beta} + a^2 F_{\alpha}{}^{\beta}{}_{,\beta}F^{\alpha\gamma}{}_{,\gamma}\right] \tag{7}$$

with a an arbitrary constant. The physical interpretation [11] of the term involving the derivatives of the field strength $F_{\alpha\beta}$ is closely related to the theory proposed by F. Bopp [20,21] and independently discovered by Landé and Thomas [22–24] as an attempt in the early 1940s to shed some light on the issue of infinite self-energy of the electron in quantum electrodynamics.

In the case where only a magnetic field is present, Eq. (7) simplifies to

$$\mathcal{L}_{\text{E.M.}} = -\frac{1}{8\pi}[\vec{H}^2 + (\vec{\nabla}\times\vec{H})^2], \tag{8}$$

where $\vec{H} = \vec{\nabla}\times\vec{A}$. In order to determine the potential $\vec{A}$ to be used in the following calculations, we shall assume that $\mathcal{L}_{\text{E.M.}}$ in Eq. (8) has the same functional form as in the classical Landau problem, namely

$$\mathcal{L}_{\text{E.M.}} = -\frac{1}{8\pi}[c_0 + c_1(x^2 + y^2)], \tag{9}$$

with c_0 and c_1 constants. Taking a time-independent vector potential

$$\vec{A} = \left(-\frac{1}{2}By, \frac{1}{2}Bx, a_0(x^2+y^2)\right), \tag{10}$$

we have that

$$\mathcal{L}_{\text{E.M.}} = -\frac{1}{8\pi}[B^2 + 4a_0^2(x^2+y^2) + 16a^2a_0^2]. \tag{11}$$

In these expressions a_0 is an arbitrary constant. We should stress that the form of the vector potential $\vec{A}$ in Eq. (10) is the simplest one in the sense that the contribution to the electromagnetic energy due to the higher derivative term is just a constant. Even though it is not the only possible choice, it is appealing because of the fact that it is to some extent a minimal modification of a vector potential producing a magnetic field along the z-axis.

A straightforward calculation shows that the Hamiltonian describing the Landau problem is given by

$$H = \frac{1}{2m}\left(p_x + \frac{qB}{2}y\right)^2 + \frac{1}{2m}\left(p_y - \frac{qB}{2}x\right)^2 + \frac{1}{2m}[p_z - qa_0(x^2+y^2)]^2 + \frac{1}{8\pi}[B^2 + 4a_0^2(x^2+y^2) + 16a^2a_0^2]. \tag{12}$$

Therefore, we are led to consider the splitting

$$H = H_{xyz} + H_{L_z} + \frac{1}{8\pi}[B^2 + 16a^2a_0^2] \tag{13}$$

with

$$H_{xyz} = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) + \frac{1}{2}\left(m\omega_L^2 + \frac{a_0^2}{\pi}\right)(x^2+y^2), - \frac{qa_0}{m}(x^2+y^2)p_z + \frac{q^2a_0^2}{2m}(x^2+y^2)^2, \qquad H_{L_z} = -\omega_L(xp_y - yp_x) = -\omega_L L_z. \tag{14}$$

It is not difficult to see that $[H_{xyz}, H_{L_z}] = 0$. In consequence, we can look for a common set of eigenfunctions for both Hamiltonians. For the eigenfunctions we shall use the Ansatz

$$\psi(x, y, z) = R(\rho)e^{in\phi}e^{ikz}. \tag{15}$$

The Schrödinger equation in this case takes the form

$$\left[\partial_\rho^2 + \frac{1}{\rho}\partial_\rho - \frac{n^2}{\rho^2} - \frac{m^2\tilde{\omega}_L^2}{\hbar^2}\rho^2 - \frac{q^2a_0^2}{\hbar^2}\rho^4\right]R = \epsilon R, \tag{16}$$

where

$$m\tilde{\omega}_L^2 := m\omega_L^2 + \frac{a_0^2}{\pi} - 2\frac{qa_0}{m}\hbar k,$$
$$\epsilon := -\frac{2m}{\hbar^2}\left(E_{xyz} - \frac{\hbar^2 k^2}{2m}\right). \tag{17}$$

The characteristic length for the harmonic oscillator in this case is given by $d_c := \sqrt{\hbar/m\tilde{\omega}_L}$. In terms of the variable $u := \rho/d_c$ and writing $R(u) = u^{|n|}H(u)$, we have then

$$[u^2\partial_u^2 + (2|n| + 1)u\partial_u - u^4 - \alpha u^6]H = \tilde{\epsilon}u^2 H, \tag{18}$$

where

$$\alpha := \frac{q^2 a_0^2}{\hbar^2} d_c^6 = \frac{q^2 a_0^2}{\hbar^2}\frac{\hbar^3}{m^3\tilde{\omega}_L^3},$$
$$\tilde{\epsilon} := \epsilon d_c^2 = -\frac{2}{\hbar\tilde{\omega}_L}\left(E_{xyz} - \frac{\hbar^2 k^2}{2m}\right). \tag{19}$$

Dimensional analysis shows that the parameter α has no units. Equation (18) describes an anharmonic quartic oscillator. Analytic solutions are not known for this system and only perturbation procedures are available in order to obtain the energy levels as a series expansion on the coupling constant [25–28]. However, in the weak coupling limit $|a_0| \ll 1$, we can consider only terms of first order in a_0. It follows directly from Eq. (16) that the energy levels will be given by

$$E = (N + 1)\hbar\tilde{\omega}_{L,k} - n\hbar\tilde{\omega}_{L,k} + \frac{\hbar^2 k^2}{2m}, \tag{20}$$

with

$$\tilde{\omega}_{L,k} = \omega_L\sqrt{1 - 2\frac{qa_0}{m^2\omega_L^2}\hbar k} \approx \omega_L - \frac{qa_0}{m^2\omega_L}\hbar k$$
$$= \omega_L\left(1 - \frac{qa_0}{\hbar}kd^4\right). \tag{21}$$

In obtaining the second line in Eq. (21), we have implicitly restricted the set of values of k such that the second term under the square root in that expression is small, namely $a_0\hbar k \ll qB^2/8$. The eigenfunctions can be obtained from Eq. (3) with the appropriate changes. We should notice that the contribution of the higher derivative term to the total energy in Eq. (13) is not necessarily small due to the presence of the parameter a.

IV. NONCOMMUTATIVE GENERALIZED ELECTRODYNAMICS CORRECTION

In this section we shall discuss a noncommutative model based on the Hamiltonian given by Eq. (12). The commutation relation to be considered is the same as in Section II, namely $[x, y]_\star = i\theta$ with θ a constant. The energy eigenvalue equation is in this case given by the expression

$$H \star \psi(x, y, z) = E\psi(x, y, z), \tag{22}$$

with H being the Hamiltonian of Eq. (12) but where we omit the constant contributions from now on.

After using the Bopp shifts of Eq. (4), we find to lowest order on the deformation parameter θ

$$H'_{xyz} = \frac{1}{2m'}(p_x^2 + p_y^2) + \frac{1}{2}\left(m'\omega_L'^2 + \frac{a_0^2}{\pi}\right)(x^2 + y^2)$$
$$+ \frac{1}{2m}p_z^2 - \frac{qa_0}{m}(x^2 + y^2)p_z + \frac{q^2 a_0^2}{2m}(x^2 + y^2)^2$$
$$+ \frac{q\theta}{m}[p_z - qa_0(x^2 + y^2)]L_z,$$
$$H'_{L_z} = -\left(\omega'_L + \frac{a_0^2}{2\pi}\theta\right)L_z. \tag{23}$$

It is interesting to note that there is a value

$$\theta_0 := -\frac{qB}{2}\frac{1}{\frac{q^2B^2}{8\hbar} + \frac{a_0^2}{2\pi}}, \tag{24}$$

for which the contribution from H'_{L_z} can be set to vanish. In the strong field limit $B \to \infty$, we have $|\theta_0| \sim \frac{4\hbar}{|q|B} \ll 1$. Furthermore we have in general that $[H'_{xyz}, H'_{L_z}] = 0$.

The common eigenfunctions for H'_{xyz} and H'_{L_z} can be found using the same Ansatz as the one used in Eq. (15). We have

$$\left[\partial_\rho^2 + \frac{1}{\rho}\partial_\rho - \frac{n^2}{\rho^2} - \frac{m'^2\hat{\omega}_L^2}{\hbar^2}\rho^2\right]R = \epsilon' R, \tag{25}$$

where

$$\hat{\omega}_L^2 := \omega_L'^2 - 2\frac{qa_0}{m'^2}(\hbar k + qn\theta),$$
$$\epsilon' := -\frac{2m'}{\hbar^2}\left(E'_{xyz} - \frac{\hbar^2k^2}{2m} - \frac{q\theta}{m}n\hbar k\right). \tag{26}$$

From this we obtain then

$$E = (N + 1)\hbar\hat{\omega}_{L,k,n} - n\hbar\hat{\omega}_{L,k,n} + \frac{\hbar^2k^2}{2m} + \frac{q\theta}{m}n\hbar k, \tag{27}$$

with

$$\hat{\omega}_{L,k,n} = \omega_L - \frac{qa_0}{m^2\omega_L}\hbar k + \omega_L\theta\left[\frac{qB}{4\hbar} - \frac{a_0}{B}\left(3k + \frac{4n}{B}\right)\right]. \tag{28}$$

We should notice that for the value

$$\theta'_0 := -\frac{1}{3}\frac{qB}{\hbar}\frac{\hbar^2}{m^2\omega_L^2} = -\frac{4}{3}\frac{\hbar}{qB} \tag{29}$$

of the deformation parameter, the terms proportional to k in the frequency $\hat{\omega}_{L,k,n}$ cancel out so that

$$\hat{\omega}_{L,k,n} = \omega_L\left[1 + \theta'_0\left(\frac{qB}{4\hbar} - \frac{4na_0}{B^2}\right)\right]. \tag{30}$$

This result is indeed due to the suitable combination in the Hamiltonian of the terms arising from the Bopp shifts together with the contributions of the A_z component of the vector potential in Eq. (10).

We should say some words about the expression in Eq. (10) for the vector potential $\vec{A}$, which has been used throughout this work. If we consider a general time-independent vector potential $\vec{A}$

$$\vec{A} = (A_x, A_y, A_z), \tag{31}$$

then

$$c_0 + c_1(x^2 + y^2) = (\partial_y A_z - \partial_z A_y)^2 + (\partial_x A_z - \partial_z A_x)^2(B + \partial_x A_y - \partial_y A_x)^2 + a^2(\partial^2_{xy} A_y + \partial^2_{xz} A_z - \Delta_{yz} A_x)^2 + a^2(\partial^2_{xy} A_x + \partial^2_{yz} A_z - \Delta_{xz} A_y)^2 + a^2(\partial^2_{xz} A_x + \partial^2_{yz} A_y - \Delta_{xy} A_z)^2 \tag{32}$$

gives the condition to obtain an electromagnetic Lagrangian $\mathcal{L}_{\mathrm{E.M.}}$ of the form given in Eq. (9). Here we have used the short notation $\Delta_{ij} := \partial_i^2 + \partial_j^2$.

Even though Eq. (10) is not the only possible solution of Eq. (32), it is appealing because of the following two facts: first, it is to some extent a minimal modification of a vector potential producing a magnetic field along the z-axis. Second, and perhaps more important, it produces terms such that the operators H'_{xyz} and H'_{L_z} commute and hence, it allows that common eigenfunctions can be found for both operators. It would be certainly more complicated and it is not clear at first sight if such a result can be obtained by including a z dependence on the components A_x, A_y, and A_z of the vector potential.

It remains the issue of the dependence of $\hat{\omega}_{L,k,n}$ on the magnetic quantum number n. Except for $n = 0$, the value of a_0 can be chosen in an appropriate way in order to have $\hat{\omega}_{L,k,n} = \omega_L$ for at least one value of n. However, for strong magnetic fields we recover the noncommutative frequency ω'_L, which is due to the particular form in which the magnetic field B along the z-axis enters into the additional terms associated with Podolsky's generalized electrodynamics.

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